Some New Results on the Convergence of Degenerate Elliptic and Parabolic Equations

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In this paper we study the convergence of the Cauchy-Dirichlet problems for a sequence of parabolic operators
\[ P_h = \lambda_h \frac{\partial}{\partial t} - \text{div}(a_h(x,t) \cdot D) \]
where the matrices of the coefficients \( a_h(x,t) \) verify the following degenerate elliptic condition
\[ \lambda_h(x)|\xi|^2 \leq (a_h(x,t) \cdot \xi, \xi) \leq L\lambda_h(x)|\xi|^2, \]
being \( (\lambda_h)_h \) a sequence of weights satisfying a uniform Muckenhoupt’s condition in \( h \). When \( a_h = a_h(x) \) we compare this result with the analogous results for the sequence of operators \( A_h = -\text{div}(a_h(x) \cdot D) \) and \( Q_h = \frac{\partial}{\partial t} - \text{div}(a_h(x) \cdot D) \)

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1. Introduction

The asymptotic behaviour, as \( h \to \infty \), of the equations
\[ A_h u = f \quad \text{and} \quad Q_h u = f \quad (h = 1, 2, \ldots) \]
has been widely studied where
\[ A_h = -\text{div} (a_h(x) \cdot D), \quad Q_h = \frac{\partial}{\partial t} - \text{div} (a_h(x,t) \cdot D), \]
\[ a_h(x) = [a_{h,ij}(x)]_{i,j=1}^{n} \quad \text{or} \quad a_h(x,t) = [a_{h,ij}(x,t)]_{i,j=1}^{n} \]
are matrices of measurable functions defined respectively on a bounded open set \( \Omega \) of \( \mathbb{R}^n \) or on a bounded open cylinder \( \Omega \times (0, T) \) of \( \mathbb{R}^{n+1} \) with \( a_h \) satisfying in both cases the classical ellipticity condition
\[ \lambda_0|\xi|^2 \leq \sum_{i,j=1}^{n} a_{h,ij}(x_1\xi_1) \xi_j \leq \Lambda_0|\xi|^2 \quad \text{for a.e. } x \in \Omega, \forall \xi \in \mathbb{R}^n, \forall h \in \mathbb{N}, \]
for a.e. \( x \in \Omega \) if \( a_h = a_h(x) \) and for a.e. \( (x, t) \in \Omega \times (0, T) \) if \( a_h = a_h(x,t) \), for suitable positive constants \( 0 < \lambda_0 \leq \Lambda_0 \). We recall among all some important results due to E. De Giorgi, S. Spagnolo, F. Murat, L. Tartar and O. A. Oleinik (see for instance [11], [26], [27], [29], [6] for the elliptic case and [5], [28], [33] for the parabolic one). We recall that also the non linear case has been considered later (see, for instance, [29], [1]).

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Arising from some physical applications (see, for instance, [23]) even the asymptotic behaviour of some class of degenerate operators has been considered, operators like those in (1) when the matrices \( a_h \)'s satisfy the following degenerate ellipticity condition (note that \( a_{h,i,j} \) may depend on \( x \) or on \((x,t)\), while \( \lambda_h \) depend only on \( x \))

\[
\lambda_h(x)|\xi|^2 \leq \sum_{i,j=1}^n a_{h,i,j}(x)\xi_i \xi_j \leq L\lambda_h(x)|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \forall h
\]

(3)

for a.e. \( x \in \Omega \) if \( a_h = a_h(x) \) and for a.e. \((x,t) \in \Omega \times (0,T)\) if \( a_h = a_h(x,t) \), with suitable weight functions \( \lambda_h \) (i.e. nonnegative, locally summable functions on \( \mathbb{R}^n \)) and a suitable constant \( L \geq 1 \). Some interesting results in this setting are obtained in many cases. The first ones are obtained in the elliptic case, first when the matrices \( a_h \)'s are symmetric and verify (3) with

\[
\lambda_h = \lambda \quad \text{a.e. in } \Omega, \text{ for every } h
\]

for a fixed weight \( \lambda \) verifying suitable integrability conditions together with its inverse \( \lambda^{-1} \) (see [18]), then in the homogenization case (see, for instance, [19], [8]), i.e. the sequence of matrices \( (a_h)_h \) verifying (3) has the form

\[
a_h(x) = a(hx) \quad (x \in \mathbb{R}^n)
\]

with \( a(x) = [a_{i,j}(x)]_{i,j=1,...,n} \) symmetric matrix of measurable periodic functions on \( \mathbb{R}^n \) and \( \lambda_h(x) = \lambda(hx) \) for a suitable fixed periodic weight \( \lambda \).

Other results, which apply not only to the variational case (i.e. when the matrices are symmetric), are those obtained when each one of the \( a_h \)'s verifies (3) with a weight \( \lambda_h \) verifying a uniform Muckenhoupt's condition with respect to \( h \), the condition \( A_p(K) \) where \( 1 < p < +\infty \), (see [9], [10] for elliptic operators, [22], [21] for parabolic operators), i.e. there exists \( K \geq 1 \) such that

\[
\left( \frac{1}{|Q|} \int_Q \lambda_h dx \right)^{1/p} \left( \frac{1}{|Q|} \int_Q \lambda_h^{-1/(p-1)} dx \right)^{p-1} \leq K \quad A_p(K)
\]

for every cube \( Q \subset \mathbb{R}^n \) with faces parallel to the coordinate planes, for every \( h \). This class is consider with \( p = 2 \) for the elliptic case and with \( p = 1 + 2/n \) for the parabolic one, i.e. for a strict subclass of \( A_2 \) when \( n \geq 3 \).

In this setting, i.e. given two sequences of matrices \( a_h = a_h(x,t) \) and weights \( \lambda_h = \lambda_h(x) \) verifying (3) (with \( a_{h,i,j} = a_{h,i,j}(x,t) \)) with \( \lambda_h \in A_2(K) \) for every \( h \), in this paper we study the asymptotic behaviour of parabolic operators of the form

\[
\mathcal{P}_{\lambda_h,a_h} = \lambda_h(x) \frac{\partial}{\partial t} - \text{div} \left( a_h(x,t) \cdot \nabla \right).
\]

(4)

Given a matrix \( a = a(x,t) \) whose coefficients are in \( L^1_{\text{loc}}(\mathbb{R}^n \times (0,T)) \), we say that it belongs to the class \( \mathcal{N}_{\Omega \times (0,T)}(L,M,K,C) \) if there exists a weight \( \lambda = \lambda(x) \) such that \( a \) satisfies the following properties with \( p = 1 + 2/n \) (\( n \geq 2 \), to the class \( \mathcal{M}_{\Omega \times (0,T)}(L,M,K,C) \) if it satisfies the following properties with \( p = 2 \) (and by \( \mathcal{N}_{\Omega}(L,M,K,C) \) and \( \mathcal{M}_{\Omega}(L,M,K,C) \)
Theorem 1.1. Let $\Omega$ be a bounded open set with Lipschitz boundary, $T > 0$, $a_h = [a_{h,ij}(x,t)]_{i,j=1}^n$ a sequence of matrices with $a_{h,ij} \in L^1_{\text{loc}}(\mathbb{R}^n \times (0,T))$ belonging to $\mathcal{M}_{\Omega \times (0,T)} (L,M,K,C)$ for some positive constants $L,M,K,C$. Say $u_h$ the unique solution of the Cauchy-Dirichlet problem (see Definitions 3.5 and 4.1)

$$
(P_h) \begin{cases}
\lambda_h \frac{\partial u}{\partial t} - \text{div} \ (a_h(x,t) \cdot Du) = f & \text{in } \Omega \times (0,T) \\
u = 0 & \text{on } (\partial \Omega \times (0,T)) \cup (\Omega \times \{0\}).
\end{cases}
$$

Then there exist a subsequence $(a_{h_k})$, a matrix $a_\infty = [a_{\infty,ij}(x,t)]_{i,j=1}^n \in \mathcal{M}_{\Omega \times (0,T)} (L',M',K,C')$ for suitable constants $L',M',C'$ and a weight $\lambda_\infty \in A_2(K)$ such that for every datum $f \in L^2(0,T;L^n(\Omega))$

$$
\lambda_{h_k} \rightarrow \lambda_\infty \text{ in } L^1_{\text{loc}}(\mathbb{R}^n)-\text{weak}
$$

and the subsequence $(u_{h_k})$ of the solutions of $(P_{h_k})$ satisfy

$$
u_{h_k} \rightarrow u \quad \text{in } L^2(0,T;L^1(\Omega))$$
$$a_{h_k} \cdot Du_{h_k} \rightarrow a_\infty \cdot Du \quad \text{in } L^2(0,T;L^1(\Omega))^n-\text{weak},$$

where $u$ denotes the solution of the problem

$$
\begin{cases}
\lambda_\infty \frac{\partial u}{\partial t} - \text{div} \ (a_\infty(x,t) \cdot Du) = f & \text{in } \Omega \times (0,T) \\
u = 0 & \text{on } (\partial \Omega \times (0,T)) \cup (\Omega \times \{0\}).
\end{cases}
$$

The result above turns out to be an extension of a well-known classic parabolic $G$-compactness result (see Definition 4.1 and [26], [27], [28], [5], [33]).

In the degenerate setting, analogous results are contained in [10] for Dirichlet problems where the operators are $A_h$ defined by matrices in $\mathcal{M}_\Omega$ and in [22] for Cauchy-Dirichlet problems where the parabolic operators are $Q_h$, but defined by matrices in the class $\mathcal{N}_{\Omega \times (0,T)} \subset \mathcal{M}_{\Omega \times (0,T)}$.

Another interesting result is the equivalence between the elliptic $G$-convergence and the
parabolic $G$-convergence of the operators $P_{\lambda_h,a_h}$ (see Definition 5.5 and Theorem 5.6). This turns to be an extension to the degenerate case of the classical result contained in [5] (only for matrices independent of $t$).

The difference with the classical case, i.e., when (2) holds, is that we have to consider parabolic operators like (4) to obtain this equivalence. We also state the equivalence between the elliptic $G$-convergence and the $G$-convergence of operators $Q_h$, but only for matrices in the class $N_{\Omega}$ (Theorem 5.8), and so, in this case, the equivalence between the parabolic $G$-convergence of operators $Q_h$ and $P_{\lambda_h,a_h}$.

The sharpness of the assumptions, in the framework of Muckenhoupt’s weights, will follow from some considerations made in the fifth section.

We recall that operators like (4) have already been considered to study the regularity of the solutions (see, for instance, [2], [3], [14], [15], but also [24] (Chap. 5, sect. 6), [32] (Theorem 2.3), [21] for the homogenization).

