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Invariant Convex Sets in Operator Lie Algebras

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In this paper we study the closed convex subsets of Lie algebras of bounded linear operators on a Hilbert space that are invariant under the corresponding group of unitary operators. We will give a family f_j of convex function such, that for each closed convex invariant set C there are real numbers c_j satisfying

$$C = \{X : (\forall j) f_j(X) \leq c_j\}.$$

1. Introduction

Let $\mathfrak{k}$ be a finite dimensional compact Lie algebra, $\mathfrak{t}$ a Cartan subalgebra and $K = \exp\mathfrak{k}$ a corresponding Lie group. Further let $\mathfrak{W} := N_K(\mathfrak{t})/Z_K(\mathfrak{t})$ denote the Weyl group and $p_\mathfrak{t}$ denote the projection onto $\mathfrak{t}$ that is orthogonal with respect to the Cartan-Killing form. Then we have for every closed convex $\mathrm{Ad}(K)$-invariant subset C of $\mathfrak{k}$:

(1) $p_\mathfrak{t}(C) = C \cap \mathfrak{t}$.
(2) $p_\mathfrak{t}(C^o) = C^o \cap \mathfrak{t} = \mathrm{int}_\mathfrak{t}(C \cap \mathfrak{t})$.
(3) $C = \overline{\mathrm{Ad}(K).(C \cap \mathfrak{t})}$.

One way to prove these assertions makes heavy use of the Kostant Convexity Theorem, which states that for every $X \in \mathfrak{t}$ we have

$$p_\mathfrak{t}(\mathrm{Ad}(K).X) = \mathrm{conv}(\mathfrak{W}.X).$$

In [5] a generalization of the Kostant Convexity Theorem for certain infinite dimensional Lie algebras was given. All but finitely many simple compact Lie algebras are isomorphic to classical matrix Lie algebras. So in [5] the corresponding Lie subalgebras of the algebra of bounded linear operators on a Hilbert space $\mathfrak{H}$ were studied. These were the Lie algebra $\mathfrak{u}(\mathfrak{H})$ of skew-hermitian operators on $\mathfrak{H}$, further

$$\mathfrak{uo}(I_c) = \{X \in \mathfrak{u}(\mathfrak{H}) : X^* I_c + I_c X = 0\},$$

where I_c is a conjugation, that is an antilinear real isometry on $\mathfrak{H}$ satisfying $I_c^2 = \mathbf{1}$, and

$$\mathfrak{usp}(I_a) = \{X \in \mathfrak{u}(\mathfrak{H}) : X^* I_a + I_a X = 0\},$$

where I_a is an anticonjugation, that is an antilinear real isometry on $\mathfrak{H}$ satisfying $I_a^2 = -\mathbf{1}$. The corresponding maximal unitary subgroup for $\mathfrak{u}(\mathfrak{H})$ is $\mathfrak{U}(\mathfrak{H})$, the group of unitary operators on $\mathfrak{H}$. For $\mathfrak{uo}(I_c)$ we get

$$\mathrm{UO}(I_c) := \{X \in \mathfrak{U}(\mathfrak{H}) : X^* I_c X = I_c\}$$

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and for $\mathfrak{usp}(I_a)$ we obtain

$$\mathrm{USp}(I_a) := \{X \in \mathfrak{U}(\mathfrak{H}) : X^* I_a X = I_a\}.$$

In the Lie algebra $\mathfrak{u} \in \{\mathfrak{u}(\mathfrak{H}), \mathfrak{uo}(I_c), \mathfrak{usp}(I_a)\}$ we define a *split Cartan subalgebra* $\mathfrak{t}$ to be a maximal abelian subalgebra that can be simultaneously diagonalized. Then $p_{\mathfrak{t}}$ denotes the projection onto $\mathfrak{t}$ which is given as the projection on the diagonal with respect to an orthonormal basis in which $\mathfrak{t}$ consists of diagonal operators. We define the Weyl group to be the group $\mathfrak{W} := N_{\mathfrak{U}}(\mathfrak{t})/Z_{\mathfrak{U}}(\mathfrak{t})$, where $\mathfrak{U}$ denotes the maximal unitary group corresponding to $\mathfrak{u}$. Then we have for every $X \in \mathfrak{t}$

$$\overline{p_{\mathfrak{t}}(\{U^* X U : U \in \mathfrak{U}\})} = \overline{\mathrm{conv}(\mathfrak{W}.X)}.$$

With this theorem we generalize the results for finite dimensional compact Lie algebras mentioned in the beginning to our infinite dimensional Lie algebras. We will also give a family $\{f_k : k \in K\}$ of invariant convex functions on $\mathfrak{u}$ such, that for each closed convex $\mathfrak{U}$-invariant subset C of $\mathfrak{u}$ there exist real numbers c_k with

$$C = \{X \in \mathfrak{u} : (\forall k \in K)\, f_k(X) \leq c_k\}.$$

We begin by recalling the main results from [5] in Section 2 and Section 3. In the infinite dimensional setting our generalized convexity theorem allows us to describe the set $\overline{p_{\mathfrak{t}}(\{U^* X U : U \in \mathfrak{U}\})}$ only if the skew-hermitian operator X is diagonalizable, which need not be the case.

Therefore we introduce in Section 4 the two families $\{L_k : k \in \mathbb{N}\}$ and $\{\aleph^+, \aleph^- : \aleph \in \mathfrak{M}\}$ of convex functionals that will help us to control closed convex $\mathfrak{U}$-invariant sets.

In Section 5 we will show that each operator $A \in \mathfrak{u}$ can be approximated by diagonalizable operators for which $\{L_k : k \in \mathbb{N}\}$ and $\{\aleph^+, \aleph^-, \aleph \in \mathfrak{M}\}$ have the same values.

In Section 6 we will use this method to describe the sets $p_{\mathfrak{t}}(\{U^* A U : U \in \mathfrak{U}\})$, where $A \in \mathfrak{u}$ is a not necessarily diagonalizable operator.

In Section 7 we collect the remaining tools necessary to finally show in Section 8 that for each closed convex $\mathfrak{U}$-invariant subsets C of $\mathfrak{u}$

(1) $p_{\mathfrak{t}}(C) = C \cap \mathfrak{t}$.
(2) $p_{\mathfrak{t}}(C^o) = C^o \cap \mathfrak{t} = \mathrm{int}_{\mathfrak{t}}(C \cap \mathfrak{t})$.
(3) $C = \overline{\mathrm{conv}\, \mathfrak{U}.(C \cap \mathfrak{t})}$.

This implies that each such set C can be reconstructed from its intersection with the Cartan subspace $\mathfrak{t}$.

In Section 8 we show that each closed convex $\mathfrak{U}$-invariant subset of $\mathfrak{h}$ can be described by using only $\mathfrak{F} := \{L_k : k \in \mathbb{N}\} \cup \{\aleph^+, \aleph^-, \aleph \in \mathfrak{M}\}$. We get

$$\overline{\mathrm{conv}\{U^* A U : U \in \mathfrak{U}\}} = \{B \in \mathfrak{u} : (\forall f \in \mathfrak{F})\, f(iB) \leq f(iA), f(-iB) \leq f(-iA)\}.$$

For arbitrary closed convex $\mathfrak{U}$-invariant sets it will be necessary to use generalized convex combinations of the elements of $\mathfrak{F}$.

2. The Weyl Group Orbits

In this section we recall the main results from [4] that describe the closed convex hull of the Weyl group orbit. This will be the basis of our further calculations.

Definition 2.1.

(1) Let J be an arbitrary infinite set. We define the Banach space

$$l^\infty(J) := \{(a_j)_{j\in J} \in \mathbb{R}^J : \sup_{j\in J} |a_j| < \infty\}$$

equipped with the norm $\|(a_j)_{j\in J}\| := \|(a_j)_{j\in J}\|_\infty := \sup_{j\in J} |a_j|$. For $A \subseteq l^\infty(J)$ we denote by $\overline{A}$ the closure of A with respect to the norm $\|.\|_\infty$.

(2) In $l^\infty(J)$ we have the closed subspace

$$c_0(J) := \{(a_j)_{j\in J} : (\forall \varepsilon > 0) \#\{a_j : |a_j| \geq \varepsilon\} < \infty\}.$$

(3) We denote by $\mathfrak{S}(J)$ the group of all bijections of the set J. Then $\mathfrak{S}(J)$ acts on $l^\infty(J)$ by permutation of the entries, that is for $\sigma \in \mathfrak{S}(J)$ and $a = (a_j)_{j\in J}$ we get $(\sigma.a)_j = a_{\sigma(j)}$, which is a right action.

(4) We write $\mathbb{Z}_2 := \{-1, 1\}$ for the group of units of the integers. The group $\mathbb{Z}_2^J$ acts on $l^\infty(J)$ by component wise multiplication. This way we get a right action of $\mathfrak{W}_2(J) := \mathbb{Z}_2^J \rtimes \mathfrak{S}(J)$ on $l^\infty(J)$.

(5) For $n \in \mathbb{N}$ we write $\mathfrak{S}(n) := \mathfrak{S}(\{1, \ldots, n\})$ and $\mathfrak{W}_2(n) := \mathfrak{W}_2(\{1, \ldots, n\})$.

Definition 2.2. For a given set K we write $\#K$ for the cardinality of K. Let $a \in l^\infty(J)$.

(1) For $k \in \mathbb{N}$ we define

$$L_k(a) := \sup \Big\{ \sum_{j\in E} a_j : E \subseteq J, \#E = k \Big\}.$$

(2) We write $\mathfrak{M}$ for the set of infinite cardinal numbers $\aleph$ satisfying $\aleph \leq \#J$. We write $\aleph_0 := \#\mathbb{N}$.

(3) For $\aleph \in \mathfrak{M}$ we write $\aleph + 1 := \min\{\aleph' \in \mathfrak{M} : \aleph' > \aleph\}$. Further we write

$$\mathfrak{M}^* := \{\aleph \in \mathfrak{M} : (\exists \aleph' \in \mathfrak{M})\, \aleph = \aleph' + 1\}.$$

(4) For $x \in \mathbb{R}$ we define

$$o_a(x) := \min_{U\in\mathcal{U}(x)} \#\{j \in J : a_j \in U\},$$

where $\mathcal{U}(x)$ is the set of all neighborhoods of x. We note that this minimum always exists, as $\mathfrak{M}$ is well ordered.

(5) For every $\aleph \in \mathfrak{M}$ the set $\{x \in \mathbb{R} : o_a(x) \geq \aleph\}$ is obviously closed and bounded, so it is compact. Therefore we define for $\aleph \in \mathfrak{M}$

$$\begin{aligned} \aleph^+(a) &:= \max\{x \in \mathbb{R} : o_a(x) \geq \aleph\} \\ \aleph^-(a) &:= \min\{x \in \mathbb{R} : o_a(x) \geq \aleph\}, \end{aligned}$$

the maximal and minimal cluster points of order at least $\aleph$. In particular we get $\aleph_0^+(a) = \limsup a$ and $\aleph_0^-(a) = \liminf a$.

Definition 2.3. For an $a = (a_j)_{j\in J} \in l^\infty(J)$ we define $\overline{a} := (\overline{a}_j)_{j\in J} \in l^\infty(J)$, $\underline{a} := (\underline{a}_j)_{j\in J} \in l^\infty(J)$ and $a' := (a'_j)_{j\in J} \in l^\infty(J)$ by

$$\begin{aligned} \overline{a}_j &:= \max\{a_j, \limsup a\} - \limsup a, \\ \underline{a}_j &:= \min\{a_j, \liminf a\} - \liminf a, \\ a'_j &:= a_j - \overline{a}_j - \underline{a}_j. \end{aligned}$$

Further we define $\widehat{|a|} := |a| - \overline{|a|}$.

Then $\overline{a}$, $\underline{a}$ and $\widehat{|a|}$ lie in $c_0(J)$, in particular they have countable support.

Lemma 2.4 ([4, Lemma 2.17.]). *Let $a \in l^\infty(J)$. Then we have for every $k \in \mathbb{N}$:*

(1) $L_k(a) = L_k(\overline{a}) + k\aleph_0^+(a)$;
(2) $L_k(-a) = L_k(-\underline{a}) - k\aleph_0^-(a)$.

We also have that

$$\lim_{k\to\infty} \tfrac{1}{k} L_k(a) = \aleph_0^+(a) \qquad\qquad \lim_{k\to\infty} -\tfrac{1}{k} L_k(-a) = \aleph_0^-(a).$$

and for $\aleph_0 < \aleph \notin \mathfrak{M}^*$ we have

$$\aleph^+(a) = \inf\{(\aleph')^+(a) : \aleph' < \aleph\} \qquad \aleph^-(a) = \sup\{(\aleph')^-(a) : \aleph' < \aleph\}.$$

Proposition 2.5 ([5, Proposition 2.6]). *Let $a \in c_0(J)$. Then $b \in \overline{\mathrm{conv}(\mathfrak{S}(J)a)}$ if and only if for all $k \in \mathbb{N}$ we have:*

(1) $L_k(b) \le L_k(a)$;
(2) $L_k(-b) \le L_k(-a)$.

Proposition 2.6 ([5, Proposition 2.8]). *Let $a \in c_0(J)$. Then for $b \in l^\infty(J)$ the following are equivalent:*

(1) $b \in \overline{\mathrm{conv}(\mathfrak{W}_2(J)a)}$
(2) $|b| \in \overline{\mathrm{conv}(\mathfrak{S}(J)|a|)}$.
(3) $L_k(|b|) \le L_k(|a|)$ *for all* $k \in \mathbb{N}$.

Theorem 2.7. *Let $a \in l^\infty(J)$. Then*

$$\begin{aligned} \overline{\mathrm{conv}(\mathfrak{S}(J).a)} = &= \overline{\mathrm{conv}(\mathfrak{S}(J).\underline{a})} + \overline{\mathrm{conv}(\mathfrak{S}(J).a')} + \overline{\mathrm{conv}(\mathfrak{S}(J).\overline{a})} \\ &= \overline{\mathrm{conv}(\mathfrak{S}(J).\underline{a})} + \left([\aleph_0^-(a), \aleph_0^+(a)]^J \cap \bigcap_{\aleph\in\mathfrak{M}^*} \mathcal{O}_a(\aleph) \right) + \overline{\mathrm{conv}(\mathfrak{S}(J).\overline{a})} \\ &= \{b \in l^\infty(J) : (\forall k \in \mathbb{N})\, L_k(b) \le L_k(a),\ L_k(-b) \le L_k(-a), \\ &\qquad (\forall \aleph \in \mathfrak{M}^*)\, \aleph^+(b) \le \aleph^+(a),\ \aleph^-(b) \ge \aleph^-(a)\}. \end{aligned}$$

$$\begin{aligned} \overline{\mathrm{conv}(\mathfrak{W}_2(J)a)} &= \overline{\mathrm{conv}(\mathfrak{W}_2(J)\overline{|a|})} + \overline{\mathrm{conv}(\mathfrak{W}_2(J)\widehat{|a|})} \\ &= \overline{\mathrm{conv}(\mathfrak{W}_2(J)\overline{|a|})} + \left([-\aleph_0^+(|a|), \aleph_0^+(|a|)]^J \cap \bigcap_{\aleph\in\mathfrak{M}^*} \mathcal{O}^+_{|a|}(\aleph) \right) \\ &= \{b \in l^\infty(J) : (\forall k \in \mathbb{N})\, L_k(|b|) \le L_k(|a|),\ (\forall \aleph \in \mathfrak{M}^*)\, \aleph^+(|b|) \le \aleph^+(|a|)\}. \end{aligned}$$

Proof. This is [5, Theorem 5.1, Lemma 5.3, Theorem 6.3 and Lemma 6.4]. □

3. The Classical Infinite Dimensional Lie Algebras

The main result of [4] was a generalization of the Kostant Convexity Theorem for compact Lie algebras.

Theorem 3.1 (The Kostant Convexity Theorem for Compact Lie Algebras). *Let $\mathfrak{k}$ be a compact Lie algebra and K a corresponding Lie group. Further let $\mathfrak{t}$ denote a Cartan subalgebra of $\mathfrak{k}$, $\mathfrak{W} := N_K(\mathfrak{t})/Z_K(\mathfrak{t})$ the Weyl group and $p\colon \mathfrak{k} \to \mathfrak{t}$ the projection onto $\mathfrak{t}$ that is orthogonal with respect to the Cartan-Killing form. Then for every $X \in \mathfrak{t}$*

$$p(\mathrm{Ad}(K).X) = \mathrm{conv}(\mathfrak{W}.X).$$

Proof. The original proof can be found in [3], or a very short and recent proof in [7]. $\square$

In this section we want to introduce the infinite dimensional setting to which we want to generalize this theorem. A compact Lie algebra $\mathfrak{k}$ is more or less a direct sum of compact real forms of simple complex Lie algebras. These, with finitely many exceptions, are classical matrix algebras. Therefore we will use the straightforward generalization of these matrix algebras to infinite dimensions and look at their hermitian real forms. We will also investigate the Cartan subspaces and the Weyl group. We write $\mathfrak{H}$ for a Hilbert space and $B(\mathfrak{H})$ resp. $\mathfrak{gl}(\mathfrak{H})$ for the Lie algebra of bounded linear operators on $\mathfrak{H}$.

Definition 3.2. We denote by $\mathbf{1}$ the identity map or the corresponding matrix.

(1) An anti-linear operator I_c on $\mathfrak{H}$ is called a *conjugation*, if $\langle I_c.v, I_c.w\rangle = \overline{\langle v, w\rangle}$ for all $v, w \in \mathfrak{H}$ and $I_c^2 = \mathbf{1}$.

(2) An anti-linear operator I_a on $\mathfrak{H}$ is called an *anticonjugation*, if $\langle I_a.v, I_a.w\rangle = \overline{\langle v, w\rangle}$ for all $v, w \in \mathfrak{H}$ and $I_a^2 = -\mathbf{1}$.

For an infinite set J we write $2J := J \,\dot\cup\, (-J)$ and $2J + 1 := J \,\dot\cup\, \{0\} \,\dot\cup\, -J$, where $-J$ is a identical copy of J, but disjoint with it, and $j \mapsto -j$ is a bijection from J to $-J$.

Lemma 3.3 ([1, Appendix I]). (1) *Let I_c be a conjugation on $\mathfrak{H} = l^2(J)$, where J is not necessarily infinite. Then there exists an orthonormal basis $\{e_j : j \in J\}$ of $\mathfrak{H}$ such that $I_c(e_j) = e_j$ for all $\{j \in J\}$, called an I_c basis of type zero.*

For an infinite set J we also have:

(a) *There exists an orthonormal basis $\{e_j : j \in 2K\}$ with $I_c(e_j) = e_{-j}$ for all $j \in 2K$. This basis is called an I_c-basis of type one.*

(b) *There exists an orthonormal basis $\{e_j : j \in 2K + 1\}$ such that $I_c(e_j) = e_{-j}$ for all $j \in 2K + 1$. This basis is called an I_c-basis of type two.*

In both cases K is an infinite set with $\#K = \#J$.

(2) *Let I_a be an anticonjugation on $\mathfrak{H}$. Then there exists an orthonormal basis $\{e_j : j \in 2K\}$ such that*

$$I_a(e_j) = \begin{cases} -e_{-j} & j \in J \\ e_{-j} & j \in -J. \end{cases}$$

Such a basis is called an I_a-basis.

