Journal of Convex Analysis
Volume 9 (2002), No. 1, 31–54

Some New Results on the Convergence of Degenerate Elliptic and Parabolic Equations

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Received June 15, 2000
Revised manuscript received January 22, 2001

In this paper we study the convergence of the Cauchy-Dirichlet problems for a sequence of parabolic operators $\mathcal{P}_h = \lambda_h \frac{\partial}{\partial t} - \mathrm{div}(a_h(x,t)\cdot D)$ where the matrices of the coefficients $a_h(x,t)$ verify the following degenerate elliptic condition

$$\lambda_h(x)|\xi|^2 \leq (a_h(x,t)\cdot\xi,\xi) \leq L\lambda_h(x)|\xi|^2,$$

being $(\lambda_h)_h$ a sequence of weights satisfying a uniform Muckenhoupt's condition in h. When $a_h = a_h(x)$ we compare this result with the analogous results for the sequence of operators $A_h = -\mathrm{div}(a_h(x)\cdot D)$ and $\mathcal{Q}_h = \frac{\partial}{\partial t} - \mathrm{div}(a_h(x)\cdot D)$

Keywords: Jumping problems, variational inequalities, nonsmooth critical point theory

1991 Mathematics Subject Classification: 35D05, 58J05

1. Introduction

The asymptotic behaviour, as $h \to \infty$, of the equations

$$A_h u = f \quad \text{and} \quad \mathcal{Q}_h u = f \qquad (h = 1, 2...)$$

has been widely studied where

$$A_h = -\mathrm{div}\,(a_h(x)\cdot D)\,, \qquad \mathcal{Q}_h = \frac{\partial}{\partial t} - \mathrm{div}\,(a_h(x,t)\cdot D)\,, \tag{1}$$

$a_h(x) = [a_{h,ij}(x)]_{i,j=1}^n$ or $a_h(x,t) = [a_{h,ij}(x,t)]_{i,j=1}^n$ are matrices of measurable functions defined respectively on a bounded open set Ω of $\mathbb{R}^n$ or on a bounded open cylinder $\Omega \times (0,T)$ of $\mathbb{R}^{n+1}$ with a_h satisfying in both cases the classical ellipticity condition

$$\lambda_0|\xi|^2 \leq \sum_{i,j=1}^n a_{h,ij}\xi_i\xi_j \leq \Lambda_0|\xi|^2 \quad \text{for a.e. } x \in \Omega, \forall\, \xi \in \mathbb{R}^n, \forall\, h \in \mathbb{N}, \tag{2}$$

for a.e. $x \in \Omega$ if $a_h = a_h(x)$ and for a.e. $(x,t) \in \Omega \times (0,T)$ if $a_h = a_h(x,t)$, for suitable positive constants $0 < \lambda_0 \leq \Lambda_0$. We recall among all some important results due to E. De Giorgi, S. Spagnolo, F. Murat, L. Tartar and O. A. Oleinik (see for instance [11], [26], [27], [29], [6] for the elliptic case and [5], [28], [33] for the parabolic one). We recall that also the non linear case has been considered later (see, for instance, [29], [1]).

ISSN 0944-6532 / \$ 2.50

Arising from some phisycal applications (see, for instance, [23]) even the asymptotic behaviour of some class of *degenerate* operators has been considered, operators like those in (1) when the matrices a_h's satisfy the following degenerate ellipticity condition (note that $a_{h,ij}$ may depend on x or on (x,t), while λ_h depend only on x)

$$\lambda_h(x)|\xi|^2 \leq \sum_{i,j=1}^{n} a_{h,ij}\xi_i\xi_j \leq L\lambda_h(x)|\xi|^2\,, \qquad \forall\, \xi \in \mathbb{R}^n, \forall\, h \tag{3}$$

for a.e. $x \in \Omega$ if $a_h = a_h(x)$ and for a.e. $(x,t) \in \Omega \times (0,T)$ if $a_h = a_h(x,t)$, with suitable *weight* functions λ_h (i.e. nonnegative, locally summable functions on $\mathbb{R}^n$) and a suitable constant $L \geq 1$. Some interesting results in this setting are obtained in many cases. The first ones are obtained in the elliptic case, first when the matrices a_h's are symmetric and verify (3) with

$$\lambda_h = \lambda \qquad \text{a.e. in } \Omega, \text{ for every } h$$

for a fixed weight λ verifying suitable integrability conditions together with its inverse λ^{-1} (see [18]), then in the *homogenization* case (see, for instance, [19], [8]), i.e. the sequence of matrices $(a_h)_h$ verifying (3) has the form

$$a_h(x) = a(hx) \qquad (x \in \mathbb{R}^n)$$

with $a(x) = [a_{ij}(x)]_{i,j=1,\ldots n}$ symmetric matrix of measurable periodic functions on $\mathbb{R}^n$ and $\lambda_h(x) = \lambda(hx)$ for a suitable fixed periodic weight λ.
Other results, which apply not only to the variational case (i.e. when the marices are symmetric), are those obtained when each one of the a_h's verifies (3) with a weight λ_h verifying a uniform Muckenhoupt's condition with respect to h, the condition $A_p(K)$ where $1 < p < +\infty$, (see [9], [10] for elliptic operators, [22], [21] for parabolic operators), i. e. there exists $K \geq 1$ such that

$$\left(\frac{1}{|Q|}\int_Q \lambda_h dx\right)\left(\frac{1}{|Q|}\int_Q \lambda_h^{-1/(p-1)} dx\right)^{p-1} \leq K \qquad\qquad A_p(K)$$

for every cube $Q \subset \mathbb{R}^n$ with faces parallel to the coordinate planes, for every h. This class is consider with $p = 2$ for the elliptic case and with $p = 1 + 2/n$ for the parabolic one, i.e. for a strict subclass of A_2 when $n \geq 3$.

In this setting, i.e. given two sequences of matrices $a_h = a_h(x,t)$ and weights $\lambda_h = \lambda_h(x)$ verifying (3) (with $a_{h,ij} = a_{h,ij}(x,t)$) with $\lambda_h \in A_2(K)$ for every h, in this paper we study the asymptotic behaviour of parabolic operators of the form

$$\mathcal{P}_{\lambda_h,a_h} = \lambda_h(x)\frac{\partial}{\partial t} - \operatorname{div}\,(a_h(x,t)\cdot D)\,. \tag{4}$$

Given a matrix $a = a(x,t)$ whose coefficients are in $L^1_{loc}(\mathbb{R}^n \times (0,T))$, we say that it belongs to the class $\mathcal{N}_{\Omega\times(0,T)}(L,M,K,C)$ if there exists a weight $\lambda = \lambda(x)$ such that a satisfies the following properties with $p = 1 + 2/n$ ($n \geq 2$), to the class $\mathcal{M}_{\Omega\times(0,T)}(L,M,K,C)$ if it satisfies the following properties with $p = 2$ (and by $\mathcal{N}_\Omega(L,M,K,C)$ and $\mathcal{M}_\Omega(L,M,K,C)$

the corresponding subclasses of matrices $a = a(x)$ depending only on x):

$$\begin{aligned}
&\lambda(x)|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \leq L\lambda(x)|\xi|^2 \\
&\Big|\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_j\eta_j\Big| \leq M\Big(\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j\Big)^{1/2}\Big(\sum_{i,j=1}^{n} a_{ij}(x,t)\eta_i\eta_j\Big)^{1/2} \\
&\lambda \in A_p(K), \qquad \int_\Omega \lambda dx + \Big(\int_\Omega \lambda^{-1/(p-1)}dx\Big)^{p-1} \leq C
\end{aligned} \tag{5}$$

for a.e. $(x,t) \in \Omega \times (0,T)$ and for every $\xi, \eta \in \mathbb{R}^n$. Given a matrix a, we denote by Λ_a the class of the weights which satisfy (5).

The main results contained in the paper are in the fourth and fifth sections. The first one is the following (see Theorem 4.3).

Theorem 1.1. *Let Ω be a bounded open set with Lipschitz boundary, $T > 0$, $a_h = [a_{h,ij}(x,t)]_{i,j=1}^n$ a sequence of matrices with $a_{h,ij} \in L^1_{loc}(\mathbb{R}^n \times (0,T))$ belonging to $\mathcal{M}_{\Omega\times(0,T)}$ (L, M, K, C) for some positive constants L, M, K, C. Say u_h the unique solution of the Cauchy-Dirichlet problem (see Definitions 3.5 and 4.1)*

$$(P_h)\begin{cases} \lambda_h \dfrac{\partial u}{\partial t} - \text{div}\,(a_h(x,t)\cdot Du) = f & \text{in } \Omega\times(0,T) \\ u = 0 & \text{on } (\partial\Omega\times(0,T))\cup(\Omega\times\{0\}). \end{cases}$$

Then there exist a subsequence $(a_{h_k})_k$, a matrix $a_\infty = [a_{\infty,ij}(x,t)]_{i,j=1}^n \in \mathcal{M}_{\Omega\times(0,T)}(L', M', K, C')$ for suitable constants L', M', C' and a weight $\lambda_\infty \in A_2(K)$ such that for every datum $f \in L^2(0,T;L^n(\Omega))$

$$\lambda_{h_k} \to \lambda_\infty \text{ in } L^1_{loc}(\mathbb{R}^n)\text{-weak}$$

and the subsequence $(u_{h_k})_k$ of the solutions of (P_{h_k}) satisfy

$$\begin{aligned} &u_{h_k} \to u && \text{in } L^2(0,T;L^1(\Omega)) \\ &a_{h_k}\cdot Du_{h_k} \to a_\infty \cdot Du && \text{in } L^2(0,T;L^1(\Omega))^n\text{-weak}, \end{aligned}$$

where u denotes the solution of the problem

$$\begin{cases} \lambda_\infty \dfrac{\partial u}{\partial t} - \text{div}\,(a_\infty(x,t)\cdot Du) = f & \text{in } \Omega\times(0,T) \\ u = 0 & \text{on } (\partial\Omega\times(0,T))\cup(\Omega\times\{0\}). \end{cases}$$

The result above turns out to be an extension of a well-known classic parabolic G-compactness result (see Definition 4.1 and [26], [27], [28], [5], [33]).

In the degenerate setting, analogous results are contained in [10] for Dirichlet problems where the operators are A_h defined by matrices in $\mathcal{M}_\Omega$ and in [22] for Cauchy-Dirichlet problems where the parabolic operators are $\mathcal{Q}_h$, but defined by matrices in the class $\mathcal{N}_{\Omega\times(0,T)} \subset \mathcal{M}_{\Omega\times(0,T)}$.

Another interesting result is the equivalence between the elliptic G-convergence and the

parabolic G-convergence of the operators $\mathcal{P}_{\lambda_h,a_h}$ (see Definition 5.5 and Theorem 5.6). This turns to be an extension to the degenerate case of the classical result contained in [5] (only for matrices independent of t).
The difference with the classical case, i. e. when (2) holds, is that we have to consider parabolic operators like (4) to obtain this equivalence. We also state the equivalence between the elliptic G-convergence and the G-convergence of operators $\mathcal{Q}_h$, but only for matrices in the class $\mathcal{N}_\Omega$ (Theorem 5.8), and so, in this case, the equivalence between the parabolic G-convergence of operators $\mathcal{Q}_h$ and $\mathcal{P}_{\lambda_h,a_h}$.
The sharpness of the assumptions, in the framework of Muckenhoupt's weights, will follow from some considerations made in the fifth section.
We recall that operators like (4) have already been considered to study the regularity of the solutions (see, for instance, [2], [3], [14], [15], but also [24] (Chap. 5, sect. 6), [32] (Theorem 2.3), [21] for the homogenization).

