Journal of Convex Analysis
Volume 9 (2002), No. 1, 287–290

Lipschitzian Characterizations of Finite Dimensional Banach Spaces

Dumitru Popa
*Dept. of Math, Univ. of Constanta,
Constanta, Romania
dpopa@univ-ovidius.ro*

Received January 20, 2000
Revised manuscript received May 28, 2001

We give equivalent formulations of finite dimensional Banach spaces in terms of Lipschitzian functions..

Keywords: Performance function, multipliers, stability, convex like functions, measurable integrands, richness, integral functional, growth conditions

1991 Mathematics Subject Classification: 46E30, 28A20, 49B, 60B12

In the papers [1] and [2] tight connections between the sequential properties of a Banach space and properties of convex functions on that space are established. In [4] it is shown that there is a sequence $(x_n)_{n\in N}$ of norm one in l_2 and a convex function $f : l_2 \longrightarrow \mathbb{R}$ that is NOT Lipschitz in a neighborhood of that sequence. In this note we complete the circle of ideas pursued in [1], [2] and [4]. Our first lemma presents a way to construct convex and continuous functions, and it shows that if a function is Lipschitz on a open set then it must satisfy a "boundary boundedness" condition, see Lemma 1 c) below.

Lemma 1. *Let $\varphi_n : [0,\infty) \to [0,\infty)$, be a sequence of C^1 functions such that φ_n and $\varphi_n^{'}$ are increasing functions for each $n \in N$ and let $0 < a < \infty$ be such that the series $\sum_{n=1}^{\infty} \varphi_n(a)$ is convergent. Let X be a Banach space, $p_n : X \to \mathbb{R}$ a sequence of seminorms on X with: $p_n(x) \to 0$, for each $x \in X$ and $p_n(x) \leq \| x \|$, for each $x \in X$. Let $f : X \to \mathbb{R}$ be defined by $f(x) = \sum_{n=1}^{\infty} \varphi_n(p_n(x))$. Then:*

a) f is a continuous and convex function on X.

b) If $0 < M < \infty$ is such that $\sum_{n=1}^{\infty} \varphi_n(M) < \infty$, then f is bounded on $\overline{B}(0,M)$.

c) If f is Lipschitz on a open subset $G \subset X$, then there exist $L > 0$, such that $\sum_{n=1}^{\infty} p_n(x)\varphi_n^{'}(p_n(x)) \leq L \| x \|$, for each x in the closure $\overline{G}$ of G.

d) If $0 < M < \infty$ is such that $\sum_{n=1}^{\infty} \varphi_n^{'}(M) < \infty$, then f is Lipschitz on $B(0,M)$.

Proof. Let $x \in X$. Then there exists $n_0 \in \mathbb{N}$ such that $p_n(x) \leq a$ for all $n \geq n_0$. Then $\varphi_n(p_n(x)) \leq \varphi_n(a)$ for all $n \geq n_0$ because each φ_n is increasing. Since the series $\sum_{n=1}^{\infty} \varphi_n(a)$ is convergent we obtain the convergence of the series $\sum_{n=1}^{\infty} \varphi_n(p_n(x))$.

ISSN 0944-6532 / \$ 2.50

a) The convexity of the function f is clear because φ_n is convex and increasing for each $n \in \mathbb{N}$. The continuity of f is again clear because it is lower semicontinuous (as a sum of positive lsc functions), convex, finite valued everywhere, and a well-known consequence of Baire category theorem is that a lsc convex function is continuous on the interior of its domain in a Banach space.

b) $$0 \leq f(x) = \sum_{n=1}^{\infty} \varphi_n(p_n(x)) \leq \sum_{n=1}^{\infty} \varphi_n(\| x \|) \leq \sum_{n=1}^{\infty} \varphi_n(M), \text{ for } \| x \| \leq M.$$

c) Let $L > 0$ be such that $| f(x) - f(y) | \leq L \| x - y \|$, for each $x, y \in G$. Let $x \in G$. Since G is open, there exists $\varepsilon > 0$, such that $\overline{B}(x, \varepsilon) \subset G$. Then: $(1 + \varepsilon t)x$, $x \in \overline{B}(x, \varepsilon)$, where $t = \frac{1}{\|x\|+1}$ and from the above relation we obtain:

$$| f((1 + \varepsilon t)x) - f(x) | \leq L \| (1 + \varepsilon t)x - x \| = L\varepsilon t \| x \| . \tag{1}$$

Since $\varphi_n^{'}$ is increasing we have $\varphi_n(b) - \varphi_n(a) \geq (b - a)\varphi_n^{'}(a)$, for $a, b \in \mathbb{R}$ with $0 \leq a \leq b$. Using this inequality we have

$$\begin{aligned} f((1 + \varepsilon t)x) - f(x) &= \sum_{n=1}^{\infty} [\varphi_n(p_n(1 + \varepsilon t)x)] - \varphi_n p_n(x)] \\ &= \sum_{n=1}^{\infty} [\varphi_n((1 + \varepsilon t)p_n(x)) - \varphi_n(p_n(x))] \\ &\geq \sum_{n=1}^{\infty} \varepsilon t p_n(x)\varphi_n^{'}(p_n(x)) \end{aligned} \tag{2}$$