For a given conjugation I_c we define

$$\mathfrak{o}(I_c) := \{X \in B(\mathfrak{H}) : X^* I_c + I_c X = 0\},$$

where X^* denotes the adjoint operator to X. For a given anticonjugation I_a we define

$$\mathfrak{sp}(I_a) := \{X \in B(\mathfrak{H}) : X^* I_a + I_a X = 0\}.$$

We call the Lie algebras $\mathfrak{gl}(\mathfrak{H})$, $\mathfrak{o}(I_c)$ and $\mathfrak{sp}(I_a)$ the *classical (infinite dimensional) Lie algebras.*

For a classical Lie algebra $\mathfrak{g}$ we obtain an infinite dimensional analog to the finite dimensional compact real forms by intersecting $\mathfrak{g}$ with the algebra $\mathfrak{u}(\mathfrak{H})$ of skew-hermitian operators on $\mathfrak{H}$. However we have for the set $\mathfrak{h}(\mathfrak{H})$ of hermitian operators on $\mathfrak{H}$ that $\mathfrak{h}(\mathfrak{H}) = i\mathfrak{u}(\mathfrak{H})$. The generalized Kostant Convexity theorem for $\mathfrak{g} \cap \mathfrak{u}(\mathfrak{H})$ is equivalent to the one for $\mathfrak{g} \cap \mathfrak{h}(\mathfrak{H})$. Since the notation in the latter case will be easier, these will be the objects we will be looking at. We therefore define the *hermitian real forms*

$$\mathfrak{h}(\mathfrak{H}), \qquad \mathfrak{ho}(I_c) := \mathfrak{o}(I_c) \cap \mathfrak{h}(\mathfrak{H}), \qquad \mathfrak{hsp}(I_a) := \mathfrak{sp}(I_a) \cap \mathfrak{h}(\mathfrak{H}).$$

We write $\mathfrak{h}(J)$ for the set of hermitian $J \times J$ matrices corresponding to a hermitian operator on the Hilbert space $l^2(J)$. In an I_c-basis of type one we have

$$\mathfrak{ho}(I_c) = \mathfrak{ho}(2J) := \{A \in \mathfrak{h}(2J) : A^t R + RA = 0\},$$

where A^t denotes the matrix transposed to A and R is the operator given by the $2J \times 2J$-matrix

$$R = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}.$$

In an I_c-basis of type two we have

$$\mathfrak{ho}(I_c) = \mathfrak{ho}(2J+1) := \{A \in \mathfrak{h}(2J+1) : A^t R' + R'A = 0\},$$

where R' is the operator given by the $2J+1 \times 2J+1$-matrix

$$R' = \begin{pmatrix} & & \mathbf{1} \\ & 1 & \\ \mathbf{1} & & \end{pmatrix}.$$

Finally in an I_a-basis we have

$$\mathfrak{hsp}(I_a) = \mathfrak{hsp}(J) := \{A \in \mathfrak{h}(2J) : A^t Q + QA = 0\},$$

where Q is the operator corresponding to the $2J \times 2J$-matrix

$$Q = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}.$$

Now we need an analog for the maximal compact group K. We define the corresponding *unitary groups*. We write $\mathfrak{U}(\mathfrak{H})$ for the group of all unitary operators on $\mathfrak{H}$, further

$$\begin{aligned} \mathrm{UO}(I_c) &:= \{U \in \mathfrak{U}(\mathfrak{H}) : U^* I_c U = I_c\} \\ \mathrm{USp}(I_a) &:= \{U \in \mathfrak{U}(\mathfrak{H}) : U^* I_a U = I_a\}. \end{aligned}$$

Now we introduce an analog to the Cartan subalgebras in finite dimensional Lie algebras.

Definition 3.4. Let $\mathfrak{h} \in \{\mathfrak{h}(\mathfrak{H}), \mathfrak{ho}(I_c), \mathfrak{hsp}(I_a)\}$. A $*$-invariant subspace $\mathfrak{t}$ of $\mathfrak{h}$ is called a (*splitting*) *Cartan subspace* if

(1) The subspace $\mathfrak{t}$ is a maximal abelian Lie algebra in $\mathfrak{h}$.
(2) There exists an orthonormal basis in which each element of $\mathfrak{t}$ is diagonal.

The condition that $\mathfrak{t}$ is simultaneously diagonalizable may seem a little artificial at first. We will see later why it was necessary.

Lemma 3.5 ([5, Lemma 10.1]). (1) *In $\mathfrak{h}(\mathfrak{H})$ let $\mathfrak{t}$ denote the algebra of diagonal operators with respect to some orthonormal basis. Then $\mathfrak{t}$ is a Cartan subspace of $\mathfrak{h}(\mathfrak{H})$ and every Cartan subalgebra of $\mathfrak{h}(\mathfrak{H})$ is conjugate to $\mathfrak{t}$ under $\mathfrak{U}(\mathfrak{H})$.*

(2) *In $\mathfrak{ho}(I_c)$ let $\mathfrak{t}_1$ denote the algebra of operators diagonal with respect to some given I_c-basis of type one and $\mathfrak{t}_2$ denote the algebra of operators diagonal with respect to some given I_c-basis of type two. Then $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are Cartan subspaces of $\mathfrak{ho}(I_c)$ that are not conjugate under* $\mathrm{UO}(I_c)$ *and every other Cartan subspace of $\mathfrak{o}(I_c)$ is conjugate to one of them under* $\mathrm{UO}(I_c)$.

(3) *In $\mathfrak{hsp}(I_a)$ let $\mathfrak{t}$ denote the algebra of operators that are diagonal with respect to a given I_a-basis. Then $\mathfrak{t}$ is a Cartan subspace of $\mathfrak{sp}(I_a)$ and every Cartan subspace of $\mathfrak{sp}(I_a)$ is conjugate to $\mathfrak{t}$ under* $\mathrm{USp}(I_a)$.

Lemma 3.6. *Let $\mathfrak{h} \in \{\mathfrak{h}(\mathfrak{H}), \mathfrak{ho}(I_c), \mathfrak{hsp}(I_a)\}$ and let $X \in \mathfrak{h}$ be diagonalizable. Then X is contained in a Cartan subspace.*

If $\mathfrak{h} = \mathfrak{ho}(I_c)$ then X is contained in a Cartan subspace of each conjugacy class if and only if X has an infinite dimensional kernel.

Proof. This follows from [5, Lemma 8.3 and Lemma 9.3]. $\square$

As we have seen, there exists for every Cartan subspace $\mathfrak{t}$ of $\mathfrak{h} \in \{\mathfrak{h}(\mathfrak{H}), \mathfrak{ho}(I_c), \mathfrak{hsp}(I_a)\}$ an up to permutation unique orthonormal basis, such that $\mathfrak{t}$ consists of the diagonal operators with respect to that basis. We will denote by $p_\mathfrak{t}$ the projection on the diagonal with respect to this basis, which will give us a projection $p_\mathfrak{t} \colon \mathfrak{h} \to \mathfrak{t}$.

In the space $\mathfrak{h}(J)$ of hermitian $J \times J$ matrices we look at the *Cartan subspace* $\mathfrak{t} = \mathfrak{d}(J)$ of diagonal matrices. This space equipped with the operator norm is a Banach space canonically isomorphic to the Banach space $l^\infty(J)$. We denote by

$$p : \mathfrak{h}(J) \to l^\infty(J)$$

the projection on the diagonal and by $\mathrm{diag}(a)$ the diagonal matrix with diagonal a, where $a \in l^\infty(J)$. In this case we have $p_\mathfrak{t} = p$.

In $\mathfrak{hsp}(J)$ we have the maximal abelian subalgebra

$$\mathfrak{t} = \mathfrak{dsp}(J) := \left\{ \mathrm{diag}_\mathfrak{d}(a) := \begin{pmatrix} \mathrm{diag}(a) & \\ & -\mathrm{diag}(a) \end{pmatrix} : a \in l^\infty(J) \right\}.$$

The Lie algebra $\mathfrak{dsp}(J)$ equipped with the operator norm is canonically isomorphic to $l^\infty(J)$. We have the projection

$$p_\mathfrak{d} : B(l^2(2J)) \to l^\infty(J); \ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \frac{1}{2}(p(A) - p(D))$$

where $p : B(l^2(J)) \to l^\infty(J)$ is as above. Then $p_{\mathfrak{d}}(\mathrm{diag}_{\mathfrak{d}}(a)) = a$ for all $a \in l^\infty(J)$. In this case we have $p_{\mathfrak{t}} = \mathrm{diag}_{\mathfrak{d}} \circ p_{\mathfrak{d}}$.

We get one conjugacy class of Cartan subspaces of $\mathfrak{ho}(I_c)$ by looking at the Cartan subspace

$$\mathfrak{t} = \mathfrak{do}(2J) := \{\mathrm{diag}_{\mathfrak{d}}(a) : a \in l^\infty(J)\}.$$

in $\mathfrak{ho}(2J)$. The Lie algebra $\mathfrak{do}(2J)$ equipped with the operator norm is canonically isomorphic to $l^\infty(J)$. We have again that $p_{\mathfrak{t}} = \mathrm{diag}_{\mathfrak{d}} \circ p_{\mathfrak{d}}$.

To obtain the other conjugacy class of Cartan subspaces, we define

$$\mathrm{diag}'_{\mathfrak{d}}(a) : l^\infty(J) \to \mathfrak{o}(2J+1),\ a \mapsto \begin{pmatrix} \mathrm{diag}(a) & & \\ & 0 & \\ & & -\mathrm{diag}(a) \end{pmatrix}$$

and consider

$$\mathfrak{t} := \mathfrak{do}(2J+1) := \{\mathrm{diag}'_{\mathfrak{d}}(a) : a \in l^\infty(J)\}.$$

in $\mathfrak{uo}(2J+1)$. The Lie algebra $\mathfrak{do}(2J+1)$ equipped with the operator norm is canonically isomorphic to $l^\infty(J)$. We define

$$p'_{\mathfrak{d}} : B(l^2(2J+1)) \to l^\infty\ ;\ \begin{pmatrix} A & v & B \\ u^t & e & w^t \\ C & y & D \end{pmatrix} \mapsto \frac{1}{2}(p(A) - p(D)).$$

Then $p'_{\mathfrak{d}} \circ \mathrm{diag}'_{\mathfrak{d}}(a) = a$ for all $a \in l^\infty(J)$. We get $p_{\mathfrak{t}} = \mathrm{diag}'_{\mathfrak{d}} \circ p'_{\mathfrak{d}}$.

Definition 3.7. Let $\mathfrak{h}$ denote a hermitian real form, $\mathfrak{t} \subseteq \mathfrak{h}$ a Cartan subspace and $\mathfrak{U}$ the corresponding maximal unitary group. Then we define the *Weyl group* to be the group

$$\mathfrak{W} := N_{\mathfrak{U}}(\mathfrak{t})/Z_{\mathfrak{U}}(\mathfrak{t}),$$

where $N_{\mathfrak{U}}(\mathfrak{t})$ is the normalizer and $Z_{\mathfrak{U}}(\mathfrak{t})$ is the centralizer of $\mathfrak{t}$ in $\mathfrak{U}$.

Lemma 3.8 ([5, Lemma 10.3]). (1) *For $\mathfrak{h} = \mathfrak{h}(\mathfrak{H})$ we have $\mathfrak{W} = \mathfrak{S}(J)$.*
(2) *For $\mathfrak{h} \in \{\mathfrak{ho}(I_c), \mathfrak{hsp}(I_a)\}$ we have $\mathfrak{W} = \mathfrak{W}_2(J)$.*

In both cases $\mathfrak{W}$ acts on $\mathfrak{t}$ canonically.

It is interesting to note that in $\mathfrak{ho}(I_c)$ the Weyl group is the same for both conjugacy classes of Cartan subspaces. Now we are able to formulate the main result of [5].

Theorem 3.9. *Let $\mathfrak{h} \in \{\mathfrak{h}(\mathfrak{H}), \mathfrak{ho}(I_c), \mathfrak{hsp}(I_a)\}$ and $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{h}$. Further let $p_{\mathfrak{t}} \colon \mathfrak{h} \to \mathfrak{t}$ denote the projection onto $\mathfrak{t}$, $\mathfrak{U}$ the corresponding maximal unitary group and $\mathfrak{W}$ the Weyl group. Then*

$$\overline{p_{\mathfrak{t}}(\mathfrak{U}.X)} = \overline{\mathrm{conv}(\mathfrak{W}.X)}.$$

for every $X \in \mathfrak{t}$.

Now we want to study the closed convex subsets C of a classical Lie algebra $\mathfrak{g}$ that are invariant under the respective unitary group. In particular we will show that $p_{\mathfrak{t}}(C) = C \cap \mathfrak{t}$ for every closed convex $\mathfrak{U}$-invariant subset of $\mathfrak{h}$ and we will show that the map $C \mapsto C \cap \mathfrak{t}$ is a bijection between the $\mathfrak{U}$-invariant closed convex subset of $\mathfrak{h}$ and the $\mathfrak{W}$-invariant closed convex subsets of $\mathfrak{t}$.

4. Invariant Convex Functions

In this section we introduce the convex invariant functions L_k and $\pm\aleph^{\pm}$ and show that they are an extension of the functions L_k and $\pm\aleph^{\pm}$, that were defined on $l^\infty(J)$ resp. a Cartan subspace, to all of $\mathfrak{h}(\mathfrak{H})$. These functions will be an important tool to describe closed convex invariant subsets of $\mathfrak{h}(\mathfrak{H})$. We write $\sigma(A)$ for the spectrum of the operator A.

Lemma 4.1. *Let $\mathfrak{c}$ be a C^*-algebra and $\pi\colon B(\mathfrak{H}) \to \mathfrak{c}$ be a Lie algebra-$*$-homomorphism, that is $\pi(X^*) = \pi(X)^*$ for all $X \in B(\mathfrak{H})$. Further we assume that $\pi(\mathbf{1}) = K\mathbf{1}_\mathfrak{c}$ for some positive $K \in \mathbb{R}$. Then*

$$\sup \sigma(\pi(A)) = \|\pi(A + \lambda\mathbf{1})\| - \lambda K = \|\pi(A) + \lambda K\mathbf{1}_\mathfrak{c}\| - \lambda K$$

for all $A \in \mathfrak{h}(\mathfrak{H})$ and $\lambda > \frac{1}{K}\|\pi(A)\|$.

Proof. We note that with $A \in \mathfrak{h}(\mathfrak{H})$ we also have $A + \nu\mathbf{1} \in \mathfrak{h}(\mathfrak{H})$ for all $\nu \in \mathbb{R}$. Therefore $\pi(A + \nu\mathbf{1}) = \pi(A) + \nu K\mathbf{1}_\mathfrak{c}$ is hermitian in $\mathfrak{c}$. Further we recall that for each hermitian operator A holds $\|A\| = r(A)$, where $r(A)$ denotes the spectral radius.

If $\lambda > \frac{1}{K}\|\pi(A)\|$ we have that $\pi(A + \lambda\mathbf{1}) = \pi(A) + \lambda K\mathbf{1}_\mathfrak{c}$ has nonnegative spectrum and therefore

$$\begin{aligned}\|\pi(A + \lambda\mathbf{1})\| &= \sup \sigma(\pi(A + \lambda\mathbf{1})) = \sup \sigma(\pi(A) + \lambda K\mathbf{1}_\mathfrak{c}) \\ &= \sup \sigma(\pi(A)) + \lambda K.\end{aligned}$$

□

For $k \in \mathbb{N}$ the map $\pi_k\colon B(\mathfrak{H}) \to B(\Lambda^k(\mathfrak{H}))$ given by

$$\pi_k(X)(v_1 \wedge \ldots \wedge v_k) := (X.v_1) \wedge v_2 \wedge \ldots \wedge v_k + \ldots + v_1 \wedge \ldots \wedge v_{k-1} \wedge (X.v_k)$$

is a Lie algebra-$*$-homomorphism. In particular $\pi_k(\mathbf{1}) = k\mathbf{1}$.

For any $\aleph \in \mathfrak{M}$ we define the two sided $*$-invariant ideal $R_\aleph$ of $B(\mathfrak{H})$

$$R_\aleph := \overline{\{X \in B(\mathfrak{H}) : \dim(X.\mathfrak{H}) < \aleph\}},$$

where dim is the Hilbert space dimension. The Banach space $B(\mathfrak{H})/R_\aleph$ equipped with the norm

$$\|X + R_\aleph\|_\aleph := \inf_{Y \in R_\aleph} \|X - Y\|$$

is a C^* algebra and

$$\pi_\aleph : B(\mathfrak{H}) \to B(\mathfrak{H})/R_\aleph \ ; \ X \mapsto X + R_\aleph$$

is a Lie algebra-$*$-homomorphism with $\pi_\aleph(\mathbf{1}) = \mathbf{1}$.

Definition 4.2. For $A \in \mathfrak{h}(\mathfrak{H})$ we define the functions

$$\begin{aligned}L_k(A) &:= \sup \sigma(\pi_k(A)) \\ \aleph^+(A) &:= \sup \sigma(\pi_\aleph(A)) \\ \aleph^-(A) &:= -\aleph^+(-A) \inf \sigma(\pi_\aleph(A))\end{aligned}$$

for all $k \in \mathbb{N}$ and $\aleph \in \mathfrak{M}$.

There is another way to obtain the functions $\aleph^+$ and $\aleph^-$. Let $A \in \mathfrak{h}(\mathfrak{H})$. We recall that for every hermitian operator $A \in B(\mathfrak{H})$ there exists an operator-valued spectral measure P on the spectrum $\sigma(A)$ of A such that

$$A = \int_{\sigma(A)} x dP(x).$$

We can extend P to a spectral measure on all of $\mathbb{R}$ by setting $P(M) = 0$ for all measurable sets $M \subseteq \mathbb{R} \backslash \sigma(A)$. Then we can define for every $x \in \mathbb{R}$

$$o_A(x) := \min_{U \in \mathcal{U}(x)} \dim(P(U).\mathfrak{H}),$$

where $\mathcal{U}(x)$ is the set of all open neighborhoods of x. Obviously the set $\{x \in \mathbb{R} : o_A(x) \geq \aleph\}$ is closed and bounded, therefore compact.

Lemma 4.3. *If $A = \operatorname{diag}(a)$ for $a \in l^\infty(J)$ with respect to the orthonormal basis $\{e_j : j \in J\}$, then we have $o_A(x) = o_a(x)$.*

Proof. If $A = \operatorname{diag}(a)$, then $\sigma(A) = \overline{\{a_j : j \in J\}}$ and for $U \subseteq \mathbb{R}$ we have $P(U).\mathfrak{H} = \overline{\operatorname{span}\{e_j : a_j \in U\}}$. Therefore

$$\dim(P(U).\mathfrak{H}) = \dim(\overline{\operatorname{span}\{e_j : a_j \in U\}}) = \#\{j : a_j \in U\}.$$

Now the assertion follows from the definition of $o_a(x)$. □

Lemma 4.4. *For $A \in \mathfrak{h}(\mathfrak{H})$ and $\aleph \in \mathfrak{M}$ we have*

$$\begin{aligned} \aleph^+(A) &= \max\{x \in \mathbb{R} : o_A(x) \geq \aleph\} \\ \aleph^-(A) &= \min\{x \in \mathbb{R} : o_A(x) \geq \aleph\}. \end{aligned}$$

Proof. We write $s := \max\{x \in \mathbb{R} : o_A(x) \geq \aleph\}$. For $\varepsilon > 0$ we have that $\dim(P([s+\varepsilon, \|A\|]).\mathfrak{H}) < \aleph$, otherwise we could use a bisection argument to find some $x \in [s+\varepsilon, \|A\|]$ satisfying $o_A(x) \geq \aleph$. Therefore $\int_{s+\varepsilon}^{\infty} (x+\lambda) dP(x) \in R_\aleph$ and we get for $\lambda \geq \max\{\|\pi_\aleph(A)\|_\aleph, \|A\|\}$

$$\begin{aligned} \aleph^+(A) &= \|A + \lambda \mathbf{1} + R_\aleph\|_\aleph - \lambda \leq \left\| A + \lambda \mathbf{1} - \int_{s+\varepsilon}^{\infty} (x+\lambda) dP(x) \right\| - \lambda \\ &= \left\| \int_{-\lambda}^{s+\varepsilon} (x+\lambda) dP(x) \right\| - \lambda \leq s + \varepsilon + \lambda - \lambda = s + \varepsilon. \end{aligned}$$

Therefore $\aleph^+(A) \leq s$.