2. Notations and preliminary results

We will denote by $Q$ a generic (open or closed) cube of $\mathbb{R}^n$ with faces parallel to the coordinate planes, by $cQ$ the cube concentric with $Q$ and having side length $c$ times that of $Q$ and by $B(x,r)$ the open ball of $\mathbb{R}^n$ centered in $x$ with radius $r$. The symbols $(\cdot,\cdot), |E|, \int_E f \, dx$ and $p'$ will indicate respectively the scalar product of $\mathbb{R}^n$, the Lebesgue measure of the set $E$, the mean value of $f$ on $E$ (i.e. $|E|^{-1} \int_E \lambda(x) \, dx$) and the conjugate $p' = p/(p-1)$ of $p$.

Let $\lambda$ be a weight on $\mathbb{R}^n$, that is

$$\lambda > 0 \text{ a.e. in } \mathbb{R}^n \quad \text{and} \quad \lambda, \lambda^{-1} \in L^1_{\text{loc}}(\mathbb{R}^n),$$

then for every bounded open set of $\mathbb{R}^n$ we define

$$L^2(\Omega,\lambda) = \left\{ u \in L^1_{\text{loc}}(\Omega) \mid u\lambda^{1/2} \in L^2(\Omega) \right\}$$

and the space

$$H^1(\Omega,\lambda) = \left\{ u \in W^{1,1}_{\text{loc}}(\Omega) \mid u \in L^2(\Omega,\lambda) \text{ and } |Du| \in L^2(\Omega,\lambda) \right\}.$$  

It is easy to verify that the space $H^1(\Omega,\lambda)$ endowed with the topology induced by the norm

$$\|u\|_{H^1(\Omega,\lambda)} = \left( \int_{\Omega} (u^2 + |Du|^2) \lambda \, dx \right)^{1/2}$$

is a separable Hilbert space. We will denote by $H^1_0(\Omega,\lambda)$ the closure of $C_0^1(\Omega)$ in the topology of $H^1(\Omega,\lambda)$, by $H^{-1}(\Omega,\lambda)$ its dual space.

**Definition 2.1.** Let $p > 1$, $K \geq 1$ and let $\lambda$ be a weight on $\mathbb{R}^n$. We will say that $\lambda$ belongs to the Muckenhoupt class $A_p(K)$ if

$$\left( \int_Q \lambda \, dx \right) \left( \int_Q \lambda^{-1/(p-1)} \, dx \right)^{p-1} \leq K \quad \text{for every cube } Q \subset \mathbb{R}^n.$$  

Moreover we define $A_p = \cup_{K \geq 1} A_p(K)$ and $A_\infty = \cup_{p>1} A_p$. 

Remark. By definition we have at once that $A_p(K) \subset A_q(K)$ if $1 < p < q < \infty$. Moreover if $\lambda \in A_p$, then $\lambda^{-\frac{1}{p-1}} \in A_{p'}$.

$A_p$ weights verify the following higher sommability property (see [4] and [7]): for every $K \geq 1$, for every $p > 1$, there exist two positive constants $c = c(n, p, K)$ and $\delta = \delta(n, p, K)$ (depending only on $n, p$ and $K$) such that

$$
\left( \int_Q \lambda^{1+\delta} dx \right)^{\frac{1}{1+\delta}} \leq c \left( \int_Q \lambda dx \right), \quad \left( \int_Q \lambda^{-\frac{1}{p-1}} dx \right)^{\frac{1}{p-1}} \leq c \left( \int_Q \lambda^{\frac{1}{p-1}} dx \right)
$$

(6)

for every cube $Q$ and $\lambda \in A_p(K)$. $A_p$ weights also verify the doubling property, i.e. if $\lambda \in A_p(K)$ for every $t > 0$ there exists a constant $c = c(t, n, p, K)$ (depending only on $t, n, p, K$) such that

$$
\lambda(tQ) \leq c\lambda(Q)
$$

(7)

for every cube $Q$ of $\mathbb{R}^n$ (see for instance [13]). We recall that (see Theorem 1.4 in [25] and Proposition 1.2 in [7]), if $\lambda \in A_2$ and $\Omega$ is a bounded open set with Lipschitz boundary,

$$
H^1_0(\Omega, \lambda) = H^1(\Omega, \lambda) \cap W^{1,1}_0(\Omega).
$$

Moreover (see [12]) the following Poincaré’s inequality holds: there exists a constant $c = c(n, K, \Omega)$ such that

$$
\int_{\Omega} u^2 \lambda dx \leq c \int_{\Omega} |Du|^2 \lambda dx
$$

for every $\lambda \in A_2(K)$ and $u \in H^1_0(\Omega, \lambda)$. Then, from now on, if $\lambda \in A_2(K)$, by $H^1_0(\Omega, \lambda)$ we will denote the closure of $C^0_0(\Omega)$ with respect to the norm

$$
\|v\|_{H^1_0(\Omega, \lambda)} \triangleq \left( \int_{\Omega} |Du|^2 \lambda dx \right)^{1/2}.
$$

(8)

By (6) and (7), it can be easily proved that, if $\lambda \in A_2(K)$ taking $\sigma \triangleq \delta/(2 + \delta)$, where $\delta$ is the constant in (6), there exist two positive constants $c_i = c_i(n, K, \Omega)$ (depending only on $n, K, \Omega$), $i = 1, 2$, such that

$$
L^{1+\sigma'}(\Omega) \subset L^2(\Omega, \lambda) \subset L^{1+\sigma}(\Omega),
$$

$$
c_1 \|\lambda^{-1}\|_{L^1(\Omega)}^{1/2} \|v\|_{L^{1+\sigma}(\Omega)} \leq \|v\|_{L^2(\Omega, \lambda)} \leq c_2 \|\lambda\|_{L^1(\Omega)}^{1/2} \|v\|_{L^{1+\sigma'}(\Omega)}.
$$

(9)

Hence it follows that there exist two positive constants $c_i = c_i(n, K, \Omega)$, $i = 3, 4$, such that

$$
W^{1,1+\sigma'}_0(\Omega) \subset H^1_0(\Omega, \lambda) \subset W^{1,1+\sigma}_0(\Omega), \quad H^{-1}(\Omega, \lambda) \subset W^{-1,1+\sigma}_0(\Omega)
$$

$$
c_3 \|\lambda^{-1}\|_{L^1(\Omega)}^{1/2} \|u\|_{W^{1,1+\sigma}_0(\Omega)} \leq \|u\|_{H^1_0(\Omega, \lambda)} \leq c_4 \|\lambda\|_{L^1(\Omega)}^{1/2} \|u\|_{W^{1,1+\sigma'}_0(\Omega)}
$$

(10)

for every $\lambda \in A_2(K)$ and for every $u \in W^{1,1+\sigma'}_0(\Omega)$, where $\|\cdot\|_{H^1_0(\Omega, \lambda)}$ denotes the norm in (8).
Moreover, for every $\lambda \in A_2$, $H^1(\Omega, \lambda)$ continuously embeds in $W^{1,1}(\Omega)$, which continuously embeds in $L^{n/(n-1)}(\Omega)$ if $n \geq 2$ (and in $L^2(\Omega)$ if $\Omega$ is an real interval) and then there exists a constant $c = (n, \Omega)$ (depending only on $n, \Omega$) such that for every $f \in L^a(\Omega)$

$$\|f\|_{H^{-1}(\Omega, \lambda)} \leq c \left( \int_{\Omega} \lambda^{-1} dx \right)^{1/2} \|f\|_{L^a(\Omega)}. \quad (11)$$

If $n = 1$ the analogous estimate holds with $f \in L^2(\Omega)$.

Finally we recall some classical results concerning abstract parabolic equations (see, for instance, [17], [16], [31] and [33]). Let $V$ be a real reflexive Banach space and let $H$ a Hilbert space for which we have the classical triple

$$V \subset H \subset V'$$

with continuous and dense embeddings. Moreover consider a family of linear operators

$$A(t) : V \longrightarrow V', \quad 0 \leq t \leq T$$

such that

$$\begin{align*}
&\{ t \mapsto \langle A(t)\varphi, \psi \rangle_{V' \times V} \text{ is measurable on } [0, T] \} \\
&\|\varphi\|_V^2 \leq \langle A(t)\varphi, \varphi \rangle_{V' \times V} \leq L\|\varphi\|_V^2 \\
&|\langle A(t)\varphi, \psi \rangle_{V' \times V}| \leq M\|A(t)\varphi\|_V^{1/2}\|A(t)\psi\|_V^{1/2} \quad (12)
\end{align*}$$

for every $\varphi, \psi \in V$. We define now

$$H \equiv L^2(0, T; H), \quad V \equiv L^2(0, T; V), \quad W \equiv \{ v \in V \mid v' \in V' \} \quad (13)$$

and $V' = L^2(0, T; V')$ the dual space of $V$, endowed with the standard norms

$$\|u\|_{L^2(0, T; X)} = \left( \int_0^T \|u(t)\|_X^2 \, dt \right)^{1/2}, \quad \|u\|_W \equiv \left( \|u\|_V^2 + \|u'\|_{V'}^2 \right)^{1/2},$$

(with $X = H, V, V'$) where $v'$ denotes the distributional derivative of $v$. It is well known (see Theorem 3.1, chap. 1, in [17]) that

$$W \subset C^0([0, T]; H), \quad (14)$$

and there exists a constant $c$, depending only on $T$ and on the norms of the embeddings $V \subset H, H \subset V'$, such that

$$\max_{t \in [0, T]} \|u(t)\|_H \leq c \|u\|_W,$$

hence we can introduce the space

$$W^0 = \{ u \in W \mid u(0) = 0 \}, \quad (15)$$

and we recall that $C^\infty([0, T]; V)$ is dense in $(W, \| \cdot \|_W)$ (see, for instance, [31], chap. 23). Now define the abstract operators

$$A : V \longrightarrow V' \quad \text{where} \quad Au(t) \equiv A(t)u(t) \quad 0 \leq t \leq T, \quad (16)$$

$$P : W \longrightarrow V', \quad (Pu)(t) = u'(t) + Au(t) \quad 0 \leq t \leq T. \quad (17)$$

and recall the following classical result (see, for instance, [17], Theorem 4.1, chap. 3).
Theorem 2.2. Let $P : W \rightarrow V'$ be the operator defined in (17) and suppose that the family of operators $A(t)$ $(0 \leq t \leq T)$ verifies (12). Then