2. Notations and preliminary results

We will denote by Q a generic (open or closed) cube of $\mathbb{R}^n$ with faces parallel to the coordinate planes, by cQ the cube concentric with Q and having side length c times that of Q and and by $B(x,r)$ the open ball of $\mathbb{R}^n$ centered in x with radius r. The symbols $(\cdot,\cdot)$, $|E|$, $\fint_E f dx$ and p' will indicate respectively the scalar product of $\mathbb{R}^n$, the Lebesgue measure of the set E, the mean value of f on E (i. e. $|E|^{-1}\int_E \lambda(x)dx$) and the conjugate $p' = p/(p-1)$ of p.
Let λ be a *weight on* $\mathbb{R}^n$, that is

$$\lambda > 0 \ \text{ a.e. in } \mathbb{R}^n \quad \text{and} \quad \lambda, \lambda^{-1} \in L^1_{loc}(\mathbb{R}^n),$$

then for every bounded open set of $\mathbb{R}^n$ we define

$$L^2(\Omega,\lambda) = \left\{ u \in L^1_{loc}(\Omega) \mid u\lambda^{1/2} \in L^2(\Omega) \right\}$$

and the space

$$H^1(\Omega,\lambda) = \left\{ u \in W^{1,1}_{loc}(\Omega) \;\middle|\; u \in L^2(\Omega,\lambda) \text{ and } |Du| \in L^2(\Omega,\lambda) \right\}.$$

It is easy to verify that the space $H^1(\Omega,\lambda)$ endowed with the topology induced by the norm

$$\|u\|_{H^1(\Omega,\lambda)} = \left(\int_\Omega (u^2 + |Du|^2)\lambda dx \right)^{1/2}$$

is a separable Hilbert space. We will denote by $H^1_0(\Omega,\lambda)$ the closure of $C^1_0(\Omega)$ in the topology of $H^1(\Omega,\lambda)$, by $H^{-1}(\Omega,\lambda)$ its dual space.

Definition 2.1. Let $p > 1$, $K \geq 1$ and let λ be a weight on $\mathbb{R}^n$. We will say that λ *belongs to the Muckenhoupt class* $A_p(K)$ if

$$\left(\fint_Q \lambda dx \right) \left(\fint_Q \lambda^{-1/(p-1)} dx \right)^{p-1} \leq K \qquad \textit{for every cube } Q \subset \mathbb{R}^n.$$

Moreover we define $A_p = \cup_{K\geq 1} A_p(K)$ and $A_\infty = \cup_{p>1} A_p$.

Remark. By definition we have at once that $A_p(K) \subset A_q(K)$ if $1 < p < q < \infty$. Moreover if $\lambda \in A_p$, then $\lambda^{-\frac{1}{p-1}} \in A_{p'}$.

A_p weights verify the following higher sommability property (see [4] and [7]): for every $K \geq 1$, for every $p > 1$, there exist two positive constants $c = c(n,p,K)$ and $\delta = \delta(n,p,K)$ (depending only on n, p and K) such that

$$\left(\fint_Q \lambda^{1+\delta} dx \right)^{\frac{1}{(1+\delta)}} \leq c \left(\fint_Q \lambda dx \right), \qquad \left(\fint_Q \lambda^{-\frac{1+\delta}{p-1}} dx \right)^{\frac{1}{(1+\delta)}} \leq c \left(\fint_Q \lambda^{-\frac{1}{p-1}} dx \right) \tag{6}$$

for every cube Q and $\lambda \in A_p(K)$. A_p weights also verify the *doubling property*, i.e. if $\lambda \in A_p(K)$ for every $t > 0$ there exists a constant $c = c(t,n,p,K)$ (depending only on t,n,p,K) such that

$$\lambda(tQ) \leq c\lambda(Q) \tag{7}$$

for every cube Q of $\mathbb{R}^n$ (see for instance [13]). We recall that (see Theorem 1.4 in [25] and Proposition 1.2 in [7]), if $\lambda \in A_2$ and Ω is a bounded open set with Lipschitz boundary,

$$H_0^1(\Omega,\lambda) = H^1(\Omega,\lambda) \cap W_0^{1,1}(\Omega).$$

Moreover (see [12]) the following Poincaré's inequality holds: there exists a constant $c = c(n,K,\Omega)$ such that

$$\int_\Omega u^2 \lambda dx \leq c \int_\Omega |Du|^2 \lambda dx$$

for every $\lambda \in A_2(K)$ and $u \in H_0^1(\Omega,\lambda)$. Then, from now on, if $\lambda \in A_2(K)$, by $H_0^1(\Omega,\lambda)$ we will denote the closure of $C_0^1(\Omega)$ with respect to the norm

$$\|v\|_{H_0^1(\Omega,\lambda)} \stackrel{\text{def}}{=} \left(\int_\Omega |Dv|^2 \lambda dx \right)^{1/2}. \tag{8}$$

By (6) and (7), it can be easily proved that, if $\lambda \in A_2(K)$ taking $\sigma \stackrel{\text{def}}{=} \delta/(2+\delta)$, where δ is the constant in (6), there exist two positive constants $c_i = c_i(n,K,\Omega)$ (depending only on n,K,Ω), $i = 1,2$, such that

$$\begin{gathered} L^{(1+\sigma)'}(\Omega) \subset L^2(\Omega,\lambda) \subset L^{1+\sigma}(\Omega), \\ c_1 \, \|\lambda^{-1}\|_{L^1(\Omega)}^{-1/2} \|v\|_{L^{1+\sigma}(\Omega)} \leq \|v\|_{L^2(\Omega,\lambda)} \leq c_2 \, \|\lambda\|_{L^1(\Omega)}^{1/2} \|v\|_{L^{(1+\sigma)'}(\Omega)}. \end{gathered} \tag{9}$$

Hence it follows that there exist two positive constants $c_i = c_i(n,K,\Omega)$, $i = 3,4$, such that

$$\begin{gathered} W_0^{1,(1+\sigma)'}(\Omega) \subset H_0^1(\Omega,\lambda) \subset W_0^{1,1+\sigma}(\Omega), \qquad H^{-1}(\Omega,\lambda) \subset W^{-1,1+\sigma}(\Omega) \\ \\ c_3 \|\lambda^{-1}\|_{L^1(\Omega)}^{-1/2} \|u\|_{W_0^{1,1+\sigma}(\Omega)} \leq \|u\|_{H_0^1(\Omega,\lambda)} \leq c_4 \|\lambda\|_{L^1(\Omega)}^{1/2} \|u\|_{W_0^{1,(1+\sigma)'}(\Omega)} \end{gathered} \tag{10}$$

for every $\lambda \in A_2(K)$ and for every $u \in W_0^{1,(1+\sigma)'}(\Omega)$, where $\| \cdot \|_{H_0^1(\Omega,\lambda)}$ denotes the norm in (8).

Moreover, for every $\lambda \in A_2$, $H^1(\Omega, \lambda)$ continuously embeds in $W^{1,1}(\Omega)$, which continuously embeds in $L^{n/(n-1)}(\Omega)$ if $n \geq 2$ (and in $L^2(\Omega)$ if Ω is an real interval) and then there exists a constant $c = (n, \Omega)$ (depending only on n, Ω) such that for every $f \in L^n(\Omega)$

$$\|f\|_{H^{-1}(\Omega,\lambda)} \leq c \left(\int_\Omega \lambda^{-1} dx \right)^{1/2} \|f\|_{L^n(\Omega)}. \tag{11}$$

If $n = 1$ the analogous estimate holds with $f \in L^2(\Omega)$.

Finally we recall some classical results concerning abstract parabolic equations (see, for instance, [17], [16], [31] and [33]). Let V be a real reflexive Banach space and let H a Hilbert space for which we have the classical triple

$$V \subset H \subset V'$$

with continuous and dense embeddings. Moreover consider a family of linear operators

$$A(t) : V \longrightarrow V', \qquad 0 \leq t \leq T$$

such that

$$\begin{cases} t \mapsto \langle A(t)\varphi, \psi \rangle_{V' \times V} & \text{is measurable on } [0,T] \\ \|\varphi\|_V^2 \leq \langle A(t)\varphi, \varphi \rangle_{V' \times V} \leq L \|\varphi\|_V^2 & \\ |\langle A(t)\varphi, \psi \rangle_{V' \times V}| \leq M \langle A(t)\varphi, \varphi \rangle_{V' \times V}^{1/2} \langle A(t)\psi, \psi \rangle_{V' \times V}^{1/2} & \end{cases} \tag{12}$$

for every $\varphi, \psi \in V$. We define now

$$\mathcal{H} \stackrel{\text{def}}{=} L^2(0,T;H), \quad \mathcal{V} \stackrel{\text{def}}{=} L^2(0,T;V), \quad \mathcal{W} \stackrel{\text{def}}{=} \{v \in \mathcal{V} \mid v' \in \mathcal{V}'\} \tag{13}$$

and $\mathcal{V}' = L^2(0,T;V')$ the dual space of $\mathcal{V}$, endowed with the standard norms

$$\|u\|_{L^2(0,T;X)} = \left(\int_0^T \|u(t)\|_X^2 dt \right)^{1/2}, \quad \|u\|_{\mathcal{W}} \stackrel{\text{def}}{=} \left(\|u\|_{\mathcal{V}}^2 + \|u'\|_{\mathcal{V}'}^2 \right)^{1/2},$$

(with $X = H, V, V'$) where v' denotes the distributional derivative of v. It is well known (see Theorem 3.1, chap. 1, in [17]) that

$$\mathcal{W} \subset C^0([0,T]; H), \tag{14}$$

and there exists a constant c, depending only on T and on the norms of the embeddings $V \subset H$, $H \subset V'$, such that

$$\max_{t \in [0,T]} \|u(t)\|_H \leq c \, \|u\|_{\mathcal{W}},$$

hence we can introduce the space

$$\mathcal{W}^0 = \{u \in \mathcal{W} \mid u(0) = 0\}, \tag{15}$$

and we recall that $C^\infty([0,T]; V)$ is dense in $(\mathcal{W}, \|\cdot\|_{\mathcal{W}})$ (see, for instance, [31], chap. 23). Now define the abstract operators

$$\mathbb{A} : \mathcal{V} \longrightarrow \mathcal{V}' \qquad \text{where} \qquad \mathbb{A}u\ (t) \stackrel{\text{def}}{=} A(t)u(t) \qquad 0 \leq t \leq T, \tag{16}$$

$$\mathcal{P} \ : \ \mathcal{W} \longrightarrow \mathcal{V}', \qquad (\mathcal{P}u)\ (t) \ = \ u'(t) + \mathbb{A}u(t) \qquad 0 \leq t \leq T. \tag{17}$$

and recall the following classical result (see, for instance, [17], Theorem 4.1, chap. 3).