On the other hand we get for $\varepsilon > 0$ that $\dim(P([s-\varepsilon, \infty[).\mathfrak{H}) \geq \aleph$. We choose $\lambda \geq \max\{\|\pi_\aleph(A)\|_\aleph, \|A\|\}$. Then we get for every $v \in P([s-\varepsilon, \infty[).\mathfrak{H}$ that $\|(A+\lambda\mathbf{1})v\| \geq s - \varepsilon + \lambda$, therefore $\|A + \lambda \mathbf{1}\|_\aleph \geq s - \varepsilon + \lambda$ and

$$\aleph^+(A) = \|A + \lambda \mathbf{1}\|_\aleph - \lambda \geq s - \varepsilon.$$

This proves the first assertion. The second assertion follows from the first one. □

The set $\{x \in \mathbb{R} : o_A(x) \geq \aleph\}$ can be viewed as the essential spectrum of A of order $\aleph$. For $\aleph = \aleph_0$ we get the usual essential spectrum.

We define

$$M_A(\aleph) := [\aleph^-(A), \aleph^+(A)],$$

the smallest closed interval that contains the whole essential spectrum of A of order $\aleph$.

Now we want to see that L_k and $\aleph^\pm$ are in fact an extension of the functions we already know.

Lemma 4.5. *Let $A = \operatorname{diag}(a)$ for $a \in l^\infty(J)$ with respect to the orthonormal basis $\{e_j : j \in J\}$. Then we have*

$$\begin{array}{rcl} L_k(A) & = & L_k(a) \\ \aleph^+(A) & = & \aleph^+(a) \\ \aleph^-(A) & = & \aleph^-(a). \end{array}$$

Proof. If A is diagonal with respect to the orthonormal basis $\{e_j : j \in J\}$, then $\pi_k(A)$ is diagonal with respect to the orthonormal basis

$$\{e_{j_1} \wedge \ldots \wedge e_{j_k} : j_1 < \ldots < j_k \in J\}$$

of $\Lambda^k(\mathfrak{H})$, where $<$ denotes an arbitrary total order on J. Then

$$\sigma(\pi_k(A)) = \overline{\{a_{j_1} + \ldots + a_{j_k} : j_1 < \ldots < j_k \in J\}}$$

and therefore

$$L_k(A) = \sup(\sigma(\pi_k(A))) \sup_{j_1 < \ldots < j_k} a_{j_1} + \ldots + a_{j_k} \sup_{E \in \mathfrak{E}_k} \sum_{j \in E} a_j L_k(a).$$

The other assertions follow immediately from Lemma 4.3 and Lemma 4.4. $\square$

Lemma 4.6. *Let $A = \operatorname{diag}_{\mathfrak{d}}(a)$ or $A = \operatorname{diag}_{\mathfrak{d}'}(a)$ for $a \in l^\infty(J)$ with respect to the orthonormal basis $\{e_j : j \in 2J\}$ resp. $\{e_j : j \in 2J+1\}$. Then we have*

$$\begin{array}{rcl} L_k(A) & = & L_k(|a|) \\ \aleph^+(A) & = & \aleph^+(|a|) \\ \aleph^-(A) & = & -\aleph^+(A) \quad = \quad -\aleph^+(|a|). \end{array}$$

Proof. We have

$$\operatorname{diag}_{\mathfrak{d}}(a) = \operatorname{diag}(a, -a) = \operatorname{diag}(\sigma(|a|, -|a|))$$

for some $\sigma \in \mathfrak{S}(2J)$. So the assertion follows immediately from Lemma 4.5. The same argument works for $\operatorname{diag}_{\mathfrak{d}'}(a)$. $\square$

Lemma 4.7. *The functions*

$$\begin{array}{rcll} A & \mapsto & \frac{1}{k} L_k(A) & k \in \mathbb{N} \\ A & \mapsto & \frac{1}{k} L_k(-A) & k \in \mathbb{N} \\ A & \mapsto & \aleph^+(A) & \aleph \in \mathfrak{M} \\ A & \mapsto & -\aleph^-(A) & \aleph \in \mathfrak{M} \end{array}$$

are contracting, convex and $\mathfrak{U}(\mathfrak{H})$-invariant.

Proof. The $\mathfrak{U}(\mathfrak{H})$-invariance follows immediately from the definition of the respective functions.

To see the convexity we use Lemma 4.1 to see that

$$\tfrac{1}{k}L_k(A) = \lim_{n\to\infty} \tfrac{1}{k}\|\pi_k(A+n\mathbf{1})\| - n, \quad \aleph^+(A) = \lim_{n\to\infty} \|\pi_\aleph(A+n\mathbf{1})\| - n.$$

So these functions are pointwise limits of convex functions and therefore convex. The convexity of the other two functions now follows immediately.

All we have left to show is that these functions are contracting. We pick $A, B \in \mathfrak{h}(\mathfrak{H})$ and $\lambda \geq \max\{\|\pi_k(A)\|, \|\pi_k(B)\|\}$. Then we get with Lemma 4.1

$$\begin{aligned} |L_k(A) - L_k(B)| &= |\|\pi_k(A+\lambda\mathbf{1})\| - \|\pi_k(B+\lambda\mathbf{1})\|| \\ &\leq \|\pi_k(A+\lambda\mathbf{1}) - \pi_k(B+\lambda\mathbf{1})\| \\ &= \|\pi_k(A-B)\| \leq k\|A-B\| \end{aligned}$$

and the assertion follows. The calculation for $\aleph^+$ is identical. This finishes the proof. $\square$

Definition 4.8. (1) We write

$$\begin{aligned} k^+ &: \mathfrak{h}(\mathfrak{H}) \to \mathbb{R}\ ;\ A \mapsto \tfrac{1}{k}L_k(A) \\ k^- &: \mathfrak{h}(\mathfrak{H}) \to \mathbb{R}\ ;\ A \mapsto \tfrac{1}{k}L_k(-A) \end{aligned}$$

and define

$$\begin{aligned} \mathfrak{F} &:= \{k^+, k^- : k \in \mathbb{N}\} \cup \{\aleph^+, -\aleph^- : \aleph \in \mathfrak{M}^*\} \\ \mathfrak{F}^+ &:= \{k^+ : k \in \mathbb{N}\} \cup \{\aleph^+ : \aleph \in \mathfrak{M}^*\} \end{aligned}$$

(2) On $\mathfrak{F}$ and $\mathfrak{F}^+$ we define an order relation $\leq$ the following way:

$$\begin{aligned} k^+ &\leq m^+ && \text{for all } k \leq m \in \mathbb{N}. \\ k^+ &\leq \aleph^+ && \text{for all } k \in \mathbb{N},\ \aleph \in \mathfrak{M}^*. \\ \aleph_1^+ &\leq \aleph_2^+ && \text{for all } \aleph_1 \leq \aleph_2 \in \mathfrak{M}^*. \\ k^- &\leq m^- && \text{for all } k \leq m \in \mathbb{N}. \\ k^- &\leq -\aleph^- && \text{for all } k \in \mathbb{N},\ \aleph \in \mathfrak{M}^*. \\ -\aleph_1^- &\leq -\aleph_2^- && \text{for all } \aleph_1 \leq \aleph_2 \in \mathfrak{M}^*. \end{aligned}$$

Lemma 4.9. *Let $A \in \mathfrak{h}(\mathfrak{H})$. Then $f(A) \geq g(A)$ holds for all $f, g \in \mathfrak{F}$ with $f \leq g$.*

Proof. As all functions $f \in \mathfrak{F}$ are continuous, it is sufficient to prove the assertion for diagonalizable operators. Because of Lemma 4.5 we can therefore restrict ourselves to sequences $a \in l^\infty(J)$.

It follows immediately from the definition that $\aleph_1^+(a) \geq \aleph_2^+(a)$ whenever $\aleph_1 \leq \aleph_2$. We get from Lemma 2.4 that

$$k^+(A) = \frac{1}{k}L_k(a) = \frac{1}{k}L_k(\overline{a}) + \aleph_0^+(a) \geq \aleph_0^+(a)$$

as $\overline{a}$ by definition is a nonnegative sequence. Using induction and again Lemma 2.4, we see that in order to show $k^+(A) \geq m^+(A)$ for all diagonalizable $A \in \mathfrak{h}(\mathfrak{H})$ and $k \leq m \in \mathbb{N}$

it is sufficiet to show that $\frac{1}{k}L_k(a) \geq \frac{1}{k+1}L_{k+1}(a)$ for a nonnegative sequence $a \in c_0(J)$. As a is nonnegative and converges to 0 we can find the $k+1$ largest entries $a_1 \geq \ldots \geq a_{k+1}$ of a. Then

$$\begin{aligned}\tfrac{1}{k+1}L_{k+1}(a) &= \tfrac{1}{k+1}\sum_{j=1}^{k+1} a_j = \tfrac{1}{k+1}\sum_{j=1}^{k}(a_j + \tfrac{1}{k}a_{k+1}) \leq \tfrac{1}{k+1}\sum_{j=1}^{k}(1+\tfrac{1}{k})a_j \\ &= \tfrac{1}{k}\sum_{j=1}^{k} a_j = \tfrac{1}{k}L_k(a).\end{aligned}$$

The second half of the inequalities follows immediately from the first. □

Lemma 4.10. *Let $\mathfrak{h} \in \{\mathfrak{h}(\mathfrak{H}), \mathfrak{ho}(I_c), \mathfrak{hsp}(I_a)\}$, $\mathfrak{t} \subseteq \mathfrak{h}$ a Cartan subspace and $\mathfrak{W}$ the corresponding Weyl group. For $X \in \mathfrak{t}$ we have*

$$\overline{\mathrm{conv}(\mathfrak{W}.X)} = \{Y \in \mathfrak{t} : (\forall f \in \mathfrak{F})\, f(Y) \leq f(X)\}.$$

If $\mathfrak{h} \in \{\mathfrak{ho}(I_c), \mathfrak{hsp}(I_a)\}$ we also get

$$\overline{\mathrm{conv}(\mathfrak{W}.X)} = \{Y \in \mathfrak{t} : (\forall f \in \mathfrak{F}^+)\, f(Y) \leq f(X)\}.$$

Proof. This is an immediate consequence of Theorem 2.7 considering Lemma 4.5 and Lemma 4.6. □

Lemma 4.11. *For $A \in \mathfrak{h}(\mathfrak{H})$ we have*

$$\aleph_0^+(A) = \lim_{k\to\infty} k^+(A), \qquad -\aleph_0^-(A) = \lim_{k\to\infty} k^-(A)$$

and for $\aleph_0 < \aleph \notin \mathfrak{M}^*$ we have

$$\aleph^+(A) = \inf_{\aleph'<\aleph}(\aleph')^+(A), \qquad \aleph^-(A) = \sup_{\aleph'<\aleph}(\aleph')^-(A).$$

Proof. If A is unitarily diagonalizable, this follows immediately from Lemma 2.4. As the unitarily diagonalizable operators lie dense in $\mathfrak{h}(\mathfrak{H})$ the general assertion follows from the fact that $k^\pm$ and $\aleph^\pm$ are contractions according to Lemma 4.7 and therefore an equicontinuous family of functions. □

5. Approximation of Nondiagonalizable Operators

Now we want to approximate nondiagonalizable hermitian operators by diagonalizable ones in such a way that the invariant convex functions studied in the last section do not change. This will be an important tool to study closed convex $\mathfrak{U}$-invariant sets.

Let $A = \int_{\sigma(A)} x dP(x)$ be hermitian. We have $\sigma(A) \subseteq \mathbb{R}$. We define

$$\sigma_p(A) = \{x \in \sigma(A) : P(\{x\}) \neq 0\},$$

the *point spectrum* of A, and

$$\mathfrak{H}_p : \overline{\bigoplus_{x\in\sigma_p(A)} P(\{x\}).\mathfrak{H}}, \qquad \mathfrak{H}_c : \mathfrak{H}_p^\perp.$$

These are A-invariant subspaces. Obviously $\sigma_p(A|_{\mathfrak{H}_c}) = \emptyset$. We define $\sigma_c(A) := \sigma(A|_{\mathfrak{H}_c})$, the *continuous spectrum* of A.

Lemma 5.1. *Let $A \in B(\mathfrak{H})$ be hermitian. Then for every $\varepsilon > 0$ there exists a diagonalizable operator $A_\varepsilon \in B(\mathfrak{H})$ satisfying*

(1) $\mathfrak{H}_p$ *and* $\mathfrak{H}_c$ *are* A_ε*-invariant subspaces.*
(2) $A_\varepsilon|_{\mathfrak{H}_p} = A|_{\mathfrak{H}_p}$
(3) $\|A - A_\varepsilon\| \leq \varepsilon$
(4) $M_A(\aleph) = M_{A_\varepsilon}(\aleph)$ *for all* $\aleph \in \mathfrak{M}$.
(5) $\sigma(A_\varepsilon|_{\mathfrak{H}_c}) \subseteq M_A(\aleph_0)$.

Proof. We first assume that $\sigma_p(A) = \emptyset$. We write $A = \int_{\sigma(A)} x dP(x)$. We consider Lemma 2.4 to obtain

$$
\begin{aligned}
](\#J)^+(A), \aleph_0^+(A)] &= \dot{\bigcup_{\aleph \in \mathfrak{M}\setminus\{\#J\}}}](\aleph+1)^+(A), \aleph^+(A)] \\
[\aleph_0^-(A), (\#J)^-(A)[&= \dot{\bigcup_{\aleph \in \mathfrak{M}\setminus\{\#J\}}} [\aleph^-(A), (\aleph+1)^-(A)[.
\end{aligned}
$$

Now we define

$$
\begin{aligned}
\mathfrak{M}^+ &:= \{\aleph \in \mathfrak{M}\setminus\{\#J\} : (\aleph+1)^+(A) < \aleph^+(A)\} \\
\mathfrak{M}^- &:= \{\aleph \in \mathfrak{M}\setminus\{\#J\} : (\aleph+1)^-(A) > \aleph^+(A)\}.
\end{aligned}
$$

We note that $\mathfrak{M}^+$ and $\mathfrak{M}^-$ are countable. For $\aleph \in \mathfrak{M}^+$ we choose an increasing step function $S_\varepsilon^{\aleph,+}$ on $](\aleph+1)^+(A), \aleph^+(A)]$ with the following properties:

(1) $\|S_\varepsilon^{\aleph,+} - \mathrm{id}\,\|_\infty \leq \varepsilon$
(2) $S_\varepsilon^{\aleph,+}(]\aleph^+(A) - \delta, \aleph^+(A)]) = \{\aleph^+(A)\}$ for some $\delta > 0$
(3) $S_\varepsilon^{\aleph,+}(x) > (\aleph+1)^+(A)$ for all $x \in](\aleph+1)^+(A), \aleph^+(A)]$.

By combining the $S_\varepsilon^{\aleph,+}$, we get a function S_ε^+ on $](\#J)^+(A), \aleph_0^+(A)]$.

We pick for $\aleph \in \mathfrak{M}^-$ on $[\aleph^-(A), (\aleph+1)^-(A)[$ an increasing step function $S_\varepsilon^{\aleph,-}$ with the following properties:

(1) $\|S_\varepsilon^{\aleph,-} - \mathrm{id}\,\|_\infty \leq \varepsilon$
(2) $S_\varepsilon^{\aleph,-}([\aleph^-(A), \aleph^-(A) + \delta[) = \{\aleph^-(A)\}$ for some $\delta > 0$
(3) $S_\varepsilon^{\aleph,-}(x) < (\aleph+1)^-(A)$ for all $x \in [\aleph^-(A), (\aleph+1)^-(A)[$.

By combining the $S_\varepsilon^{\aleph,-}$, we get a function S_ε^- on $[\aleph_0^-(A), (\#J)^-(A)[$.

Finally we pick on $[(\#J)^-(A), (\#J)^+(A)]$ an increasing step function S'_ε satisfying

(1) $\|S'_\varepsilon - \mathrm{id}\,\|_\infty \leq \varepsilon$
(2) $S'_\varepsilon([(\#J)^-(A), (\#J)^-(A) + \delta[) = \{(\#J)^-(A)\}$ for some $\delta > 0$
(3) $S'_\varepsilon(](\#J)^+(A) - \delta, (\#J)^+(A)]) = \{(\#J)^+(A)\}$ for some $\delta > 0$

If $(\#J)^-(A) = (\#J)^+(A)$ then S'_ε is constant.

Combining S'_ε, S_ε^+ and S_ε^-, we get the function S_ε on $[\aleph_0^-(A), \aleph_0^+(A)]$. It follows from our construction that there exists a countable partition of $[\aleph_0^-(A), \aleph_0^+(A)]$ into intervals such that S_ε is a constant on each of the subsets. Therefore S_ε is P-measurable.

We have $\sigma(A) \subseteq [\aleph_0^-(A), \aleph_0^+(A)]$ because of $\sigma_p(A) = \emptyset$. So the operator

$$A_\varepsilon := \int_{\sigma(A)} S_\varepsilon(x) dP(x)$$

is well defined and diagonalizable and satisfies the required properties.

If $\sigma_p(A) \neq \emptyset$ we write $A = A|_{\mathfrak{H}_p} \oplus A|_{\mathfrak{H}_c}$. Then we can find for $A_c := A|_{\mathfrak{H}_c}$ an operator $A_{c,\varepsilon} \in B(\mathfrak{H}_c)$ satisfying $\|A_c - A_{c,\varepsilon}\| \leq \varepsilon$ and $M_{A_c}(\aleph) = M_{A_{c,\varepsilon}}(\aleph)$ for all $\aleph \in \mathfrak{M}$. We define $A_\varepsilon := A|_{\mathfrak{H}_p} \oplus A_{c,\varepsilon}$. Then A_ε is diagonalizable.

We observe that for every operator $B \oplus C \in B(\mathfrak{H}_p) \oplus B(\mathfrak{H}_c)$ we have

$$\begin{aligned} \|B \oplus C\| &= \max\{\|B\|, \|C\|\}, \\ \aleph^+(B \oplus C) &= \max\{\aleph^+(B), \aleph^+(C)\}, \\ \aleph^-(B \oplus C) &= \min\{\aleph^-(B), \aleph^-(C)\} \end{aligned}$$

for all $\aleph \in \mathfrak{M}$. From this it follows immediately that A_ε has the required properties. $\square$

Lemma 5.2. *Let $A = \int_{\sigma(A)} x dP(x) \in \mathfrak{h}(\mathfrak{H})$. We define*

$$\overline{A} = \int_{]\aleph_0^+(A), \infty[} (x - \aleph_0^+(A)) dP(x)$$

We have:

(1) $\overline{A}|_{\mathfrak{H}_c} = 0$, *and $\overline{A}$ is determined by $A|_{\mathfrak{H}_p}$.*

(2) $\overline{A} = \mathrm{diag}(\overline{a})$ *for $\overline{a} \in c_0(J)$. The sequence $\overline{a}$ is uniquely determined by A up to a permutation of the entries.*

(3) *If $A = \mathrm{diag}(a)$ for $a \in l^\infty(J)$, then the sequence $\overline{a}$ obtained as above coincides with the one from Definition 2.3*

(4) *For $A \in \mathfrak{h}(\mathfrak{H})$ we have*

$$L_k(A) = L_k(\overline{A}) + k\aleph_0^+(A) = L_k(\overline{a}) + k\aleph_0^+(A)$$

for all $k \in \mathbb{N}$.

(5) *For $A \in \mathfrak{h}(\mathfrak{H})$ and A_ε as in Lemma 5.1 we have*

$$\overline{A_\varepsilon} = \overline{A} \qquad \text{and} \qquad L_k(A_\varepsilon) = L_k(A)$$

for all $k \in \mathbb{N}$.

