Journal of Convex Analysis
Volume 9 (2002), No. 1, 259–268

Coincidence Theorems for Convex Functions

Joël Benoist
Faculté des Sciences, LACO, URA CNRS 1586, Université de Limoges,
123 avenue Albert Thomas, 87060 Limoges Cedex, France
benoist@unilim.fr

Aris Daniilidis*
Laboratoire de Mathématiques Appliquées, CNRS ERS 2055,
Université de Pau et des Pays de l'Adour, Av. de l'Université, 64000 Pau, France
aris.daniilidis@univ-pau.fr

Received June 8, 2000
Revised manuscript received May 21, 2001

The paper aims at creating a new insight into our perception of convexity by focusing on two fundamental problems: the coincidence of two functions (at least one being convex) upon an information on a dense set and the clarification of the relation between convexity and Fenchel subdifferential. Various results are established into these directions. Several examples are also illustrated showing that some rather unexpected situations can often occur.

Keywords: Convex function, coincidence, subdifferential

1991 Mathematics Subject Classification: 52A41, 49J52, 26E99

1. Introduction

If X is a Banach space, would you say that two lower semicontinuous (in short lsc) convex functions $g^1, g^2 : X \to \mathbb{R} \cup \{+\infty\}$, equal on a dense subset of their domain, necessarily coincide? In other words, can we determine a lsc convex function if we know its values on a dense set? In Proposition 3.4, we show that in infinite dimensions the answer is negative even if it is assumed that $g^1 \leq g^2$ and that both functions are positively homogeneous. Motivated by these considerations, we introduce the class $\mathcal{G}_1(X)$ of lsc convex functions g that do not admit any non trivial lsc majorant f coinciding with g on a dense subset of $\operatorname{dom} g$. We show that $\mathcal{G}_1(\mathbb{R}^d)$ coincides with the set of lsc convex functions (see Corollary 3.7). This is not the case in infinite dimensions, since - as we prove in Theorem 3.8 - a lsc convex function with a dense domain belongs to the class $\mathcal{G}_1(X)$ if, and only if, its domain is equal to X.

We also investigate the relation between Fenchel subdifferential and convexity by focusing on the following question: given a lsc function $f : X \to \mathbb{R} \cup \{+\infty\}$, can we conclude that f is necessarily convex whenever the domain of the Fenchel subdifferential of f is dense in X? This assertion is true in finite dimensions but fails impressively as soon as we consider infinite dimensional spaces (see Proposition 4.3). An interesting relevant question is the following. Given a lsc convex function g and a lsc function f such that the domain of the Fenchel subdifferential of f is dense in the domain of g and such that the closed convex

*The research of the second author was supported by the TMR post-doctoral grant ERBFMBI CT 983381.

ISSN 0944-6532 / $ 2.50

envelope of f is equal to g, can we conclude that $f = g$? If $\mathcal{G}_2(X)$ denotes the class of lsc convex functions g for which the latter conclusion is true for all f, we show in Proposition 4.5 that $\mathcal{G}_2(X)$ strictly contains $\mathcal{G}_1(X)$. Finally we obtain the analogue of Theorem 3.8 for the class $\mathcal{G}_2(X)$. More precisely, we show in Theorem 4.8 that a lsc convex and positively homogeneous function with a dense domain belongs to the class $\mathcal{G}_2(X)$ if, and only if, its domain is equal to X.

2. Preliminaries

In the sequel, let $\mathbb{N}$ denote the set of strictly positive integers, X a Banach space and X^* its dual. For any $x \in X$ and $p \in X^*$ we denote by $\langle p, x\rangle$ the value of p at x. Given a function $f : X \to \mathbb{R} \cup \{+\infty\}$, we denote by $\operatorname{dom} f = \{x \in X : f(x) \in \mathbb{R}\}$ its domain. Throughout this article, we shall deal with proper (i.e. $\operatorname{dom} f$ is nonempty) and lsc functions. Let us recall that the Fenchel subdifferential ∂f of any such function f at the point $x \in \operatorname{dom} f$ is defined as follows

$$\partial f(x) = \{p \in X^* : f(y) - f(x) \geq \langle p, y - x\rangle, \ \forall y \in X\}. \tag{1}$$

If $x \notin \operatorname{dom} f$, we set $\partial f(x) = \emptyset$. We denote by $\operatorname{dom} \partial f = \{x \in X : \partial f(x) \neq \emptyset\}$ the domain of the subdifferential of f. We recall that whenever f is convex, $\operatorname{dom} \partial f$ is dense in $\operatorname{dom} f$. In fact, the following result holds (see for example [5, Theorem 3.17]).