(i) $P : W \rightarrow V'$ is linear and continuous and

$$||Pu||_{V'} \leq \sqrt{2ML}\|u\|_W \text{ for every } u \in W,$$

(ii) $\langle Pu, u \rangle \geq \|u\|^2_{V'} \text{ for every } u \in W^0,$

(iii) $P : W^0 \rightarrow V'$ is an isomorphism such that

$$\|u\|_W \leq (ML + 2)||Pu||_{V'} \text{ for every } u \in W^0.$$

From now on we will actually consider these spaces when $H = L^2(\Omega, \lambda)$ and $V = H^1_0(\Omega, \lambda)$ endowed with the topology induced by the norm $\| \cdot \|_{H^1_0(\Omega, \lambda)}$ in (8) where $\Omega$ is a bounded open set of $\mathbb{R}^n$ with Lipschitz boundary and $\lambda \in A_2$, and we will denote by $H(0, T; \Omega)$, $V_\lambda(0, T; \Omega)$, $V'_\lambda(0, T; \Omega)$ and $W_\lambda(0, T; \Omega)$ (and for sake of simplicity we will omit $(0, T; \Omega)$ if there is no ambiguity) the corresponding abstract spaces like those introduced in (13). As usual we get the evolution triple

$$V_\lambda \subset H \equiv H' \subset V'_\lambda$$

and $\|f\|_{V'_\lambda} \leq \|f\|_H$ for every $f \in H$ and for every weight $\lambda \in A_2$ and, if $u \in H$ and $v \in V_\lambda$, we have that

$$\langle u, v \rangle_{V'_\lambda \times V_\lambda} \overset{\text{def}}{=} \langle u, v \rangle_\lambda = \langle u, v \rangle_H = \int_0^T \int_\Omega u(x, t)v(x, t)\lambda(x)dxdt. \quad (18)$$

As in (14) we obtain that $W_\lambda(0, T; \Omega) \subset C^0([0, T]; L^2(\Omega, \lambda))$; indeed by (7) it follows that there exists a positive constant $c = c(n, \Omega, K)$ (depending only on $n, \Omega, K$) such that

$$\max_{t \in [0, T]} \|u(t)\|_{L^2(\Omega, \lambda)} \leq c \|u\|_{W_\lambda}. \quad (19)$$

for every $u \in W_\lambda$, for every $\lambda \in A_2(K)$. Therefore we introduce $W_\lambda^0(0, T; \Omega)$ as done in (15). By (19) and (9)

$$W_\lambda \subset C^0([0, T]; L^{1+\sigma}(\Omega)) \quad \text{and} \quad \max_{t \in [0, T]} \|u(t)\|_{L^{1+\sigma}(\Omega)} \leq c\|\lambda^{-1}\|_{L^1(\Omega)}^{1/2} \|u\|_{W_\lambda}, \quad (20)$$

where $c = c(n, \Omega, K)$ depends only on $n, \Omega, K$.

Finally observe that if two weights $\lambda$ and $\tilde{\lambda}$ are comparable in $\Omega$, that is there exist constants $c_1, c_2 > 0$ such that

$$c_1 \tilde{\lambda}(x) \leq \lambda(x) \leq c_2 \tilde{\lambda}(x) \quad \text{a.e. in } \Omega, \quad (21)$$

the norms induced by $\lambda$ and $\tilde{\lambda}$ are equivalent and in particular

$$\sqrt{c_1}\|u\|_{V'_\lambda} \leq \|u\|_{V'_\lambda} \leq \sqrt{c_2}\|u\|_{V'_\lambda}, \quad \sqrt{c_1}\|f\|_{V'_\lambda} \leq \|f\|_{V'_\lambda} \leq \sqrt{c_2}\|f\|_{V'_\lambda}. \quad (22)$$
3. Compactness type results in weighted Sobolev spaces

In this section we are going to prove some compactness type results for a sequence of functions \((v_h)_h\), each one belonging to a suitable weighted Sobolev space depending on \(h\). There results are needed to prove the main result in the fourth section.

First we begin by proving an extension of a classical compactness result due to J. L. Lions (see [16], chap. 1, Theorem 5.1).

**Theorem 3.1.** Let \(\Omega\) be a bounded open set of \(\mathbb{R}^n\). Consider a sequence of weights \((\lambda_h)_h\) and a sequence of functions \(v_h \in W_{\lambda_h}\) (see section 2) such that there exist three constants \(K \geq 1\) and \(c_1, c_2 > 0\) for which

\[
(\lambda_h)_h \subset A_2(K), \quad \int_\Omega \lambda_h^{-1} dx \leq c_1 \quad \text{for every } h;
\]

and

\[
\|v_h\|_{W_{\lambda_h}} \leq c_2 \quad \text{for every } h \in \mathbb{N}.
\]

Then there exists a function \(v \in L^2(0, T; L^1(\Omega))\) such that, up to subsequences,

(i) \(v_h \rightharpoonup v\) in \(L^2(0, T; L^1(\Omega))\).

If moreover \(\lambda_h \to \lambda\), \(\lambda_h^{-1} \to \tilde{\lambda}^{-1}\) in \(L^1(\Omega)\)-weak with \(\lambda\) and \(\tilde{\lambda}\) comparable weights (see (21)) then \(v \in L^2(0, T; L^2(\Omega, \lambda))\) and, up to subsequences, the following facts hold:

(ii) \(\int_0^T \int_\Omega v_h^2 \lambda_h dxdt \to \int_0^T \int_\Omega v^2 \lambda dxdt;\)

(iii) \(\int_0^T \int_\Omega \varphi v_h \lambda_h dxdt \to \int_0^T \int_\Omega \varphi v \lambda dxdt \quad \text{for every } \varphi \in C_c^\infty(\Omega \times (0, T)).\)

**Proof.** Denote for simplicity by \(W_h\) the space \(W_{\lambda_h}\). First of all observe that if you given two reflexive and separable Banach spaces \(U\) and \(V\), \(U\) dense in \(V\), a Hilbert space \(H\) such that \(V \subset H\) with dense and continuous embedding, one has that

\[
C^1([0, T]; U) \quad \text{is dense in} \quad \{z \in L^2(0, T; V) \mid z' \in L^2(0, T; V')\}
\]

(see [31], ex. in chap. 23). Consider the space

\[
\mathbb{X} = C^1([0, T]; W^{1,1(1+\sigma)'}_0(\Omega))
\]

which then is dense in every \(W_h\). Hence we can suppose the sequence \((v_h)_h\) is in \(\mathbb{X}\), otherwise we could approximate it as follows

\[
\|v_h - z_h\|_{W_h} \leq 1/h
\]

with \((z_h)_h \subset \mathbb{X}\). For every \(z \in \mathbb{X}\) consider

\[
z_\varepsilon(x, t) \overset{def}{=} \int_{\mathbb{R}^n} \rho_\varepsilon(x - y) \tilde{z}(y, t) \quad (x, t) \in \mathbb{R}^n \times [0, T]
\]

where \((\rho_\varepsilon)_{\varepsilon > 0}\) is a family of mollifiers defined by a radial function and \(\tilde{z}(\cdot, t)\) denotes the function \(z(\cdot, t)\) extended to zero outside of \(\Omega\). Observe that \((z_\varepsilon)' = (z')_\varepsilon\).

(i) Now considering the functions \(v_{h, \varepsilon}\) one can prove, following the proof of Theorem 3.1 in [22], that for every \(\varepsilon > 0\) the sequence \((v_{h, \varepsilon})_h\) is bounded in

\[
\mathcal{Z} = \{z \in L^2(0, T; W^{1,1(1+\sigma)'}_0(\Omega)) \mid z' \in L^2(0, T; W^{-1,1+\sigma}(\Omega))\}
\]
where \( \tilde{\Omega} \) is a bounded open set containing \( \bar{\Omega} \). Then by Theorem 5.1, chap. 1, in [16], one obtain the compactness of \((v_{h, \varepsilon})_h\) in \(L^2(0, T; L^2(\Omega))\).

(ii) Then we show the second point. First of all, since \((\lambda^{-1}_h)_h\) is bounded in \(L^1(\Omega)\), by (10) we have that there exists \(u \in L^2(0, T; W_0^{1,1+\sigma}(\Omega))\) such that \(v_h \to u\) in \(L^2(0, T; W_0^{1,1+\sigma}(\Omega))\) weak and by point (i) we deduce \(u = v\). Moreover, by Lemma 2.13 in [21] and since \(\lambda\) and \(\lambda\) are comparable, we have

\[
v \in L^2(0, T; L^2(\Omega, \tilde{\lambda})) = L^2(0, T; L^2(\Omega, \lambda)).
\]  

(24)

Since \(\lambda_h \to \lambda\) in \(L^1(\Omega)\) we have that

\[
\int_{\Omega} \lambda_h dx \leq c \quad \text{for every } h \in \mathbb{N}.
\]  

(25)

Then we have

\[
\int_0^T \int_{\Omega} (v_{h,\varepsilon}^2\lambda_h - v^2\lambda) dx dt = \int_0^T \int_{\Omega} (v_h^2 - v_{h,\varepsilon}^2)\lambda_h dx dt +
\]

\[
+ \int_0^T \int_{\Omega} (v_{h,\varepsilon}^2\lambda_h - v_{\varepsilon}^2\lambda) dx dt + \int_0^T \int_{\Omega} (v_{\varepsilon}^2 - v^2)\lambda dx dt.
\]  

(26)

By Proposition 2.12 in [21] and (25), the first and the third term in the right of (26) are \(O(\varepsilon^2)\) uniformly in \(h\). As the central term is concerned we have:

\[
|v_{h,\varepsilon}(x, t) - v_{\varepsilon}(x, t)| = \left| \int_{\mathbb{R}^n} (\bar{v}_h(\xi, t) - \bar{v}(\xi, t))\rho_{\varepsilon}(x - \xi) d\xi \right|
\]

\[
\leq \|\rho_{\varepsilon}\|_{\infty} \int_{\Omega} |v_h(\xi, t) - v(\xi, t)| d\xi
\]  

(27)

by which we obtain the existence of a constant \(c_1 = c_1(\rho, \varepsilon, \sigma, \Omega)\) such that

\[
\int_0^T \|v_{h,\varepsilon}(t) - v_{\varepsilon}(t)\|_{\infty}^2 dt \leq c_1 \int_0^T \left( \int_{\Omega} |v_h(x, t) - v(x, t)| dx \right)^2 dt.
\]  

By (i) we have that \(v_h \to v\) in \(L^2(0, T; L^1(\Omega))\) and then we can conclude that for every \(\varepsilon\)

\[
\|v_{h,\varepsilon} - v_{\varepsilon}\|_{L^2(0,T;L^\infty(\Omega))} \xrightarrow[h]{} 0.
\]  

(28)

Moreover from (27) we derive that, with fixed \(\varepsilon\), \(v_{h,\varepsilon}^2(t) \to v_{\varepsilon}^2(t)\) in \(L^\infty(\Omega)\) for a.e. \(t \in [0, T]\). We then obtain

\[
\int_{\Omega} v_{h,\varepsilon}^2(x, t)\lambda_h dx \to \int_{\Omega} v_{\varepsilon}^2\lambda dx \quad \text{for a.e. } t \in [0, T].
\]  