Proof. (1) This follows from $]\aleph_0^+(A), \infty[\cap \sigma(A) \subseteq \sigma_p(A)$.

(2) This is a consequence of the fact that $\overline{A}$ is compact and normal.

(3) This is an immediate consequence of Lemma 4.5.

(4) & (5) Now we choose $A \in \mathfrak{h}(\mathfrak{H})$ and pick A_ε as in Lemma 5.1. Then we have $\aleph_0^+(A_\varepsilon) = \aleph_0^+(A)$ and because of $A_\varepsilon|_{\mathfrak{H}_p} = A|_{\mathfrak{H}_p}$ and $\sigma(A_\varepsilon|_{\mathfrak{H}_c}) \subseteq M_A(\aleph_0)$ we get $\overline{A_\varepsilon} = \overline{A}$.

Lemma 2.4, Lemma 4.5 and 1) then show that

$$L_k(A_\varepsilon) = L_k(\overline{A_\varepsilon}) + k\aleph_0^+(A_\varepsilon) = L_k(\overline{A}) + k\aleph_0^+(A).$$

This is true for all $\varepsilon > 0$ and L_k is continuous. Therefore it follows that $L_k(A) = L_k(\overline{A}) + k\aleph_0^+(A)$ and as a consequence $L_k(A_\varepsilon) = L_k(A)$. $\square$

Lemma 5.3. *Let $A \in \mathfrak{h}(\mathfrak{H})$. Then there exists a diagonalizable operator $A_\varepsilon \in \mathfrak{h}(\mathfrak{H})$ satisfying*

(1) $f(A_\varepsilon) = f(A)$ *for all* $f \in \mathfrak{F}$.
(2) $\|A - A_\varepsilon\| \leq \varepsilon$.

Proof. We choose A_ε as in Lemma 5.1. Then the assertion follows immediately from the properties of A_ε and Lemma 5.2. □

Now we want to show that for $A \in \mathfrak{ho}(I_c)$ ($A \in \mathfrak{hsp}(I_a)$) we can even find $A_\varepsilon \in \mathfrak{ho}(I_c)$ ($A_\varepsilon \in \mathfrak{hsp}(I_a)$).

Proposition 5.4. *Let I be a conjugation or anticonjugation on $\mathfrak{H}$. If the normal operator $A = \int_{\sigma(A)} x dP(x)$ satisfies $A^*I + IA = 0$, then we have*

$$P(U)I = IP(-U)$$

for every P-measurable $U \subseteq \mathbb{R}$.

Proof. We have $p(A^*) = p(A)^*$ for every polynomial function p with real coefficients. Therefore $p(A^*)I = Ip(-A)$. Further we have $p(A) = \int_{\sigma(a)} p(x)dP(x)$.

The mapping

$$L^\infty(\sigma(A)) \to B(\mathfrak{H}) \; ; \; f \mapsto f(A) := \int_{\sigma(A)} f(x)dP(x)$$

is a W^*-algebra homomorphism. In particular it is continuous with respect to the weak-$*$-topologies on the respective spaces. Since $\sigma(A)$ is compact, the polynomials with real coefficients lie weak-$*$-dense in $L^\infty(\sigma(A), \mathbb{R})$. As $L^\infty(\sigma(A))$ equipped with the weak-$*$-topology is metrizable, there exists a sequence $(p_n)_{n\in\mathbb{N}}$ of polynomials with real coefficients that converges to χ_U, the characteristic function of U, in the weak-$*$-topology. In particular we have

$$\begin{aligned} P(U) &= P(U)^* = \int_{\sigma(A)} \overline{\chi_U(x)}\, dP(x) = \int_{\sigma(A)} \overset{*}{\lim_{n\to\infty}} \overline{p_n(x)}\, dP(x) = \overset{*}{\lim_{n\to\infty}} p_n(A^*), \\ P(-U) &= \int_{\sigma(A)} \chi_U(-x)\, dP(x) = \overset{*}{\lim_{n\to\infty}} \int_{\sigma(A)} p_n(-x)\, dP(x) = \overset{*}{\lim_{n\to\infty}} p_n(-A), \end{aligned}$$

where $\overset{*}{\lim}$ denotes the limit in the weak-$*$-topology. According to [6, Theorem 1.7.8.] the mappings $B \mapsto BI$ and $B \mapsto IB$ are weak-$*$-continuous. So we get for every trace class operator S on $\mathfrak{H}$

$$\langle S, P(U)I \rangle = \lim_{n\to\infty} \langle S, p_n(A^*)I \rangle = \lim_{n\to\infty} \langle S, Ip_n(-A) \rangle = \langle S, IP(-U) \rangle \,.$$

Now the assertion follows. □

Corollary 5.5. *Let $A \in \mathfrak{h}$ with $\mathfrak{h} \in \{\mathfrak{ho}(I_c), \mathfrak{hsp}(I_a)\}$. Then*

(1) $\aleph^-(A) = -\aleph^+(A)$ *for all* $\aleph \in \mathfrak{M}$.
(2) $L_k(-A) = L_k(A)$ *for all* $k \in \mathbb{N}$.

Proof. (1) follows immediately from Lemma 4.4 and Proposition 5.4.

(2) We have $\overline{-A} = -\int_{]-\infty,-\aleph_0^+(A)[} x dP(x)$. Further $\mathrm{rk}P(\{-x\}) = \mathrm{rk}P(\{x\})$ for all $x \in \mathbb{R}$ because of Proposition 5.4 . So the assertion follows from (1) and Lemma 5.2. $\square$

Corollary 5.6. *Let $A \in \mathfrak{h}$ with $\mathfrak{h} \in \{\mathfrak{ho}(I_c), \mathfrak{hsp}(I_a)\}$. Then*

$$\{B \in \mathfrak{h} : (\forall f \in \mathfrak{F}) f(B) \leq f(A)\} = \{B \in \mathfrak{h} : (\forall f \in \mathfrak{F}^+) f(B) \leq f(A)\}$$

Lemma 5.7. *Let I be a conjugation or anticonjugation on $\mathfrak{H}$ and A a hermitian operator satisfying $A^*I + IA = 0$. Then for every $\varepsilon > 0$ there exists a diagonalizable hermitian operator A_ε satisfying*

(1) $A_\varepsilon^* I + IA_\varepsilon = 0$
(2) $\|A - A_\varepsilon\| \leq \varepsilon$
(3) $f(A_\varepsilon) = f(A)$ *for all* $f \in \mathfrak{F}$

Proof. We write $A = \int_{s(A)} x dP(x)$ and define the mutually orthogonal subspaces

$$\mathfrak{H}_- := P(]-\infty, 0[).\mathfrak{H} \qquad \mathfrak{H}_0 := P(\{0\}).\mathfrak{H} \qquad \mathfrak{H}_+ := P(]0, \infty[).\mathfrak{H}$$

of $\mathfrak{H}$. With Proposition 5.4 we get that $I.\mathfrak{H}_+ = \mathfrak{H}_-$ and $\mathfrak{H}_0$ is I-invariant. Then $\aleph^+(A) = \aleph^+(A|_{\mathfrak{H}_+})$ and $\aleph^-(A) = \aleph^-(A|_{\mathfrak{H}_-})$ are a consequence of Lemma 4.4 and Corollary 5.5.

We pick an $\varepsilon > 0$. By applying Lemma 5.1 to $A|_{\mathfrak{H}_+}$, we get a diagonalizable operator A_ε^+ on $\mathfrak{H}_+$ satisfying in particular

(1) $A_\varepsilon^+|_{\mathfrak{H}_+ \cap \mathfrak{H}_p} = A|_{\mathfrak{H}_+ \cap \mathfrak{H}_p}$
(2) $\|A_\varepsilon^+ - A|_{\mathfrak{H}_+}\| \leq \varepsilon$
(3) $\aleph^+(A_\varepsilon^+) = \aleph^+(A)$ for all $\aleph \in \mathfrak{M}$.
(4) $\aleph^-(A_\varepsilon^+) \geq 0$ for all $\aleph \in \mathfrak{M}$.

Now we define

$$A_\varepsilon.v := \begin{cases} A_\varepsilon^+.v & v \in \mathfrak{H}_+ \\ 0 & v \in \mathfrak{H}_0 \\ \eta I A_\varepsilon^+.(I.v) & v \in \mathfrak{H}_- \end{cases},$$

where $\eta = -1$ if I is a conjugation and $\eta = 1$ if I is an anticonjugation. The operator A_ε satisfies

(1) $A|_{\mathfrak{H}_p} = A_\varepsilon|_{\mathfrak{H}_p}$.
(2) $A_\varepsilon^* I + IA_\varepsilon = 0$.
(3) $\|A - A_\varepsilon\| \leq \varepsilon$
(4) $M_A(\aleph) = M_{A_\varepsilon}(\aleph)$ for all $\aleph \in \mathfrak{M}$.
(5) $\sigma(A_\varepsilon|_{\mathfrak{H}_c}) \subseteq M_A(\aleph_0)$.

So all that is left to show is that $k^+(A_\varepsilon) = k^+(A)$ and $k^-(A_\varepsilon) = k^-(A)$ for all $k \in \mathbb{N}$. This follows immediately from Lemma 5.2. $\square$

6. The set $p_\mathfrak{t}(\mathfrak{U}.A)$

Our next goal is to describe for $\mathfrak{h} \in \{\mathfrak{h}(\mathfrak{H}), \mathfrak{ho}(I_c), \mathfrak{hsp}(I_a)\}$ the subsets $p_\mathfrak{t}(\mathfrak{U}.A)$, where $A \in \mathfrak{h}$, $\mathfrak{U}$ is the corresponding unitary group and $p_\mathfrak{t}$ the projection on the Cartan subspace

$\mathfrak{t}$. If A can be conjugate into $\mathfrak{t}$ under $\mathfrak{U}$, this problem is solved by Theorem 3.9. However this assumption requires in particular that the operator A is diagonalizable. In this section we will give a description of the set $p_{\mathfrak{t}}(\mathfrak{U}.A)$ for $\mathfrak{h} \in \{\mathfrak{h}(\mathfrak{H}), \mathfrak{hsp}(I_a)\}$ and arbitrary $A \in \mathfrak{h}$.

Lemma 6.1. *On the set $\mathcal{C}_0(\mathfrak{B})$ of closed bounded subsets of a Banach space $\mathfrak{B}$ we have the Hausdorff metric*

$$d_H(A,B) := \max\left\{\sup_{a\in A}\operatorname{dist}(a,B), \sup_{b\in B}\operatorname{dist}(b,A)\right\},$$

where $\operatorname{dist}(x,Y) := \inf_{y\in Y}\|x-y\|$. *Then we have for any closed subgroup G of the group of isometric isomorphisms of $\mathfrak{B}$ that the maps*

$$\begin{aligned} \mathfrak{B} &\to \mathcal{C}_0(\mathfrak{B}) \ ; \ x \mapsto \overline{G.x} \\ \mathfrak{B} &\to \mathcal{C}_0(\mathfrak{B}) \ ; \ x \mapsto \overline{\operatorname{conv}(G.x)} \end{aligned}$$

are Lipschitz continuous with constant 1, *that is*

$$\begin{aligned} d_H(\overline{G.x}, \overline{G.y}) &\leq \|x-y\| \\ d_H(\overline{\operatorname{conv}(G.x)}, \overline{\operatorname{conv}(G.y)}) &\leq \|x-y\| \end{aligned}$$

for all $x, y \in \mathfrak{B}$.

Proof. We choose $x \neq y \in \mathfrak{B}$ and $\varepsilon > 0$. For every $a \in \overline{G.x}$ we can find a $g \in G$ such that $\|a - g.x\| \leq \varepsilon$. Then we get

$$\|a - g.y\| \leq \|a - g.x\| + \|g.x - g.y\| \leq \|x-y\| + \varepsilon.$$

Therefore $\operatorname{dist}(a, \overline{G.y}) \leq \|x-y\| + \varepsilon$. We conclude that $\operatorname{dist}(a, \overline{G.y}) \leq \|x-y\|$ and therefore

$$\sup_{a\in\overline{G.x}} \operatorname{dist}(a, \overline{G.y}) \leq \|x-y\|.$$

Our considerations were symmetric in x and y, so we get the first assertion.

This implies that $\overline{G.x} \subseteq \overline{G.y} + B$, where B denotes the Ball with center 0 and radius $\|x-y\|$. It follows that $\overline{G.x} \subseteq \overline{\operatorname{conv}(G.y)} + B$ and, as the set on the right hand side is closed and convex, $\overline{\operatorname{conv}(G.x)} \subseteq \overline{\operatorname{conv}(G.y)} + B$. The same way we obtain $\overline{\operatorname{conv}(G.y)} \subseteq \overline{\operatorname{conv}(G.x)} + B$, which proves the second assertion. □

Lemma 6.2. *Let $\mathfrak{g}$ be a classical Lie algebra and $\mathfrak{h} := \mathfrak{g} \cap \mathfrak{h}(\mathfrak{H})$. Further let $\mathfrak{t}$ be a Cartan subspace of $\mathfrak{h}$, $p_{\mathfrak{t}} \colon \mathfrak{h} \to \mathfrak{t}$ the projection onto $\mathfrak{t}$, and $\mathfrak{U}$ the unitary group corresponding to $\mathfrak{g}$.*

Let $A_n \in \mathfrak{h}$ be a sequence converging to $A \in \mathfrak{h}$. If there exists a closed set $C_0 \subseteq \mathfrak{t}$ such that $\overline{p(\mathfrak{U}.A_n)} = C_0$ for all $n \in \mathbb{N}$, then we have $\overline{p(\mathfrak{U}.A)} = C_0$ as well.

Proof. Obviously the sets $\mathfrak{U}.A_n$ and $\mathfrak{U}.A$ are bounded. The map $C \mapsto \overline{p_{\mathfrak{t}}(C)}$ is continuous with respect to the Hausdorff metric, as defined in Lemma 6.1. Further we have $\overline{p_{\mathfrak{t}}(\overline{C})} = \overline{p_{\mathfrak{t}}(C)}$. So we get with Lemma 6.1, that

$$\overline{p(\mathfrak{U}.A)}\overline{p(\overline{\mathfrak{U}.A})}\overline{p(\mathfrak{U}.(\lim_{n\to\infty} A_n))} \lim_{n\to\infty} \overline{p(\overline{\mathfrak{U}.A_n})} \lim_{n\to\infty} \overline{p(\mathfrak{U}.A_n)} = \lim_{n\to\infty} C_0 C_0,$$

where the limit was taken in the Hausdorff metric. □

Now we can prove one of our main results.

Theorem 6.3. *Let $A \in \mathfrak{h}(\mathfrak{H})$ and let $\mathfrak{t}$ denote a Cartan subspace. Then we have*

$$\overline{p_{\mathfrak{t}}(\mathfrak{U}(\mathfrak{H}).A)} = \{B \in \mathfrak{t} : (\forall f \in \mathfrak{F})\, f(B) \leq f(A)\}.$$

Proof. According to Lemma 3.5, there exists an orthonormal basis such that $\mathfrak{t}$ consists of the operators in $\mathfrak{h}(\mathfrak{H})$ that are diagonal with respect to this basis. With Lemma 5.3 we can find $A_\varepsilon \in B(\mathfrak{H})$ such that A_ε is diagonalizable and satisfies $f(A_\varepsilon) = f(A)$ for all $f \in \mathfrak{F}$. As A_ε is unitarily diagonalizable, there exists a $U \in \mathfrak{U}(\mathfrak{H})$ such, that $U.A_\varepsilon = \operatorname{diag}(a_\varepsilon)$ for $a_\varepsilon \in l^\infty(J)$. So we get with Theorem 3.9 and Lemma 4.10 and because of the $\mathfrak{U}(\mathfrak{H})$-invariance of the functions $f \in \mathfrak{F}$:

$$\begin{aligned}
\overline{p(\mathfrak{U}(\mathfrak{H}).A_\varepsilon)} &= \overline{p(\mathfrak{U}(\mathfrak{H}).(U^{-1}.\operatorname{diag}(a_e)))} = \overline{p(\mathfrak{U}(\mathfrak{H}).\operatorname{diag}(a_e))} \\
&= \{\operatorname{diag}(b) : b \in l^\infty(J), (\forall f \in \mathfrak{F})\, f(b) \leq f(a_e)\} \\
&= \{B \in \mathfrak{t} : (\forall f \in \mathfrak{F})\, f(B) \leq f(A_\varepsilon)\} \\
&= \{B \in \mathfrak{t} : (\forall f \in \mathfrak{F})\, f(B) \leq f(A)\}
\end{aligned}$$

Now Lemma 6.2 finishes the proof. □

We can show a similar result for $\mathfrak{hsp}(I_a)$.

Theorem 6.4. *Let $A \in \mathfrak{hsp}(I_a)$ and let $\mathfrak{t}$ denote a Cartan-Subspace. We have*

$$\overline{p_{\mathfrak{t}}(\mathrm{USp}(I_a).A)} = \{B \in \mathfrak{t} : (\forall f \in \mathfrak{F}^+)\, f(B) \leq f(A)\}.$$

Proof. With Lemma 5.7 we can find $A_\varepsilon \in \mathfrak{hsp}(I_a)$ such that A_ε is unitarily diagonalizable and satisfies

(1) $\|A - A_\varepsilon\| \leq \varepsilon$
(2) $f(A_\varepsilon) = f(A)$ for all $f \in \mathfrak{F}$.

According to Lemma 3.5, there exists an I_a-basis of $\mathfrak{H}$ such that $\mathfrak{t}$ consists of the operators in $\mathrm{hsp}(I_a)$ diagonal with respect to that basis. In particular we can find a $U \in \mathrm{USp}(I_a)$ satisfying $U.A_\varepsilon = \operatorname{diag}_{\mathfrak{d}}(a_\varepsilon) \in \mathfrak{t}$, where $a_\varepsilon \in l^\infty(J)$. The functions $f \in \mathfrak{F}$ are $\mathfrak{U}(\mathfrak{H})$-invariant and therefore $\mathrm{USp}(I_a)$-invariant. So we get with Theorem 3.9 and Lemma 4.10

$$\begin{aligned}
\overline{p_{\mathfrak{t}}(\mathrm{USp}(I_a).A_\varepsilon)} &= \overline{p_{\mathfrak{t}}(\mathrm{USp}(I_a).(U^{-1}.\operatorname{diag}_{\mathfrak{d}}(a_\varepsilon)))} \\
&= \overline{p_{\mathfrak{t}}(\mathrm{USp}(I_a).\operatorname{diag}_{\mathfrak{d}}(a_\varepsilon))} \\
&= \{\operatorname{diag}_{\mathfrak{d}}(b) : b \in l^\infty(J), (\forall f \in \mathfrak{F}) f(|b|) \leq f(|a_\varepsilon|)\} \\
&= \{B \in \mathfrak{t} : (\forall f \in \mathfrak{F}) f(B) \leq f(A_\varepsilon)\} \\
&= \{B \in \mathfrak{t} : (\forall f \in \mathfrak{F}) f(B) \leq f(A)\}
\end{aligned}$$

Now Lemma 6.2 and Corollary 5.6 finish the proof. □

The results of Theorem 6.3 and Theorem 6.4 look remarkably similar to Theorem 2.7, which shows that the generalization we chose for L_k and $\aleph^\pm$ was a good one.

7. Reconstructing Closed Convex Invariant Sets

In this section we show that each closed convex $\mathfrak{U}$-invariant subset of $\mathfrak{h}$ is determined by its intersection with a Cartan subspace $\mathfrak{t}$. While we collect the necessary tools we also show that for each $A \in \mathfrak{h}$ we have

$$\overline{\mathrm{conv}(\mathfrak{U}.A)} = \{B \in \mathfrak{h} : (\forall f \in \mathfrak{F})\, f(B) \leq f(A)\}.$$

Lemma 7.1. *For every operator $A \in \mathfrak{h}$, $\mathfrak{h} \in \{\mathfrak{h}(J), \mathfrak{hsp}(I_a)\}$, and every $\varepsilon > 0$ there exists a diagonalizable operator $A_\varepsilon \in \mathfrak{h}$ satisfying*

(1) $\|A - A_\varepsilon\| \leq \varepsilon$.
(2) $\overline{p_\mathfrak{t}(\mathfrak{U}.A)} = \overline{p_\mathfrak{t}(\mathfrak{U}.A_\varepsilon)}$.