Proposition 2.1. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc convex function. Then* $\operatorname{dom} \partial f$ *is f-graphically dense in* $\operatorname{dom} f$*, that is, for every $x \in \operatorname{dom} f$, there exists a sequence $\{x^n\}$ in* $\operatorname{dom} \partial f$ *converging to x such that the sequence $\{f(x^n)\}$ converges to $f(x)$.*

Let us further recall that the closed convex hull of the function f, denoted $\overline{co}\, f$, is defined as the greatest lsc convex function majorized by f. It is well known that its epigraph coincides with the closed convex hull of the epigraph of f. Let us remark that $\overline{co}\, f$ takes its values in $\mathbb{R} \cup \{+\infty\}$ if, and only if, there exists an affine continuous function minimizing f on X. It is known (see [4] for example) that for a proper lsc function $f : X \to \mathbb{R} \cup \{+\infty\}$ one has $f = \overline{co}\, f$ on $\operatorname{dom} \partial f$. This fact can be seen in the following simple way. Fix $x \in \operatorname{dom} \partial f$, take any $p \in \partial f(x)$ and consider the function $g = p + (f(x) - \langle p, x\rangle)$. It follows from (1) that $f \geq g$. Since g is lsc convex, we get that $f \geq \overline{co}\, f \geq g$. The result follows from the observation that $f(x) = g(x)$.

3. Uniquely determined majorants of convex functions

In this section we are interested in the question of determination of a convex function, by means of an information for its values on a dense set.

A relevant, but more specific question is the following. Let $g : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc convex function. If f is lsc and D is a dense subset of $\operatorname{dom} g$ then

$$\left.\begin{array}{r} f \geq g \\ \\ f = g \text{ on } D \end{array}\right\} \implies f = g\ ? \tag{2}$$

Let us first tackle (2) for the special case where $D = \operatorname{dom} \partial g$.

Lemma 3.1. *Let $g : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc convex function. Then for every lsc function f satisfying $f \geq g$ we have*

$$f\,|_{\operatorname{dom}\partial g} = g\,|_{\operatorname{dom}\partial g} \implies f = g.$$

Proof. Since $f \geq g$, it clearly suffices to show that f coincides with g on $\operatorname{dom} g$. By Proposition 2.1, for every $x \in \operatorname{dom} g$, there exists a sequence $\{x^n\}$ in $\operatorname{dom}\partial g$ converging to x such that the sequence $\{g(x^n)\}$ converges to $g(x)$. By our hypothesis, $f(x^n) = g(x^n)$ for all n. Since f is lsc at x, letting $n \to +\infty$ this last equality yields $f(x) \leq g(x)$. Recalling that $f \geq g$, we conclude that $f(x) = g(x)$. □

We now have the following proposition.

Proposition 3.2. *Let $g : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc convex function and let D be a subset of* $\operatorname{dom} g$ *which is g-graphically dense in* $\operatorname{dom}\partial g$ *(that is, any point in* $\operatorname{dom}\partial g$ *is the g-graphical limit of a sequence in D). Then for every lsc function f satisfying $f \geq g$ we have*

$$f\,|_D = g\,|_D \implies f = g.$$

Proof. In view of Lemma 3.1, it suffices to show that f coincides with g on $\operatorname{dom}\partial g$. Let any $x \in \operatorname{dom}\partial g$. Then there exists a sequence $\{x^n\}$ in D converging to x with $\{g(x^n)\}$ converging to $g(x)$. Then $f(x^n) = g(x^n)$ for all $n \geq 1$, and letting $n \to +\infty$ we obtain $f(x) \leq g(x)$, since f is lsc at x. Recalling that $f \geq g$, we conclude that $f(x) = g(x)$. □

The following corollary is a direct consequence of Proposition 3.2.