Theorem 2.2. *Let $\mathcal{P} : \mathcal{W} \to \mathcal{V}'$ be the operator defined in* (17) *and suppose that the family of operators $A(t)$ $(0 \leq t \leq T)$ verifies* (12)*. Then*

(*i*) $\mathcal{P} : \mathcal{W} \longrightarrow \mathcal{V}'$ *is linear and continuous and*

$$\|\mathcal{P}u\|_{\mathcal{V}'} \leq \sqrt{2}ML\|u\|_{\mathcal{W}} \qquad \text{for every } u \in \mathcal{W},$$

(*ii*) $\langle \mathcal{P}u, u \rangle \geq \|u\|^2_{\mathcal{V}}$ *for every* $u \in \mathcal{W}^0$,

(*iii*) $\mathcal{P} : \mathcal{W}^0 \longrightarrow \mathcal{V}'$ *is an isomorphism such that*

$$\|u\|_{\mathcal{W}} \leq (ML + 2)\|\mathcal{P}u\|_{\mathcal{V}'} \qquad \text{for every } u \in \mathcal{W}^0.$$

From now on we will actually consider these spaces when $H = L^2(\Omega, \lambda)$ and $V = H^1_0(\Omega, \lambda)$ endowed with the topology induced by the norm $\|\cdot\|_{H^1_0(\Omega,\lambda)}$ in (8) where Ω is a bounded open set of $\mathbb{R}^n$ with Lipschitz boundary and $\lambda \in A_2$, and we will denote by $\mathcal{H}(0, T; \Omega)$, $\mathcal{V}_\lambda(0, T; \Omega)$, $\mathcal{V}'_\lambda(0, T; \Omega)$ and $\mathcal{W}_\lambda(0, T; \Omega)$ (and for sake of simplicity we will omit $(0, T; \Omega)$ if there is no ambiguity) the corresponding abstract spaces like those introduced in (13). As usual we get the evolution triple

$$\mathcal{V}_\lambda \subset \mathcal{H} \equiv \mathcal{H}' \subset \mathcal{V}'_\lambda$$

and $\|f\|_{\mathcal{V}'_\lambda} \leq \|f\|_{\mathcal{H}}$ for every $f \in \mathcal{H}$ and for every weight $\lambda \in A_2$ and, if $u \in \mathcal{H}$ and $v \in \mathcal{V}_\lambda$, we have that

$$\langle u, v \rangle_{\mathcal{V}'_\lambda \times \mathcal{V}_\lambda} \overset{\text{def}}{=} \langle u, v \rangle_\lambda = (u, v)_{\mathcal{H}} = \int_0^T \int_\Omega u(x, t)v(x, t)\lambda(x)dxdt. \tag{18}$$

As in (14) we obtain that $\mathcal{W}_\lambda(0, T; , \Omega) \subset C^0([0, T]; L^2(\Omega, \lambda))$; indeed by (7) it follows that there exists a positive constant $c = c(n, \Omega, K)$ (depending only on n, Ω, K) such that

$$\max_{t\in[0,T]} \|u(t)\|_{L^2(\Omega,\lambda)} \leq c\ \|u\|_{\mathcal{W}_\lambda}. \tag{19}$$

for every $u \in \mathcal{W}_\lambda$, for every $\lambda \in A_2(K)$. Therefore we introduce $\mathcal{W}^0_\lambda(0, T; \Omega)$ as done in (15). By (19) and (9)

$$\mathcal{W}_\lambda \subset C^0([0, T]; L^{1+\sigma}(\Omega) \qquad \text{and} \qquad \max_{t\in[0,T]} \|u(t)\|_{L^{1+\sigma}(\Omega)} \leq c\|\lambda^{-1}\|^{1/2}_{L^1(\Omega)}\|u\|_{\mathcal{W}_\lambda}, \tag{20}$$

where $c = c(n, \Omega, K)$ depends only on n, Ω, K.
Finally observe that if two weights λ and $\tilde{\lambda}$ are *comparable* in Ω, that is there exist constants $c_1, c_2 > 0$ such that

$$c_1\tilde{\lambda}(x) \leq \lambda(x) \leq c_2\tilde{\lambda}(x) \qquad \text{a.e. in } \Omega, \tag{21}$$

the norms induced by λ and $\tilde{\lambda}$ are equivalent and in particular

$$\sqrt{c_1}\|u\|_{\mathcal{V}_{\tilde{\lambda}}} \leq \|u\|_{\mathcal{V}_\lambda} \leq \sqrt{c_2}\|u\|_{\mathcal{V}_{\tilde{\lambda}}}, \qquad \sqrt{c_1}\|f\|_{\mathcal{V}'_\lambda} \leq \|f\|_{\mathcal{V}'_{\tilde{\lambda}}} \leq \sqrt{c_2}\|f\|_{\mathcal{V}'_\lambda}. \tag{22}$$

3. Compactness type results in weighted Sobolev spaces

In this section we are going to prove some compactness type results for a sequence of functions $(v_h)_h$, each one belonging to a suitable weighted Sobolev space depending on h. There results are needed to prove the main result in the fourth section.
First we begin by proving an extension of a classical compactness result due to J. L. Lions (see [16], chap. 1, Theorem 5.1).

Theorem 3.1. *Let Ω be a bounded open set of $\mathbb{R}^n$. Consider a sequence of weights $(\lambda_h)_h$ and a sequence of functions $v_h \in \mathcal{W}_{\lambda_h}$ (see section 2) such that there exist three constants $K \geq 1$ and $c_1, c_2 > 0$ for which*

$$(\lambda_h)_h \subset A_2(K), \qquad \int_\Omega \lambda_h^{-1} dx \leq c_1 \quad \textit{for every } h;$$

and

$$\|v_h\|_{\mathcal{W}_{\lambda_h}} \leq c_2 \qquad \textit{for every } h \in \mathbb{N}.$$

Then there exists a function $v \in L^2(0,T;L^1(\Omega))$ such that, up to subsequences,

(i) $\quad v_h \to v$ *in* $L^2(0,T;L^1(\Omega))$.

If moreover $\lambda_h \to \lambda$, $\lambda_h^{-1} \to \tilde{\lambda}^{-1}$ in $L^1(\Omega)$-weak with λ and $\tilde{\lambda}$ comparable weights (see (21)) then $v \in L^2(0,T;L^2(\Omega,\lambda))$ and, up to subsequences, the following facts hold:

(ii) $\int_0^T \int_\Omega v_h^2 \lambda_h dxdt \to \int_0^T \int_\Omega v^2 \lambda dxdt;$

(iii) $\int_0^T \int_\Omega v_h \varphi \lambda_h dxdt \to \int_0^T \int_\Omega v \varphi \lambda dxdt \qquad$ *for every* $\varphi \in C_c^\infty(\Omega \times (0,T))$.

Proof. Denote for simplicity by $\mathcal{W}_h$ the space $\mathcal{W}_{\lambda_h}$. First of all observe that if you given two reflexive and separable Banach spaces U and V, U dense in V, a Hilbert space H such that $V \subset H$ with dense and continuous embedding, one has that

$$C^1\big([0,T];U\big) \qquad \text{is dense in} \qquad \big\{z \in L^2(0,T;V) \mid z' \in L^2(0,T;V')\big\}$$

(see [31], ex. in chap. 23). Consider the space

$$\mathbb{X} = C^1\big([0,T];W_0^{1,(1+\sigma)'}(\Omega)\big)$$

which then is dense in every $\mathcal{W}_h$. Hence we can suppose the sequence $(v_h)_h$ is in $\mathbb{X}$, otherwise we could approximate it as follows

$$\|v_h - z_h\|_{\mathcal{W}_h} \leq 1/h$$

with $(z_h)_h \subset \mathbb{X}$. For every $z \in \mathbb{X}$ consider

$$z_\varepsilon(x,t) \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} \rho_\varepsilon(x-y)\bar{z}(y,t) \qquad (x,t) \in \mathbb{R}^n \times [0,T] \tag{23}$$

where $(\rho_\varepsilon)_{\varepsilon>0}$ is a family of mollifiers defined by a radial function and $\bar{z}(\cdot,t)$ denotes the function $z(\cdot,t)$ extended to zero outside of Ω. Observe that $(z_\varepsilon)' = (z')_\varepsilon$.
(i) Now considering the functions $v_{h,\varepsilon}$ one can prove, following the proof of Theorem 3.1 in [22], that for every $\varepsilon > 0$ the sequence $(v_{h,\varepsilon})_h$ is bounded in

$$\mathcal{Z} = \big\{z \in L^2(0,T;W_0^{1,(1+\sigma)'}(\Omega)) \mid z' \in L^2(0,T;W^{-1,1+\sigma}(\tilde{\Omega}))\big\}$$

where $\tilde{\Omega}$ is a bounded open set containing $\bar{\Omega}$. Then by Theorem 5.1, chap. 1, in [16], one obtain the compactness of $(v_{h,\varepsilon})_h$ in $L^2(0,T;L^2(\Omega))$.
(ii) Then we show the second point. First of all, since $(\lambda_h^{-1})_h$ is bounded in $L^1(\Omega)$, by (10) we have that there exists $u \in L^2(0,T;W_0^{1,1+\sigma}(\Omega))$ such that $v_h \to u$ in $L^2(0,T;W_0^{1,1+\sigma}(\Omega))$-weak and by point (i) we deduce $u = v$. Moreover, by Lemma 2.13 in [21] and since $\tilde{\lambda}$ and λ are comparable, we have

$$v \in L^2(0,T;L^2(\Omega,\tilde{\lambda})) = L^2(0,T;L^2(\Omega,\lambda)). \tag{24}$$

Since $\lambda_h \to \lambda$ in $L^1(\Omega)$ we have that

$$\int_\Omega \lambda_h dx \leq c \qquad \text{for every } h \in \mathbb{N}. \tag{25}$$

Then we have

$$\begin{aligned}\int_0^T \int_\Omega (v_h^2\lambda_h - v^2\lambda)dxdt &= \int_0^T \int_\Omega (v_h^2 - v_{h,\varepsilon}^2)\lambda_h dxdt + \\ &+ \int_0^T \int_\Omega (w_{h,\varepsilon}^2\lambda_h - v_\varepsilon^2\lambda)dxdt + \int_0^T \int_\Omega (v_\varepsilon^2 - v^2)\lambda dxdt.\end{aligned} \tag{26}$$

By Proposition 2.12 in [21] and (25), the first and the third term in the right of (26) are $O(\varepsilon^2)$ uniformly in h. As the central term is concerned we have:

$$\begin{aligned}|v_{h,\varepsilon}(x,t) - v_\varepsilon(x,t)| &= \Big| \int_{\mathbb{R}^n} \big(\bar{v}_h(\xi,t) - \bar{v}(\xi,t)\big)\rho_\varepsilon(x-\xi)d\xi \Big| \\ &\leq \|\rho_\varepsilon\|_\infty \int_\Omega |v_h(\xi,t) - v(\xi,t)|d\xi\end{aligned} \tag{27}$$

by which we obtain the existence of a constant $c_1 = c_1(\rho,\varepsilon,\sigma,\Omega)$ such that

$$\int_0^T \|v_{h,\varepsilon}(t) - v_\varepsilon(t)\|_\infty^2 dt \leq c_1 \int_0^T \Big(\int_\Omega |v_h(x,t) - v(x,t)|dx\Big)^2 dt.$$

By (i) we have that $v_h \to v$ in $L^2(0,T;L^1(\Omega))$ and then we can conclude that for every ε

$$\|v_{h,\varepsilon} - v_\varepsilon\|_{L^2(0,T;L^\infty(\Omega))} \underset{h}{\longrightarrow} 0. \tag{28}$$

Moreover from (27) we derive that, with fixed ε, $v_{h,\varepsilon}^2(t) \to v_\varepsilon^2(t)$ in $L^\infty(\Omega)$ for a.e. $t \in [0,T]$. We then obtain

$$\int_\Omega v_{h,\varepsilon}^2(x,t)\lambda_h dx \to \int_\Omega v_\varepsilon^2 \lambda dx \qquad \text{for a.e. } t \in [0,T].$$