Moreover the functions \(t \mapsto \int_{\Omega} v_{h,\varepsilon}^2(x, t)\mu_t dx\) are equibounded in \(L^1(0, T)\). Infact by the proof of (i) we have that \((v_{h,\varepsilon})_h\) is bounded in \(Z\) and using (14) with \(V = W_0^{1,1+\sigma}(\Omega)\) and \(H = L^2(\Omega, \lambda_h)\) and \(Z\) in the place of \(\mathcal{W}\) (thanks to (9)) obtain that there exists \(c_2\) such that

\[
\max_{t \in [0, T]} \|v_{h,\varepsilon}(t)\|_{L^2(\Omega, \lambda_h)} \leq c_2
\]
so that
\[ \int_0^T \int_\Omega v_{h,\varepsilon}^2 \lambda_h dx - \int_\Omega \int_0^T v_{\varepsilon}^2 \lambda dx dt \to 0. \]

(iii) It is an immediate consequence of the following estimates: if \( \varphi \in C^\infty_c(\Omega \times (0, T)) \)
\[ |\int_0^T \int_\Omega v_{h} \varphi \lambda_h dx dt - \int_0^T \int_\Omega v \varphi \lambda dx dt| = 
= |\int_0^T \int_\Omega (v_h - v_{h,\varepsilon}) \varphi \lambda dx dt + \int_0^T \int_\Omega (v_{h,\varepsilon} \varphi - v_{\varepsilon} \varphi) \lambda dx dt + \int_0^T \int_\Omega (v_{\varepsilon} - v) \varphi \lambda dx dt| 
\leq \left( \int_0^T \int_\Omega (v_h - v_{h,\varepsilon})^2 \lambda_h \right)^{1/2} \left( \int_0^T \int_\Omega \varphi^2 \lambda_h \right)^{1/2} + \int_0^T \int_\Omega (v_{h,\varepsilon} \varphi - v_{\varepsilon} \varphi) \lambda dx dt + 
+ \left( \int_0^T \int_\Omega (v_{\varepsilon} - v)^2 \lambda dx dt \right)^{1/2} \left( \int_0^T \int_\Omega \varphi^2 \lambda dx dt \right)^{1/2}. \]

By Proposition 2.12 in [21] and (28) we have that for every fixed \( \varepsilon \),
\[ \int_0^T \int_\Omega (v_{h,\varepsilon} \varphi \mu_h - v_{\varepsilon} \varphi \mu) dx dt \to \int_0^T \int_\Omega uv\lambda dx dt. \]

Then for fixed \( \varepsilon \) letting \( h \) go to infinity we conclude. \( \square \)

In addition to this “strong” result, a “weak” result holds. That is the following proposition.

**Proposition 3.2.** Suppose \((u_h)_h\) and \((v_h)_h\) two sequences, \( u_h \in L^2(0, T; L^2(\Omega, \lambda_h)) \), \( v_h \in W_{\lambda_h} \), where
\[ (\lambda_h)_h \subset A_2(K), \quad \int_\Omega \lambda_h dx \leq c_1 \quad \text{and} \quad \int_\Omega \lambda^{-1}_h dx \leq c_2 \quad \text{for every} \quad h \]
\[ \lambda_h \to \lambda, \quad \lambda^{-1}_h \to \tilde{\lambda}^{-1} \quad \text{in} \quad L^1(\Omega)\text{-weak}, \quad \lambda \text{ and } \tilde{\lambda} \text{ comparable}. \]

Suppose that
\[ \|u_h\|_{L^2(0, T; L^2(\Omega, \lambda_h))} \leq c_3 \quad \text{and} \quad \|v_h\|_{W_{\lambda_h}} \leq c_4. \]

Then there exist two functions \( u, v \in L^2(0, T; L^2(\Omega, \lambda)) \) such that, up to subsequences,
\[ \int_0^T \int_\Omega u_h v_h \lambda_h dx dt \to \int_0^T \int_\Omega uv \lambda dx dt. \]

**Proof.** It is sufficient writing
\[ \int_0^T \int_\Omega v_{h} u_h \lambda_h dx dt - \int_0^T \int_\Omega v \varphi \lambda dx dt = \int_0^T \int_\Omega (v_h u_h \lambda_h - v_{h,\varepsilon} u_h \lambda_h) dx dt + 
+ \int_0^T \int_\Omega (v_{h,\varepsilon} u \varphi \lambda_h - v_{\varepsilon} u \lambda) dx dt + \int_0^T \int_\Omega (v_{\varepsilon} u \lambda - v u \lambda) dx dt, \]
where the functions $v_{h,\varepsilon}$ and $v_\varepsilon$ are the approximations analogous to those defined in (23), and then estimate the three terms. By Proposition 2.12 in [21] we estimate the first term
\[ \left| \int_0^T \int_\Omega (v_h u_h \lambda_h - v_{h,\varepsilon} u_h \lambda_h) dx dt \right| \leq \left( \int_0^T \int_\Omega u_h^2 \lambda_h dx dt \int_0^T \int_\Omega |v_h - v_{h,\varepsilon}|^2 \lambda_h dx dt \right)^{1/2} \leq c \varepsilon \]
and the third one in the same way. As regards the central term we have that, by (6), there exists $\sigma \in \mathbb{R}$ such that
\[ \int_0^T \|u_h \lambda_h\|^2_{L^{1+\sigma}(\Omega)} dt \leq c \]
and then $(u_h \lambda_h)_h$ is, up to subsequences, weakly convergent in $L^2(0,T;L^1(\Omega))$ to a function $w$. As done in Proposition 2.1 in [31] one can prove that there exists $u \in L^2(0,T;L^2(\Omega,\lambda))$ such that $w = u \lambda$. Since, by (28), $v_{h,\varepsilon}$ strongly converge to $v_\varepsilon$ in $L^2(0,T;L^\infty(\Omega))$ we conclude. \hfill \QED

**Remark.** From Theorem 3.1 and Proposition 3.2 we obtain at once a compactness result for functions $v_h$ independent of $t$.

The proof of the following proposition can be derived from the corresponding Proposition 3.8 in [22].

**Proposition 3.3.** Let $K, \bar{K} \geq 1$; let $\Omega$ be a bounded open set of $\mathbb{R}^n$ with Lipschitz boundary, let $(\lambda_h)_h$ be a sequence in $A_2(K)$ and let $v_h \in \mathcal{W}_h$ ($h = 1,2,...$) be a sequence for which there exists a positive constant $c$ such that $\|v_h\|_{\mathcal{W}_h}, \|v'_{h}\|_{\mathcal{W}_h} \leq c$. Assume that there exist two comparable weights (see (21)) $\lambda \in A_2(K)$ and $\tilde{\lambda} \in A_2(\bar{K})$ such that
\[ \lambda_h \rightarrow \lambda \quad \text{and} \quad \lambda_h^{-1} \rightarrow \tilde{\lambda}^{-1} \quad \text{in } L^1(\Omega)-\text{weak}. \]

Then there exists $\sigma > 0$ (see (10)) and a function $v$ such that
\begin{enumerate}[(i)]
  \item $v_h \rightarrow v$ in $L^2(0,T;W^{1,1+\sigma}(\Omega))-\text{weak}$ and $v'_h \rightarrow v'$ in $L^2(0,T;W^{-1,1+\sigma}(\Omega))-\text{weak}$ (up to a subsequence);
  \item $v \in \mathcal{V}_\lambda$ and $\|v\|_{\mathcal{V}_\lambda} \leq \lim \inf_{h \rightarrow \infty} \|v_h\|_{\mathcal{W}_h}$;
  \item $v' \in \mathcal{V}'_\lambda$ and $\|v'\|_{\mathcal{V}'_\lambda} \leq \lim \inf_{h \rightarrow \infty} \|v'_h\|_{\mathcal{W}'_h}$;
  \item if $f \in L^2(0,T;L^n(\Omega))$ then
    \[ \|f\|_{\mathcal{V}'_\lambda} \leq \lim \inf_{h \rightarrow \infty} \|f\|_{\mathcal{W}'_h} \leq \lim \sup_{h \rightarrow \infty} \|f\|_{\mathcal{W}'_h} \leq \|f\|_{\mathcal{V}'_\lambda}, \]
  \item if $f_h \in L^2(0,T;L^2(\Omega,\lambda_h))$ such that $\|f_h\|_{L^2(0,T;L^2(\Omega,\lambda_h))} \leq c$, then there exists $f \in L^2(0,T;L^2(\Omega,\lambda)) = L^2(0,T;L^2(\Omega,\tilde{\lambda}))$ such that
    \[ \|f\|_{\mathcal{V}'_\lambda} \leq \lim \inf_{h \rightarrow \infty} \|f_h\|_{\mathcal{W}'_h} \leq \lim \sup_{h \rightarrow \infty} \|f_h\|_{\mathcal{W}'_h} \leq \|f\|_{\mathcal{V}'_\lambda}. \]
\end{enumerate}

We introduce now the class of coefficients matrices of these parabolic equations and the definition of weak solution of Cauchy-Dirichlet problem to prove successively some compactness results on the solutions (see also [33]).
Definition 3.4. Let \( L, M, K, T \) be positive constants, with \( L, M, K \geq 1 \), let \( \Omega \) be a bounded open set of \( \mathbb{R}^n \). We denote by \( \mathcal{M}_{\Omega \times (0,T)}(L, M, K, C) \) the class of matrices \( a(x) = [a_{ij}(x)]_{i,j=1}^{n} \) of order \( n \) with \( a_{ij} \in L^1_{loc}(\mathbb{R}^n \times (0,T)) \), \( i, j = 1, \ldots, n \), for which there exists \( \lambda \), weight on \( \mathbb{R}^n \), such that

\[
\lambda(x)|\xi|^2 \leq \left( a(x, t) : \xi, \xi \right) \leq \lambda \lambda(x)|\xi|^2, \quad \text{for a.e.} \quad (x, t) \in \Omega \times (0,T) \quad \text{and for every} \quad \xi \in \mathbb{R}^n, \tag{S.1}
\]

\[
\left| \left( a(x, t) : \xi, \eta \right) \right| \leq M \left( a(x, t) : \xi, \xi \right)^{1/2} \left( a(x, t) : \eta, \eta \right)^{1/2}, \quad \text{for a.e.} \quad (x, t) \in \Omega \times (0,T) \quad \text{and for every} \quad \xi, \eta \in \mathbb{R}^n, \tag{S.2}
\]

\[
\lambda \in \mathcal{A}_2(K), \quad \int_{\Omega} \lambda dx + \int_{\Omega} \lambda^{-1} dx \leq C. \tag{S.3}
\]

Remark. Consider a sequence of matrices \( \xi \in \mathcal{M}_{\Omega \times (0,T)}(L, M, K, C) \) and \( \lambda \) a sequence of weights \( \lambda \in \mathcal{A}_2(K) \). If \( \xi \in \mathcal{M}_{\Omega \times (0,T)}(L, M, K, C) \) then, by Remark 2.7 in [21], we have the existence of two weights \( \lambda \) and \( \lambda' \) such that, up to a subsequence, \( \lambda_h \to \lambda \) and \( \lambda'^{-1} \to \lambda^{-1} \) in \( L^1_{loc}(\mathbb{R}^n) \).