Proof. This follows immediately from Lemma 5.3 and Theorem 6.3, resp., Lemma 5.7 and Theorem 6.4. □

Lemma 7.2. *Let $A \in \mathfrak{h}$, $\mathfrak{h} \in \{\mathfrak{h}(J), \mathfrak{hsp}(I_a)\}$. Then*

$$p_\mathfrak{t}(\mathfrak{U}.A) \subseteq \overline{\mathrm{conv}(\mathfrak{U}.A)}.$$

Proof. With Lemma 7.1 we can find for every $\varepsilon > 0$ a diagonalizable $\tilde{A}_\varepsilon \in \mathfrak{h}$ satisfying $\|A - \tilde{A}_\varepsilon\| \leq \varepsilon$ and $\overline{p_\mathfrak{t}(\mathfrak{U}.A)} = \overline{p_\mathfrak{t}(\mathfrak{U}.\tilde{A}_\varepsilon)}$. In particular there exists an $A_\varepsilon \in \mathfrak{t}$ and $U_\varepsilon \in \mathfrak{U}$ such that $\tilde{A}_\varepsilon = U_\varepsilon^* A_\varepsilon U_\varepsilon$ and therefore, according to Theorem 3.9,

$$\overline{p_\mathfrak{t}(\mathfrak{U}.\tilde{A}_\varepsilon)} = \overline{\mathrm{conv}(\mathfrak{W}.A_\varepsilon)}.$$

For $B \in \overline{p_\mathfrak{t}(\mathfrak{U}.A)} = \overline{\mathrm{conv}(\mathfrak{W}A_\varepsilon)}$ we can find $\lambda_1, \ldots, \lambda_n \in [0,1]$ satisfying $\sum_{j=1}^n \lambda_j = 1$ and $w_1, \ldots, w_n \in \mathfrak{W}$ such that $\|B - \sum_{j=1}^n \lambda_j w_j A_\varepsilon\|_\infty \leq \varepsilon$. For every w_j there exists a $W_j \in \mathfrak{U}$ satisfying $w_j A_\varepsilon = W_j^{-1} A_\varepsilon W_j$. So we get

$$\begin{aligned}
& \left\| B - \sum_{j=1}^n \lambda_j (U_\varepsilon W_j)^{-1} A (U_\varepsilon W_j) \right\| \\
\leq & \left\| B - \sum_{j=1}^n \lambda_j (U_\varepsilon W_j)^{-1} \tilde{A}_\varepsilon (U_\varepsilon W_j) \right\| + \left\| \sum_{j=1}^n \lambda_j (U_\varepsilon W_j)^{-1} (\tilde{A}_\varepsilon - A)(U_\varepsilon W_j) \right\| \\
\leq & \left\| B - \sum_{j=1}^n \lambda_j W_j^{-1} A_\varepsilon W_j \right\| + \sum_{j=1}^n \lambda_j \|(U_\varepsilon W_j)^{-1} (\tilde{A}_\varepsilon - A)(U_\varepsilon W_j)\| \\
\leq & \ 2\varepsilon.
\end{aligned}$$

As ε was arbitrary, this finishes the proof. □

Now we can obtain another main result of this chapter, the description of the sets $\overline{\mathrm{conv}(\mathfrak{U}(\mathfrak{H}).A)}$ with the functions $f \in \mathfrak{F}$.

Theorem 7.3. *Let $A \in \mathfrak{h}(\mathfrak{H})$. Then $B \in \overline{\mathrm{conv}(\mathfrak{U}(\mathfrak{H}).A)}$ if and only if $f(B) \leq f(A)$ for all $f \in \mathfrak{F}$.*

Proof. Lemma 4.7 tells us that L_k, $L_k(-\,.\,)$, $\aleph^+$ and $-\aleph^-$ are continuous convex $\mathfrak{U}(\mathfrak{H})$-invariant functions. This immediately shows one inclusion. We assume now that $f(B) \le f(A)$ for all $f \in \mathfrak{F}$.

Let $\varepsilon > 0$. We approximate B with an operator B_ε as in Lemma 7.1, that is $\|B - B_\varepsilon\| \le \varepsilon$ and $f(B) = f(B_\varepsilon)$ for all $f \in \mathfrak{F}$. As B_ε is diagonalizable, we can find a Cartan subspace $\mathfrak{t}$ such that $B_\varepsilon \in \mathfrak{t}$. If $p_\mathfrak{t}$ is the corresponding projection, we get with Theorem 6.3 that $B_\varepsilon \in \overline{p_\mathfrak{t}(\mathfrak{U}(\mathfrak{H}).A)}$. So we obtain $B_\varepsilon \in \overline{\operatorname{conv}(\mathfrak{U}(\mathfrak{H}).A)}$ from Lemma 7.2. As $\|B_\varepsilon - B\| \le \varepsilon$, we conclude that $B \in \overline{\operatorname{conv}(\mathfrak{U}(\mathfrak{H}).A)}$ which proves the assertion. □

Corollary 7.4. *Let $\mathfrak{h} \in \{\mathfrak{h}(\mathfrak{H}), \mathfrak{hsp}(I_a), \mathfrak{ho}(I_c)\}$, $\mathfrak{t} \subseteq \mathfrak{h}$ a Cartan subspace, and $p_\mathfrak{t}$ the projection onto $\mathfrak{t}$. Then we get for every $A \in \mathfrak{h}$*

$$f(p_\mathfrak{t}(A)) \le f(A)$$

for all $f \in \mathfrak{F}$.

Proof. With Lemma 3.5 we can find an orthonormal basis $\{e_j : j \in J\}$ of $\mathfrak{H}$ such that each element of $\mathfrak{t}$ is diagonal with respect to that basis. We denote by $\tilde{\mathfrak{t}} \subseteq \mathfrak{h}(\mathfrak{H})$ the space of all operators diagonal with respect to this basis. because of the maximality of $\mathfrak{t}$ in $\mathfrak{h}$ we then have $\tilde{\mathfrak{t}} \cap \mathfrak{h} = \mathfrak{t}$ and therefore $p_{\tilde{\mathfrak{t}}}|_\mathfrak{h} = p_\mathfrak{t}$. On the other hand it follows immediately from Lemma 7.2 and Theorem 7.3 that $f(p_{\tilde{\mathfrak{t}}}(A)) \le f(A)$ for all $A \in \mathfrak{h} \subseteq \mathfrak{h}(\mathfrak{H})$. This proves the assertion. □

Theorem 7.5. *Let $\mathfrak{h} \in \{\mathfrak{h}(J), \mathfrak{hsp}(I_a)\}$ and C be a $\mathfrak{U}$-invariant closed convex subset of $\mathfrak{h}$. Then*

$$p_\mathfrak{t}(C) = C \cap \mathfrak{t}.$$

Proof. Obviously we have $C \cap \mathfrak{t} \subseteq p_\mathfrak{t}(C)$. With the help of Lemma 7.2 we get

$$\begin{aligned}
p_\mathfrak{t}(C) &= p_\mathfrak{t}\Big(\bigcup_{X\in C} \overline{\operatorname{conv}(\mathfrak{U}.X)}\Big) \bigcup_{X\in C} p_\mathfrak{t}\Big(\overline{\operatorname{conv}(\mathfrak{U}.X)}\Big) \subseteq \bigcup_{X\in C} \overline{p_\mathfrak{t}\left(\operatorname{conv}(\mathfrak{U}.X)\right)} \\
&\subseteq \bigcup_{X\in C} \overline{\operatorname{conv} p_\mathfrak{t}(\mathfrak{U}.X)} \subseteq \bigcup_{X\in C} \overline{\operatorname{conv}\Big(\overline{\operatorname{conv}(\mathfrak{U}.X)} \cap \mathfrak{t}\Big)} = \bigcup_{X\in C} \overline{\operatorname{conv}(\mathfrak{U}.X)} \cap \mathfrak{t} \\
&= \Big(\bigcup_{X\in C} \overline{\operatorname{conv}(\mathfrak{U}.X)}\Big) \cap \mathfrak{t} = C \cap \mathfrak{t}.
\end{aligned}$$

□

Theorem 7.6. *Let $\mathfrak{h} \in \{\mathfrak{h}(J), \mathfrak{hsp}(I_a)\}$ and $C \subseteq \mathfrak{h}$ be a closed convex $\mathfrak{U}$-invariant subset. Then*

$$C = \overline{\mathfrak{U}.(C \cap \mathfrak{t})}.$$

Proof. Obviously we have $\overline{\mathfrak{U}.(C \cap \mathfrak{t})} \subseteq C$. To show the other inclusion, we pick $X \in C$ and $\varepsilon > 0$. With Lemma 7.1 we can find a diagonalizable operator $X_\varepsilon \in \mathfrak{h}$ satisfying $\|X_\varepsilon - X\| \le \varepsilon$ and $\overline{p_\mathfrak{t}(\mathfrak{U}.X_\varepsilon)} = \overline{p_\mathfrak{t}(\mathfrak{U}.X)}$. As X_ε is diagonalizable, we can find a $U_\varepsilon \in \mathfrak{U}$ such that $U_\varepsilon.X_\varepsilon \in \mathfrak{t}$. Now we get with Theorem 7.5

$$U_\varepsilon.X_\varepsilon \in \overline{p_\mathfrak{t}(\mathfrak{U}.X_\varepsilon)} = \overline{p_\mathfrak{t}(\mathfrak{U}.X)} \subseteq \overline{p_\mathfrak{t}(\overline{\operatorname{conv}(\mathfrak{U}.X)})} = \overline{\overline{\operatorname{conv}(\mathfrak{U}.X)} \cap \mathfrak{t}} = \overline{\operatorname{conv}(\mathfrak{U}.X)} \cap \mathfrak{t} \subseteq C \cap \mathfrak{t}.$$

Therefore $X_\varepsilon \in \mathfrak{U}.(C \cap \mathfrak{t})$ for every $\varepsilon > 0$. This finishes the proof. □

The only case left out in the last two theorems is $\mathfrak{h} = \mathfrak{ho}(I_c)$. This case is a little more complicated as we have two conjugacy classes of Cartan subalgebras. We recall the definition of the two nonconjugate Cartan subalgebras $\mathfrak{t}_1$ and $\mathfrak{t}_2$ from Lemma 3.5.

Lemma 7.7. *Let $a \in l^\infty(J)$. Then there exists an $\tilde{a} \in l^\infty(J)$ with infinitely many entries equal to 0 satisfying $\overline{\mathrm{conv}(\mathfrak{W}_2.a)} = \overline{\mathrm{conv}(\mathfrak{W}_2.\tilde{a})}$*

Proof. There exists a bijection $\rho : J \cup \mathbb{N} \to J$. We define

$$\tilde{a}_j := \begin{cases} a_{\rho^{-1}(j)} & \rho^{-1}(j) \in J \\ 0 & \text{else} \end{cases} .$$

Then $\tilde{a}$ has infinitely many entries equal to 0 and obviously $\overline{a} = \overline{\tilde{a}}$ and $\aleph^+(|a|) = \aleph^+(|\tilde{a}|)$ for all $\aleph \in \mathfrak{M}$. So $\tilde{a}$ has the required properties. $\square$

Proposition 7.8. *Let $A \in \mathfrak{ho}(I_c)$. Further let $\mathfrak{t} \subseteq \mathfrak{ho}(I_c)$ denote a Cartan subspace. Then we have*

$$\begin{aligned} \overline{p_{\mathfrak{t}}(\mathrm{conv}(\mathrm{UO}(I_c).A)} &= \overline{\mathrm{conv}(\mathrm{UO}(I_c).A))} \cap \mathfrak{t} \\ &= \{B \in \mathfrak{t} : (\forall f \in \mathfrak{F}^+)\, f(B) \leq f(A)\} \end{aligned}$$

Proof. We assume that $\mathfrak{t} = \mathfrak{t}_1$. The proof for the case $\mathfrak{t} = \mathfrak{t}_2$ is identical. We write

$$\mathfrak{C}(A) = \{B \in \mathfrak{t} : (\forall f \in \mathfrak{F})\, f(B) \leq f(A)\}.$$

According to Corollary 5.6

$$\mathfrak{C}(A) = \{B \in \mathfrak{t} : (\forall f \in \mathfrak{F}^+)\, f(B) \leq f(A)\}.$$

We obviously have $\overline{\mathrm{conv}(\mathrm{UO}(I_c).A)} \cap \mathfrak{t}_1 \subseteq \overline{p_{\mathfrak{t}_1}(\mathrm{UO}(I_c).A)}$. It follows from Lemma 4.7 and Corollary 7.4 that $\overline{p_{\mathfrak{t}_1}(\mathrm{UO}(I_c).A)} \subseteq \mathfrak{C}(A)$. So all that is left to show is that $\mathfrak{C}(A) \subseteq \overline{\mathrm{conv}(\mathrm{UO}(I_c).A)} \cap \mathfrak{t}_1$.

We pick $X \in \mathfrak{C}(A)$ and choose an $\varepsilon > 0$. With Lemma 5.7 we can find $A_\varepsilon \in \mathfrak{ho}(I_c)$ such that A_ε is diagonalizable and satisfies $f(A_\varepsilon) = f(A)$ for all $f \in \mathfrak{F}$. By Lemma 3.6, the operator A_ε can be conjugated into $\mathfrak{t}_1$ or $\mathfrak{t}_2$ under $\mathrm{UO}(I_c)$. If there exists a $U \in \mathrm{UO}(I_c)$ such that $U.A_\varepsilon \in \mathfrak{t}_1$ we obtain with Lemma 4.10 that $X \in \overline{\mathrm{conv}(\mathfrak{W}_2(J).(U.A_\varepsilon))} \subseteq \overline{\mathrm{conv}(\mathrm{UO}(I_c).A_\varepsilon)}$.

Now we assume there exists only a $U \in \mathrm{UO}(I_c)$ such that $U.A_\varepsilon \in \mathfrak{t}_2$. With Lemma 7.7 we can find an $\tilde{A} \in \mathfrak{t}_2$ with infinite dimensional kernel such that $f(\tilde{A}) = f(U.A_\varepsilon)$ for all $f \in \mathfrak{F}$. In particular we get

$$\tilde{A} \in \overline{\mathrm{conv}(\mathfrak{W}_2.(U.A_\varepsilon))} \subseteq \overline{\mathrm{conv}(\mathrm{UO}(I_c).A_\varepsilon)}.$$

With Lemma 3.6 we can find a $U' \in \mathrm{UO}(I_c)$ such that $U'.\tilde{A} \in \mathfrak{t}_1$ and therefore because of Lemma 4.10

$$\begin{aligned} X &\in \overline{\mathrm{conv}(\mathfrak{W}_2.(U'.\tilde{A}))} \subseteq \overline{\mathrm{conv}(\mathrm{UO}(I_c).\tilde{A})} \\ &\subseteq \overline{\mathrm{conv}(\mathrm{UO}(I_c).\overline{\mathrm{conv}(\mathrm{UO}(I_c).A_\varepsilon)})} = \overline{\mathrm{conv}(\mathrm{UO}(I_c).A_\varepsilon)}. \end{aligned}$$

So $X \in \overline{\mathrm{conv}(\mathrm{UO}(I_c).A_\varepsilon)}$. We can find $\lambda_1, \ldots, \lambda_m \in [0,1]$, $\lambda_1 + \ldots + \lambda_m = 1$ and $U_1, \ldots, U_m \in \mathrm{UO}(I_c)$ such that $\|X - \sum_{i=1}^m \lambda_i U_i.A_\varepsilon\| \leq \varepsilon$ and

$$\begin{aligned}\left\|X - \sum_{i=1}^m \lambda_i U_i.A\right\| &\leq \left\|X - \sum_{i=1}^m \lambda_i U_i.A_\varepsilon\right\| + \left\|\sum_{i=1}^m \lambda_i U_i.(A_\varepsilon - A)\right\| \\ &\leq \varepsilon + \sum_{i=1}^m \lambda_i \|U_i.(A_\varepsilon - A)\| \leq 2\varepsilon.\end{aligned}$$

As ε was arbitrary this finishes the proof. □

Proposition 7.9. *Let $C \subseteq \mathfrak{ho}(I_c)$ be closed, convex and* $\mathrm{UO}(I_c)$*-invariant. Further let $\mathfrak{t}$ denote a Cartan subalgebra of $\mathfrak{ho}(I_c)$. Then*

$$C = \overline{\mathrm{conv}(\mathrm{UO}(I_c).(C \cap \mathfrak{t}))}.$$

In particular we get $p_\mathfrak{t}(C) = C \cap \mathfrak{t}$.

Proof. We can assume that $\mathfrak{t} = \mathfrak{t}_1$, as the proof is identical in the other case.

Clearly $\overline{\mathrm{conv}(\mathrm{UO}(I_c).(C \cap \mathfrak{t}))} \subseteq C$. So we pick $A \in C$ and choose $\varepsilon > 0$. With Lemma 5.7 we can find $A_\varepsilon \in \mathfrak{ho}(I_c)$ such that A_ε is diagonalizable and satisfies $\|A - A_\varepsilon\| \leq \varepsilon$ and $f(A_\varepsilon) = f(A)$ for all $f \in \mathfrak{F}$. By Lemma 3.6 the operator A_ε can be conjugated into $\mathfrak{t}_1$ or $\mathfrak{t}_2$ under $\mathrm{UO}(I_c)$. If there exists a $U \in \mathrm{UO}(J)$ such that $U.A_\varepsilon \in \mathfrak{t}_1$ we have with Proposition 7.8 that $U.A_\varepsilon \in \overline{\mathrm{conv}(\mathrm{UO}(I_c).A)} \cap \mathfrak{t}_1 \subseteq C \cap \mathfrak{t}_1$ and therefore $A_\varepsilon \in \overline{\mathrm{UO}(I_c).(C \cap \mathfrak{t}_1)}$.

Now we assume there exists only a $U \in \mathrm{UO}(J)$ such that $U.A_\varepsilon \in \mathfrak{t}_2$. With Lemma 7.7 we can find a $\tilde{A} \in \mathfrak{t}_2$ with infinite dimensional Kernel such that $f(\tilde{A}) = f(U.A_\varepsilon)$ for all $f \in \mathfrak{F}$. In particular we get

$$U.A_\varepsilon \in \overline{\mathrm{conv}(\mathfrak{W}_2.\tilde{A})} \subseteq \overline{\mathrm{conv}(\mathrm{UO}(I_c).\tilde{A})}.$$

With Lemma 3.6 we can now find a $U' \in \mathrm{UO}(I_c)$ such that $U'.\tilde{A} \in \mathfrak{t}_1$. With Proposition 7.8 we get $U'.\tilde{A} \in \overline{\mathrm{conv}(\mathrm{UO}(I_c).A)} \cap \mathfrak{t}_1 \subseteq C \cap \mathfrak{t}_1$ and therefore

$$\begin{aligned}A_\varepsilon &\in U^{-1}.\overline{\mathrm{conv}(\mathrm{UO}(I_c).\tilde{A})} \subseteq \overline{\mathrm{conv}(\mathrm{UO}(I_c).\tilde{A})} \\ &= \overline{\mathrm{conv}(\mathrm{UO}(I_c).((U')^{-1}.\tilde{A}))} \subseteq \overline{\mathrm{conv}\,\mathrm{UO}(I_c).(C \cap \mathfrak{t}_1)}.\end{aligned}$$

As ε was arbitrary this implies $A \in \overline{\mathrm{conv}(\mathrm{UO}(I_c).(C \cap \mathfrak{t}_1))}$.