Corollary 3.3. *Let $g : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc convex function such that $g\,|_{\operatorname{dom}\partial g}$ is continuous and let D be a dense subset of* $\operatorname{dom}\partial g$. *Then for every lsc function f satisfying $f \geq g$ we have*

$$f\,|_D = g\,|_D \implies f = g.$$

Remark.

1. As observed by the referee, the assumptions of Lemma 3.1 (as well as those of Proposition 3.2 and Corollary 3.3) yield $\partial g(x) \subset \partial f(x)$, for all $x \in X$. Thanks to the convexity of g, this inclusion still holds even if ∂ is remplaced by any abstract subdifferential in the sense of [8, page 35], which relates directly to results concerning integration of subdifferentials (see [6], [8], [9], [3] and references therein, as well as [7, Theorem 24.9] for the convex case).

2. The assumption "$g\,|_{\operatorname{dom}\partial g}$ is continuous" adapted in Corollary 3.3 is strictly weaker than the relative continuity of g on $\operatorname{dom} g$. We refer to [2] for further details and a dual characterization of this property.

The above results establish positive answers for the assertion (2) provided that the dense set D satisfies certain conditions. The following proposition shows that without these conditions, assertion (2) may fail even if both functions g^1 and g^2 are convex and positively homogeneous. As usual, $\ell^2(\mathbb{N})$ denotes the Hilbert space of square summable sequences and $\{e^i\}$ the canonical basis.

Proposition 3.4. *There exist two distinct positively homogeneous lsc convex functions $g^1, g^2 : \ell^2(\mathbb{N}) \to \mathbb{R} \cup \{+\infty\}$ such that $g^1 \leq g^2$ and $g^1 = g^2 < +\infty$ on an open dense subset of $\ell^2(\mathbb{N})$.*

Proof. Consider the functions $g^1, g^2 : \ell^2(\mathbb{N}) \to \mathbb{R}\cup\{+\infty\}$ defined for any $x = (x_i) \in \ell^2(\mathbb{N})$ as follows:

$$g^1(x) = \sup_{i\geq 1}\{\, i\,|x_i|\,\} \quad \text{and} \quad g^2(x) = \max\ \{\ 2|x_1|\ ,\ g^1(x)\ \}.$$

It follows directly that both functions are lsc convex and positively homogeneous and that $g^1 \leq g^2$. Let us also note that $g^1(e^1) \neq g^2(e^1)$ and consequently $g^1 \neq g^2$. Moreover, it is easily seen that $g^1 = g^2 < +\infty$ on the subset

$$D = \{(x_i) \in \ell^2(\mathbb{N}) :\ \exists i \in \mathbb{N}, |\,x_1\,| < \ \frac{i}{2}\,|\,x_i\,|\}.$$

Since D is an open dense subset of $\ell^2(\mathbb{N})$, the assertion is established. □

In the sequel we consider the question (2) globally, in the sense that we are interested to lsc convex functions g for which the assertion holds true simultaneously for all dense subsets D of $\operatorname{dom} g$. This class is introduced in the following definition.

Definition 3.5. We say that a proper lsc convex function $g : X \to \mathbb{R} \cup \{+\infty\}$ belongs to the class $\mathcal{G}_1(X)$, if g does not admit any non trivial lsc majorant that coincides with g on a dense set of $\operatorname{dom} g$.

Proposition 3.2 guarantees that $\mathcal{G}_1(X)$ contains all convex continuous functions on X. The following result shows that, more generally, $\mathcal{G}_1(X)$ contains also all lsc convex functions g with $\operatorname{int}\operatorname{dom} g \neq \emptyset$.

Proposition 3.6. *If $g : X \to \mathbb{R} \cup \{+\infty\}$ is a proper lsc convex function satisfying $\operatorname{int}\operatorname{dom} g \neq \emptyset$, then $g \in \mathcal{G}_1(X)$.*

Proof. Let f be a lsc majorant of g such that $f = g$ on a dense subset D of $\operatorname{dom} g$. Let $D_1 = D \cap \operatorname{int}\operatorname{dom}\,\partial g$. Thanks to the continuity of g on $\operatorname{int}\operatorname{dom}\, g$ it is easily seen that D_1 is g-graphically dense in $\operatorname{dom}\partial g$, whence $f = g$, in view of Proposition 3.2. □

Taking the relative interior, one gets the following corollary.