Moreover the functions $t \mapsto \int_\Omega v_{h,\varepsilon}^2(x,t)\mu_h dx$ are equibounded in $L^1(0,T)$. Infact by the proof of (i) we have that $(v_{h,\varepsilon})_h$ is bounded in $\mathcal{Z}$ and using (14) with $V = W_0^{1,(1+\sigma)'}(\Omega)$ and $H = L^2(\Omega,\lambda_h)$ and $\mathcal{Z}$ in the place of $\mathcal{W}$ (thanks to (9)) obtain that there exists c_2 such that

$$\max_{t\in[0,T]} \|v_{h,\varepsilon}(t)\|_{L^2(\Omega,\lambda_h))} \leq c_2$$

so that

$$\int_0^T \Big| \int_\Omega v_{h,\varepsilon}^2 \lambda_h dx - \int_\Omega v_\varepsilon^2 \lambda dx \Big| dt \to 0.$$

(iii) It is an immediate consequence of the following estimates: if $\varphi \in C_c^\infty(\Omega \times (0,T))$

$$\begin{aligned}
&\Big| \int_0^T \int_\Omega v_h \varphi \lambda_h dxdt - \int_0^T \int_\Omega v \varphi \lambda dxdt \Big| = \\
&= \Big| \int_0^T \int_\Omega (v_h - v_{h,\varepsilon}) \varphi \lambda_h dxdt + \int_0^T \int_\Omega (v_{h,\varepsilon} \varphi \lambda_h - v_\varepsilon \varphi \lambda) dxdt + \int_0^T \int_\Omega (v_\varepsilon - v) \varphi \lambda dxdt \Big| \\
&\le \Big(\int_0^T \int_\Omega (v_h - v_{h,\varepsilon})^2 \lambda_h \Big)^{1/2} \Big(\int_0^T \int_\Omega \varphi^2 \lambda_h \Big)^{1/2} + \int_0^T \int_\Omega (v_{h,\varepsilon} \varphi \lambda_h - v_\varepsilon \varphi \lambda) dxdt + \\
&\qquad + \Big(\int_0^T \int_\Omega (v_\varepsilon - v)^2 \lambda dxdt \Big)^{1/2} \Big(\int_0^T \int_\Omega \varphi^2 \lambda dxdt \Big)^{1/2}.
\end{aligned}$$

By Proposition 2.12 in [21] and (28) we have that for every fixed ε,

$$\int_0^T \int_\Omega (v_{h,\varepsilon} \varphi \mu_h - v_\varepsilon \varphi \mu) dxdt \xrightarrow[h]{} 0.$$

Then for fixed ε letting h go to infinity we conclude. □

In addition to this "strong" result, a "weak" result holds. That is the following proposition.

Proposition 3.2. *Suppose $(u_h)_h$ and $(v_h)_h$ two sequences, $u_h \in L^2(0,T; L^2(\Omega, \lambda_h))$, $v_h \in \mathcal{W}_{\lambda_h}$, where*

$$\begin{gathered}
(\lambda_h)_h \subset A_2(K), \qquad \int_\Omega \lambda_h dx \le c_1 \quad \text{and} \quad \int_\Omega \lambda_h^{-1} dx \le c_2 \qquad \text{for every } h \\
\lambda_h \to \lambda, \quad \lambda_h^{-1} \to \tilde\lambda^{-1} \qquad \text{in } L^1(\Omega)\text{-weak}, \qquad \lambda \text{ and } \tilde\lambda \text{ comparable.}
\end{gathered} \tag{29}$$

Suppose that

$$\|u_h\|_{L^2(0,T;L^2(\Omega,\lambda_h))} \le c_3 \quad \text{and} \quad \|v_h\|_{\mathcal{W}_{\lambda_h}} \le c_4.$$

Then there exist two functions $u, v \in L^2(0,T; L^2(\Omega, \lambda))$ such that, up to subsequences,

$$\int_0^T \int_\Omega u_h v_h \lambda_h dxdt \ \to \ \int_0^T \int_\Omega uv\lambda dxdt.$$

Proof. It is sufficient writing

$$\begin{aligned}
\int_0^T \int_\Omega v_h u_h \lambda_h dxdt - \int_0^T \int_\Omega vu\varphi\lambda dxdt \ &= \int_0^T \int_\Omega (v_h u_h \lambda_h - v_{h,\varepsilon} u_h \lambda_h) dxdt + \\
&+ \int_0^T \int_\Omega (v_{h,\varepsilon} u_h \varphi \lambda_h - v_\varepsilon u \lambda) dxdt + \int_0^T \int_\Omega (v_\varepsilon u \lambda - vu\lambda dxdt\,,
\end{aligned}$$

where the functions $v_{h,\varepsilon}$ and v_ε are the approximations analogous to those defined in (23), and then estimate the three terms. By Proposition 2.12 in [21] we estimate the first term

$$\Big|\int_0^T\int_\Omega (v_h u_h\lambda_h - v_{h,\varepsilon}u_h\lambda_h)dxdt\Big| \leq \Big(\int_0^T\int_\Omega u_h^2\lambda_h dxdt \int_0^T\int_\Omega |v_h - v_{h,\varepsilon}|^2\lambda_h dxdt\Big)^{1/2} \leq c\,\varepsilon$$

and the third one in the same way. As regards the central term we have that, by (6), there exists $\sigma \in \mathbb{R}$ such that

$$\int_0^T \|u_h\lambda_h\|^2_{L^{1+\sigma}(\Omega)}dt \leq c$$

and then $(u_h\lambda_h)_h$ is, up to subsequences, weakly convergent in $L^2(0,T;L^1(\Omega))$ to a function w. As done in Proposition 2.1 in [31] one can prove that there exists $u \in L^2(0,T;L^2(\Omega,\lambda))$ such that $w = u\lambda$. Since, by (28), $v_{h,\varepsilon}$ strongly converge to v_ε in $L^2(0,T;L^\infty(\Omega))$ we conclude. $\square$

Remark. From Theorem 3.1 and Proposition 3.2 we obtain at once a compactness result for functions v_h independent of t.

The proof of the following proposition can be derived from the corrisponding Proposition 3.8 in [22].

Proposition 3.3. *Let $K, \widetilde{K} \geq 1$; let Ω be a bounded open set of $\mathbb{R}^n$ with Lipschitz boundary, let $(\lambda_h)_h$ be a sequence in $A_2(K)$ and let $v_h \in \mathcal{W}_{\lambda_h}$ $(h = 1,2,...)$ be a sequence for which there exists a positive constant c such that $\|v_h\|_{\mathcal{W}_{\lambda_h}} \leq c$. Assume that there exist two comparable weights (see (21)) $\lambda \in A_2(K)$ and $\tilde{\lambda} \in A_2(\widetilde{K})$ such that*

$$\lambda_h \longrightarrow \lambda \qquad \text{and} \qquad \lambda_h^{-1} \longrightarrow \tilde{\lambda}^{-1} \ \text{ in } L^1(\Omega)\text{−weak.}$$

Then there exists $\sigma > 0$ (see (10)) and a function v such that

(i) $v_h \to v$ in $L^2(0,T;W_0^{1,1+\sigma}(\Omega))$-weak and $v_h' \to v'$ in $L^2(0,T;W^{-1,1+\sigma}(\Omega))$-weak (up to a subsequence);

(ii) $v \in \mathcal{V}_{\tilde{\lambda}}$ and $\|v\|_{\mathcal{V}_{\tilde{\lambda}}} \leq \liminf_{h\to\infty}\|v_h\|_{\mathcal{V}_{\lambda_h}}$;

(iii) $v' \in \mathcal{V}_\lambda'$ and $\|v'\|_{\mathcal{V}_\lambda'} \leq \liminf_{h\to\infty}\|v_h'\|_{\mathcal{V}_{\lambda_h}'}$;

(iv) if $f \in L^2(0,T;L^n(\Omega))$ then

$$\|f\|_{\mathcal{V}_\lambda'} \leq \liminf_{h\to\infty}\|f\|_{\mathcal{V}_{\lambda_h}'} \leq \limsup_{h\to\infty}\|f\|_{\mathcal{V}_{\lambda_h}'} \leq \|f\|_{\mathcal{V}_{\tilde{\lambda}}'}\,,$$

(v) if $f_h \in L^2(0,T;L^2(\Omega,\lambda_h))$ such that $\|f_h\|_{L^2(0,T;L^2(\Omega,\lambda_h))} \leq c$, then there exists $f \in L^2(0,T;L^2(\Omega,\lambda)) = L^2(0,T;L^2(\Omega,\tilde{\lambda}))$ such that

$$\|f\|_{\mathcal{V}_\lambda'} \leq \liminf_{h\to\infty}\|f_h\|_{\mathcal{V}_{\lambda_h}'} \leq \limsup_{h\to\infty}\|f_h\|_{\mathcal{V}_{\lambda_h}'} \leq \|f\|_{\mathcal{V}_{\tilde{\lambda}}'}.$$

We introduce now the class of coefficients matrices of these parabolic equations and the definition of weak solution of Cauchy-Dirichlet problem to prove successively some compactness results on the solutions (see also [33]).

Definition 3.4. Let L, M, K, C, T be positive constants, with $L, M, K \geq 1$, let Ω be a bounded open set of $\mathbb{R}^n$. We denote by $\mathcal{M}_{\Omega\times(0,T)}(L, M, K, C)$ the class of matrices $a(x,t) = [a_{ij}(x,t)]_{i,j=1}^n$ of order n with $a_{ij} \in L^1_{loc}(\mathbb{R}^n \times (0,T))$, $i, j = 1, ...n$, for which there exists λ, weight on $\mathbb{R}^n$, such that

(S.1) $\lambda(x)|\xi|^2 \leq \big(a(x,t)\cdot\xi, \xi\big) \leq L\lambda(x)|\xi|^2$, for a.e. $(x,t) \in \Omega \times (0,T)$ and for every $\xi \in \mathbb{R}^n$,

(S.2) $|\big(a(x,t)\cdot\xi, \eta\big)| \leq M\big(a(x,t)\cdot\xi, \xi\big)^{1/2}\big(a(x,t)\cdot\eta, \eta\big)^{1/2}$, for a.e. $(x,t) \in \Omega \times (0,T)$ and for every $\xi, \eta \in \mathbb{R}^n$,

(S.3) $\lambda \in A_2(K), \qquad \int\limits_\Omega \lambda dx + \int\limits_\Omega \lambda^{-1} dx \leq C.$

Given $a \in \mathcal{M}_{\Omega\times(0,T)}(L, M, K, C)$ we will denote by Λ_a the set of A_2 weights for which (S.1)-(S.3) hold.
By $\mathcal{N}_{\Omega\times(0,T)}(L, M, K, C)$ we denote the class of matrices $a(x,t) = [a_{ij}(x,t)]_{i,j=1}^n$ with $a_{ij} \in L^1_{loc}(\mathbb{R}^n \times (0,T))$, $i, j = 1, ...n$, for which there exists λ, weight on $\mathbb{R}^n$, such that (S.1), (S.2) and

(S.3)$'$ $\lambda \in A_{1+2/n}(K), \qquad \int\limits_\Omega \lambda dx + \Big(\int\limits_\Omega \lambda^{-n/2} dx\Big)^{2/n} \leq C$

hold (if $n = 1$ we define the class $\mathcal{N}_{\Omega\times(0,T)}(L, M, K, C) = \mathcal{M}_{\Omega\times(0,T)}(L, M, K, C)$). By $\mathcal{M}_\Omega(L, M, K, C)$ and $\mathcal{N}_\Omega(L, M, K, C)$ we denote the corresponging subclasses of matrices independent of t.