The last assertion is an immediate consequence of Proposition 7.8. □

Now we can prove a result similar to Theorem 7.3 for all $\mathfrak{h} \in \{\mathfrak{h}(\mathfrak{H}), \mathfrak{ho}(I_c), \mathfrak{hsp}(I_a)\}$.

Theorem 7.10. (1) *Let $A \in \mathfrak{h}(\mathfrak{H})$. Then $B \in \overline{\mathrm{conv}(\mathfrak{U}(\mathfrak{H}).A)}$ if and only if $f(B) \leq f(A)$ for all $f \in \mathfrak{F}$.*

(2) *Let $A \in \mathfrak{h}$, $\mathfrak{h} \in \{\mathfrak{ho}(I_c), \mathfrak{hsp}(I_a)\}$. Then $B \in \overline{\mathrm{conv}(\mathfrak{U}.A)}$ if and only if $f(B) \leq f(A)$ for all $f \in \mathfrak{F}^+$.*

Proof. The first assertion is Theorem 7.3. All that is left to prove is the second assertion.

Lemma 4.7 tells us that $f \in \mathfrak{F}^+$ are continuous convex $\mathfrak{U}$-invariant functions. This immediately shows one inclusion. We assume that $f(B) \leq f(A)$ for all $f \in \mathfrak{F}^+$.

Let $\varepsilon > 0$. We use Lemma 5.7 to approximate B with a diagonalizable operator $B_\varepsilon \in \mathfrak{h}$ that satisfies $\|B_\varepsilon - B\| \leq \varepsilon$ and $f(B_\varepsilon) = f(B)$ for all $f \in \mathfrak{F}^+$. As B_ε is diagonalizable, we can use Lemma 3.6 to find a Cartan subspace $\mathfrak{t}$ such that $B_\varepsilon \in \mathfrak{t}$. Let $p_\mathfrak{t}$ denote the corresponding projection. We get with Theorem 6.4, resp. Proposition 7.8, that $B_\varepsilon \in \overline{p_\mathfrak{t}(\mathfrak{U}.A)}$ and therefore by Theorem 7.5 resp. Proposition 7.9 the relation $B_\varepsilon \in \overline{\mathrm{conv}(\mathfrak{U}.A)}$. As $\|B_\varepsilon - B\| \leq \varepsilon$, we conclude that $B \in \overline{\mathrm{conv}(\mathfrak{U}.A)}$ which proves the assertion. □

Proposition 7.11. *Let* $\mathfrak{h} \in \{\mathfrak{h}(\mathfrak{H}), \mathfrak{ho}(I_c), \mathfrak{hsp}(I_a)\}$. *Then we have for every closed convex set* $C \subseteq \mathfrak{h}$

$$C^o \cap \mathfrak{t} = p_\mathfrak{t}(C^o) = \mathrm{int}_\mathfrak{t}(C \cap \mathfrak{t}).$$

where C^o *denotes the interior of* C *and* $\mathrm{int}_\mathfrak{t}(C \cap \mathfrak{t})$ *denotes the interior of* $C \cap \mathfrak{t}$ *in* $\mathfrak{t}$.

Proof. The inclusion $C^o \cap \mathfrak{t} \subseteq p_\mathfrak{t}(C^o)$ is obvious and $p_\mathfrak{t}(C^o) \subseteq \mathrm{int}_\mathfrak{t}(C \cap \mathfrak{t})$ follows from $p_\mathfrak{t}(C) = C \cap \mathfrak{t}$ (Theorem 7.5 resp. Proposition 7.9) and the Open Mapping Theorem. So all we have to show is $\mathrm{int}_\mathfrak{t}(C \cap \mathfrak{t}) \subseteq C^o \cap \mathfrak{t}$.

We choose $X \in \mathrm{int}_\mathfrak{t}(C \cap \mathfrak{t})$. Then there exists an $\varepsilon > 0$ such, that

$$\{Y \in \mathfrak{t} : \|Y - X\| \leq \varepsilon\} \subseteq C.$$

If $\mathfrak{h} = \mathfrak{h}(\mathfrak{H})$ we can write $X = \mathrm{diag}(x)$ for $x \in l^\infty(J)$.

If there exists an $\aleph \in \mathfrak{M}$ with $\aleph^+(x) = \aleph^-(x)$ we define

$$\aleph_1 := \min\{\aleph \in \mathfrak{M} : \aleph^+(x) = \aleph^-(x)\}, \qquad J_1 := \{j \in J : x_j \in]\aleph_1^+(x) - \tfrac{\varepsilon}{4}, \aleph_1^+(x) + \tfrac{\varepsilon}{4}[\},$$

write J_1 as the disjoint union of two sets I^+ and I^- with $\#I^+ = \#I^- = \#J_1 = \#J$, and define $\delta := \aleph_1^+(x) = \aleph_1^-(x)$. Else we let $J_1 := I^+ := I^- := \emptyset$ and choose $\delta \in](\#J)^-(x), (\#J)^+(x)[$ arbitrary. Then we define $y \in l^\infty(J)$ by

$$y_j := \begin{cases} x_j + \varepsilon & x_j \geq \delta, j \notin J_1 \\ x_j + \frac{3}{4}\varepsilon & j \in I^+ \\ x_j - \frac{3}{4}\varepsilon & j \in I^- \\ x_j - \varepsilon & x_j < \delta, j \notin J_1 \end{cases}.$$

For $Y := \mathrm{diag}(y)$ we have $f(Y) \geq f(X) + \frac{\varepsilon}{4}$ for all $f \in \mathfrak{F}$.

If $\mathfrak{h} \neq \mathfrak{h}(\mathfrak{H})$ we can assume $X = \mathrm{diag}_\mathfrak{d}(x)$ for $x \in l^\infty(J)$. The case $X = \mathrm{diag}'_\mathfrak{d}(x)$ is identical. Then we define $y \in l^\infty(J)$ via

$$y_j := \begin{cases} x_j + \varepsilon & x_j \geq 0 \\ x_j - \varepsilon & x_j < 0 \end{cases}.$$

and get for $Y := \mathrm{diag}_\mathfrak{d}(y)$ that $f(Y) = f(X) + \varepsilon$ for all $f \in \mathfrak{F}$.

So in both cases we obtain an $Y \in \mathfrak{t}$ with $f(Y) \geq f(X) + \frac{\varepsilon}{4}$ for all $f \in \mathfrak{F}$ and $\|Y - X\| \leq \varepsilon$, implying $Y \in C$. Since all $f \in \mathfrak{F}$ are contracting, we obtain with Theorem 7.10 that

$$\{X' \in \mathfrak{h} : \|X' - X\| \leq \tfrac{\varepsilon}{4}\} \subseteq \overline{\mathrm{conv}(\mathfrak{U}.Y)} \subseteq C.$$

This implies $X \in C^o$, which proves the assertion. □

Now we summarize our results on the reconstructability of closed convex $\mathfrak{U}$-invariant sets $C \subseteq \mathfrak{h}$.

Theorem 7.12. *Let $\mathfrak{h} \in \{\mathfrak{h}(J), \mathfrak{hsp}(I_a), \mathfrak{ho}(I_c)\}$. Further let $\mathfrak{t}$ denote a Cartan subalgebra of $\mathfrak{h}$, $p_\mathfrak{t} \colon \mathfrak{h} \to \mathfrak{t}$ the corresponding projection, $\mathfrak{U}$ the corresponding unitary group and $\mathfrak{W}$ the corresponding Weyl group.*

We have for every closed convex $\mathfrak{U}$-invariant set $C \subseteq \mathfrak{h}$

(1) $p_\mathfrak{t}(C) = C \cap \mathfrak{t}$.
(2) $p_\mathfrak{t}(C^o) = C^o \cap \mathfrak{t} = \mathrm{int}_\mathfrak{t}(C \cap \mathfrak{t})$.
(3) $C = \overline{\mathrm{conv}(\mathfrak{U}.(C \cap \mathfrak{t}))}$.

In particular the map $p_\mathfrak{t}$ induces a bijection between the $\mathfrak{U}$-invariant closed convex subsets of $\mathfrak{h}$ and the $\mathfrak{W}$-invariant closed convex subsets of $\mathfrak{t}$.

Proof. (1)–(3) have already been proven. Let C be a closed convex $\mathfrak{U}$-invariant subset of $\mathfrak{h}$. It follows from Theorem 7.6 or Proposition 7.9 that $C = \overline{\mathrm{conv}(\mathfrak{U}.(C \cap \mathfrak{t}))}$. Therefore the map $C \mapsto p_\mathfrak{t}(C)$ is injective. All that is left to show is that this map is surjective. In other words we have to show that for every $\mathfrak{W}$-invariant closed convex subset $C_\mathfrak{t}$ of $\mathfrak{t}$ there exists a $\mathfrak{U}$-invariant closed convex subset C of $\mathfrak{h}$ satisfying $C_\mathfrak{t} = p_\mathfrak{t}(C) = C \cap \mathfrak{t}$.

We pick such a set $C_\mathfrak{t}$ and define $C := \overline{\mathrm{conv}(\mathfrak{U}.C_\mathfrak{t})}$. Obviously we have $C_\mathfrak{t} \subseteq p_\mathfrak{t}(C)$. On the other hand we get with the help of Theorem 3.9 for every $X \in C_\mathfrak{t}$

$$p_\mathfrak{t}(\mathfrak{U}.X) \subseteq \overline{\mathrm{conv}(\mathfrak{W}.X)} \subseteq C_\mathfrak{t}.$$

Now the last assertion follows. □

Remark 7.13. We would like to remark that in the case $\mathfrak{h} = \mathfrak{ho}(I_c)$ we no longer get the assertion $C = \overline{\mathfrak{U}.(C \cap \mathfrak{t})}$ that was true in the other cases. To see this, we consider the following example:

We let $\mathfrak{t} = \mathfrak{t}_1$ be a Cartan subalgebra of type one. We choose an I_c-basis of type one such that $\mathfrak{t}_1$ is the algebra of diagonal operators. Then we have $\mathfrak{t}_1 = \{\mathrm{diag}_\mathfrak{d}(a) : a \in l^\infty(J)\}$. If we pick $b_j = 1$ for all $j \in J$ we get

$$B := \mathrm{diag}_\mathfrak{d}(b) = \begin{pmatrix} \mathbf{1} & \\ & -\mathbf{1} \end{pmatrix} \in \mathfrak{ho}(2J).$$

Obviously we get for every $v \in l^2(2J)$ that $\|B.v\| = \|v\|$. We choose $A \in \mathfrak{t}_2$ satisfying $(\#J)^+ \geq 1$. We obtain $B \in C := \overline{\mathrm{conv}(\mathrm{UO}(I_c).A)}$ from Theorem 7.10.

We can find for every $X \in C \cap \mathfrak{t}_2$ at least one $0 \neq v \in l^2(2J)$ satisfying $X.v = 0$. So we have for every $U \in \mathrm{UO}(I_c)$ that

$$\begin{aligned} \|(B - U^{-1}XU).(U^{-1}v)\| &= \|B(U^{-1}v) - U^{-1}Xv\| \\ &= \|B(U^{-1}v)\| = \|U^{-1}v\| \end{aligned}$$

and therefore $\|B - U.X\| \geq 1$. Thus we get $B \notin \overline{\mathrm{UO}(I_c).(C \cap \mathfrak{t}_2)}$.

In a similar way we can obtain a $B \in \overline{\mathrm{conv}(\mathrm{UO}(I_c).\mathfrak{t}_1)} \backslash \overline{\mathrm{UO}(I_c).\mathfrak{t}_1}$. This implies that for closed convex $\mathfrak{W}$-invariant sets $C_\mathfrak{t} \subseteq \mathfrak{t}$ the sets $\overline{\mathrm{UO}(I_c).(C_\mathfrak{t})}$ need not be convex.

Remark 7.14. Now we can also understand why it was necessary to demand that our Cartan subalgebras be simultaneously diagonalizable. The reason for this is that Theorem 7.12 need not be true without this restraint.

To see this, we look at $\mathfrak{H} = L^2([0,1])$ with respect to the Lebesgue measure. Then $\mathfrak{t} := L^\infty([0,1])$ acts on $\mathfrak{H}$ by multiplication and is a maximal abelian subspace of $\mathfrak{h}(\mathfrak{H})$, the space of hermitian bounded linear operators on $L^2([0,1])$. Now we look at the closed convex subset C of $\mathfrak{h}$ consisting of all the hermitian compact operators. We get $C \cap \mathfrak{t} = \{0\}$. This follows from the fact that each $X \in C$ is diagonalizable and each eigenspace of X except the one for the eigenvalue 0 is finite dimensional. The elements of $\mathfrak{t}$ do not have finite dimensional eigenspaces.

This illustrates that closed convex sets are not reconstructible from their intersection with arbitrary maximal abelian subalgebras. As this was an essential property in the finite dimensional case, we had to include the additional condition in our definition.

We conclude the section with an observation about general continuous convex $\mathfrak{U}$-invariant functions.

Proposition 7.15. (1) *Let $\mathfrak{h} \in \{\mathfrak{h}(\mathfrak{H}), \mathfrak{hsp}(I_a)\}$. Further let $\mathfrak{t}$ be a Cartan subspace and $\mathfrak{U}$ the corresponding maximal unitary group.*
Let C be a closed convex $\mathfrak{U}$-invariant subset of $\mathfrak{h}$. A continuous $\mathfrak{U}$-invariant function $f\colon C \to \mathbb{R}$ is convex if and only if its restriction $f|_{C\cap\mathfrak{t}}\colon C \cap \mathfrak{t} \to \mathbb{R}$ is convex.
(2) *There exists a continuous $\mathrm{UO}(I_c)$-invariant function $f\colon \mathfrak{ho}(I_c) \to \mathbb{R}$ that is not convex but has a convex restriction $f|_{\mathfrak{t}}\colon \mathfrak{t} \to \mathbb{R}$.*

Proof. (1) One inclusion is obvious. So we assume now that $f|_{C\cap\mathfrak{t}}$ is convex. We define $\mathfrak{h}' := \mathfrak{h} \oplus \mathbb{R}$, $\mathfrak{t}' := \mathfrak{t} \oplus \mathbb{R}$, $C' := C \oplus \mathbb{R}$, where all these sets are equipped with the product topology, and

$$pr : \mathfrak{h}' \to \mathfrak{t}' \ ; \ (X, \eta) \mapsto (p_{\mathfrak{t}}(X), \eta).$$

Then $\mathfrak{U}$ acts on $\mathfrak{h}'$, and $\mathfrak{W}$, the corresponding Weyl group, acts on $\mathfrak{t}'$ by acting on the first component. Further we define $E := \{(X, \eta) \in C' : f(X) \leq \eta\}$, the epigraph of f. The set E is closed and $\mathfrak{U}$-invariant. We know that f is convex if and only if E is convex. By our assumption we have that $E \cap \mathfrak{t}$ is closed, convex and $\mathfrak{W}$-invariant.

We define for $\eta \in \mathbb{R}$

$$\begin{aligned} E_\eta &:= \{X \in C : (X, \eta) \in E\} = \{X \in C : f(X) \leq \eta\}, \\ E_\eta^1 &:= \overline{\{X \in C : f(X) < \eta\}}. \end{aligned}$$

Then $E_\eta^1 \subseteq E_\eta$ and both sets are closed and $\mathfrak{U}$-invariant. The set E_η^1 has dense interior in C. Therefore we get with Theorem 7.6 that $E_\eta^1 = \overline{\mathfrak{U}.(E_\eta^1 \cap \mathfrak{t})}$. So we have that

$$E^1 := \bigcup_{\eta\in\mathbb{R}} E_\eta^1 \times \{\eta\} \subseteq \overline{\mathfrak{U}.(E^1 \cap \mathfrak{t})}.$$

On the other hand, if $(X, \eta) \in E$ for $X \in C$, $\eta \in \mathbb{R}$, then $f(X) \leq \eta$, so we have $f(X) < \eta + \varepsilon$ for all $\varepsilon > 0$ and $(X, \eta + \varepsilon) \in E_{\eta+\varepsilon}^1 \times \{\eta + \varepsilon\} \subseteq E^1$. Therefore E^1 is dense in E. From the closedness of E we now get

$$E \subseteq \overline{\mathfrak{U}.(E^1 \cap \mathfrak{t}')} \subseteq \overline{\mathfrak{U}.(E \cap \mathfrak{t}')}$$

and with Theorem 3.9

$$\begin{aligned} pr(E) &\subseteq \overline{\bigcup_{\eta\in\mathbb{R}} p_{\mathfrak{t}}(E_\eta^1)\times\{\eta\}} = \overline{\bigcup_{\eta\in\mathbb{R}} p_{\mathfrak{t}}\left(\overline{\mathfrak{U}.(E_\eta^1\cap\mathfrak{t})}\right)\times\{\eta\}} \subseteq \overline{\bigcup_{\eta\in\mathbb{R}} p_{\mathfrak{t}}\left(\overline{\mathfrak{U}.(E_\eta\cap\mathfrak{t})}\right)\times\{\eta\}} \\ &\subseteq \overline{\bigcup_{\eta\in\mathbb{R}} \overline{\mathrm{conv}(\mathfrak{W}(E_\eta\cap\mathfrak{t}))}\times\{\eta\}} = \overline{\bigcup_{\eta\in\mathbb{R}} (E_\eta\cap\mathfrak{t})\times\{\eta\}} = E\cap\mathfrak{t}', \end{aligned}$$

so we have $pr(E) = E\cap\mathfrak{t}'$. We define

$$F := \bigcap_{U\in\mathfrak{U}} U.pr^{-1}(E\cap\mathfrak{t}).$$

The set F is closed, convex and $\mathfrak{U}$-invariant. So $F_\eta := \{X\in C : (X,\eta)\in F\}$ is closed, convex and $\mathfrak{U}$-invariant for every $\eta\in\mathbb{R}$. In particular $F_\eta\cap\mathfrak{t}$ is $\mathfrak{W}$-invariant, closed and convex. Then we get from Theorem 7.6 that $F_\eta = \overline{\mathfrak{U}.\,(F_\eta\cap\mathfrak{t})}$ for every $\eta\in\mathbb{R}$ and therefore $F = \overline{\mathfrak{U}.\,(F\cap\mathfrak{t}')}$. Now we get with Theorem 7.5 that

$$pr(F) = \bigcup_{\eta\in\mathbb{R}} p_{\mathfrak{t}}(F_\eta)\times\{\eta\} = \bigcup_{\eta\in\mathbb{R}} (F_\eta\cap\mathfrak{t})\times\{\eta\} = F\cap\mathfrak{t}'.$$

As E is $\mathfrak{U}$-invariant, we have $E\subseteq F$ and therefore $pr(E)\subseteq pr(F)$. On the other hand we get from the definition of F that $pr(F)\subseteq pr(E)$, therefore $pr(F) = pr(E)$. So we have

$$E = \overline{\mathfrak{U}.(E\cap\mathfrak{t}')} = \overline{\mathfrak{U}.pr(E)} = \overline{\mathfrak{U}.pr(F)} = \overline{\mathfrak{U}.(F\cap\mathfrak{t}')} = F.$$

As F is convex, this proves the assertion.

(2) We define

$$f(X) := -\mathrm{dist}(X, \overline{\mathrm{UO}(I_c).\mathfrak{t}}).$$

From Remark 7.13 we get an $X\in\mathfrak{ho}(I_c)$ with $f(X)<0$. As $f(-X) = f(X)$, we have

$$f(0) = 0 > \tfrac{1}{2}(f(X)+f(-X)),$$

so f is not convex. Obviously f is continuous and $\mathrm{UO}(I_c)$-invariant. For $X\in\mathfrak{t}$ we have $f(X)=0$, so f is convex on $\mathfrak{t}$. Therefore f is the desired counterexample. □

8. Controlling Invariant Convex Sets

In this section we want to see how we can describe closed convex $\mathfrak{U}$-invariant subsets with the functions $f\in\mathfrak{F}$. Unless states otherwise we assume throughout this section that $\mathfrak{h}\in\{\mathfrak{h}(\mathfrak{H}), \mathfrak{ho}(I_c), \mathfrak{hsp}(I_a)\}$ and $\mathfrak{U}$ is the corresponding unitary group.