Corollary 3.7. *$\mathcal{G}_1(\mathbb{R}^d)$ coincides with the class of all proper lsc convex functions.*

The following theorem gives information on $\mathcal{G}_1(X)$ in the infinite dimensional case.

Theorem 3.8. *Let $g : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc convex function with a dense domain. Then $g \in \mathcal{G}_1(X)$ if, and only if, $\operatorname{dom} g = X$.*

Proof. Suppose that $\operatorname{dom} g \neq X$. Since $\operatorname{dom} g$ is convex and dense in X, it follows easily that $\operatorname{int}\operatorname{dom} g = \emptyset$. Hence g takes at least one infinite value in every neighborhood of each point of X. Since the function g is lsc at every x in X, we conclude that for every integer n the set

$$D_n = \{x \in \operatorname{dom} g : g(x) \geq n\}$$

is dense in X. Let us now take $n > \inf g$ and consider the lsc function $f : X \to \mathbb{R}\cup\{+\infty\}$ defined by

$$f(x) = \max\{g(x), n\}.$$

It is easily seen that f violates (2), i.e. $f \geq g$, $f = g$ on the dense set D_n and $f \neq g$. Hence $g \notin \mathcal{G}_1(X)$, which proves the "necessity" part. The "sufficiency" part is a direct consequence of Proposition 3.6, since $\operatorname{dom} g = X$ implies that $\operatorname{dom} \partial g = X$. $\square$

4. Fenchel subdifferential and convexity

In this section we investigate the relation between Fenchel subdifferential and convexity. The central question in this section is whether the non-emptiness of the Fenchel subdifferential of a lsc function f on a dense set guarantees the convexity of f.

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc function. If we assume that f is convex, then Proposition 2.1 asserts that $\operatorname{dom} \partial f$ is dense in $\operatorname{dom} f$. Let us observe that the converse assertion is not true even if $X = \mathbb{R}$ as shows the example below:

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or } 1; \\ \\ +\infty & \text{elsewhere.} \end{cases}$$

Indeed, the function f is obviously lsc, non-convex and $\operatorname{dom} f = \operatorname{dom} \partial f = \{0, 1\}$.

However, assuming that $\operatorname{dom} f$ is convex, the following proposition ensures the converse in finite dimensions.

Proposition 4.1. *Let $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc function with a convex domain. Then f is convex if, and only if, $\operatorname{dom} \partial f$ is dense in $\operatorname{dom} f$.*

Proof. The "necessity" part follows from Proposition 2.1. For the "sufficiency" part, assume that f is lsc and that $D = \operatorname{dom} \partial f$ is dense in $\operatorname{dom} f$. Setting $g = \overline{co} f$ we obviously have that g is lsc convex, $f \geq g$ and $f \mid_D = g \mid_D$. Since $\operatorname{dom} f$ is convex, it follows that $\operatorname{dom} f$ (and subsequently also $\operatorname{dom} \partial f$) is dense in $\operatorname{dom} g$. This finishes the proof in view of Corollary 3.7. $\square$

In case where the lsc function f has a dense domain in X, the converse assertion becomes:

$$\text{if } \operatorname{dom} \partial f \text{ is dense in } X, \text{ is } f \text{ convex ?} \tag{3}$$

This question was first considered in [1] where the following positive result was established (see [1, Section 3]).

Proposition 4.2. *Let f be a lsc function such that $\operatorname{dom} \partial f$ is dense in X. Suppose that at least one of the conditions (a), (b) or (c) is satisfied:*

(a) $X = \mathbb{R}^d$;

(b) $\operatorname{dom} f = X$;

(c) *∂f has a locally bounded selection on $\operatorname{dom} \partial f$, i.e., for every $x \in X$ there exist $M > 0$ and $r > 0$ such that for all $y \in \operatorname{dom} \partial f$,*

$$\|y - x\| \leq r \implies \exists p \in \partial f(y) : \|p\| \leq M.$$

Then f is a convex continuous function.

The following proposition completes the above results by exhibiting an example showing that, in infinite dimensions, (3) is not true without additional assumptions. Let us note that in the forthcoming example the domain of the constructed nonconvex function f is convex as was the case in Proposition 4.1.

