Journal of Convex Analysis
Volume 9 (2002), No. 2, 601–607

Lipschitzian Behavior of the Fenchel Biconjugacy

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Received June 29, 2000
Revised manuscript received October 02, 2001

We present sufficient conditions ensuring that the Legendre-Young-Fenchel biconjugacy is locally Lipschizian. An application to the continuity of some operations on the space of subsets of an Euclidean space are presented.

Keywords: Bipolar, conjugacy, biconjugate, convex set, excess, Fenchel transform, Hausdorff distance, Minkowski difference, support function

1991 Mathematics Subject Classification: 26E25, 49J53, 52A20, 54C60, 52A05

In the sequel, unless otherwise stated, X is a finite dimensional Euclidean vector space of dimension d, with closed unit ball U and unit sphere S. We denote by H the set of positively homogeneous functions $h : X \to \mathbb{R}$ which are bounded on U. Clearly, H is a Banach space when endowed with the norm given by

$$\|h\| = \sup_{x \in U} |h(x)| .$$

The Legendre-Young-Fenchel conjugacy $f \mapsto f^*$, with

$$f^*(x^*) := \sup_{x \in X} \left(\langle x^*, x \rangle - f(x) \right),$$

is a fundamental tool of convex analysis. Its continuity for various topologies or convergences has been extensively studied during the last decades. One of the most classical convergences used in this context is epiconvergence, which corresponds to the Painlevé-Kuratowski convergence of the epigraphs of the functions [15]; in the infinite dimensional case, it has several variants: Mosco convergence, Joly convergence or slice convergence, bounded-hemi-convergence or Attouch-Wets convergence...). Here we are interested in the continuity of the biconjugacy $f \mapsto f^{**} = (f^*)^*$ for the topology induced by the norm of H. We limit our study to this case for the sake of simplicity and for the needs of the application we have in view. As a matter of fact, our study has been motivated by the hope of a simplified approach to a differentiability result of Silin [17]. This result relies on delicate estimates ([17, Lemma 2.20], [13]) which are by-passed when one uses the criteria for Lipschitz behavior of $h \mapsto h^{**}$ we present here.

1. Local Lipschitz behavior

Although the norm of H induces the usual dual norm on X^*, the topology associated to it does not corresponds to the topologies used in variational analysis. Let us compare it

ISSN 0944-6532 / \$ 2.50

with the topology of bounded Hausdorff (or bounded hemiconvergence). Recall that the latter is defined through the family of écarts

$$d_n(f,g) := \sup\{|d\left((x,r),\operatorname{epi} f\right) - d\left((x,r),\operatorname{epi} g\right)| : (x,r) \in X \times \mathbb{R},\ \|x\| \le n, |r| \le n\},$$

where $\operatorname{epi} f$ and $\operatorname{epi} g$ are the epigraphs of f and g respectively. Equivalently, setting for two nonempty subsets A, B of $X \times \mathbb{R}$ and $p > 0$

$$e_p(A,B) := \sup\{d(a,B) : a \in A \cap (pU \times [-p,p])\},$$

it can be shown that a net $(f_i)_{i\in I}$ in $\overline{\mathbb{R}}^X$ converges to $f \in \overline{\mathbb{R}}^X$ iff for each $p > 0$ one has $(e_p(\operatorname{epi} f_i, \operatorname{epi} f))_{i\in I} \to 0$, $(e_p(\operatorname{epi} f, \operatorname{epi} f_i))_{i\in I} \to 0$ (see [1] Prop. 2.1 for instance).

Lemma 1.1. *The topology induced on H by the norm defined above is stronger than the bounded Hausdorff topology. For each $k > 0$ the two topologies coincide on the subset H_k of $h \in H$ which are Lipschitzian with rate k.*

Proof. Let f, g in H. Then, for any $p > 0$ and any $(x,r) \in (\operatorname{epi} f) \cap (pU \times [-p,p])$ we have

$$\inf\{\max(\|x-y\|, |r-s|) : (y,s) \in \operatorname{epi} g\} \le |r - (g(x) + r - f(x))| \le p\,\|f-g\|,$$

hence $e_p(\operatorname{epi} f, \operatorname{epi} g) \le p\,\|f-g\|$ and, similarly, $e_p(\operatorname{epi} g, \operatorname{epi} f) \le p\,\|f-g\|$.

Now, given $k > 1$ and $f, g \in H_k$, for each $x \in U$ and each $\delta > e_k(\operatorname{epi} f, \operatorname{epi} g)$ we can find $(y,s) \in \operatorname{epi} g$ such that $\max(\|x-y\|, |f(x)-s|) < \delta$. Then

$$g(x) \le g(y) + k\delta \le s + k\delta \le f(x) + (k+1)\delta.$$

Similarly, for $\delta > e_k(\operatorname{epi} g, \operatorname{epi} f)$ and $x \in U$ we have $f(x) \le g(x) + (k+1)\delta$. Thus $\|f-g\| \le (k+1)\max(e_k(\operatorname{epi} f, \operatorname{epi} g), e_k(\operatorname{epi} g, \operatorname{epi} f))$. □