Now (1) and (2) imply $\sum_{n=1}^{\infty} p_n(x)\varphi_n^{'}(p_n(x)) \leq L \| x \|$, for each $x \in G$. A continuity argument shows that the above inequality is still true for each x in the closure $\overline{G}$.

d) Since $\varphi_n^{'}$ is increasing we have: if $a, b \in \mathbb{R}$, $a \geq 0$, $b \geq 0$, then $| \varphi_n(a) - \varphi_n(b) | \leq | b - a | \max\{\varphi_n^{'}(a), \varphi_n^{'}(b)\}$. Let $x, y \in B(0, M)$, then

$$\begin{aligned} | f(x) - f(y) | &\leq \sum_{n=1}^{\infty} | \varphi_n(p_n(x)) - \varphi_n(p_n(y)) | \\ &\leq \sum_{n=1}^{\infty} | p_n(x) - p_n(y) | \max\{\varphi_n^{'}(p_n(x)), \varphi_n^{'}(p_n(y))\} \\ &\leq \sum_{n=1}^{\infty} | p_n(x) - p_n(y) | \varphi_n^{'}(M) \leq \sum_{n=1}^{\infty} \varphi_n^{'}(M) \| x - y \| . \end{aligned}$$

□

In the following theorem the equivalence ii), vii) has been proved in [1] in a different manner.

Theorem 2. *Let X be a Banach space. Then the following are equivalent:*

i) *X is finite dimensional;*

ii) *weak star and norm convergence agree sequentially in X^*;*

iii) *each convex and continuous function $f : X \longrightarrow \mathbb{R}$ has the property: If $A \subset X$ is a bounded subset on which f is bounded, then f is Lipschitzian on some neighborhood of A;*

iv) *each convex and continuous function $f : X \longrightarrow \mathbb{R}$ has the property: If $A \subset X$ is a bounded subset on which f is bounded, then f is Lipschitzian on A;*

v) *each convex and continuous function $f : X \longrightarrow \mathbb{R}$ has the property: If $A \subset X$ is a bounded subset, then f is Lipschitzian on some neighborhood of A;*

vi) *each convex and continuous function $f : X \longrightarrow \mathbb{R}$ has the property: If $A \subset X$ is a bounded subset, then f is Lipschitzian on A;*

vii) *each convex and continuous function $f : X \longrightarrow \mathbb{R}$ has the property: If $A \subset X$ is a bounded subset, then f is bounded on A;*

viii) *each convex and continuous function $f : X \longrightarrow \mathbb{R}$ is Lipschitzian on the open unit ball;*

ix) *each convex and continuous function $f : X \longrightarrow \mathbb{R}$ is bounded on the open unit ball.*

Proof. The equivalence *i)* $\Leftrightarrow$ *ii)* is the Jossefsson-Nissenzweig Theorem, see [3], p. 219. The implications: *i)* $\Rightarrow$ *iii)* $\Rightarrow$ *iv)*, *i)* $\Rightarrow$ *v)* $\Rightarrow$ *vi)* $\Rightarrow$ *viii)*, *i)* $\Rightarrow$ *vii)* $\Rightarrow$ *ix)* are clear or well known. For the implications *iv)* $\Rightarrow$ *ii)* and *viii)* $\Rightarrow$ *ii)*, let us suppose that *ii)* is not true. Then there exist $(x_n^*)_{n\in N} \subset X^*$ such that: $x_n^* \longrightarrow 0$ weak*, (i.e. $x_n^*(x) \longrightarrow 0$, for each $x \in X$), but $\|x_n^*\| = 1$, for each $n \in \mathbb{N}$. Let $p_n(x) =\mid x_n^*(x) \mid$. Now define $f : X \longrightarrow \mathbb{R}$, by $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2}[p_n(x)]^{n^3}$. Let us observe that the function f is bounded on the closed unit ball. If f is Lipschitz on the open unit ball, then lemma 1 c) shows that we must have for some $L > 0$; $\sum_{n=1}^{\infty} \frac{n^3}{n^2}[p_n(x)]^{n^3} \leq L \parallel x \parallel$, for each $\parallel x \parallel \leq 1$. Hence: $n[p_n(x)]^{n^3} \leq L \parallel x \parallel$ for each $\parallel x \parallel \leq 1$, and each $n \in \mathbb{N}$. Thus $n\|x_n^*\|^{n^3} \leq L$, $n \leq L$ for each $n \in \mathbb{N}$, which is clearly a contradiction. For the implication *ix)* $\Rightarrow$ *ii)* let us suppose that *ii)* is not true. Then there exist $(x_n^*)_{n\in N} \subset X^*$ such that: $x_n^* \longrightarrow 0$ weak*, but $\|x_n^*\| = 1$, for each $n \in \mathbb{N}$. Let $p_n(x) =\mid x_n^*(x) \mid$. Now define $f : X \longrightarrow \mathbb{R}$, by $f(x) = \sum_{n=1}^{\infty} n[p_n(x)]^n$. Lemma 1 a) ensures that f is a convex and continuous function. If f is bounded on the open unit ball, we must have for some $L > 0$: $\sum_{n=1}^{\infty} n[p_n(x)]^n \leq L$, for each $\parallel x \parallel \leq 1$. Hence: $n[p_n(x)]^n \leq L$, for each $\parallel x \parallel \leq 1$, and each $n \in \mathbb{N}$. Then $n\|x_n^*\|^n \leq L$, $n \leq L$ for each $n \in \mathbb{N}$, which is clearly a contradiction. We remark that Lemma 1 d), shows that the above functions are also Lipschitzian on each open ball $B(0, \varepsilon)$, for each $0 < \varepsilon < 1$. □