Proposition 4.3. *In the Banach space $X = \ell^2(\mathbb{N})$, there exists a proper lsc non-convex function with a dense domain of Fenchel subdifferential.*

Proof. Let us consider the functions $g, f : \ell^2(\mathbb{N}) \to \mathbb{R}\cup\{+\infty\}$ defined for $x = (x_i) \in \ell^2(\mathbb{N})$ by

$$g(x) = \sup_{i\neq 1} \{ \, i \, |x_i| \, \} \quad \text{and} \quad f(x) = \max\left\{\chi_{(-1,1)}(x_1), \, g(x)\right\},$$

where $\chi_{(-1,1)}(x_1) = 0$ if $x_1 \in (-1, 1)$, and $\chi_{(-1,1)}(x_1) = 1$ if not. The functions f, g are clearly lsc and g is convex. Considering the restriction of f to the one-dimensional subspace $\mathbb{R}e^1$ we deduce that f is not convex. Set

$$U = \{(x_i) \in \ell^2(\mathbb{N}) : \exists i \in \mathbb{N}, |\, x_i \,| > \frac{1}{i}\}.$$

We obviously have

$$f \geq g \geq 0 \quad \text{and} \quad f \,|_U = g \,|_U \, . \tag{4}$$

Let us remark that the domain of g contains the subspace of almost everywhere null sequences, thus it is dense in $\ell^2(\mathbb{N})$. Since g is a lsc convex function, it follows that $\operatorname{dom} \partial g$ is dense in $\operatorname{dom} g$, and consequently also in $\ell^2(\mathbb{N})$. On the other hand, we conclude from (4) that $\operatorname{dom} \partial g \cap U \subset \operatorname{dom} \partial f \cap U$. Since U is open and dense, it follows that $\operatorname{dom} \partial f$ is dense in X. ☐

Motivated by the above example, let us consider the following general problem. Given a lsc convex function $g : X \to \mathbb{R} \cup \{+\infty\}$ and a lsc function $f : X \to \mathbb{R} \cup \{+\infty\}$, is it true that

$$\left.\begin{array}{r} \overline{co} f = g \\ \\ \operatorname{dom} \partial f \text{ dense in } \operatorname{dom} g \end{array}\right\} \implies f = g\,? \tag{5}$$

Analogously to Definition 3.5, we introduce the following class of lsc convex functions.

Definition 4.4. We say that a lsc convex function $g : X \to \mathbb{R} \cup \{+\infty\}$ belongs to the class $\mathcal{G}_2(X)$, if g does not admit any non trivial lsc f such that $\overline{co}\, f = g$ and $\operatorname{dom} \partial f$ is dense in $\operatorname{dom} g$.

It is easily seen that assertion (5) is related with (2). In particular, if $g \in \mathcal{G}_1(X)$, then obviously (5) holds and $g \in \mathcal{G}_2(X)$. Hence, for any Banach space X, we have

$$\mathcal{G}_1(X) \subset \mathcal{G}_2(X). \tag{6}$$

In finite dimensions, in view of Corollary 3.7, $\mathcal{G}_1(X) = \mathcal{G}_2(X)$ and both classes coincide with the class of all proper lsc convex functions. The following proposition shows that inclusion (6) can be strict in infinite dimensions.

Proposition 4.5. *Consider the lsc convex function $g : \ell^2(\mathbb{N}) \to \mathbb{R} \cup \{+\infty\}$ defined for every $x = (x_i) \in \ell^2(\mathbb{N})$ by*

$$g(x) = \sum_{i=1}^{+\infty} |\, x_i \,|^{3/2} \, .$$

Then $g \in \mathcal{G}_2(X)$, but $g \notin \mathcal{G}_1(X)$. More precisely, for every lsc function $f : \ell^2(\mathbb{N}) \to \mathbb{R} \cup \{+\infty\}$, the following implication is satisfied