Remark. Given a sequence of matrices $(a_h)_h \subset \mathcal{M}_{\Omega\times(0,T)}(L, M, K, C)$ and a sequence of weights $(\lambda_h)_h$, $\lambda_h \in \Lambda_{a_h}$, by Remark 2.7 in [21] we have the existence of two weights λ and $\tilde{\lambda}$ such that, up to a subsequence, $\lambda_h \to \lambda$ and $\lambda_h^{-1} \to \tilde{\lambda}^{-1}$ in $L^1_{loc}(\mathbb{R}^n)$.

Given $a \in \mathcal{M}_{\Omega\times(0,T)}(L, M, K, C)$ and $\lambda \in \Lambda_a$ we define a parabolic operator as follows:

$$\mathbb{A} = \mathbb{A}_a : \mathcal{V}_\lambda \to \mathcal{V}'_\lambda \quad \text{and} \quad \mathcal{P} = \mathcal{P}_{\lambda,a} : \mathcal{W}_\lambda \to \mathcal{V}'_\lambda.$$

where $\mathbb{A}$ and $\mathcal{P}$ are respectively the abstract operators defined in (16) and (17) with $A(t) = -\text{div}\,(a(\cdot,t)\cdot D) : H^1_0(\Omega,\lambda) \to H^{-1}(\Omega,\lambda)$ verifying (12) with $V = H^1_0(\Omega,\lambda)$.

Remark. Observe that if $\Lambda_a \neq \emptyset$, the operators belonging to the family $(\mathcal{P}_{\lambda,a})_{\lambda\in\Lambda_a}$ are defined on the same space. Indeed, by (S.1), $\lambda_1, \lambda_2 \in \Lambda_a$ are comparable (see (21) and (22)), but $\mathcal{P}_{\lambda_1,a} \neq \mathcal{P}_{\lambda_2,a}$.

Definition 3.5. Let $L, M, K \geq 1$, $C, T > 0$, let Ω be a bounded open set of $\mathbb{R}^n$, $a \in \mathcal{M}_{\Omega\times(0,T)}(L, M, K, C)$, $\lambda \in \Lambda_a$ and $f \in \mathcal{V}'_\lambda$. Then we call a function $u \in \mathcal{W}^0_\lambda$ *solution of the Cauchy-Dirichlet problem*

$$\begin{cases} \lambda\frac{\partial}{\partial t}u - \text{div}(a\cdot Du) = f & \text{in } \Omega\times(0,T) \\ u = 0 & \text{on } \big(\partial\Omega\times(0,T)\big)\cup\big(\Omega\times\{0\}\big) \end{cases} \tag{30}$$

if

$$(\mathcal{P}_{\lambda,a}u)(t) = u'(t) + (\mathbb{A}_a u)(t) = f(t) \qquad \text{for a.e. } t \in [0,T]\,. \tag{31}$$

Remark. Given a matrix $a \in \mathcal{M}_{\Omega\times(0,T)}(L, M, K, C)$ then, by Theorem 2.2 (iii), we obtain that for every $\lambda \in \Lambda_a$ and $f \in \mathcal{V}'_\lambda$ there exists a unique solution of the problem (30).

Given a sequences $(a_h)_h$ of matrices in $\mathcal{M}_{\Omega\times(0,T)}(L,M,K,C)$ and $(\lambda_h)_h$ of weights, $\lambda_h \in \Lambda_{a_h}$ for every $h \in \mathbb{N}$, consider the following sequence of problems

$$\begin{cases} \lambda_h(x)\frac{\partial}{\partial t}u - \operatorname{div}(a_h(x,t)\cdot Du) = f & \text{in } \Omega\times(0,T) \\ u = 0 & \text{on } \big(\partial\Omega\times(0,T)\big)\cup\big(\Omega\times\{0\}\big) \end{cases} \tag{32}$$

We will write (32) in the abstract way (here $\frac{d}{dt} : \mathcal{W}^0_{\lambda_h} \to \mathcal{V}'_h$ is to be intended as in (18))

$$\begin{cases} \mathcal{P}_{\lambda_h,a_h}u := \dfrac{d}{dt}u + \mathbb{A}_{a_h}u = f \\ u \in \mathcal{W}^0_{\lambda_h}\,. \end{cases} \tag{33}$$

The proof of the following theorem can be derived following the analogous one in [21].

Theorem 3.6. *Let L, M, K, C, T be positive constants with $L, M, K \geq 1$ and let Ω be a bounded open set of $\mathbb{R}^n$. Let $(a_h)_h$ be a sequence in $\mathcal{M}_{\Omega\times(0,T)}(L,M,K,C)$, $(\lambda_h)_h$ a sequence with $\lambda_h \in \Lambda_{a_h}$ and $f \in L^2(0,T;L^n(\Omega))$. Consider the parabolic equations* (33) *and their solutions u_h. Then*

(i) there exists a positive constant $c = c(L,M,K,C,\Omega)$ (depending only on L,M,K,C,Ω) such that

$$\|u_h\|_{\mathcal{W}_{\lambda_h}} \leq c\|f\|_{L^2(0,T;L^n(\Omega))};$$

(ii) say λ and u the weak limits, up to a subsequence, respectively of $(\lambda_h)_h$ in $L^1(\Omega)$-weak and of $(u_h)_h$ in $L^2(0,T;W^{1,1}(\Omega))$-weak; we have that

$$U_h(t) = \int_\Omega u_h^2(x,t)\lambda_h(x)dx \;\to\; U(t) = \int_\Omega u^2(x,t)\lambda dx \qquad \text{in } C^0\big([0,T]\big).$$

Finally we state a weighted compensated compactness type result which extends well-known classical results (see, for instance, [20], [30], [29], [28], [9] and [10]), whose proof can be obtained in completely analogous way to the corresponding result in [22] using Proposition 3.2.

Theorem 3.7. *Suppose that Ω is a bounded open set of $\mathbb{R}^n$, $K \geq 1$, $(\lambda_h)_h$ is a sequence in $A_2(K)$, $\lambda \in A_2$, such that*

$$\int_\Omega \lambda_h^{-1}dx \leq c_1 \qquad \text{for every } h \in \mathbb{N}.$$

Consider a sequence of functions $u_h \in \mathcal{W}_{\lambda_h}$ ($h = 1,2,...$) and a function $u \in \mathcal{W}_\lambda$ such that

$$\|u_h\|_{\mathcal{W}_{\lambda_h}} \leq c_2 \qquad \text{for every } h, \qquad u_h \to u \qquad \text{in } L^2(0,T;L^1(\Omega))\,.$$

Consider a sequence of vector functions $\alpha_h \in L^2(0,T;(L^2(\Omega,\lambda_h^{-1}))^n)$ ($h = 1,2,...$) and $\alpha \in L^2((0,T);(L^2(\Omega,\lambda^{-1}))^n)$ such that

$$\|\alpha_h\|_{L^2((0,T);(L^2(\Omega,\lambda_h^{-1}))^n)} \leq c_3\,, \qquad \alpha_h \to \alpha \qquad \text{in } L^2(0,T;(L^1(\Omega))^n)-\text{weak}.$$

Assume further that

$$\lambda_h\frac{\partial u_h}{\partial t} - \text{div}\,(\alpha_h) = f \in L^2(0,T;L^n(\Omega)) \qquad \text{on } C^1_0(\Omega\times(0,T))$$

for every $h \in \mathbb{N}$. *Then*

$$\big(\alpha_h, Du_h\big) \to \big(\alpha, Du\big) \qquad \textit{in } D'(\Omega \times (0,T)).$$

4. The definition of G-convergence and a G-compactness result

In this section we will introduce the definition of G-convergence for a class of degenerate parabolic operators and get a G-compactness result (Theorem 4.3).

Definition 4.1. Let L, M, K, C, T be positive constants with $L, M, K \geq 1$, let Ω be a bounded open set of $\mathbb{R}^n$ and let a_h $(h = 1, 2, ...)$ and a be matrices in $\mathcal{M}_{\Omega\times(0,T)}(L, M, K, C)$ and $\lambda_h \in \Lambda_{a_h}$ and $\lambda \in \Lambda_a$ (see Definition 3.4). We say that the sequence $\mathcal{P}_{\lambda_h,a_h}$ *G-converges to* $\mathcal{P}_{\lambda,a}$ *in* Ω $\times(0,T)$, and we write

$$\mathcal{P}_{\lambda_h,a_h} \xrightarrow{\ G\ } \mathcal{P}_{\lambda,a} \qquad \text{in } \Omega \times (0,T),$$

if for every $f \in L^2(0,T; L^n(\Omega))$ it results that

$$\begin{aligned} u_h &\to u && \text{in } L^2(0,T; L^1(\Omega)) \\ a_h \cdot Du_h &\to a \cdot Du && \text{in } L^2(0,T; L^1(\Omega))^n\textit{-weak}, \end{aligned}$$

where u_h and u denote respectively the solutions (see Definition 3.5) of

$$\begin{cases} \mathcal{P}_{\lambda_h,a_h} v = \lambda_h \frac{\partial v}{\partial t} - \operatorname{div}(a_h \cdot Dv) = f & \text{in } \Omega \times (0,T) \\ v = 0 & \text{on } \big(\partial\Omega \times (0,T)\big) \cup \big(\Omega \times \{0\}\big), \end{cases}$$

$$\begin{cases} \mathcal{P}_{\lambda,a} v = \lambda \frac{\partial v}{\partial t} - \operatorname{div}(a \cdot Dv) = f & \text{in } \Omega \times (0,T) \\ v = 0 & \text{on } \big(\partial\Omega \times (0,T)\big) \cup \big(\Omega \times \{0\}\big). \end{cases}$$

Remark. In the classic case, i.e. when $\lambda \in \Lambda_a$ is a constant $\lambda_0 > 0$ the space $H_0^1(\Omega, \lambda_0)$ is $H_0^1(\Omega)$ and we get the classic definition of G-convergence (see, for instance, [26], [27], [5], [28] and [33]).

The proof of the following proposition can be obtained following the corresponding one for elliptic operators (see Proposition 2.9 in [10]).

Proposition 4.2. *Let* L, M, K, C, T, T_1, T_2 *be positive constants with* $L, M, K \geq 1$, Ω *be a bounded open set of* $\mathbb{R}^n$ *with Lipschitz boundary. Consider* $(a_h)_{h\in\mathbb{N}}$ *a sequence in* $\mathcal{M}_{\Omega\times(0,T)}$ $(L, M, K,\ C)$ *and* $\lambda_h \in \Lambda_{a_h}$ *a sequence of weights. If, for every interval* $I_i = [0, T_i]$ $(i = 1, 2)$ $I_1 \subseteq I_2 \subseteq [0, T]$ *and for every bounded open set* Ω_i $(i = 1, 2)$ *of* $\mathbb{R}^n$ *with Lipschitz boundaries with* $\Omega_1 \subseteq \Omega_2 \subseteq \Omega$, *we have*

$$\mathcal{P}_{\lambda_h,a_h} \xrightarrow{\ G\ } \mathcal{P}_{\mu_i,b_i} \qquad \textit{on } \Omega_i \times I_i,\ (i = 1, 2)$$

for suitable $b_i \in \mathcal{M}_{\Omega_i\times I_i}(L_i, M_i, K_i, C_i)$ *and* $\mu_i \in \Lambda_{b_i}$ $(i = 1, 2)$, *then* $b_1(x,t) = b_2(x,t)$ *a.e. in* $\Omega_1 \times I_1$ *and* $\mu_1 = \mu_2$ *a.e. in* Ω_1.