Given a sequence \( \lambda \in \mathcal{A}_2(K) \) we define a parabolic operator as follows:

\[
\mathcal{A} = \mathcal{A}_a : \mathcal{V}_\lambda \to \mathcal{V}_\lambda', \quad \text{and} \quad \mathcal{P} = \mathcal{P}_{\lambda,a} : \mathcal{W}_\lambda \to \mathcal{V}_\lambda',
\]

where \( \mathcal{A} \) and \( \mathcal{P} \) are respectively the abstract operators defined in (16) and (17) with \( A(t) = -\text{div} \left( a(t) : \xi, \xi \right) \) verifying (12) with \( V = H^1_{00}(\Omega, \lambda) \).

Remark. Observe that if \( \mathcal{A}_a \neq \emptyset \), the operators belonging to the family \( \mathcal{P}_{\lambda,a,\lambda} \in \mathcal{A}_a \) are defined on the same space. Indeed, by (S.1), \( \lambda_1, \lambda_2 \in \mathcal{A}_a \) are comparable (see (21) and (22)), but \( \mathcal{P}_{\lambda_1, a} \neq \mathcal{P}_{\lambda_2, a} \).

Definition 3.5. Let \( L, M, K \geq 1, C, T > 0 \), let \( \Omega \) be a bounded open set of \( \mathbb{R}^n \), \( a \in \mathcal{M}_{\Omega \times (0,T)}(L, M, K, C) \), \( \lambda \in \mathcal{A}_a \) and \( f \in \mathcal{V}_\lambda' \). Then we call a function \( u \in \mathcal{W}^0_{\lambda} \) solution of the Cauchy-Dirichlet problem

\[
\begin{cases}
\lambda \frac{\partial u}{\partial t} - \text{div}(a \cdot Du) = f & \text{in } \Omega \times (0,T) \\
u = 0 & \text{on } (\partial \Omega \times (0,T)) \cup (\Omega \times \{0\})
\end{cases} \tag{30}
\]

if

\[
\mathcal{P}_{\lambda,a} u(t) = u'(t) + (\mathcal{A}_a u)(t) = f(t) \quad \text{for a.e.} \quad t \in [0,T]. \tag{31}
\]

Remark. Given a matrix \( a \in \mathcal{M}_{\Omega \times (0,T)}(L, M, K, C) \) then, by Theorem 2.2 (iii), we obtain that for every \( \lambda \in \mathcal{A}_a \) and \( f \in \mathcal{V}_\lambda' \) there exists a unique solution of the problem (30).
Given a sequence \( (a_h)_h \) of matrices in \( M_{\Omega \times (0,T)}(L,M,K,C) \) and \( (\lambda_h)_h \) of weights, \( \lambda_h \in \Lambda_{a_h} \) for every \( h \in \mathbb{N} \), consider the following sequence of problems

\[
\begin{cases}
\lambda_h(x) \frac{\partial}{\partial t} u - \text{div}(a_h(x,t) \cdot Du) = f & \text{in } \Omega \times (0,T) \\
u = 0 & \text{on } (\partial \Omega \times (0,T)) \cup (\Omega \times \{0\})
\end{cases}
\]  

(32)

We will write (32) in the abstract way (here \( \frac{d}{dt} : \mathcal{W}_{\lambda_h}^0 \rightarrow \mathcal{V}_h^0 \) is to be intended as in (18))

\[
\begin{cases}
P_{\lambda_h,a_h} u := \frac{d}{dt} u + A_{a_h} u = f \\
u \in \mathcal{W}_{\lambda_h}^0.
\end{cases}
\]  

(33)

The proof of the following theorem can be derived following the analogous one in [21].

**Theorem 3.6.** Let \( L, M, K, C, T \) be positive constants with \( L, M, K \geq 1 \) and let \( \Omega \) be a bounded open set of \( \mathbb{R}^n \). Let \( (a_h)_h \) be a sequence in \( M_{\Omega \times (0,T)}(L,M,K,C) \), \( (\lambda_h)_h \) a sequence with \( \lambda_h \in \Lambda_{a_h} \) and \( f \in L^2(0,T;L^n(\Omega)) \). Consider the parabolic equations (33) and their solutions \( u_h \). Then

(i) there exists a positive constant \( c = c(L,M,K,C,\Omega) \) (depending only on \( L,M,K,C,\Omega \)) such that

\[
\|u_h\|_{\mathcal{W}_{\lambda_h}} \leq c\|f\|_{L^2(0,T;L^n(\Omega))};
\]

(ii) say \( \lambda \) and \( u \) the weak limits, up to a subsequence, respectively of \( (\lambda_h)_h \) in \( L^1(\Omega) \)-weak and of \( (u_h)_h \) in \( L^2(0,T;W^{1,1}(\Omega)) \)-weak; we have that

\[
U_h(t) = \int_{\Omega} u_h^2(x,t) \lambda_h(x) dx \rightarrow U(t) = \int_{\Omega} u^2(x,t) \lambda dx \quad \text{in } C^0([0,T]).
\]

Finally we state a weighted compensated compactness type result which extends well-known classical results (see, for instance, [20], [30], [29], [28], [9] and [10]), whose proof can be obtained in completely analogous way to the corresponding result in [22] using Proposition 3.2.

**Theorem 3.7.** Suppose that \( \Omega \) is a bounded open set of \( \mathbb{R}^n \), \( K \geq 1 \), \( (\lambda_h)_h \) is a sequence in \( A_2(K) \), \( \lambda \in A_2 \), such that

\[
\int_{\Omega} \lambda_h^{-1} dx \leq c_1 \quad \text{for every } h \in \mathbb{N}.
\]

Consider a sequence of functions \( u_h \in \mathcal{W}_{\lambda_h} \) \((h = 1,2,...)\) and a function \( u \in \mathcal{W}_{\lambda} \) such that

\[
\|u_h\|_{\mathcal{W}_{\lambda_h}} \leq c_2 \quad \text{for every } h, \quad u_h \rightarrow u \quad \text{in } L^2(0,T;L^1(\Omega)).
\]

Consider a sequence of vector functions \( \alpha_h \in L^2(0,T;(L^2(\Omega,\lambda_h^{-1}))^n) \) \((h = 1,2,...)\) and \( \alpha \in L^2((0,T);(L^2(\Omega,\lambda^{-1}))^n) \) such that

\[
\|\alpha_h\|_{L^2((0,T);(L^2(\Omega,\lambda_h^{-1}))^n)} \leq c_3, \quad \alpha_h \rightharpoonup \alpha \quad \text{in } L^2(0,T;(L^1(\Omega))^n) \text{-weak}.
\]

Assume further that

\[
\lambda_h \frac{\partial u_h}{\partial t} - \text{div } (\alpha_h) = f \in L^2(0,T;L^n(\Omega)) \quad \text{on } C^1_0(\Omega \times (0,T))
\]
for every $h \in \mathbb{N}$. Then
\[
(\alpha_h, Du_h) \to (\alpha, Du) \quad \text{in } D'(\Omega \times (0, T)).
\]

4. The definition of $G$-convergence and a $G$-compactness result

In this section we will introduce the definition of $G$-convergence for a class of degenerate parabolic operators and get a $G$-compactness result (Theorem 4.3).

**Definition 4.1.** Let $L, M, K, C, T$ be positive constants with $L, M, K \geq 1$, let $\Omega$ be a bounded open set of $\mathbb{R}^n$ and let $a_h (h = 1, 2, ...)$ and $a$ be matrices in $\mathcal{M}_{\Omega \times (0, T)}(L, M, K, C)$ and $\lambda_h \in \Lambda_{a_h}$ and $\lambda \in \Lambda_a$ (see Definition 3.4). We say that the sequence $\mathcal{P}_{\lambda_h, a_h}$ $G$-converges to $\mathcal{P}_{\lambda, a}$ in $\Omega \times (0, T)$, and we write
\[
\mathcal{P}_{\lambda_h, a_h} \xrightarrow{G} \mathcal{P}_{\lambda, a} \quad \text{in } \Omega \times (0, T),
\]
if for every $f \in L^2(0, T; L^n(\Omega))$ it results that
\[
\begin{align*}
\omega_h &\to \omega \quad \text{in } L^2(0, T; L^1(\Omega)) \\
a_h \cdot Du_h &\to a \cdot Du \quad \text{in } L^2(0, T; L^1(\Omega))^n \text{--weak},
\end{align*}
\]
where $\omega_h$ and $\omega$ denote respectively the solutions (see Definition 3.5) of
\[
\begin{align*}
\mathcal{P}_{\lambda_h, a_h} v &= \lambda_h \frac{\partial v}{\partial t} - \text{div}(a_h \cdot Dv) = f & \text{in } \Omega \times (0, T) \\
v &= 0 & \text{on } (\partial \Omega \times (0, T)) \cup (\Omega \times \{0\}),
\end{align*}
\]
\[
\begin{align*}
\mathcal{P}_{\lambda, a} v &= \lambda \frac{\partial v}{\partial t} - \text{div}(a \cdot Dv) = f & \text{in } \Omega \times (0, T) \\
v &= 0 & \text{on } (\partial \Omega \times (0, T)) \cup (\Omega \times \{0\}).
\end{align*}
\]

**Remark.** In the classic case, i.e. when $\lambda \in \Lambda_a$ is a constant $0 > \lambda_0 > 0$ the space $H_0^1(\Omega, \lambda_0)$ is $H_0^1(\Omega)$ and we get the classic definition of $G$-convergence (see, for instance, [26], [27], [5], [28] and [33]).

The proof of the following proposition can be obtained following the corresponding one for elliptic operators (see Proposition 2.9 in [10]).

**Proposition 4.2.** Let $L, M, K, C, T, T_1, T_2$ be positive constants with $L, M, K \geq 1$, $\Omega$ be a bounded open set of $\mathbb{R}^n$ with Lipschitz boundary. Consider $(a_h)_{h \in \mathbb{N}}$ a sequence in $\mathcal{M}_{\Omega \times (0, T)}(L, M, K, C)$ and $\lambda_h \in \Lambda_{a_h}$ a sequence of weights. If, for every interval $I_i = [0, T_i]$ $(i = 1, 2)$ $I_1 \subseteq I_2 \subseteq [0, T]$ and for every bounded open set $\Omega_i$ $(i = 1, 2)$ of $\mathbb{R}^n$ with Lipschitz boundaries with $\Omega_1 \subseteq \Omega_2 \subseteq \Omega$, we have
\[
\mathcal{P}_{\lambda_h, a_h} \xrightarrow{G} \mathcal{P}_{\mu_i, b_i} \quad \text{on } \Omega_i \times I_i, \quad (i = 1, 2)
\]
for suitable $b_i \in \mathcal{M}_{\Omega_i \times I_i}(L_i, M_i, K_i, C_i)$ and $\mu_i \in \Lambda_{b_i}$ $(i = 1, 2)$, then $b_1(x, t) = b_2(x, t)$ a.e. in $\Omega_1 \times I_1$ and $\mu_1 = \mu_2$ a.e. in $\Omega_1$.