$$\overline{co} f = g \Longrightarrow f = g.$$

Proof. Let us observe that

$$g(x) = \sum_{i=1}^{+\infty} \varphi(x_i), \tag{7}$$

where the function $\varphi : \mathbb{R} \to \mathbb{R}$ defined for all $t \in \mathbb{R}$ by $\varphi(t) = |\,t\,|^{3/2}$ is convex, differentiable on $\mathbb{R}$ and twice continuously differentiable on $\mathbb{R} \setminus \{0\}$ with $\varphi''(t) > 0$, for all $t \neq 0$. Clearly the function g is lsc and convex. Since $\varphi(0) = 0$, the domain of g contains the subspace of almost everywhere null sequences, thus it is dense in $\ell^2(\mathbb{N})$. Considering the sequence $x = \left(\sqrt[3]{(\frac{1}{i})^2}\right)_{i \geq 1}$, we conclude that $\operatorname{dom} g \neq \ell^2(\mathbb{N})$. Hence according to Theorem 3.8 we have $g \notin \mathcal{G}_1(X)$.

Let us now prove that $g \in \mathcal{G}_2(X)$. To this end, let $f : \ell^2(\mathbb{N}) \to \mathbb{R} \cup \{+\infty\}$ be a lsc function satisfying $\overline{co} f = g$ (the other assumption in (5) that $\operatorname{dom} \partial f$ is dense in $\operatorname{dom} g$ will be superfluous in the sequel). By Lemma 3.1, in order to show that $f = g$ it suffices to ensure that $f\,|_{\operatorname{dom} \partial g} = g\,|_{\operatorname{dom} \partial g}$.

Let us show that for every $x \in \operatorname{dom} \partial g$ and $p = (p_i) \in \partial g(x)$, there exists $m > 0$ such that for all $u \in X$ with $\|u\| \leq 1$ we have

$$g(x + u) \geq g(x) + \langle p, u \rangle + m\ \|u\|^2. \tag{8}$$

Indeed, let us fix $x \in \operatorname{dom} \partial g$ and $p = (p_i) \in \partial g(x)$. Applying (1) for $y = x + te^i$ (where $t \in \mathbb{R}$ and $\{e^i\}$ is the canonical basis), we have $\varphi(x_i + t) \geq\ \varphi(x_i) +\ p_i t$. This inequality shows that $\ p_i \in \partial\varphi(x_i)$. Since φ is convex and differentiable we conclude that

$$p_i = \varphi'(x_i). \tag{9}$$

Choose $m = \frac{1}{2} \min \{\varphi''(t) :\ \ |t|\ \ \leq \|x\| + 1,\ t \neq 0\}$; obviously $m > 0$. Consider now any $u = (u_i) \in \ell^2(\mathbb{N})$ with $\|u\| \leq 1$. Then for all $i \geq 1$ a direct calculation gives

$$\varphi(x_i + u_i) = \varphi(x_i) + u_i\varphi'(x_i) + \int_{x_i}^{x_i+u_i} (x_i + u_i - t)\varphi''(t)\ dt, \tag{10}$$

where the above integral is defined as the sum of the generalized integrals $\lim\limits_{\delta \to 0^+} \int_{x_i}^{-\delta} (x_i{+}u_i{-}t)$ $\varphi''(t)\ dt$ and $\lim\limits_{\delta \to 0^+} \int_{\delta}^{x_i+u_i} (x_i + u_i - t)\ \varphi''(t)\ dt$ in case that $x_i \leq 0 \leq x_i + u_i$ (and analogously, if $x_i + u_i \leq 0 \leq x_i$). (Note that if $x_i(x_i + u_i) > 0$, then (10) is nothing but the Taylor's integration formula for the twice differentiable function φ on the segment $[x_i, x_i + u_i]$ (respectively on $[x_i + u_i, x_i]$, if $u_i < 0$)).

Let us remark that for all $i \geq 1$

$$\int_{x_i}^{x_i+u_i} (x_i + u_i - t)\varphi''(t)\ dt \geq 2m \int_{x_i}^{x_i+u_i} (x_i + u_i - t)\ dt = m\ u_i^2.$$

Combining with (10) we obtain

$$\varphi(x_i + u_i) \geq \varphi(x_i) + u_i\varphi'(x_i) + m\ u_i^2. \tag{11}$$

Adding the above inequalities for all i and recalling (7) and (9), we obtain

$$g(x+u) \geq g(x) + \langle p, u\rangle + m\ \|u\|^2.$$

Hence (8) holds and the proof finishes in view of the following lemma.