Let us remark that in the book [5] chapter 34, boundedness issues of biconjugates are studied.

In the end we indicate also an another way to construct examples of continuous and convex functions as is given in [1] Lemma 2.1. In [1] Lemma 2.1. is used as a technical tool in order to obtain further results and characterizations of Banach spaces.

Lemma 3. *With the same notation as in Lemma 1, we have:*

a) If W is a bounded subset of X such that $p_n \to 0$, uniformly on W, and if $0 < M < \infty$

is such that $\sum_{n=1}^{\infty} \varphi_n^{'}(M) < \infty$, *then* f *is Lipschitz on the set* $W + B(0,\varepsilon)$, *for each* $0 < \varepsilon < M$.

b) If W *is a bounded subset of* X *such that sequence* $(p_n)_{n\in\mathbb{N}}$, *does not converge uniformly on* W *and* $c > 0$ *is such that* $\lim_{n\to\infty} \varphi_n(c) = \infty$, *then there exist* $\lambda > 0$, *such that* f *is unbounded on the set* λW.

Proof.

a) Let $n_0 \in \mathbb{N}$ be such that $p_n(w) \leq M - \varepsilon$, $\forall n \geq n_0$, $\forall w \in W$. Hence for $n \geq n_0$, $x \in W + B(0,\varepsilon)$, $p_n(x) \leq M$. Now if x, $y \in W + B(0,\varepsilon)$, then:

$$\begin{aligned} \mid f(x) - f(y) \mid &\leq \sum_{n=1}^{\infty} \mid \varphi_n(p_n(x)) - \varphi_n(p_n(y)) \mid \\ &\leq \sum_{n=1}^{\infty} \mid p_n(x) - p_n(y) \mid \max\{\varphi_n^{'}(p_n(x)), \varphi_n^{'}(p_n(y))\} \\ &\leq \sum_{n=1}^{n_0} \mid p_n(x) - p_n(y) \mid \varphi_n^{'}(L) + \sum_{n=n_0+1}^{\infty} \mid p_n(x) - p_n(y) \mid \varphi_n^{'}(M) \\ &\leq [\sum_{n=1}^{n_0} \varphi_n^{'}(L) + \sum_{n=n_0+1}^{\infty} \varphi_n^{'}(M)] \parallel x - y \parallel \end{aligned}$$

Where above we denote $L > 0$, such that $W + B(0,\varepsilon) \subset B(0,L)$.

b) Since the sequence $(p_n)_{n\in\mathbb{N}}$, does not converge uniformly on W, passing to a subsequence we have for some $\delta > 0$, there exist $w_k \in W$ such that $p_{n_k}(w_k) > \delta$, for each $k \in \mathbb{N}$. Let be $\lambda = \frac{c}{\delta} > 0$, $x_k = \frac{cw_k}{\delta} \in \lambda W$ and $f(x_k) \geq \varphi_{n_k}(p_{n_k}(x_k)) \geq \varphi_{n_k}(c)$, for each $k \in \mathbb{N}$. Since $\varphi_{n_k}(c) \to \infty$, we obtain $f(x_k) \to \infty$. □

Acknowledgements. This paper has written when the author has been a TEMPUS grant in the period 2.05 - 26.07, 1996, a I.A.E Aix en Provence, France. We thank also the referees for carefully reading of the manuscript, their comments and useful remarks and also for showing us the reference [5].

References

[1] J. M. Borwein, S. Fitzpatrick, J. Vanderwerff: Examples of convex functions and classifications of normed spaces, Journal of Convex Analysis 1 (1994) 61–73.

[2] J. M. Borwein: A survey of examples of convex functions and the classification of Banach spaces, in: Recent Developments in Optimization: VII Colloque Franco-Allemand d'Optimisation, Lecture Notes Economic and Mathematical Systems, R. Durier and C. Michelot (eds.), Springer-Verlag (1995) 260–271.

[3] J. Diestel: Sequences and Series in Banach Spaces, Graduate Texts in Mathematics (1984).

[4] G. Pritchard, G. Gurkan, A. Y. Ozge: A note on locally Lipschitzian functions, Math. Programming 71, series A (1995) 369–370.

[5] S. Simons: Minimax and Monotonicity, Lecture Notes Math. Vol. 1963, Springer-Verlag (1998).