For a special case we already have such a description. We recall Theorem 7.10, that states that for $A\in\mathfrak{h}$

$$\overline{\mathrm{conv}(\mathfrak{U}(\mathfrak{H}).A)} = \{B\in\mathfrak{h} : (\forall f\in\mathfrak{F})\, f(B)\le f(A)\}.$$

In the case $\mathfrak{h}\neq\mathfrak{h}(\mathfrak{H})$ the conditions for $f\in\mathfrak{F}\backslash\mathfrak{F}^+$ were redundant. This shows that sets of the form $\overline{\mathrm{conv}(\mathfrak{U}.A)}$ for $A\in\mathfrak{h}$ can be controlled solely with the functions in $\mathfrak{F}$ resp. $\mathfrak{F}^+$. We want to examine now what can be obtained for arbitrary closed convex $\mathfrak{U}$-invariant sets. To do so we first have to develop the necessary tools.

Definition 8.1. We define a map ρ the following way: If $\mathfrak{h} = \mathfrak{h}(\mathfrak{H})$ we define

$$\rho : \mathfrak{h}(\mathfrak{H}) \to l^\infty(\mathfrak{F}) \ ; \ A \mapsto (f(A))_{f\in\mathfrak{F}}$$

and else

$$\rho : \mathfrak{h} \to l^\infty(\mathfrak{F}^+) \ ; \ A \mapsto (f(A))_{f\in\mathfrak{F}^+}.$$

It will turn out that ρ is a characteristic map for closed convex $\mathfrak{U}$-invariant subsets of $\mathfrak{h}$. The following lemma gives a first illustration of this fact.

Lemma 8.2. *Let $F\colon \mathfrak{h} \to \mathbb{R}$ be a continuous convex $\mathfrak{U}$-invariant function. Then there exists a function $\tilde{F}\colon l^\infty(\mathfrak{F}) \to \mathbb{R}$ resp. $\tilde{F}\colon l^\infty(\mathfrak{F}^+) \to \mathbb{R}$ such that $F = \tilde{F} \circ \rho$.*

Proof. We have to show that $F(A) = F(B)$ holds whenever $f(A) = f(B)$ for all $f \in \mathfrak{F}$ resp. $f \in \mathfrak{F}^+$. But in this case we get from Theorem 7.10 that

$$B \in \overline{\mathrm{conv}(\mathfrak{U}.A)} \qquad \text{and} \qquad A \in \overline{\mathrm{conv}(\mathfrak{U}.B)}.$$

So we have $F(B) \le F(A)$ and $F(A) \le F(B)$. This proves the assertion. □

Now we want to examine the map ρ closer. We start with a technical lemma.

Lemma 8.3. (1) *We have $\#\mathfrak{M} \le \#J$.*
(2) *For each $\aleph \in \mathfrak{M}^*$ we choose sets $M_\aleph, M_\aleph^+, M_\aleph^-$ of cardinality $\aleph$. Then*

$$\#J = \#\Big(\mathbb{N} \,\dot\cup \dot{\bigcup_{\aleph\in\mathfrak{M}^*}} M_\aleph\Big) = \#\Big(\mathbb{N}_1 \,\dot\cup\, \mathbb{N}_2 \,\dot\cup \dot{\bigcup_{\aleph\in\mathfrak{M}^*}} (M_\aleph^+ \,\dot\cup\, M_\aleph^-)\Big),$$

where $\mathbb{N}_1$ and $\mathbb{N}_2$ are identical copies of $\mathbb{N}$.

Proof. (1) For each ordinal number α we write $c(\alpha) := \#A$, where A is an ordered set with $\mathrm{ord}(A) = \alpha$. Obviously this definition is independent of the choice of A. According to the Well-ordering Theorem there exists for each cardinal number $\aleph$ an ordinal number α with $\aleph = c(\alpha)$. We choose an α_J satisfying $c(\alpha_J) = \#J$. Then, according to [2, Theorem 4.47], the set P_{α_J} of ordinal numbers $< \alpha_J$ is well-ordered and $\mathrm{ord}(P_{\alpha_J}) = \alpha_J$, in particular $\#P_{\alpha_J} = \#J$. On the other hand we get an injective map *or* from $\mathfrak{M}\backslash\{\#J\}$ into P_{α_J} by setting $or(\aleph) := \min\{\alpha : c(\alpha) = \aleph\}$. Therefore

$$\#(\mathfrak{M}\backslash\{\#J\}) \le \#P_{\alpha_J} = \#J.$$

As J was infinite the assertion follows.

(2) Obviously

$$\#\Big(\mathbb{N} \,\dot\cup \dot{\bigcup_{\aleph\in\mathfrak{M}^*}} M_\aleph\Big) = \#\Big(\mathbb{N} \,\dot\cup\, \mathbb{N} \,\dot\cup \dot{\bigcup_{\aleph\in\mathfrak{M}^*}} (M_\aleph^+ \,\dot\cup\, M_\aleph^-)\Big).$$

So we only have to show

$$\#J = \#\Big(\mathbb{N} \,\dot\cup \dot{\bigcup_{\aleph\in\mathfrak{M}^*}} M_\aleph\Big).$$

The inequality $\ge$ follows from (1). If $\#J \in \mathfrak{M}^*$, then $\le$ is obvious. Otherwise we note that $\#(\dot\bigcup_{\aleph'\in\mathfrak{M}^*} M_{\aleph'}^+) \ge \aleph$ for each $\aleph \in \mathfrak{M}^*$, and as $\#J \notin \mathfrak{M}^*$ we have $\#J = \sup\{\aleph \in \mathfrak{M} : \aleph < \#J\}$, so the assertion follows as well. □

Definition 8.4. (1) We define $\mathfrak{R}$ as the set of all $c \in l^\infty(\mathfrak{F})$ satisfying

(R1) $(k+1)c_{(k+1)^+} + (k-1)c_{(k-1)^+} \leq 2kc_{k^+}$ for all $k \in \mathbb{N}$.
(R2) $(k+1)c_{(k+1)^-} + (k-1)c_{(k-1)^-} \leq 2kc_{k^-}$ for all $k \in \mathbb{N}$.
(R3) $c_f \geq c_g$ for all $f \leq g \in \mathfrak{F}$.
(R4) $c_{\aleph^+} + c_{-\aleph^-} \geq 0$ for all $\aleph \in \mathfrak{M}^*$.

Analogously we define $\mathfrak{R}'$ as the set of all $c \in l^\infty(\mathfrak{F}^+)$ satisfying

(R1') $(k+1)c_{(k+1)^+} + (k-1)c_{(k-1)^+} \leq 2kc_{k^+}$ for all $k \in \mathbb{N}$.
(R2') $c_f \geq c_g$ for all $f \leq g \in \mathfrak{F}^+$.
(R3') $c_{\aleph^+} \geq 0$ for all $\aleph \in \mathfrak{M}^*$.

(2) According to Lemma 8.3 we can write, up to a bijection,

$$J = \mathbb{N}_1 \dot\cup \mathbb{N}_2 \dot\cup \dot{\bigcup_{\aleph \in \mathfrak{M}^*}} (M_\aleph^+ \dot\cup M_\aleph^-),$$

where $\# M_\aleph^+ = \# M_\aleph^- = \aleph$. For $c \in \mathfrak{R}$ we construct a sequence $\tilde{c}$:

$$\tilde{c}|_{\mathbb{N}_1} = (\gamma_1^+, \gamma_2^+, \ldots) \qquad \tilde{c}|_{\mathbb{N}_2} = (-\gamma_1^-, -\gamma_2^-, \ldots),$$

where $\gamma_n^+ = nc_{n^+} - (n-1)c_{(n-1)^+}$ and $\gamma_n^- = nc_{n^-} - (n-1)c_{(n-1)^-}$. Further

$$\tilde{c}_j = c_{\aleph^+} \quad \forall j \in M_\aleph^+, \quad \tilde{c}_j = -c_{-\aleph^-} \quad \forall j \in M_\aleph^-$$

for all $\aleph \in \mathfrak{M}^*$. It is shown in Lemma 8.5 below that $\tilde{c} \in l^\infty(J)$. We define

$$h(c) := \operatorname{diag}(\tilde{c}) \in \mathfrak{h}(J).$$

If $\mathfrak{h} \neq \mathfrak{h}(J)$, then we write

$$J = \mathbb{N} \dot\cup \dot{\bigcup_{\aleph \in \mathfrak{M}^*}} M_\aleph,$$

where $\# M_\aleph = \aleph$. For $c \in \mathfrak{R}'$ we construct $\tilde{c} \in l^\infty(J)$:

$$\tilde{c}|_{\mathbb{N}} = (\gamma_1^+, \gamma_2^+, \ldots), \qquad \tilde{c}_j = c_{\aleph^+} \quad \forall j \in M_\aleph^+$$

where $\gamma_n^+ = nc_{n^+} - (n-1)c_{(n-1)^+}$ and $\aleph \in \mathfrak{M}^*$. Then we define

$$h'(c) := \operatorname{diag}_{\mathfrak{d}}(\tilde{c}) \in \mathfrak{h}.$$

If $\# J \in \mathfrak{M}^*$ we can replace (R4) by $c_{(\# J)^+} + c_{-(\# j)^-} \geq 0$ and (R3') by $c_{(\# J)^+} \geq 0$.

Lemma 8.5. *With the notation of Definition 8.4 the following assertions hold*

(1) $c_{n^+} = \frac{1}{n}(\gamma_1^+ + \ldots \gamma_n^+)$ *and* $c_{n^-} = \frac{1}{n}(\gamma_1^- + \ldots \gamma_n^-)$.
(2) $(\gamma_1^+, \gamma_2^+, \ldots)$ *and* $(\gamma_1^-, \gamma_2^-, \ldots)$ *are decreasing sequences.*
(3) $\gamma_n^+ \geq c_{\aleph^+}$ *and* $\gamma_n^- \geq c_{\aleph^-}$ *for all* $n \in \mathbb{N}, \aleph \in \mathfrak{M}^*$.
(4) $\tilde{c} \in l^\infty(J)$.
(5) *For* $c \in \mathfrak{R}$ *we have* $f(h(c)) = c_f$ *for all* $f \in \mathfrak{F}$ *and for* $c \in \mathfrak{R}'$ *we have* $f(h'(c)) = c_f$ *for all* $f \in \mathfrak{F}^+$. *In particular*

$$\rho(h(c)) = c \qquad \text{and} \qquad \rho(h'(c)) = c.$$

(6) $\rho(\mathfrak{h}(J)) = \mathfrak{R}$ *and* $\rho(\mathfrak{h}) = \mathfrak{R}'$ *for* $\mathfrak{h} \neq \mathfrak{h}(J)$.

Proof. (1) follows immediately from the definition.

(2) This is equivalent to conditions (R1) and (R2), resp., (R1').

(3) We pick $n \in \mathbb{N}$ and $\aleph \in \mathfrak{M}^*$. Then we choose $\varepsilon \geq 0$. There exists a $\lambda \in \mathbb{N}$ such that $\frac{1}{l}(\gamma_1 + \ldots + \gamma_n) \leq \varepsilon$ and

$$c_{l^+} = \frac{1}{l}\sum_{j=1}^{l}\gamma_j = \frac{1}{l}\sum_{j=1}^{n}\gamma_j + \frac{1}{l}\sum_{j=n+1}^{l}\gamma_n \leq \varepsilon + \gamma_n.$$

Therefore $\gamma_n \geq c_{n^+} - \varepsilon \geq c_{\aleph^+} - \varepsilon$ and, as ε was arbitrary, $\gamma_n \geq c_{\aleph^+}$.

The same way we get that $\gamma_n \geq c_{-\aleph^-}$.

(4) It follows immediately from (2) and (3) that $\tilde{c}|_{\mathbb{N}_1}$ and $\tilde{c}|_{\mathbb{N}_2}$ are convergent and therefore bounded. The boundedness of $\tilde{c}|_{\bigcup_{\aleph \in \mathfrak{M}^*}(M_\aleph^+ \dot\cup M_\aleph^-)}$ follows immediately from the definition.

(5) Because of (3), we get $L_k(h(c)) = \gamma_1^+ + \ldots + \gamma_k^+ = kc_{k^+}$ and therefore $k^+(h(c)) = c_{k^+}$. Analogously we get $k^-(h(c)) = c_{k^-}$.

For $\aleph \in \mathfrak{M}^*$ we have $\#(\bigcup_{\aleph' < \aleph} M_\aleph^+) < \aleph$ and therefore $\aleph^+(h(c)) = c_{\aleph^+}$. Analogously we get $-\aleph^-(h(c)) = c_{-\aleph^-}$.

The proof for $f(h'(c)) = c_f$ is identical.

(6) The inclusion $\supseteq$ follows from (5). So all we have left to show is $\subseteq$.

Let $\mathfrak{h} = \mathfrak{h}(\mathfrak{H})$ and $c = \rho(A)$. Then

$$\begin{array}{rrcl} & (k+1)c_{(k+1)^+} + (k-1)c_{(k-1)^+} & \leq & 2kc_{k^+} \\ \Leftrightarrow & L_k(A) - L_{k-1}(A) & \geq & L_{k+1}(A) - L_k(A) \\ \Leftrightarrow & L_k(\overline{a}) - L_{k-1}(\overline{a}) & \geq & L_{k+1}(\overline{a}) - L_k(\overline{a}), \end{array}$$

where $\overline{A} = \operatorname{diag}(\overline{a})$. The last equivalence follows from Lemma 5.2. As $\overline{a} \in c_0(J)$ is nonnegative, we can find the $k+1$ largest entries $a_1 \geq \ldots \geq a_{k+1}$. Then our last equality just states that $a_k \geq a_{k+1}$ which is obviously true. The same way we see $(k+1)c_{(k+1)^-} + (k-1)c_{(k-1)^-} \leq 2kc_{k^-}$. The condition $c_f \geq c_g$ for all $f \leq g \in \mathfrak{F}$ is an immediate consequence of Lemma 4.9, and the last condition is equivalent to $(\#J)^+(A) \geq (\#J)^-(A)$.

The same arguments apply for $\mathfrak{h} \in \{\mathfrak{ho}(I_c), \mathfrak{hsp}(I_a)\}$ and $c = \rho(A)$. Here the condition $c_{(\#J)^+} \geq 0$ is equivalent to $(\#J)^+(A) \geq 0$, which follows from $(\#J)^+(A) = -(\#J)^-(A)$. $\square$

Lemma 8.6. *For* $a \in l^\infty(\mathbb{N})$ *we define the sequence* $\tilde{a}$ *via*

$$\tilde{a}_n := \frac{1}{n}(a_1 + \ldots + a_n).$$

Then we have

(1) *If* a *converges, then* $\tilde{a}$ *converges and*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \tilde{a}_n.$$

(2) *Let $a^{(k)} \in l^\infty(\mathbb{N})$ be decreasing and converging for every $k \in \mathbb{N}$. If $(\tilde{a}^{(k)})_{k\in\mathbb{N}}$ converges in $l^\infty(\mathbb{N})$ then so does $(a^{(k)})_{k\in\mathbb{N}}$.*

Proof. (1) is trivial.

(2) We assume that $(\tilde{a}^{(k)})_{k\in\mathbb{N}}$ converges to $\tilde{a} \in l^\infty(\mathbb{N})$. Each $\tilde{a}^{(k)}$ converges so $\tilde{a}$ converges as well as the space of convergent sequences $c_0(\mathbb{N}) \oplus \mathbb{R}$ is a closed subspace of $l^\infty(\mathbb{N})$.

We note that each $\tilde{a}^{(k)}$ is decreasing, so $\tilde{a}$ is decreasing as well. We define now $a_n := n\tilde{a}_n - (n-1)\tilde{a}_{n-1}$ and claim that $a = (a_n)_{n\in\mathbb{N}}$ lies in $l^\infty(\mathbb{N})$ and $(a^{(k)})_{k\in\mathbb{N}}$ converges to a.

The fact that $a^{(k)}$ is decreasing is equivalent to

$$(n-1)\tilde{a}^{(k)}_{n-1} + (n+1)\tilde{a}^{(k)}_{n+1} \leq 2n\tilde{a}^{(k)}_n$$

for all $n \in \mathbb{N}$. Convergence in $l^\infty(\mathbb{N})$ implies pointwise convergence, so we have $(n-1)\tilde{a}_{n-1} + (n+1)\tilde{a}_{n+1} \leq 2n\tilde{a}_n$ for all $n \in \mathbb{N}$ and a is decreasing.

Next we show that $a^{(k)}$ converges to a pointwise. Let $\varepsilon > 0$ and $n \in \mathbb{N}$. There exists a $K \in \mathbb{N}$ such that for all $k \geq K$ we have $\|\tilde{a}^{(k)} - \tilde{a}\| \leq \frac{\varepsilon}{2(n-1)}$. Therefore we get for all $k \geq K$

$$\begin{aligned} |a^{(k)}_n - a_n| &= |n\tilde{a}^{(k)}_n - (n-1)\tilde{a}^{(k)}_{n-1} - n\tilde{a}_n + (n-1)\tilde{a}_{n-1}| \\ &\leq n|\tilde{a}^{(k)}_n - \tilde{a}_n| + (n-1)|\tilde{a}^{(k)}_{n-1} - \tilde{a}_{n-1}| \leq \varepsilon. \end{aligned}$$

We have

$$a^{(k)}_n \geq \lim_{n\to\infty} a^{(k)}_n = \lim_{n\to\infty} \tilde{a}^{(k)}_n \qquad \text{and} \qquad \lim_{n\to\infty} \tilde{a}_n = \lim_{k\to\infty}\big(\lim_{n\to\infty} \tilde{a}^{(k)}_n\big).$$

This implies that there exists a $K \in \mathbb{N}$ and $E > 0$ such that for all $k \geq K$ and $n \in \mathbb{N}$ we have $a^{(k)}_n \geq \lim_{n\to\infty} \tilde{a}_n - E$. Because of the pointwise convergence of the $a^{(k)}$ we therefore get that a has a lower bound and is therefore convergent. In particular $a \in l^\infty(\mathbb{N})$. We write $A := \lim_{n\to\infty} \tilde{a}_n$.

Now we choose an $\varepsilon > 0$. There exists an $N \in \mathbb{N}$ such that $\tilde{a}_n \leq A + \frac{\varepsilon}{2}$ for all $n \geq N$. Further we can find a $K \in \mathbb{N}$ such, that for all $k \geq K$ holds $\|\tilde{a}^{(k)} - \tilde{a}\| \leq \frac{\varepsilon}{6}$ and $|a^{(k)}_n - a_n| \leq \frac{\varepsilon}{2}$ for all $n \leq 3N$. We claim that $\|a^{(k)} - a\| \leq \varepsilon$ for all $k \geq K$. This would finish our proof.

To see this we show that $a^{(k)}_n \in [A - \frac{\varepsilon}{2}, A + \varepsilon]$ for all $n > 3N$. As $a_n \in [A, A + \frac{\varepsilon}{2}]$, this then entails $|a^{(k)}_n - a_n| \leq \varepsilon$.

If $a^{(k)}_n < A - \frac{\varepsilon}{2}$ then we get, as $a^{(k)}$ is decreasing, that $a^{(k)}_m < A - \frac{\varepsilon}{2}$ for all $m \geq n$. This implies that $\tilde{a}^{(k)}_m < A - \frac{\varepsilon}{2}$ for large enough m, which is a contradiction to $\|\tilde{a}^{(k)} - \tilde{a}\| \leq \frac{\varepsilon}{6}$, as $\tilde{a}_n \geq A$ for all $n \in \mathbb{N}$.