**Remark.** By Proposition 4.2 the uniqueness of the $G$-limit follows.
The following theorem states a precompactness result with respect to $G$-convergence for the class of operators $\mathcal{P}_{\lambda,a}$ defined by $a \in \mathcal{M}_{\Omega \times (0,T)}(L, M, K, C)$ and $\lambda \in \Lambda_a$.

We only give a scheme of the proof since the steps are the same of the corresponding proof in [22].

**Theorem 4.3.** Let $L, M, K, C, T$ be positive constants with $L, M, K \geq 1$ and let $\Omega$ be a bounded open set of $\mathbb{R}^n$ with Lipschitz boundary. Consider a sequence $(a_h)_h \in \mathbb{N} \subset \mathcal{M}_{\Omega \times (0,T)}(L, M, K, C)$ and a sequence of weights $\lambda_h \in \Lambda_{a_h}$ (see Definition 3.4). Then there exist a subsequence, still denoted by the index $h$, a matrix $a(x,t) = [a_{ij}(x,t)]_{i,j=1}^n$, $a_{ij} \in L^1_{loc}(\Omega \times (0,T))$, defined in $\Omega \times (0,T)$ and a weight $\lambda$ such that $a \in \mathcal{M}_{\Omega \times (0,T)}(\lambda^2KL, M\sqrt{L}, K, KC)$, $\lambda/K \in \Lambda_a$ and

$$\lambda_h \rightarrow \lambda \quad \text{in } L^1_{loc}(\mathbb{R}^n) \text{ weak}$$

$$\mathcal{P}_{\lambda_h,a_h} \rightarrow \mathcal{P}_{\lambda,a} \quad \text{in } \Omega \times (0,T).$$

**Proof.** First of all by Remark 2.7 in [21] we get the existence of two weights $\lambda, \tilde{\lambda}$, comparable (see (21)), such that, up to a subsequence, $\lambda_h \rightarrow \lambda$ and $\lambda_h^{-1} \rightarrow \tilde{\lambda}^{-1}$ and define three spaces $\mathcal{V}_\lambda = \mathcal{V}_\lambda'$, $\mathcal{V}'_\lambda = \mathcal{V}'_\lambda'$, $\mathcal{W}_\lambda = \mathcal{W}_\lambda$. The space $\mathcal{W}_\lambda$ is the space in which the limit parabolic operator (which we want to find) will be defined. With the derivative $\frac{d}{dt} : \mathcal{W}_\lambda^0 \rightarrow \mathcal{V}_\lambda'$ we denote the operator acting as follows (see (18)) for $\varphi \in C_c^\infty(\Omega \times (0,T))$

$$\langle \frac{d}{dt}u, \varphi \rangle_{\mathcal{V}_\lambda \times \mathcal{W}_\lambda} = \int_0^T \int_\Omega u(x,t) \frac{\partial \varphi (x,t)}{\partial t}(x)\lambda(x)dxdt.$$

In a completely similar way as in Lemma 4.7 in [22] we can find three operators

$$\mathcal{B} : \mathcal{V}'_\lambda \rightarrow \mathcal{W}_\lambda^0, \quad \mathcal{K} : \mathcal{V}'_\lambda \rightarrow \mathcal{V}'_\lambda, \quad G : \mathcal{V}'_\lambda \rightarrow L^2(0,T; (L^2(\Omega, \lambda^{-1}))')$$

such that, up to subsequences,

$$\mathcal{P}^{-1}_{\lambda_h,a_h} f \rightarrow \mathcal{B} f \quad \text{in } L^2(0,T; L^1(\Omega)) \quad \text{for every } f \in L^2(0,T; L^1(\Omega)),$$

$$a_h \cdot D(\mathcal{P}^{-1}_{\lambda_h,a_h} f) \rightarrow G f \quad \text{in } L^2(0,T; (L^1(\Omega)))\text{-weak} \quad \text{for every } f \in L^2(0,T; L^1(\Omega)),$$

$$\frac{d}{dt}(\mathcal{B} f) + \mathcal{K} f = f \quad \text{and} \quad \mathcal{K} f = -\text{div } (Gf) \quad \text{on } \mathcal{V}_\lambda \quad \text{for every } f \in \mathcal{V}'_\lambda.$$

The convergence of $\mathcal{P}^{-1}_{\lambda_h,a_h} f$ is in $L^2(0,T; L^1(\Omega))$ by Theorem 3.1, i). The sequence $a_h \cdot D(\mathcal{P}^{-1}_{\lambda_h,a_h} f)$ is bounded in $L^2(0,T; (L^{1+\sigma}(\Omega)))$ thanks to (9). Finally, Theorem 3.6, ii), is used to prove that $\mathcal{K} f = -\text{div } (Gf)$ on $\mathcal{V}_\lambda$ since we multiply $\mathcal{P}_{\lambda_h,a_h} u_h = f$ by $u_h$ (where $u_h$ is the solution to (33)) and proceed as in [22], Lemma 4.7.

Following the proof of Lemma 4.8 in [22] (using Theorem 3.7) one can prove that the operator $\mathcal{B}$ is invertible and there exists an operator $\mathcal{A} : \mathcal{V}_\lambda \rightarrow \mathcal{V}_\lambda'$ satisfying

$$\mathcal{A}(\mathcal{B} f) = \mathcal{K} f \quad \text{for every } f \in \mathcal{V}_\lambda'$$

$$\frac{1}{K}\|u\|_{\mathcal{V}_\lambda}^2 \leq \langle \mathcal{A} u, u \rangle_{\mathcal{V}_\lambda} \leq M^2L\|u\|_{\mathcal{V}_\lambda}^2 \quad \text{for every } u \in \mathcal{V}_\lambda,$$

$$\langle \mathcal{A} u, v \rangle_{\mathcal{V}_\lambda} \leq M\sqrt{L} \langle \mathcal{A} u, u \rangle_{\mathcal{V}_\lambda}^{1/2} \langle \mathcal{A} v, v \rangle_{\mathcal{V}_\lambda}^{1/2} \quad \text{for every } u, v \in \mathcal{V}_\lambda,$$
and such that
\[ B^{-1} = \mathcal{P} \equiv \frac{d}{dt} + \mathcal{A} : \mathcal{W}^0_\lambda \to \mathcal{V}^\prime_\lambda. \]

Now, following the proof of Theorem 4.5 in [22], one can prove before that
\[ \langle \mathcal{A} u, v \rangle_\lambda = \int_0^T \int_\Omega ((G \circ \mathcal{P}) u, Dv) dx dt \]
and then that there exist a matrix \( a = [a_{ij}(x, t)]_{i,j=1}^n \) such that \( a_{ij} \in L^1_{\text{loc}}(\Omega \times (0, T)) \) and
\[
\begin{align*}
\mathcal{P} \equiv \frac{d}{dt} + \mathcal{A} : \mathcal{W}^0_\lambda &\rightarrow \mathcal{V}^\prime_\lambda, \\
(1/K) \lambda(x) |\xi|^2 &\leq (a(x, t) \cdot \xi, \xi) \leq M^2 L \lambda(x) |\xi|^2, \\
(a(x, t) \cdot \xi, \eta) &\leq M \sqrt{L}(a(x, t) \cdot \xi, \xi)^{1/2} (a(x, t) \cdot \eta, \eta)^{1/2}
\end{align*}
\]
a.e. in \( \Omega \times (0, T) \), for every \( u \in \mathcal{V}_\lambda \) and for every \( \xi, \eta \in \mathbb{R}^n \).

**Corollary 4.4.** Let \( L, M, K, C \) be positive constants with \( L, M, K \geq 1 \). Consider a sequence \( (a_h(x, t))_h \) of matrices defined in \( \mathbb{R}^n \times (0, +\infty) \) for which there exists a sequence of weights \( (\lambda_h)_h \) and a cube \( Q_0 \) such that
\[
\begin{align*}
(i) &\quad \lambda_h(x) |\xi|^2 \leq (a_h(x, t) \cdot \xi, \xi) \leq L \lambda_h(x), \\
&\quad \text{for a.e. } (x, t) \in \mathbb{R}^n \times (0, +\infty), \text{ for every } \xi \in \mathbb{R}^n, \text{ for every } h \in \mathbb{N}, \\
(ii) &\quad |(a_h(x, t) \cdot \xi, \eta)| \leq M (a_h(x, t) \cdot \xi, \xi)^{1/2} (a_h(x, t) \cdot \eta, \eta)^{1/2}, \\
&\quad \text{for a.e. } (x, t) \in \mathbb{R}^n \times (0, +\infty), \text{ for every } \xi, \eta \in \mathbb{R}^n \text{ and for every } h \in \mathbb{N}, \\
(iii) &\quad \lambda_h \in A_2(K) \text{ and } \int_{Q_0} \lambda_h dx + \int_{Q_0} \lambda_h^{-1} dx \leq C \text{ for every } h \in \mathbb{N}.
\end{align*}
\]
Then there exist a subsequence, still denoted by the index \( h \), a matrix \( a(x, t) = [a_{ij}(x, t)]_{i,j=1}^n \)
\( a_{ij} \in L^1_{\text{loc}}(\mathbb{R}^n \times (0, +\infty)) \) and a weight \( \lambda \) such that
\[ a \in \mathcal{M}_{\mathbb{R}^n \times (0, +\infty)}(M^2 KL, M \sqrt{T}, K, KC), \]
\( \lambda/K \in \Lambda_a \) and
\[ \mathcal{P} \lambda_h a_h \overset{G}{\rightarrow} \mathcal{P} \lambda a \quad \text{in } \omega \times (0, \tau), \]
for every \( \omega \) bounded open set with Lipschitz boundary and for every \( \tau > 0 \).

**Proof.** By Remark 2.7 in [21] we have that \( \lambda_h \) and \( \lambda_h^{-1} \) are equibounded in \( L^1_{\text{loc}}(\mathbb{R}^n) \) (it is easy obtained by (7)) and the existence of two limit weights \( \lambda \) and \( \tilde{\lambda} \) defined in \( \mathbb{R}^n \) such that, up to a subsequence, \( \lambda_h \to \lambda \) and \( \lambda_h^{-1} \to \tilde{\lambda}^{-1} \) in \( L^1_{\text{loc}}(\mathbb{R}^n) \)-weak. Choosing the sequence of sets \((-j, j)^n \times (0, j)\) invading \( \mathbb{R}^n \times (0, +\infty) \) one can obtain the thesis by a diagonal process (use Proposition 4.2 and follow the proof of Theorem 4.10 in [22]).

If \( \lambda_h(x) = \lambda_0 > 0 \) (\( \lambda_0 \in \mathbb{R} \)) the previous \( G \)-compactness result comes to be a well-known classic \( G \)-compactness result (see Theorem 1 and Theorem 3 in [28], but also [26], [27], [5], [33]).