Remark. By Proposition 4.2 the uniqueness of the G-limit follows.

The following theorem states a precompactness result with respect to G-convergence for the class of operators $\mathcal{P}_{\lambda,a}$ defined by $a \in \mathcal{M}_{\Omega\times(0,T)}(L,M,K,C)$ and $\lambda \in \Lambda_a$.
We only give a scheme of the proof since the steps are the same of the corresponding proof in [22].

Theorem 4.3. *Let L, M, K, C, T be positive constants with $L, M, K \geq 1$ and let Ω be a bounded open set of $\mathbb{R}^n$ with Lipschitz boundary. Consider a sequence $(a_h)_{h\in\mathbb{N}} \subset \mathcal{M}_{\Omega\times(0,T)}$ (L,M,K,C) and a sequence of weights $\lambda_h \in \Lambda_{a_h}$ (see Definition 3.4). Then there exist a subsequence, still denoted by the index h, a matrix $a(x,t) = [a_{ij}(x,t)]_{i,j=1}^n$, $a_{ij} \in L^1_{loc}(\Omega\times(0,T))$, defined in $\Omega\times(0,T)$ and a weight λ such that $a \in \mathcal{M}_{\Omega\times(0,T)}(M^2KL, M\sqrt{L}, K, KC)$, $\lambda/K \in \Lambda_a$ and*

$$\lambda_h \to \lambda \qquad \text{in } L^1_{loc}(\mathbb{R}^n) - weak$$
$$\mathcal{P}_{\lambda_h,a_h} \xrightarrow{G} \mathcal{P}_{\lambda,a} \qquad \text{in } \Omega\times(0,T)\,.$$

Proof. First of all by Remark 2.7 in [21] we get the existence of two weights λ, $\tilde\lambda$, comparable (see (21)), such that, up to a subsequence, $\lambda_h \to \lambda$ and $\lambda_h^{-1} \to \tilde\lambda^{-1}$ and define three spaces $\mathcal{V}_\lambda = \mathcal{V}_{\tilde\lambda}$, $\mathcal{V}'_\lambda = \mathcal{V}'_{\tilde\lambda}$, $\mathcal{W}_\lambda = \mathcal{W}_{\tilde\lambda}$. The space $\mathcal{W}_\lambda$ is the space in which the limit parabolic operator (which we want to find) will be defined. With the derivative $\frac{d}{dt} : \mathcal{W}^0_\lambda \to \mathcal{V}'_\lambda$ we denote the operator acting as follows (see (18)) for $\varphi \in C^\infty_c(\Omega\times(0,T))$

$$\langle \frac{d}{dt}u, \varphi\rangle_{\mathcal{V}'_\lambda\times\mathcal{V}_\lambda} = \int_0^T\int_\Omega u(x,t)\frac{\partial\varphi}{\partial t}(x,t)\lambda(x)dxdt\,.$$

In a completly similar way as in Lemma 4.7 in [22] we can find three operators

$$\mathcal{B} : \mathcal{V}'_\lambda \to \mathcal{W}^0_\lambda, \quad \mathcal{K} : \mathcal{V}'_\lambda \to \mathcal{V}'_\lambda, \quad G : \mathcal{V}'_\lambda \to L^2(0,T;(L^2(\Omega,\lambda^{-1}))^n)$$

such that, up to subsequences,

$$\begin{aligned}
&\mathcal{P}^{-1}_{\lambda_h,a_h} f \to \mathcal{B}f && \text{in } L^2(0,T;L^1(\Omega)) && \text{for every } f \in L^2(0,T;L^n(\Omega)),\\
&a_h\cdot D(\mathcal{P}^{-1}_{\lambda_h,a_h} f) \to Gf && \text{in } L^2(0,T;(L^1(\Omega))^n)\text{-weak} && \text{for every } f \in L^2(0,T;L^n(\Omega)),\\
&\frac{d}{dt}(\mathcal{B}f) + \mathcal{K}f = f \quad \text{and} \quad \mathcal{K}f = -\text{div }(Gf) && \text{on } \mathcal{V}_\lambda && \text{for every } f \in \mathcal{V}'_\lambda.
\end{aligned}$$

The convergence of $\mathcal{P}^{-1}_{\lambda_h,a_h} f$ is in $L^2(0,T;L^1(\Omega))$ by Theorem 3.1, i). The sequence $a_h\cdot D(\mathcal{P}^{-1}_{\lambda_h,a_h} f)$ is bounded in $L^2(0,T;(L^{1+\sigma}(\Omega))^n)$ thanks to (9). Finally, Theorem 3.6, ii), is used to prove that $\mathcal{K}f = -\text{div }(Gf)$ on $\mathcal{V}_\lambda$ since we multiply $\mathcal{P}_{\lambda_h,a_h}u_h = f$ by u_h (where u_h is the solution to (33)) and proceed as in [22], Lemma 4.7.
Following the proof of Lemma 4.8 in [22] (using Theorem 3.7) one can prove that the operator $\mathcal{B}$ is invertible and there exists an operator $\mathbb{A} : \mathcal{V}_\lambda \to \mathcal{V}'_\lambda$ satisfying

$$\begin{aligned}
&\mathbb{A}(\mathcal{B}f) = \mathcal{K}f && \text{for every } f \in \mathcal{V}'_\lambda\\
&\frac{1}{K}\|u\|^2_{\mathcal{V}_\lambda} \leq \langle\mathbb{A}u,u\rangle_\lambda \leq M^2L\|u\|^2_{\mathcal{V}_\lambda} && \text{for every } u \in \mathcal{V}_\lambda,\\
&\langle\mathbb{A}u,v\rangle_\lambda \leq M\sqrt{L}\ \langle\mathbb{A}u,u\rangle_\lambda^{1/2}\langle\mathbb{A}v,v\rangle_\lambda^{1/2} && \text{for every } u,v \in \mathcal{V}_\lambda,
\end{aligned}$$

and such that

$$\mathcal{B}^{-1} = \mathcal{P} \stackrel{\text{def}}{=} \frac{d}{dt} + \mathbb{A} : \mathcal{W}^0_\lambda \to \mathcal{V}'_\lambda\,.$$

Now, following the proof of Theorem 4.5 in [22], one can prove before that

$$\langle \mathbb{A}u, v\rangle_\lambda = \int_0^T \int_\Omega \big((G\circ\mathcal{P})u, Dv\big)dxdt$$

and then that there exist a matrix $a = [a_{ij}(x,t)]_{i,j=1}^n$ such that $a_{ij} \in L^1_{loc}(\Omega\times(0,T))$ and

$$\begin{aligned}
&(G\circ\mathcal{P})u = a\cdot Du\\
&\frac{1}{K}\lambda(x)|\xi|^2 \le \big(a(x,t)\cdot\xi,\xi\big) \le M^2L\lambda(x)|\xi|^2,\\
&\big(a(x,t)\cdot\xi,\eta\big) \le M\sqrt{L}\big(a(x,t)\cdot\xi,\xi\big)^{1/2}\big(a(x,t)\cdot\eta,\eta\big)^{1/2}
\end{aligned}$$

a.e. in $\Omega\times(0,T)$, for every $u\in\mathcal{V}_\lambda$ and for every $\xi,\eta\in\mathbb{R}^n$. □

Corollary 4.4. *Let L, M, K, C be positive constants with $L, M, K \ge 1$. Consider a sequence $(a_h(x,t))_h$ of matrices defined in $\mathbb{R}^n\times(0,+\infty)$ for which there exists a sequence of weights $(\lambda_h)_h$ and a cube Q_0 such that*

(i) $\lambda_h(x)|\xi|^2 \le \big(a_h(x,t)\cdot\xi,\xi\big) \le L\lambda_h(x)$,
for a.e. $(x,t)\in\mathbb{R}^n\times(0,+\infty)$, for every $\xi\in\mathbb{R}^n$, for every $h\in\mathbb{N}$,

(ii) $|\big(a_h(x,t)\cdot\xi,\eta\big)| \le M\big(a_h(x,t)\cdot\xi,\xi\big)^{1/2}\big(a_h(x,t)\cdot\eta,\eta\big)^{1/2}$,
for a.e. $(x,t)\in\mathbb{R}^n\times(0,+\infty)$, for every $\xi,\eta\in\mathbb{R}^n$ and for every $h\in\mathbb{N}$,

(iii) $\lambda_h\in A_2(K)$ *and* $\int_{Q_0}\lambda_h dx + \int_{Q_0}\lambda_h^{-1}dx \le C$ *for every $h\in\mathbb{N}$.*

Then there exist a subsequence, still denoted by the index h, a matrix $a(x,t) = [a_{ij}(x,t)]_{i,j=1}^n$, $a_{ij}\in L^1_{loc}(\mathbb{R}^n\times(0,+\infty))$ and a weight λ such that $a\in\mathcal{M}_{\mathbb{R}^n\times(0,+\infty)}(M^2KL, M\sqrt{L}, K, KC)$, $\lambda/K\in\Lambda_a$ and

$$\mathcal{P}_{\lambda_h,a_h} \xrightarrow{G} \mathcal{P}_{\lambda,a} \qquad in\ \omega\times(0,\tau),$$

for every ω bounded open set with Lipschitz boundary and for every $\tau>0$.

Proof. By Remark 2.7 in [21] we have that λ_h and λ_h^{-1} are equibounded in $L^1_{loc}(\mathbb{R}^n)$ (it is easy obtained by (7)) and the existence of two limit weights λ and $\tilde\lambda$ defined in $\mathbb{R}^n$ such that, up to a subsequence, $\lambda_h\to\lambda$ and $\lambda_h^{-1}\to\tilde\lambda^{-1}$ in $L^1_{loc}(\mathbb{R}^n)$-weak. Choosing the sequence of sets $(-j,j)^n\times(0,j)$ invading $\mathbb{R}^n\times(0,+\infty)$ one can obtain the thesis by a diagonal process (use Proposition 4.2 and follow the proof of Theorem 4.10 in [22]). □

If $\lambda_h(x) = \lambda_0 > 0$ ($\lambda_0\in\mathbb{R}$) the previous G-compactness result comes to be a well-known classic G-compactness result (see Theorem 1 and Theorem 3 in [28], but also [26], [27], [5], [33]).

The sharpness of condition A_2 considered on the weights λ_h among the class of Muckenhoupt weights is shown by some counterexamples in the elliptic case and by considerations made in the following section, showing links between elliptic and parabolic convergence (see Remark 5).

Homogenization for problems like that defined in (30) is considerd in [21].