Lemma 4.6. *Let g be a proper lsc convex function on X and $x \in X$. Suppose that there exist $p \in X^*$, $r > 0$ and $m > 0$ such that for all $u \in X$ with $\|u\| \leq 1$ we have*

$$g(x+u) \geq g(x) + \langle p, u\rangle + m\ \|u\|^r. \tag{12}$$

Then, for every lsc function f such that $f \geq g$, the following implication is satisfied:

$$f(x) > g(x) \implies \overline{co} f \neq g. \tag{13}$$

Proof. Let us set

$$G(u) = g(x+u) - g(x) - \langle p, u\rangle \quad \text{and} \quad F(u) = f(x+u) - g(x) - \langle p, u\rangle.$$

Thus G is a lsc convex function with $G(0) = 0$, F is a lsc function with $F(0) > 0$ and $F \geq G$. Moreover, relation (12) becomes:

$$m\ \|u\|^r \leq G(u), \tag{14}$$

for all $u \in X$ with $\|u\| \leq 1$. To prove the assertion of the lemma, it suffices to show that $\overline{co}F \neq G$. Let us remark that if $x \in X$ with $\|x\| > 1$, we have

$$G\left(\frac{x}{\|x\|}\right) \leq \frac{1}{\|x\|}G(x) + (1 - \frac{1}{\|x\|})G(0) = \frac{1}{\|x\|}G(x),$$

since G is convex. This shows, in view of (14), that for all $\|x\| > 1$

$$m\|x\| \leq G(x). \tag{15}$$

Let us now prove that there exists $a > 0$ such that $F \geq a$. Indeed, if this were not the case, there would exist a sequence $\{x^n\}$ in X such that for all $n \geq 1$

$$F(x^n) < \frac{1}{n}. \tag{16}$$

Since $G(x^n) \leq F(x^n)$ we conclude thanks to (15) that for n large enough $\|x^n\| \leq 1$. Thus (14) yields $m\|x^n\|^r < \frac{1}{n}$. It follows that the sequence $\{x^n\}$ converges to 0. Letting $n \to \infty$ in (16), and using the fact that F is lsc we conclude that $F(0) \leq 0$. This contradiction shows that what asserted is true.

Thus $F \geq a$, which implies that $\overline{co}F \geq a$. Consequently $\overline{co}F$ and G cannot coincide at 0. This finishes the proof. $\square$

Remark 4.7. It is easily seen that, more generally, the lsc convex function $g : \ell^2(\mathbb{N}) \to \mathbb{R} \cup \{+\infty\}$ defined for all $x = (x_i) \in \ell^2(\mathbb{N})$ by $g(x) = \sum_{i=1}^{+\infty} | x_i |^p$ $(1 < \mathrm{p} < 2)$ satisfies the conclusion of Proposition 4.5.

The following result, analogous to Theorem 3.8, deals with the case of lsc positively homogeneous convex functions g with dense domain.

Theorem 4.8. *Let $g : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc positively homogeneous convex function with a dense domain. Then $g \in \mathcal{G}_2(X)$ if, and only if,* $\operatorname{dom} g = X$.

Proof. Suppose that $\operatorname{dom} g \neq X$ and let us prove that $g \notin \mathcal{G}_2(X)$. For that, let us consider the function $f : X \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f(x) = \begin{cases} \max \ \{g(x), 1\} & \text{if } g(x) > 0; \\ \\ g(x) & \text{if } g(x) \leq 0. \end{cases}$$

It is directly seen that $f \geq g$, which implies $\overline{co} f \geq g$ since g is convex, and that for all $x \in X$

$$f(x) = g(x) \Longleftrightarrow x \in D,$$

where $D = \{ \ x \in X : g(x) \notin \ (0, 1) \ \}$.