The sharpness of condition \( A_2 \) considered on the weights \( \lambda_h \) among the class of Muckenhoupt weights is shown by some counterexamples in the elliptic case and by considerations made in the following section, showing links between elliptic and parabolic convergence (see Remark 5).

Homogenization for problems like that defined in (30) is considered in [21].
5. A comparison among $G$-convergence of different operators

In this section we want to analyze the relation existing among the behaviour of different kinds of operators defined by the same matrices. By $\Omega$ we will always denote a bounded open set with Lipschitz boundary. Consider a sequence of matrices $(a_h)_h$, $a_h = a_h(x)$, contained in $\mathcal{M}_{\Omega \times (0,T)}(L,M,K,C)$ which in this case will be simply denoted by $\mathcal{M}_\Omega(L,M,K,C)$ (see Definition 3.4). The problems we consider in this section are

$$
\begin{aligned}
(I) & \quad \begin{cases}
-\text{div}(a_h(x) \cdot Dv) = f & \text{in } \Omega \\
u = 0 & \text{in } \partial \Omega,
\end{cases} \\
(II) & \quad \begin{cases}
\lambda_h \frac{\partial v}{\partial t} - \text{div}(a_h(x) \cdot Dv) = f & \text{in } \Omega \times (0,T) \\
u = 0 & \text{in } (\partial \Omega \times (0,T)) \cup (\{0\} \times (0,T)),
\end{cases} \\
(III) & \quad \begin{cases}
\frac{\partial v}{\partial t} - \text{div}(a_h(x) \cdot Dv) = f & \text{in } \Omega \times (0,T) \\
u = 0 & \text{in } (\partial \Omega \times (0,T)) \cup (\{0\} \times (0,T)),
\end{cases}
\end{aligned}
$$

For the moment we shall confine ourselves to a comparison between problems (I) and (II).

**Definition 5.1.** Consider a sequence of matrices $a, a_h \in \mathcal{M}_\Omega(L,M,K,C)$ ($h = 1, 2, 3, \ldots$), a sequence of weights $\lambda \in \Lambda_a, \lambda_h \in \Lambda_{a_h}$ and define the operators

$$
A_h = -\text{div}(a_h \cdot D) : H^1_0(\Omega, \lambda_h) \to H^{-1}(\Omega, \lambda_h), \quad A = -\text{div}(a \cdot D) : H^1_0(\Omega, \lambda) \to H^{-1}(\Omega, \lambda).
$$

We say that $A_h$ $G$-converge to $A$ in $\Omega$, and write

$$
A_h \xrightarrow{G} A \quad \text{in } \Omega,
$$

if for every $f \in L^n(\Omega)$

$$
\begin{align*}
& u_h(f) \to u(f) \quad \text{in } L^1(\Omega) \\
& a_h \cdot Du_h(f) \to a \cdot Du(f) \quad \text{in } (L^1(\Omega))^n \text{ weak}
\end{align*}
$$

where $u_h = u_h(f)$ denote the solutions of problems (I) and $u$ solves

$$
\begin{cases}
-\text{div}(a(x) \cdot Dw) = f & \text{in } \Omega \\
w = 0 & \text{on } \partial \Omega.
\end{cases}
$$

As regards the problems (I), in [10] the authors show the existence of a matrix $b_1 = b_1(x) \in \mathcal{M}_\Omega(L', M', K, C')$ (and implicitly of a weight $\lambda' \in A_2(K)$) such that (up to subsequences)

$$
A_h \xrightarrow{G} -\text{div}(b_1(x) \cdot D) \quad \text{in } \Omega.
$$

As regards the problems (II) in the previous section it is proved the existence of a weight $\lambda$ and of a matrix $b_2 \in \mathcal{M}_{\Omega \times (0,T)}(L', M', K, C')$ such that (up to subsequences)

$$
\mathcal{P}_{\lambda_h, a_h} \xrightarrow{G} \mathcal{P}_{\lambda, b_2} \quad \text{in } \Omega \times (0,T).
$$

We will show is that $b_1 = b_2 = b(x)$ and the equivalence between these two convergences. First we prove a result involving moving data which will be useful to this aim.
Consider a sequence of weights \((\lambda_h)_h\) satisfying (29). Suppose to have a sequence \(f_h \in L^2(0, T; L^2(\Omega, \lambda_h^{-1}))\) such that
\[
\|f_h\|_{L^2(0, T; L^2(\Omega, \lambda_h^{-1}))} \leq c. \tag{34}
\]

Say \(\lambda\) and \(\tilde{\lambda}^{-1}\) the weak limits in \(L^1(\Omega)\), up to a subsequence, of \(\lambda_h\) and \(\lambda_h^{-1}\) (see Remark 2.7 in [21]). By Proposition 3.2 we have that there exists \(f \in L^2(0, T; L^2(\Omega, \tilde{\lambda}^{-1}))\) such that, up to a subsequence, \((\lambda\) and \(\tilde{\lambda}\) are comparable, see (21)) for every \(\varphi \in C_c^\infty(\Omega \times (0, T))\)
\[
\int_0^T \int_\Omega f \varphi \lambda^{-1} dxdt \to \int_0^T \int_\Omega f \varphi \tilde{\lambda}^{-1} dxdt. \tag{35}
\]

**Lemma 5.2.** Let \((\lambda_h)_h\) and \((f_h)_h\) satisfy (34) and (35). Make the additional assumption
\[
\int_0^T \int_\Omega f^2 \lambda^{-1} dxdt \to \int_0^T \int_\Omega f^2 \tilde{\lambda}^{-1} dxdt
\]
for \(f \in L^2(0, T; L^2(\Omega, \tilde{\lambda}^{-1}))\). Consider a sequence of matrices \(a, a_h\) \((h = 1, 2, 3, \ldots)\) such that \(\lambda_h \in \Lambda_{a_h}\) and \(\lambda \in \Lambda_a\) (see Definition 3.4) and the sequence of operators \(P_{\lambda, a}, P_{\lambda_h, a_h}\) \((h = 1, 2, \ldots)\). If \(P_{\lambda_h, a_h}\) \(G\)-converges to \(P_{\lambda, a}\), then, given \(u_h\) and \(u\) the solutions respectively to the problems
\[
\begin{align*}
P_{\lambda_h, a_h} u_h &= f_h \\
u &= 0
\end{align*}
\quad \begin{align*}
P_{\lambda, a} u &= f \\
u &= 0
\end{align*}
\]
we have that
\[
u_h \to u \quad \text{in } L^2(0, T; L^2(\Omega)) \quad \text{and} \quad a_h \cdot Du_h \to a \cdot Du \quad \text{in } L^2(0, T; L^1(\Omega)^n) - \text{weak}.
\]

**Proof.** We simply denote by \(P_h\) the operator \(P_{\lambda_h, a_h}\) and by \(P\) the operator \(P_{\lambda, a}\). For \(\varepsilon > 0\) fixed we can find \(g \in C^1_0(\Omega \times (0, T))\) such that
\[
\|f - g\|_{L^2(0, T; L^2(\Omega, \tilde{\lambda}^{-1}))} < \varepsilon.
\]
Then we write
\[
P_h^{-1} f_h - P^{-1} f = (P_h^{-1} f_h - P_h^{-1} g) + (P_h^{-1} g - P^{-1} g) + (P^{-1} g - P^{-1} f) \tag{36}
\]
The central term in the right hand side of (36) is going to zero in \(L^2(0, T; L^1(\Omega))\) since \(P_h\) \(G\)-converges to \(P\). The last of these three terms can be estimated as follows:
\[
\|P_h^{-1} g - P^{-1} f\|_{L^2(0, T; L^1(\Omega))} \leq c \|g - f\|_{V_\lambda^{\infty}} \leq c \|g - f\|_{L^2(0, T; L^2(\Omega, \tilde{\lambda}^{-1}))} \leq c \varepsilon.
\]
We now consider the first term in (36):
\[
\|P_h^{-1} f_h - P_h^{-1} g\|_{L^2(0, T; L^1(\Omega))} \leq c \|f_h - g\|_{V_\lambda^{\infty}} \leq c \|f_h - g\|_{L^2(0, T; L^2(\Omega, \lambda_h^{-1}))}.
\]
Since
\[
\|f_h - g\|_{L^2(0, T; L^2(\Omega, \lambda_h^{-1}))}^2 = \int_0^T \int_\Omega (f_h^2 \lambda_h^{-1} + g^2 \lambda_h^{-1} - 2f_h g \lambda_h^{-1}) dxdt
\]
by hypotheses we have that
\[ \|f_h - g\|_{L^2(0,T;L^2(\Omega,\lambda_h^{-1}))}^2 \to \|f - g\|_{L^2(0,T;L^2(\Omega,\lambda^{-1}))}^2. \]

It remains to see the the momenta converge to the corresponding momentum. Analogously to (36) we write
\[ a_h \cdot D(P_h^{-1}f_h) - a \cdot D(P^{-1}f) = [a_h \cdot D(P_h^{-1}f_h) - a_h \cdot D(P_h^{-1}g)] + [a_h \cdot D(P_h^{-1}g) - a \cdot D(P^{-1}g)] + [a \cdot D(P^{-1}g) - a \cdot D(P^{-1}f)] \]

As before the central term is going to zero since \( \mathcal{P}_h \) converges to \( \mathcal{P} \). Let us estimate the last one. Since
\[ |a \cdot D(P^{-1}g) - a \cdot D(P^{-1}f)| \leq ML^{1/2}\lambda^{1/2}(a \cdot D(P^{-1}g - P^{-1}f), D(P^{-1}g - P^{-1}f))^{1/2} \]
integrating we obtain
\[
\int_0^T \int_\Omega |a \cdot D(P^{-1}g) - a \cdot D(P^{-1}f)| dx \, dt \leq c \int_0^T \int_\Omega (a \cdot D(P^{-1}g - P^{-1}f), D(P^{-1}g - P^{-1}f)) dx \, dt
\]
\[ \leq c \|P^{-1}g - P^{-1}f\|_{\mathcal{V}_h} \leq c' \|g - f\|_{\mathcal{V}_h} < c' \varepsilon. \]