5. A comparison among G-convergence of different operators

In this section we want to analyze the relation existing among the behaviour of different kinds of operators defined by the same matrices. By Ω we will always denote a bounded open set with Lipschitz boundary. Consider a sequence of matrices $(a_h)_h$, $a_h = a_h(x)$, contained in $\mathcal{M}_{\Omega\times(0,T)}(L, M, K, C)$ which in this case will be simply denoted by $\mathcal{M}_\Omega(L, M, K, C)$ (see Definition 3.4). The problems we consider in this section are

$$\text{(I)}\quad \begin{cases} -\text{div }(a_h(x)\cdot Dv) = f & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega\,, \end{cases}$$

$$\text{(II)}\quad \begin{cases} \lambda_h \frac{\partial v}{\partial t} - \text{div}(a_h(x)\cdot Dv) = f & \text{in } \Omega\times(0,T) \\ u = 0 & \text{in } (\partial\Omega\times(0,T))\cup(\{0\}\times(0,T))\,, \end{cases}$$

$$\text{(III)}\quad \begin{cases} \frac{\partial v}{\partial t} - \text{div}(a_h(x)\cdot Dv) = f & \text{in } \Omega\times(0,T) \\ u = 0 & \text{in } (\partial\Omega\times(0,T))\cup(\{0\}\times(0,T))\,, \end{cases}$$

For the moment we shall confine ourselves to a comparison between problems (I) and (II).

Definition 5.1. Consider a sequence of matrices $a, a_h \in \mathcal{M}_\Omega(L, M, K, C)$ $(h = 1, 2, 3, ...)$, a sequence of weights $\lambda \in \Lambda_a$, $\lambda_h \in \Lambda_{a_h}$ and define the operators

$$A_h = -\text{div}(a_h\cdot D) : H^1_0(\Omega,\lambda_h) \to H^{-1}(\Omega,\lambda_h)\,, \quad A = -\text{div}(a\cdot D) : H^1_0(\Omega,\lambda) \to H^{-1}(\Omega,\lambda)\,.$$

We say that A_h *G-converge to A in* Ω, and write

$$A_h \xrightarrow{G} A \qquad \text{in } \Omega\,,$$

if for every $f \in L^n(\Omega)$

$$\begin{array}{ll} u_h(f) \to u(f) & L^1(\Omega) \\ a_h\cdot Du_h(f) \to a\cdot Du(f) & (L^1(\Omega))^n\text{−weak} \end{array}$$

where $u_h = u_h(f)$ denote the solutions of problems (I) and u solves

$$\begin{cases} -\text{div}(a(x)\cdot Dw) = f & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega\,. \end{cases}$$

As regards the problems (I), in [10] the authors show the existence of a matrix $b_1 = b_1(x) \in \mathcal{M}_\Omega(L', M', K, C')$ (and implicitly of a weight $\lambda' \in A_2(K)$) such that (up to subsequences)

$$A_h \xrightarrow{G} -\text{div}(b_1(x)\cdot D) \qquad \text{in } \Omega\,.$$

As regards the problems (II) in the previous section it is proved the existence of a weight λ and of a matrix $b_2 \in \mathcal{M}_{\Omega\times(0,T)}(L', M', K, C')$ such that (up to subsequences)

$$\mathcal{P}_{\lambda_h,a_h} \xrightarrow{G} \mathcal{P}_{\lambda,b_2} \qquad \text{in } \Omega\times(0,T)\,.$$

We will show is that $b_1 = b_2 = b(x)$ and the equivalence between these two convergences. First we prove a result involving moving data which will be useful to this aim.

Consider a sequence of weights $(\lambda_h)_h$ satisfying (29). Suppose to have a sequence $f_h \in L^2(0,T;L^2\ (\Omega,\lambda_h^{-1}))$ such that

$$\|f_h\|_{L^2(0,T;L^2(\Omega,\lambda_h^{-1}))} \leq c\,. \tag{34}$$

Say λ and $\tilde{\lambda}^{-1}$ the weak limits in $L^1(\Omega)$, up to a subsequence, of λ_h and λ_h^{-1} (see Remark 2.7 in [21]). By Proposition 3.2 we have that there exists $f \in L^2(0,T;L^2(\Omega,\tilde{\lambda}^{-1}))$ such that, up to a subsequence, (λ and $\tilde{\lambda}$ are comparable, see (21)) for every $\varphi \in C_c^\infty(\Omega\times(0,T))$

$$\int_0^T\int_\Omega f_h\varphi\lambda_h^{-1}dxdt \to \int_0^T\int_\Omega f\varphi\tilde{\lambda}^{-1}dxdt\,. \tag{35}$$

Lemma 5.2. *Let $(\lambda_h)_h$ and $(f_h)_h$ satisfy* (34) *and* (35). *Make the additional assumption*

$$\int_0^T\int_\Omega f_h^2\lambda_h^{-1}dxdt \to \int_0^T\int_\Omega f^2\tilde{\lambda}^{-1}dxdt$$

for $f \in L^2(0,T;L^2(\Omega,\tilde{\lambda}^{-1}))$. Consider a sequence of matrices a, a_h ($h = 1,2,3...$) such that $\lambda_h \in \Lambda_{a_h}$ and $\lambda \in \Lambda_a$ (see Definition 3.4*) and the sequence of operators $\mathcal{P}_{\lambda,a}$, $\mathcal{P}_{\lambda_h,a_h}$ ($h = 1,2,...$). If $\mathcal{P}_{\lambda_h,a_h}$ G-converge to $\mathcal{P}_{\lambda,a}$, then, given u_h and u the solutions respectively to the problems*

$$\begin{cases}\mathcal{P}_{\lambda_h,a_h}u_h = f_h\\ u=0\end{cases} \qquad\qquad \begin{cases}\mathcal{P}_{\lambda,a}u = f\\ u=0\end{cases}$$

we have that

$$u_h \to u \quad \text{in } L^2(0,T;L^1(\Omega)) \quad \text{and} \quad a_h\cdot Du_h \to a\cdot Du \quad \text{in } L^2(0,T;L^1(\Omega)^n)-\text{weak}.$$

Proof. We simply denote by $\mathcal{P}_h$ the operator $\mathcal{P}_{\lambda_h,a_h}$ and by $\mathcal{P}$ the operator $\mathcal{P}_{\lambda,a}$.
For $\varepsilon > 0$ fixed we can find $g \in C_0^1(\Omega\times(0,T))$ such that

$$\|f-g\|_{L^2(0,T;L^2(\Omega,\tilde{\lambda}^{-1}))} < \varepsilon\,.$$

Then we write

$$\mathcal{P}_h^{-1}f_h - \mathcal{P}^{-1}f = (\mathcal{P}_h^{-1}f_h - \mathcal{P}_h^{-1}g) + (\mathcal{P}_h^{-1}g - \mathcal{P}^{-1}g) + (\mathcal{P}^{-1}g - \mathcal{P}^{-1}f). \tag{36}$$

The central term in the right hand side of (36) is going to zero in $L^2(0,T;L^1(\Omega))$ since $\mathcal{P}_h$ G-converges to $\mathcal{P}$. The last of these three terms can be estimated as follows:

$$\|\mathcal{P}^{-1}g - \mathcal{P}^{-1}f\|_{L^2(0,T;L^1(\Omega))} \leq c\,\|g-f\|_{\mathcal{V}'_\lambda} \leq c\,\|g-f\|_{L^2(0,T;L^2(\Omega,\tilde{\lambda}^{-1}))} \leq c\,\varepsilon\,.$$

We now consider the first term in (36):

$$\|\mathcal{P}_h^{-1}f_h - \mathcal{P}_h^{-1}g\|_{L^2(0,T;L^1(\Omega))} \leq c\,\|f_h-g\|_{\mathcal{V}'_{\lambda_h}} \leq c\,\|f_h-g\|_{L^2(0,T;L^2(\Omega,\lambda_h^{-1}))}\,.$$

Since

$$\|f_h-g\|^2_{L^2(0,T;L^2(\Omega,\lambda_h^{-1}))} = \int_0^T\int_\Omega (f_h^2\lambda_h^{-1} + g^2\lambda_h^{-1} - 2f_hg\lambda_h^{-1})dxdt$$

by hypotheses we have that

$$\|f_h - g\|^2_{L^2(0,T;L^2(\Omega,\lambda_h^{-1}))} \to \|f - g\|^2_{L^2(0,T;L^2(\Omega,\tilde{\lambda}^{-1}))} < \varepsilon^2 .$$

It remains to see the the *momenta* converge to the corresponding *momentum*. Analogously to (36) we write

$$\begin{aligned} a_h \cdot D(\mathcal{P}_h^{-1} f_h) - a \cdot D(\mathcal{P}^{-1} f) = [a_h \cdot D(\mathcal{P}_h^{-1} f_h) - a_h \cdot D(\mathcal{P}_h^{-1} g)] + \\ + [a_h \cdot D(\mathcal{P}_h^{-1} g) - a \cdot D(\mathcal{P}^{-1} g)] + [a \cdot D(\mathcal{P}^{-1} g) - a \cdot D(\mathcal{P}^{-1} f)] \end{aligned}$$

As before the central term is going to zero since $\mathcal{P}_h$ G-converges to $\mathcal{P}$. Let us estimate the last one. Since

$$|a \cdot D(\mathcal{P}^{-1} g) - a \cdot D(\mathcal{P}^{-1} f)| \le M L^{1/2} \lambda^{1/2} \big(a \cdot D(\mathcal{P}^{-1} g - \mathcal{P}^{-1} f), D(\mathcal{P}^{-1} g - \mathcal{P}^{-1} f)\big)^{1/2}$$

integrating we obtain

$$\begin{aligned} &\int_0^T \Big(\int_\Omega |a \cdot D(\mathcal{P}^{-1} g) - a \cdot D(\mathcal{P}^{-1} f)| dx \Big)^2 dt \le \\ &\qquad \le c \int_0^T \int_\Omega \big(a \cdot D(\mathcal{P}^{-1} g - \mathcal{P}^{-1} f), D(\mathcal{P}^{-1} g - \mathcal{P}^{-1} f)\big) dx dt \\ &\qquad \le c \, \|\mathcal{P}^{-1} g - \mathcal{P}^{-1} f\|_{\mathcal{V}_\lambda} \le c' \, \|g - f\|_{\mathcal{V}'_\lambda} < c' \, \varepsilon . \end{aligned}$$

Now we consider the first term. Consider $\Phi = (\varphi_1, ... \varphi_n)$ with $\varphi_j \in C_c^\infty(\Omega \times (0,T))$ for every j. Then

$$\begin{aligned} &\Big| \int_0^T \int_\Omega (a_h \cdot D\mathcal{P}_h^{-1}(f_h - g)), \Phi) dx dt \Big| \le \\ &\le M \Big[\int_0^T \int_\Omega (a_h \cdot D\mathcal{P}_h^{-1}(f_h - g), D\mathcal{P}_h^{-1}(f_h - g)) dx dt \int_0^T \int_\Omega (a_h \cdot \Phi, \Phi) dx dt \Big]^{1/2} \\ &\le c(L, M, \Phi) \|\mathcal{P}_h^{-1} f_h - \mathcal{P}_h^{-1} g\|_{\mathcal{W}_\lambda} \\ &\le c' \|f_h - g\|_{\mathcal{V}'_\lambda} \le c' \, \|f_h - g\|_{L^2(0,T;L^2(\Omega,\lambda_h^{-1}))} < c' \, \varepsilon . \end{aligned}$$

□

Proposition 5.3. *Consider a sequence* $(a_h)_h \subset \mathcal{M}_\Omega(L, M, K, C)$ *(see Definition* 3.4*) and a sequence of weights* $(\lambda_h)_h$ *such that* $\lambda_h \in \Lambda_{a_h}$ *for every* $h \in \mathbb{N}$*. Suppose there exist a weight* $\lambda \in A_2$ *and a matrix* $a = [a_{ij}(x,t)]_{i,j=1}^n$ *such that*

$$\lambda_h \frac{\partial}{\partial t} - \operatorname{div}(a_h(x) \cdot D) \ \xrightarrow{G} \ \lambda \frac{\partial v}{\partial t} - \operatorname{div}(a(x,t) \cdot Dv) \qquad \text{in } \Omega \times (0,T) .$$