Let us show that $\overline{co} f = g$. This equality holds clearly on D. Let us now consider any $x \notin D$ and let us set $x^1 = \frac{1}{g(x)} x$, $x^2 = 0$ and $\lambda = g(x) \in \ (0, 1)$. Since g is positively homogeneous, $g(x^1) = 1$ and $g(x^2) = 0$ which implies that both points x^1 and x^2 belong to D. Thus $\overline{co} f(x^i) = g(x^i)$ for $i = 1, 2$. Since the functions g and $\overline{co} f$ are convex, and since $x = \lambda x^1 + (1 - \lambda) x^2$ we deduce

$$\overline{co} f(x) \leq \lambda \overline{co} f(x^1) + (1 - \lambda) \overline{co} f(x^2) = \lambda g(x^1) + (1 - \lambda) g(x^2) = g(x).$$

It follows that $\overline{co} f(x) = g(x)$.

Let us show that f is lsc. Since g is lsc, $f \geq g$ and $f \mid_D = g \mid_D$, the function f is obviously lsc at every point of D. Let now any $x \notin D$ and let $\{x^n\}$ be a sequence in X converging to x. Since g is lsc at x and $g(x) > 0$, we have $g(x^n) > 0$ for n large enough, and consequently $f(x^n) = \max\{g(x^n), 1\} \geq 1 = f(x)$. This shows that f is lsc at x.

Let us show that $\operatorname{dom} \partial f$ is dense in $\operatorname{dom} g$. To this end, let us consider any $x \in \operatorname{dom} g$. Since g takes arbitrarily large finite values around x (see the proof of Theorem 3.8), there exists a sequence $\{x^n\}$ in $\operatorname{dom} g$ such that $g(x^n) > 2$ and $\|x^n - x\| < \frac{1}{n}$ for all $n \geq 1$. Using Proposition 2.1, we obtain a sequence $\{y^n\}$ in $\operatorname{dom} \partial g$ such that $g(y^n) > 2$ and $\|x^n - y^n\| < \frac{1}{n}$ for all n. Since $f \geq g$ and $f(y^n) = g(y^n)$ for all n, by definition of the Fenchel subdifferential we have $\partial g(y^n) \subset \partial f(y^n)$ for all n. It follows that the sequence $\{y^n\}$ is included in $\operatorname{dom} \partial f$. Since the sequence $\{y^n\}$ converges to x, it follows that $\operatorname{dom} \partial f$ is dense in $\operatorname{dom} g$.

Since g is positively homogeneous and takes finite arbitrarily large (hence in particular positive) values, it follows that $D \neq X$, whence $f \neq g$. Thus $g \notin \mathcal{G}_2(X)$, which proves the "necessity" part.

The "sufficiency" part is a direct consequence of Theorem 3.8 and inclusion (6). $\square$

Remark 4.9. The example of the function given in Proposition 4.5 shows that the assumption "g is positively homogeneous" is indispensable in Theorem 4.8.

References

[1] M. Bachir, A. Daniilidis, J.-P. Penot: Lower subdifferentiability and integration, Set-Valued Anal. (to appear).

[2] J. Benoist, A. Daniilidis: Dual characterizations of relative continuity of convex functions, J. Austral. Math. Soc. 70 (2001) 211–223.

[3] J. Borwein, Q. Zhu: Multifunctional and functional analytic techniques in nonsmooth analysis, in: Nonlinear Analysis, Differential Equations and Control (Montreal, QC, 1998), 61–157, NATO Sci. Ser. C, Math. Phys. Sci. 528, Kluwer Acad. Publ., Dordrecht (1999).

[4] J.-B. Hiriart-Urruty, C. Lemarechal: Convex Analysis and Minimization Algorithms I & II, Grundlehren der mathematischen Wissenschaften 306, Springer-Verlag, Berlin (1993).

[5] R. Phelps: Convex Functions, Monotone Operators and Differentiability, 2^{nd} Edition, Springer-Verlag, Berlin (1991).

[6] R. Poliquin: Integration of subdifferentials of nonconvex functions, Nonlinear Anal. 17 (1991) 385–398.

[7] R. T. Rockafellar: Convex Analysis, Princeton Mathematical Series 28, Princeton, N.J. (1970).

[8] L. Thibault, D. Zagrodny: Integration of subdifferentials of lower semicontinuous functions on Banach spaces, J. Math. Anal. Appl. 189 (1995) 33–58.

[9] Z. Wu, J. Ye: Some results on integration of subdifferentials, Nonlinear Anal. 39 (2000) 955–976.