Now we consider the first term. Consider \( \Phi = (\varphi_1, \ldots, \varphi_n) \) with \( \varphi_j \in C_c^\infty(\Omega \times (0,T)) \) for every \( j \). Then
\[
|\int_0^T \int_\Omega (a_h \cdot D\mathcal{P}_h^{-1}(f_h - g), \Phi) dx \, dt| \leq
\leq M \left[ \int_0^T \int_\Omega (a_h \cdot D\mathcal{P}_h^{-1}(f_h - g), D\mathcal{P}_h^{-1}(f_h - g)) dx \, dt \int_0^T \int_\Omega (a_h \cdot \Phi, \Phi) dx \, dt \right]^{1/2}
\leq c(L, M, \Phi) \|\mathcal{P}_h^{-1}f_h - \mathcal{P}_h^{-1}g\|_{\mathcal{V}_h}
\leq c' \|f_h - g\|_{\mathcal{V}_h} \leq c' \|f_h - g\|_{L^2(0,T;L^2(\Omega,\lambda_h^{-1}))} < c' \varepsilon. \]

\[ \]
Consider $g \in C^\infty_c(\Omega)$ and the problems

$$
A_h : H^1_0(\Omega, \lambda_h) \to H^{-1}(\Omega, \lambda_h) \\
\begin{aligned}
A_h w := -\text{div}(a_h(x) \cdot Dw) &= g & \text{in } \Omega \\
w &= 0 & \text{on } \partial \Omega
\end{aligned}
$$

and denote by $v_h(g) = v_h$ the solutions. Thanks to Remark 3 (see Theorem 3.1) we have that $A_h^{-1} g = v_h \to v$ in $L^1(\Omega)$ and

$$
\int_\Omega v_h^2 \lambda_h dx \to \int_\Omega v^2 \lambda dx \\
\int_\Omega v_h \varphi \lambda_h dx \to \int_\Omega v \varphi \lambda dx
$$

for some $v \in H^1_0(\Omega, \lambda)$. If we consider $u_h(x, t) := tv(x)$ we have that $u_h$ are the solutions of

$$
\begin{aligned}
\lambda \frac{\partial u}{\partial t} - \text{div}(a \cdot Du) &= \lambda v + tg & \text{in } \Omega \times (0, T) \\
u &= 0 & \text{in } (\partial \Omega \times (0, T)) \cup \{0\} \times (0, T)
\end{aligned}
$$

By (37) we have that the sequence of data $\lambda_h v + tg$ satisfies the hypotheses of Lemma 5.2. Since $\mathcal{P}_{\lambda_h, a_h} G$-converges to $\mathcal{P}_{\lambda, a}$ we have that the solutions $u_h$ satisfy

$$
u_h \to u \\
a_h \cdot Du_h \to a \cdot Du
$$

where $u$ is the solution of

$$
\begin{aligned}
\lambda \frac{\partial u}{\partial t} - \text{div}(a \cdot Du) &= \lambda v + tg & \text{in } \Omega \times (0, T) \\
u &= 0 & \text{in } (\partial \Omega \times (0, T)) \cup \{0\} \times (0, T)
\end{aligned}
$$

Since $u(x, t) = tv(x)$ we have that

$$
-\text{div}(a \cdot Dv) = g.
$$

Then $a(x, t) = a(x)$ and $A_h = -\text{div}(a \cdot D) \overset{G}{\to} A = -\text{div}(a \cdot D)$. By the density of $C^\infty_c(\Omega)$ in $L^n(\Omega)$ we conclude.

As a consequence we have the following result, which is immediately derived by the previous proposition and uniqueness of the $G$-limit of $A_h$ (see [10]).

**Corollary 5.4.** Consider a sequence $a_h \in \mathcal{M}_\Omega(L, M, K, C)$ with $a_h = a_h(x)$. Then there exist $a = a(x) \in \mathcal{M}_\Omega(L, M, K, C)$ and a subsequence, still denoted by the index $h$, such that

$$
\mathcal{P}_{\lambda_h, a_h} \overset{G}{\to} \mathcal{P}_{\lambda, a} \quad \text{in } \Omega \times (0, T)
$$

for every sequence $(\lambda_h)_h$ and every $\lambda$ with $\lambda_h \in \Lambda_{a_h}$ and $\lambda_h$ weakly converging to $\lambda$ in $L^1(\Omega)$.

This last result allows us to give the following definition.
**Definition 5.5.** Given a sequence \( a_h \in \mathcal{M}_\Omega(L, M, K, C) \) we say that \( a_h \) \(\text{PG-}\)converges to \( a \) in \( \Omega \times (0, T) \), and we write
\[
a_h \overset{\text{PG}}{\to} a \quad \text{in} \quad \Omega \times (0, T),
\]
if, for every sequence of weights \( (\lambda_h)_h \) weakly converging in \( L^1(\Omega) \) (say \( \lambda \) the limit) and \( \lambda_h \in \Lambda_{a_h} \) for every \( h \in \mathbb{N} \), we have that \( \mathcal{P}_{\lambda_h,a_h} \overset{G}{\to} \mathcal{P}_{\lambda,a} \) in \( \Omega \times (0, T) \). In the same way we write
\[
a_h \overset{G}{\to} a \quad \text{in} \quad \Omega
\]
if \( A_h \overset{G}{\to} A \) in \( \Omega \) where \( A_h = -\text{div}(a_h \cdot D) \) and \( A = -\text{div}(a \cdot D) \) (see Definition 5.1).

**Theorem 5.6.** Consider a sequence \( (a_h)_h \subset \mathcal{M}_\Omega(L, M, K, C) \). Then
\[
a_h \overset{\text{PG}}{\to} a \quad \text{in} \quad \Omega \times (0, T) \quad \text{if and only if} \quad a_h \overset{G}{\to} a \quad \text{in} \quad \Omega.
\]

**Proof.**
(i) Suppose \( a_h \overset{\text{PG}}{\to} a \). Then by Proposition 5.3 it is clear that \( a_h \overset{G}{\to} a \).

(ii) Now suppose to have \( a_h \overset{G}{\to} a \). By Theorem 4.3 and Proposition 5.3 we have a subsequence \( a_{h_j} \) and a matrix \( b = b(x) \) such that \( a_{h_j} \overset{\text{PG}}{\to} b \). By (i) we then have that \( a_{h_j} \overset{G}{\to} b \). By uniqueness of the limit we conclude \( a = b \). \( \square \)

**Remark.** The sharpness of the assumptions (S.3) in Definition 3.4 to obtain Theorem 4.3 follows at once by Theorem 5.6 and by counterexamples in [25], Remark 2.10 (see also Example 1 and Example 2 in [22]).

We want now to make a comparison with another family of parabolic operators, operators like
\[
\mathcal{Q}_{a_h} = \frac{\partial}{\partial t} - \text{div}(a_h(x, t) \cdot D), \tag{38}
\]
and Cauchy-Dirichlet problems (III) defined at the beginning of this section. The convergence of such a sequence of parabolic operators is considered in [22], but for matrices in \( \mathcal{N}_{\Omega \times (0,T)}(L, M, K, C) \) (see Definition 3.4). Observe that, since \( A_{1+2/n}(K) \subset A_2(K), \mathcal{N}_{\Omega \times (0,T)}(L, M, K, C) \subset \mathcal{M}_{\Omega \times (0,T)}(L, M, K, C') \). For \( a \in \mathcal{N}_{\Omega \times (0,T)}(L, M, K, C) \) let \( \lambda \in \Lambda_a \); by \( Y_\lambda \) we denote the Hilbert space defined as the closure of \( C_0^\infty(\Omega) \) with respect to the norm \( \|u\|^2 = \int_\Omega u^2 dx + \int_\Omega |Du|^2 \lambda dx \), by \( \mathcal{Y}_\lambda \) the space \( \{ u \in L^2(0,T;Y_\lambda) \mid u' \in L^2(0,T;Y'_\lambda) \} \) and by \( Z_\lambda^0 \) the space \( \{ u \in \mathcal{Y}_\lambda \mid u(0) = 0 \text{ in } L^2(0,T;L^2(\Omega)) \} \). Now, for \( (a_h)_h \subset \mathcal{N}_{\Omega \times (0,T)}(L, M, K, C) \), consider the sequence of problems (III). We have the triples
\[
\mathcal{Y}_{a_h} \subset L^2(0,T;L^2(\Omega)) \subset \mathcal{Y}'_{a_h}.
\]
We just recall the definition of convergence defined in [22] which here we will call \( \tilde{\text{PG}} \)-convergence. Let \( a_h \) \((h = 1, 2, \ldots)\) and \( a \) be matrices in \( \mathcal{N}_{\Omega \times (0,T)}(L, M, K, C) \). With the notation as in Definition 5.5, we say that the sequence \( (a_h)_h \) \( \tilde{\text{PG}} \)-converges to \( a \) in \( \Omega \times (0,T) \), and we write
\[
a_h \overset{\text{PG}}{\to} a \quad \text{in} \quad \Omega \times (0, T),
\]
if for every $f \in L^2(0,T;L^2(\Omega))$ it results that
\[
\begin{align*}
    u_h &\to u \quad \text{in } L^2(0,T;L^2(\Omega)) \\
    a_h \cdot Du_h &\to a \cdot Du \quad \text{in } L^2(0,T;L^1(\Omega))^{n} - \text{weak},
\end{align*}
\]
where $u_h$ and $u$ denote respectively the solutions of
\[
\begin{align*}
    &\left\{ \begin{array}{l}
        Q_{a_h} v = f \quad \text{in } \mathcal{Y}_{\lambda h} \\
        v \in Z^{0}_{\lambda h}
    \end{array} \right. \\
    &\left\{ \begin{array}{l}
        Q_a v = f \quad \text{in } \mathcal{Y}_{\lambda} \\
        v \in Z^{0}_{\lambda}
    \end{array} \right.
\end{align*}
\]
\hspace{1cm} (39)

In this framework it is easy to prove the following lemma, analogous to Lemma 5.2.

**Lemma 5.7.** Consider a sequence $f, (f_h)_h \subset L^2(0,T;L^2(\Omega))$ and suppose
\[
f_h \to f \quad \text{in } L^2(0,T;L^2(\Omega)).
\]

Consider a sequence of matrices $a, a_h, h = 1, 2, \ldots$, such that $a_h \tilde{P}G$-converge to $a$. Then, given $u_h$ and $u$ the solutions to the problems (39) we have that
\[
\begin{align*}
    u_h &\to u \quad \text{in } L^2(0,T;L^2(\Omega)) \quad \text{and} \\
    a_h \cdot Du_h &\to a \cdot Du \quad \text{in } L^2(0,T;L^1(\Omega))^{n} - \text{weak}.
\end{align*}
\]

The following theorem also follows at once, following the proofs of Proposition 5.3 and Theorem 5.6.

**Theorem 5.8.** Consider a sequence $(a_h)_h \subset N_{\Omega}(L,M,K,C)$. Then
\[
a_h \tilde{P}G \to a \quad \text{in } \Omega \times (0,T) \quad \text{if and only if} \quad a_h \to a \quad \text{in } \Omega.
\]

As a final result we state a corollary of Theorem 5.6 and Theorem 5.8.

**Corollary 5.9.** Consider a sequence $(a_h)_h \subset N_{\Omega \times (0,T)}(L,M,K,C)$. If $a_h = a_h(x)$ then
\[
a_h \to a \quad \text{in } \Omega, \quad a_h \tilde{P}G \to a \quad \text{in } \Omega \times (0,T), \quad a_h \to a \quad \text{in } \Omega \times (0,T)
\]
are equivalent.

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**References**