Then $a(x,t) = a(x)$ *and*

$$-\operatorname{div}(a_h(x) \cdot Dv) \ \xrightarrow{G} \ -\operatorname{div}(a(x) \cdot Dv) \qquad \text{in } \Omega .$$

Proof. Consider a sequence of matrices $a_h = a_h(x) \in \mathcal{M}_\Omega(L, M, K, C)$ and suppose the associated parabolic operators $\mathcal{P}_{\lambda_h, a_h}$ G-converge to $\mathcal{P}_{\lambda,a}$, i.e. $\frac{d}{dt} + \mathbb{A}_h$ G-converge to

$\frac{d}{dt} + \mathbb{A}$. Observe that a priori the operator $\mathbb{A} = -\mathrm{div}(a \cdot D)$ is depending also on t, that is $a = a(x,t)$.
Consider $g \in C_c^\infty(\Omega)$ and the problems

$$A_h : H_0^1(\Omega, \lambda_h) \to H^{-1}(\Omega, \lambda_h) \qquad \begin{cases} A_h w := -\mathrm{div}(a_h(x) \cdot Dw) = g & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

and denote by $v_h(g) = v_h$ the solutions. Thanks to Remark 3 (see Theorem 3.1) we have that $A_h^{-1} g = v_h \to v$ in $L^1(\Omega)$ and

$$\int_\Omega v_h^2 \lambda_h dx \to \int_\Omega v^2 \lambda dx \quad \text{and} \quad \int_\Omega v_h \varphi \lambda_h dx \to \int_\Omega v \varphi \lambda dx \tag{37}$$

for some $v \in H_0^1(\Omega, \lambda)$. If we consider $u_h(x,t) := t v_h(x)$ we have that u_h are the solutions of

$$\begin{cases} \lambda_h \frac{\partial u}{\partial t} - \mathrm{div}(a_h \cdot Du) = \lambda_h v_h + tg & \text{in } \Omega \times (0,T) \\ u = 0 & \text{in } (\partial\Omega \times (0,T)) \cup (\{0\} \times (0,T)) \,, \end{cases}$$

By (37) we have that the sequence of data $\lambda_h v_h + tg$ satisfies the hypotheses of Lemma 5.2. Since $\mathcal{P}_{\lambda_h, a_h}$ G-converge to $\mathcal{P}_{\lambda,a}$ we have that the solutions u_h satisfy

$$\begin{array}{ll} u_h \to u & L^2(0,T; L^1(\Omega)) \\ a_h \cdot Du_h \to a \cdot Du & L^2(0,T; (L^1(\Omega))^n) - \text{weak} \end{array}$$

where u is the solution of

$$\begin{cases} \lambda \frac{\partial u}{\partial t} - \mathrm{div}(a \cdot Du) = \lambda v + tg & \text{in } \Omega \times (0,T) \\ u = 0 & \text{in } (\partial\Omega \times (0,T)) \cup (\{0\} \times (0,T)) \,, \end{cases}$$

Since $u(x,t) = tv(x)$ we have that

$$-\mathrm{div}(a \cdot Dv) = g \,.$$

Then $a(x,t) = a(x)$ and $A_h = -\mathrm{div}(a_h \cdot D) \xrightarrow{G} A = -\mathrm{div}(a \cdot D)$. By the density of $C_c^\infty(\Omega)$ in $L^n(\Omega)$ we conclude. □

As a consequence we have the following result, which is immediatly derived by the previous proposition and uniqueness of the G-limit of A_h (see [10]).

Corollary 5.4. *Consider a sequence $a_h \in \mathcal{M}_\Omega(L,M,K,C)$ with $a_h = a_h(x)$. Then there exist $a = a(x) \in \mathcal{M}_\Omega(L,M,K,C)$ and a subsequence, still denoted by the index h, such that*

$$\mathcal{P}_{\lambda_h, a_h} \xrightarrow{G} \mathcal{P}_{\lambda,a} \qquad \text{in } \Omega \times (0,T)$$

for every sequence $(\lambda_h)_h$ and every λ with $\lambda_h \in \Lambda_{a_h}$ and λ_h weakly converging to λ in $L^1(\Omega)$.

This last result allows us to give the following definition.

Definition 5.5. Given a sequence $a_h \in \mathcal{M}_\Omega(L, M, K, C)$ we say that a_h *PG-converge to* a in $\Omega \times (0, T)$, and we write

$$a_h \xrightarrow{PG} a \qquad \text{in } \Omega \times (0, T)\,,$$

if, for every sequence of weights $(\lambda_h)_h$ weakly converging in $L^1(\Omega)$ (say λ the limit) and $\lambda_h \in \Lambda_{a_h}$ for every $h \in \mathbb{N}$, we have that $\mathcal{P}_{\lambda_h, a_h} \xrightarrow{G} \mathcal{P}_{\lambda, a}$ in $\Omega \times (0, T)$. In the same way we write

$$a_h \xrightarrow{G} a \qquad \text{in } \Omega$$

if $A_h \xrightarrow{G} A$ in Ω where $A_h = -\mathrm{div}(a_h \cdot D)$ and $A = -\mathrm{div}(a \cdot D)$ (see Definition 5.1).

Theorem 5.6. *Consider a sequence $(a_h)_h \subset \mathcal{M}_\Omega(L, M, K, C)$. Then*

$$a_h \xrightarrow{PG} a \text{ in } \Omega \times (0, T) \qquad \text{if and only if} \qquad a_h \xrightarrow{G} a \text{ in } \Omega\,.$$

Proof.
(i) Suppose $a_h \xrightarrow{PG} a$. Then by Proposition 5.3 it is clear that $a_h \xrightarrow{G} a$.
(ii) Now suppose to have $a_h \xrightarrow{G} a$. By Theorem 4.3 and Proposition 5.3 we have a subsequence a_{h_j} and a matrix $b = b(x)$ such that $a_{h_j} \xrightarrow{PG} b$. By (i) we then have that $a_{h_j} \xrightarrow{G} b$. By uniqueness of the limit we conclude $a = b$. □

Remark. The sharpness of the assumptions (S.3) in Definition 3.4 to obtain Theorem 4.3 follows at once by Theorem 5.6 and by counterexamples in [25], Remark 2.10 (see also Example 1 and Example 2 in [22]).

We want now to make a comparison with another family of parabolic operators, operators like

$$\mathcal{Q}_{a_h} = \frac{\partial}{\partial t} - \mathrm{div}(a_h(x, t) \cdot D)\,, \tag{38}$$

and Cauchy-Dirichlet problems (III) defined at the beginning of this section. The convergence of such a sequence of parabolic operators is considered in [22], but for matrices in $\mathcal{N}_{\Omega\times(0,T)}(L, M,\ K, C)$ (see Definition 3.4). Observe that, since $A_{1+2/n}(K) \subset A_2(K)$, $\mathcal{N}_{\Omega\times(0,T)}(L, M, K, C) \subset \mathcal{M}_{\Omega\times(0,T)}(L, M, K, C')$. For $a \in \mathcal{N}_{\Omega\times(0,T)}(L, M,\ K, C)$ let $\lambda \in \Lambda_a$: by Y_λ we denote the Hilbert space defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|^2 = \int_\Omega u^2 dx + \int_\Omega |Du|^2 \lambda dx$, by $\mathcal{Y}_\lambda$ the space $\{u \in L^2(0, T; Y_\lambda) \mid u' \in L^2(0, T; Y_\lambda')\}$ and by $\mathcal{Z}_\lambda^0$ the space $\{u \in \mathcal{Y}_\lambda \mid u(0) = 0 \text{ in } L^2(0, T; L^2(\Omega))\}$. Now, for $(a_h)_h \subset \mathcal{N}_{\Omega\times(0,T)}(L, M, K, C)$, consider the sequence of problems (III). We have the triples

$$\mathcal{Y}_{\lambda_h} \subset L^2(0, T; L^2(\Omega)) \subset \mathcal{Y}_{\lambda_h}'\,.$$

We just recall the definition of convergence defined in [22] which here we will call $\tilde{P}G$-convergence.
Let a_h $(h = 1, 2, ...)$ and a be matrices in $\mathcal{N}_{\Omega\times(0,T)}(L, M, K, C)$. With the notation as in Definition 5.5, we say that the sequence $(a_h)_h$ $\tilde{P}G$-converges to a in $\Omega \times (0, T)$, and we write

$$a_h \xrightarrow{\tilde{P}G} a \qquad \text{in } \Omega \times (0, T),$$

if for every $f \in L^2(0,T;L^2(\Omega))$ it results that

$$\begin{aligned} u_h \to u \qquad & \text{in } L^2(0,T;L^2(\Omega)) \\ a_h \cdot Du_h \to a \cdot Du \qquad & \text{in } L^2(0,T;L^1(\Omega))^n\text{—weak,} \end{aligned}$$

where u_h and u denote respectively the solutions of

$$\begin{cases} \mathcal{Q}_{a_h} v = f & \text{in } \mathcal{Y}_{\lambda_h} \\ v \in \mathcal{Z}^0_{\lambda_h} \end{cases} \qquad \begin{cases} \mathcal{Q}_{a} v = f & \text{in } \mathcal{Y}_{\lambda} \\ v \in \mathcal{Z}^0_{\lambda} \end{cases} \tag{39}$$

In this framework it is easy to prove the following lemma, analogous to Lemma 5.2.

Lemma 5.7. *Consider a sequence $f, (f_h)_h \subset L^2(0,T;L^2(\Omega))$ and suppose*

$$f_h \to f \text{ in } L^2(0,T;L^2(\Omega)).$$

Consider a sequence of matrices a, a_h, $h = 1, 2, ...,$ such that a_h $\tilde{P}G$-converge to a. Then, given u_h and u the solutions to the problems (39) *we have that*

$$\begin{aligned} u_h \to u \qquad & \text{in } L^2(0,T;L^2(\Omega)) \quad \text{and} \\ a_h \cdot Du_h \to a \cdot Du \qquad & \text{in } L^2(0,T;L^1(\Omega)^n) - \text{weak.} \end{aligned}$$

The following theorem also follows at once, following the proofs of Proposition 5.3 and Theorem 5.6.

Theorem 5.8. *Consider a sequence $(a_h)_h \subset \mathcal{N}_\Omega(L,M,K,C)$. Then*

$$a_h \xrightarrow{\tilde{P}G} a \text{ in } \Omega \times (0,T) \qquad \text{if and only if} \qquad a_h \xrightarrow{G} a \text{ in } \Omega\,.$$

As a final result we state a corollary of Theorem 5.6 and Theorem 5.8.

Corollary 5.9. *Consider a sequence $(a_h)_h \subset \mathcal{N}_{\Omega\times(0,T)}(L,M,K,C)$. If $a_h = a_h(x)$ then*

$$a_h \xrightarrow{G} a \text{ in } \Omega\,, \quad a_h \xrightarrow{PG} a \text{ in } \Omega \times (0,T)\,, \quad a_h \xrightarrow{\tilde{P}G} a \text{ in } \Omega \times (0,T)$$

are equivalent.

Acknowledgements. I wish to thank Prof. F. Serra Cassano for some fruitful discussions on the subject. I also wish to thank the anonimous Referee for some precious suggestions.

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