If $a^{(k)}_n > A + \varepsilon$, then $a^{(k)}_m > A + \varepsilon$ for all $m \leq n$. This implies that $a^{(k)}_m - a_m > \frac{\varepsilon}{2}$ for $N < m < n$. Therefore

$$\begin{aligned} |\tilde{a}^{(k)}_n - \tilde{a}_n| &= \frac{1}{n}\left|\sum_{m=1}^{n} a^{(k)}_m - a_m\right| \geq \frac{1}{n}\left|\sum_{m=N}^{n} a^{(k)}_m - a_m\right| - \frac{1}{n}\left|\sum_{m=1}^{N-1} a^{(k)}_m - a_m\right| \\ &> \frac{1}{n}(n-N)\frac{\varepsilon}{2} - \frac{1}{n}(N-1)\frac{\varepsilon}{2} \geq \frac{\varepsilon}{2n}(n-2N+1) \geq \frac{\varepsilon}{2n}(n - \frac{2}{3}n) \geq \frac{\varepsilon}{6} \end{aligned}$$

which yields a contradiction. This finishes the proof. □

Lemma 8.7. *For $a, b \in \mathfrak{R}$ we define*

$$s(b,a) = \sup_{f\in\mathfrak{F}}(\max\{b_f - a_f, 0\}).$$

Then there exists a $b' \in \mathfrak{R}$ satisfying

(1) $b'_f \leq a_f$ *for all* $f \in \mathfrak{F}$.
(2) $\|b' - b\| \leq s(b,a)$.

In the case $\mathfrak{h} \neq \mathfrak{h}(\mathfrak{H})$ the same holds if we replace $\mathfrak{R}$ by $\mathfrak{R}'$ and $\mathfrak{F}$ by $\mathfrak{F}^+$.

Proof. We start with the case $h = \mathfrak{h}(\mathfrak{H})$.

The first idea would be to define $b'_f = b_f - s(b,a)$ for all $f \in \mathfrak{F}$. Obviously we then get $b'_f \leq a_f$ for all $f \in \mathfrak{F}$ and $\|b' - b\| \leq s(b,a)$. The conditions (R1) - (R3) from Definition 8.4 are fulfilled. Only (R4) need not hold. So we have to refine our construction.

We define

$$\begin{aligned} A^+ &:= \min_{\aleph\in M^*}\{a_{\aleph^+}, b_{\aleph^+} + s(b,a)\} \\ A^- &:= \min_{\aleph\in M^*}\{a_{-\aleph^-}, b_{-\aleph^-} + s(b,a)\}. \end{aligned}$$

We have $a_{\aleph^+} + a_{-\aleph^-} \geq 0$ and $b_{\aleph^+} + s(b,a) + b_{-\aleph^-} + s(b,a) \geq 0$ for all $\aleph \in \mathfrak{M}^*$. Further

$$a_{\aleph^+} + b_{-\aleph^-} + s(b,a) \geq b_{\aleph^+} - s(b,a) + b_{-\aleph^-} + s(b,a) = b_{\aleph^+} + b_{-\aleph^-} \geq 0$$

and analogously $b_{\aleph^+} + s(b,a) + a_{-\aleph^-} \geq 0$ for all $\aleph \in \mathfrak{M}^*$. So we have

1. $A^+ + A^- \geq 0$.
2. $A^+ \leq a_{\aleph^+}$ and $A^- \leq a_{-\aleph^-}$ for all $\aleph \in \mathfrak{M}^*$.
3. $A^+ - s(b,a) \leq b_{\aleph^+}$ and $A^- - s(b,a) \leq b_{-\aleph^-}$ for all $\aleph \in \mathfrak{M}$.

We define as in Definition 8.4

$$\begin{aligned} \alpha_n^+ &:= na_{n^+} - (n-1)a_{(n-1)^+} & \beta_n^+ &:= nb_{n^+} - (n-1)b_{(n-1)^+} \\ \alpha_n^- &:= na_{n^-} - (n-1)a_{(n-1)^-} & \beta_n^- &:= nb_{n^-} - (n-1)b_{(n-1)^-}. \end{aligned}$$

Then $a_{n^\pm} = \frac{1}{n}(\alpha_1^\pm + \ldots + \alpha_n^\pm)$ and $b_{n^\pm} = \frac{1}{n}(\beta_1^\pm + \ldots + \beta_n^\pm)$. Further we get from Lemma 8.5.3 that $\alpha_n^+ \geq a_{\aleph^+}$, $\alpha_n^- \geq a_{-\aleph^-}$ and $\beta_n^+ \geq b_{\aleph^+}$, $\beta_n^- \geq b_{-\aleph^-}$ for all $\aleph \in \mathfrak{M}^*$. Now we define $b' \in l^\infty(\mathfrak{F})$ by

$$\begin{aligned} \beta_n^{+\prime} &:= \max\{\beta_n^+ - s(b,a), A^+\} & \beta_n^{-\prime} &:= \max\{\beta_n^- - s(b,a), A^-\} \\ b'_{n^+} &:= \tfrac{1}{n}(\beta_1^{+\prime} + \ldots + \beta_n^{+\prime}) & b'_{n^-} &:= \tfrac{1}{n}(\beta_1^{-\prime} + \ldots + \beta_n^{-\prime}) \\ b'_{\aleph^+} &:= \max\{b_{\aleph^+} - s(b,a), A^+\} & b'_{-\aleph^-} &:= \max\{b_{-\aleph^-} - s(b,a), A^-\}. \end{aligned}$$

for all $n \in \mathbb{N}$ and $\aleph \in \mathfrak{M}$. Then there exists an $n_0 \in \mathbb{N} \cup \{\infty\}$ such, that $\beta_n^{+\prime} \geq A^+$ for $n < n_0$ and $\beta_n^{+\prime} < A^+$ for $n \geq n_0$. So we get for $n < n_0$

$$b'_{n^+} = \tfrac{1}{n}(\beta_1^{+\prime} + \ldots + \beta_n^{+\prime}) = \tfrac{1}{n}(\beta_1^+ + \ldots + \beta_n^+) - s(b,a) = b_{n^+} - s(b,a)$$

which implies $|b'_{n^+} - b_{n^+}| \leq s(b,a)$ and $b'_{n^+} \leq a_{n^+}$. For $n \geq n_0$ we get

$$\begin{aligned} b'_{n^+} &= \tfrac{1}{n}(\beta_1^{+\prime} + \ldots + \beta_n^{+\prime}) = \tfrac{1}{n}(\beta_1^{+} + \ldots + \beta_{n_0}^{+} - n_0 s(b,a) + (n-n_0)A^+) \\ &= \tfrac{1}{n}(n_0(b_{n_0^+} - s(b,a)) + (n-n_0)A^+) \leq \tfrac{1}{n}(n_0 a_{n_0^+} + \alpha_{n_0+1}^+ + \ldots + \alpha_n^+) a_{n^+}. \end{aligned}$$

We note that $\beta_n^{+\prime} \leq \beta_n^+ + s(b,a)$. For $n < n_0$ this follows from the definition, for $n \geq n_0$ this follows from $A^+ \leq b_{\aleph^+} + s(b,a) \leq \beta_n^+ + s(b,a)$. By definition we have $\beta_n^{+\prime} \geq \beta_n^+ - s(b,a)$ and therefore $|\beta_n^{+\prime} - \beta_n^+| \leq s(b,a)$. This implies $|b_{n^+} - b'_{n^+}| \leq s(b,a)$.

The same way we see $b'_{n^-} \leq a_{n^-}$ and $|b'_{n^-} - b_{n^-}| \leq s(b,a)$. It follows from the definitions that $b'_{\pm\aleph^\pm} \leq a_{\pm\aleph^\pm}$ and $|b'_{\pm\aleph^\pm} - b_{\pm\aleph^\pm}| \leq s(b,a)$. So all that is left to show is that $b' \in \mathfrak{R}$. We check the conditions (R1) - (R4).

The sequences $(\beta_n^{+\prime})_{n\in\mathbb{N}}$ and $(\beta_n^{-\prime})_{n\in\mathbb{N}}$ are both decreasing, which implies (R1) and (R2). The condition (R3) follows immediately from our definitions, while (R4) is a consequence of $A^+ + A^- \geq 0$.

In the case $\mathfrak{h} \neq \mathfrak{h}(\mathfrak{H})$ the proof is almost identical. As we only have the index set $\mathfrak{F}^+$ we set $A^+ = 0$ and omit our definitions and calculations for $f \notin \mathfrak{F}^+$ to get to the same results. □

We say that an element in $l^\infty(\mathfrak{F})$ resp. $l^\infty(\mathfrak{F}^+)$ is nonnegative if all its entries are nonnegative. Further we denote by K^+ the cone of nonnegative elements.

Lemma 8.8. *Let $C \subseteq \mathfrak{h}$ be a closed convex $\mathfrak{U}$-invariant set. Then*

(1) $\rho^{-1}(\rho(C)) = C$.
(2) $\rho(C)$ *is closed and convex.*
(3) $\overline{(\rho(C) - K^+)} \cap \rho(\mathfrak{h}) = \rho(C)$.
(4) $\overline{\rho(C) - K^+} \cap \rho(\mathfrak{h}) = \rho(C)$.

Proof. (1) Let $X \in \rho^{-1}(\rho(C))$. Then there exists an $Y \in C$ satisfying $\rho(X) = \rho(Y)$. Theorem 7.10 now implies that $X \in \overline{\mathrm{conv}(\mathfrak{U}.Y)} \subseteq C$.

(2) The proof is almost identical in the cases $\mathfrak{h} = \mathfrak{h}(\mathfrak{H})$ and $\mathfrak{h} \neq \mathfrak{h}(\mathfrak{H})$. We start with the case $\mathfrak{h} = \mathfrak{h}(\mathfrak{H})$.

Now we assume that $c \in \overline{\rho(C)}$ for $C \subseteq \mathfrak{h}(\mathfrak{H})$ closed, convex and $\mathfrak{U}(\mathfrak{H})$-invariant. Then there exists a sequence $(c^{(n)})_{n\in\mathbb{N}} \in \rho(C)$ that converges to c. For each $c^{(n)}$ we construct $\tilde{c}^{(n)} \in l^\infty(J)$ as in Definition 8.4. Then $\mathrm{diag}(\tilde{c}^{(n)}) = h(c^{(n)})$. Because of Lemma 8.5.4 and 1) we get $h(c^{(n)}) \in C$. Analogously we get $\tilde{c} \in l^\infty(J)$ satisfying $\mathrm{diag}(\tilde{c}) = h(c)$. We claim that $\tilde{c}^{(n)}$ converges to $\tilde{c}$ and therefore $h(c^{(n)})$ converges to $h(c)$, which would prove the closedness of $\rho(C)$.

It follows immediately from the construction that

$$\lim_{n\to\infty} \tilde{c}^{(n)}\Big|_{\bigcup_{\aleph\in\mathfrak{M}^*}(M_\aleph^+ \dot\cup M_\aleph^-)} = \tilde{c}\Big|_{\bigcup_{\aleph\in\mathfrak{M}^*}(M_\aleph^+ \dot\cup M_\aleph^-)}.$$

We recall the definitions of $\tilde{c}^{(n)}|_{\mathbb{N}_1\dot\cup\mathbb{N}_2}$, $\tilde{c}|_{\mathbb{N}_1\dot\cup\mathbb{N}_2}$ and Lemma 8.5.1. Then we see that we can apply Lemma 8.6.2 twice and show that $\tilde{c}^{(n)}|_{\mathbb{N}_1}$ converges to $\tilde{c}|_{\mathbb{N}_1}$ and $\tilde{c}^{(n)}|_{\mathbb{N}_2}$ converges to $\tilde{c}|_{\mathbb{N}_2}$. Therefore $h(c)$ lies in $\overline{C} = C$ and $c = \rho(h(c)) \in \rho(C)$.

For the convexity we choose $c^1, c^2 \in \rho(C)$ and find $\tilde{c}^1, \tilde{c}^2 \in l^\infty(J)$ as in Definition 8.4. Then $\mathrm{diag}(\tilde{c}^1) = h(c^1)$ and $\mathrm{diag}(\tilde{c}^2) = h(c^2)$. With Lemma 8.5.4 and 1) we see again that $h(c^1), h(c^2) \in C$. Further we choose $\lambda \in [0,1]$. As C is convex we get that $\mathrm{diag}(\tilde{c}) \in C$, where $\tilde{c} = \lambda\tilde{c}^1 + (1-\lambda)\tilde{c}^2$. Let $c = \rho(\mathrm{diag}(\tilde{c}))$. We claim that $c = \lambda c^1 + (1-\lambda)c^2$, which would prove the assertion.

Let $\tilde{c}|_{\mathbb{N}_1} = (\gamma_1^+, \gamma_2^+, \ldots)$ and $\tilde{c}^i|_{\mathbb{N}_1} = (\gamma_1^{i+}, \gamma_2^{i+}, \ldots)$ for $i = 1, 2$. Then

$$\gamma_n^+ = \lambda\gamma_n^{1+} + (1-\lambda)\gamma_n^{2+} \leq \lambda\gamma_{n-1}^{1+} + (1-\lambda)\gamma_{n-1}^{2+} = \gamma_{n-1}^+.$$

The same way we get that $\gamma_n^+ \geq \tilde{c}_j$ for all $n \in \mathbb{N}_1$ and $j \in J\backslash\mathbb{N}_1$, therefore

$$n^+(\mathrm{diag}(\tilde{c})) = \frac{1}{n}(\gamma_1^+ + \ldots + \gamma_n^+) = \lambda n^+(h(c^1)) + (1-\lambda)n^+(h(c^2)).$$

We get $n^-(\mathrm{diag}(\tilde{c})) = \lambda n^-(h(c^1)) + (1-\lambda)n^-(h(c^2))$ the same way.

With a similar argumentation we have that $\tilde{c}_j \geq \tilde{c}_k$ for $j \in M_\aleph^+$, $k \in M_{\aleph'}^+$, $\aleph \leq \aleph'$ and as a result $\aleph^+(\tilde{c}) = \lambda\aleph^+(\tilde{c}^1) + (1-\lambda)\aleph^+(\tilde{c}^2)$ and analogously $\aleph^-(\tilde{c}) = \lambda\aleph^-(\tilde{c}^1) + (1-\lambda)\aleph^-(\tilde{c}^2)$. Therefore

$$\lambda c^1 + (1-\lambda)c^2 = \lambda\rho(h(c^1)) + (1-\lambda)\rho(h(c^2)) = \rho(\lambda h(c^1) + (1-\lambda)h(c^2)) \in \rho(C).$$

The proof in the case $\mathfrak{h} \neq \mathfrak{h}(\mathfrak{H})$ varies only in so far as we have to use the decomposition $J = \mathbb{N} \dot{\cup} \dot{\bigcup}_{\aleph\in\mathfrak{M}} M_\aleph$ and $\mathrm{diag}_{\mathfrak{d}}$ instead of diag.

(3) This follows immediately from Theorem 7.10.

(4) We choose $b \in \overline{\rho(C) - K^+} \cap \rho(\mathfrak{h})$. There exist $a_n \in \rho(C)$ and $k_n \in K^+$ such, that

$$\lim_{n\to\infty} \|a_n - k_n - b\| = 0.$$

Let $s(\,.\,,\,.\,)$ be as in Lemma 8.7. The inequality $s(b, a_n - k_n) \leq \|a_n - k_n - b\|$ implies $\lim_{n\to\infty} s(b, a_n - k_n) = 0$. Because of $s(b, a_n - k_n) \geq s(b, a_n) \geq 0$ we obtain

$$\lim_{n\to\infty} s(b, a_n) = 0.$$

According to Lemma 8.7 we can find $b_n \in \rho(\mathfrak{h})$ with $\lim_{n\to\infty} \|b_n - b\| = 0$ and $b_n \in a_n - K^+$. The latter condition combined with (3) implies $b_n \in \rho(C)$ for all $n \in \mathbb{N}$ and therefore $b \in \rho(C)$, as this set is closed. This proves the assertion. □

Lemma 8.9. *Let $A \subseteq \rho(\mathfrak{h})$ denote a closed convex set that satisfies the condition*

$$(A - K^+) \cap \rho(\mathfrak{h}) = A.$$

Then

(1) $\rho^{-1}(A)$ *is a closed convex* $\mathfrak{U}$*-invariant subset of* $\mathfrak{h}$.
(2) $\rho(\rho^{-1}(A)) = A$.

Proof. (1) As ρ is continuous, it follows that $\rho^{-1}(A)$ is closed. The $\mathfrak{U}$-invariance follows from the fact that $\rho(U^*XU) = \rho(X)$ for all $X \in \mathfrak{h}$. So all that is left to show is the convexity.

We choose $X, Y \in \rho^{-1}(A)$ and $\lambda \in [0,1]$. Then we get

$$f(\lambda X + (1-\lambda)Y) \leq \lambda f(X) + (1-\lambda) f(Y)$$

for all $f \in \mathfrak{F}$ (resp $f \in \mathfrak{F}^+$). Therefore

$$\rho(\lambda X + (1-\lambda)Y) \in \lambda\rho(X) + (1-\lambda)\rho(Y) - K^+ \subseteq A,$$

which proves the assertion.

(2) is trivial. □

Proposition 8.10. *The map ρ induces a bijection between the $\mathfrak{U}$-invariant closed convex subsets C of $\mathfrak{h}$ and the closed convex subsets A of $\rho(\mathfrak{h})$ that satisfy*

$$(A - K^+) \cap \rho(\mathfrak{h}) = A.$$

Proof. This follows immediately from Lemma 8.8 and Lemma 8.9. □

So we have seen that the map ρ is a useful tool to characterize the closed convex $\mathfrak{U}$-invariant subsets of $\mathfrak{h}$. We will use it in the following theorem.

Theorem 8.11. *Let $C \subseteq \mathfrak{h}$ be a closed convex $\mathfrak{U}$-invariant set. Then $X \in C$ if and only if*

$$(l \circ \rho)(X) \leq \sup_{Y \in C} (l \circ \rho)(Y)$$

for all continuous positive linear functions $l\colon l^\infty(\mathfrak{F}) \to \mathbb{R}$ resp. $l\colon l^\infty(\mathfrak{F}^+) \to \mathbb{R}$.

Proof. Obviously $X \in C$ implies $(l \circ \rho)(X) \leq \sup_{Y \in C} (l \circ \rho)(Y)$ for all positive functionals l. So we choose $X \notin C$. Then $\rho(X) \notin \rho(C)$ and with Lemma 8.8.4 we get $\rho(X) \notin \overline{\rho(C) - K^+}$. As $\overline{\rho(C) - K^+}$ is a closed convex set, we can use the Hahn-Banach Theorem to find a continuous linear functional l and $\alpha \in \mathbb{R}$ satisfying

$$l(Y) \leq \alpha < l(\rho(X))$$

for all $Y \in \overline{\rho(C) - K^+}$. If we choose $Y_1 \in \rho(C)$ and $Y_2 \in K^+$, then we get for all $t \in [0, \infty[$

$$\alpha \geq l(Y_1 - tY_2) = l(Y_1) - tl(Y_2).$$

This can only be true if $l(Y_2) \geq 0$. As Y_2 was arbitrary, this implies that l is positive and we have for all $Y \in C$

$$(l \circ \rho)(Y) \leq \alpha < (l \circ \rho)(X).$$

This finishes the proof. □

In the case that $l \in l^1(\mathfrak{F})$ with finite support and $\|l\| = 1$ we have that $l \circ \rho$ is just a convex combination of functions in $\mathfrak{F}$. Therefore it is intuitive to view $l \circ \rho$ as a generalized convex combination of elements in $\mathfrak{F}$.

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