In the sequel we denote by H_+ the set of $h \in H$ which are *positive definite*, i.e. for which there exists some $\alpha > 0$ such that $h(x) \ge \alpha\,\|x\|$ for each $x \in X$. We define the *positivity rate* $\pi(h)$ of $h \in H_+$ as the supremum of the set of positive numbers α satisfying this condition. Equivalently,

$$\pi(h) = \inf h(S).$$

Proposition 1.2. *Let $h \in H$ be positive definite. Then there exists $\lambda, \rho > 0$ such the mapping $f \mapsto f^{**}$ is Lipschitzian with rate λ on the open ball $B(h,\rho)$ with center h and radius ρ. More precisely, for each $\rho \in]0, \pi(h)[$ and for $\lambda = (d+1)(\pi(h)-\rho)^{-1}(\|h\|+\rho)$ one has, for any $f, g \in U(h,\rho)$,*

$$\|f^{**} - g^{**}\| \le \lambda\,\|f-g\| \tag{1}$$

Proof. Let $\alpha := \pi(h) > 0$, so that $h(x) \ge \alpha\,\|x\|$ for each $x \in X$. Let $\lambda := (d+1)(\alpha - \rho)^{-1}(\|h\|+\rho)$. Let us prove the estimate (1). For any $\rho \in]0,\alpha[$ and for any $g \in H$ such that $\|g-h\| < \rho$, setting $\beta := \alpha - \rho$, one has $g(x) \ge \beta\,\|x\|$. Moreover,

$$\|g\| < \mu := \|h\| + \rho.$$

Given $r > \beta^{-1}\mu$, let us show that for each $x \in U$ one has

$$g^{**}(x) = \inf\left\{\sum_{i=0}^{d} g(x_i) :\ x_i \in rU,\ \sum_{i=0}^{d} x_i = x\right\}. \tag{2}$$

We first observe that the sublinear envelop g^c of g is majorized by g, hence is continuous; therefore, it coincides with g^{**}. Since its epigraph is the "vertical closure" of the convex hull of the epigraph of g, from the Caratheodory theorem, for each $x \in X$ we get that

$$g^{**}(x) = g^c(x) = \inf\{t : (x,t) \in \mathrm{co}(\mathrm{epi}\, g)\} = \inf\left\{\sum_{i=0}^{d} g(x_i) :\ x_i \in X,\ \sum_{i=0}^{d} x_i = x\right\}. \tag{3}$$

We observe that for any finite family $(x_0, ..., x_d) \in X^{d+1}$ such that $\sum_{i=0}^{d} x_i = x \in U$ and $x_j \in X \backslash rU$ for some $j \in \{0, ..., d\}$ we have

$$\sum_{i=0}^{d} g(x_i) \geq g(x_j) \geq \beta \|x_j\| \geq \beta r > \mu,$$

and since $g^{**}(x) \leq g(x) \leq \mu \|x\| \leq \mu$, we can discard such a family in (3). This proves (2). Now, for any $f, g \in U(h, \rho)$ and for any finite family $(x_0, ..., x_d) \in (rU)^{d+1}$ such that $\sum_{i=0}^{d} x_i = x$ we have

$$\sum_{i=0}^{d} g(x_i) \geq \sum_{i=0}^{d} (f(x_i) - r\|f-g\|) \geq \sum_{i=0}^{d} f(x_i) - (d+1)r\|f-g\|.$$

Taking the infimum over such families, it follows that

$$g^{**}(x) \geq f^{**}(x) - (d+1)r\|f-g\|.$$

Interchanging the roles of f and g, we get

$$|f^{**}(x) - g^{**}(x)| \leq (d+1)r\|f-g\| \qquad \forall x \in U.$$

Taking the supremum over $x \in U$ and the infimum over $r > \beta^{-1}\mu$ we get the announced estimate. □

The preceding result can be extended to the set $H_+ + X^*$ of positively homogeneous functions which are the sum of a continuous linear form and of a positive definite positively homogeneous function bounded on U. We call the elements of $H_+ + X^*$ *transdefinite positive functions*. Let us characterize such functions.

Lemma 1.3. *An element h of H is in H_+ iff h^* is non positive on some ball of X^* centered at 0. An element h of H is a transdefinite positive function iff h^* is non positive on some ball of X^*.*

Proof. It is easy to check that for any $h \in H$ one has $h^* = \iota_{S(h)}$ where $S(h) := \partial h(0) := \{x^* \in X^* : x^* \leq h\}$ and ι_A is the indicator function of $A \subset X^*$ given by $\iota_A(x^*) = 0$ if $x^* \in A$, $+\infty$ else. Given $h \in H_+$ let $\alpha > 0$ be such that $h \geq \alpha\|\cdot\|$. Then $h^* \leq \alpha\iota_{U^*}(\alpha^{-1}\cdot) = \iota_{\alpha U^*}$ where U^* is the closed unit ball of X^*. Conversely, if $h^* \leq 0$ on αU^* then $h \geq h^{**} \geq \alpha\|\cdot\|$.

The second assertion follows from the relation $(g + \ell)^*(x^*) = g^*(x^* - \ell)$ for $x^* \in X^*$. □

Corollary 1.4. *The biconjugacy $f \mapsto f^{**}$ is locally Lipschitzian on the set $H_+ + X^*$ of transdefinite positive functions.*

Proof. The result is an immediate consequence of the relations $f^{**} = (f - \ell)^{**} + \ell$, $g^{**} = (g - \ell)^{**} + \ell$. □

2. Applications

In the sequel we will apply the result of the preceding section to support functions. Recall that the support function h_C of a subset C of X is defined by

$$h_C(x^*) = \sup \{\langle x^*, x\rangle : x \in C\},$$

with the usual convention $\sup \emptyset = -\infty$. We will also use the Pompeiu-Hausdorff excesses of two nonempty subsets C, D of X given by

$$e(C, D) := \sup_{x \in C} d(x, D) \text{ where } d(x, D) := \inf_{y \in D} \|x - y\|$$

and the Pompeiu-Hausdorff distance given by

$$d(C, D) := \max(e(C, D), e(D, C)).$$

A well-known relation links the two quantities when C and D are closed convex [2], [9, Lemma 2.1], [16]; we recall it now and slightly extend it to a nonconvex situation, in an obvious way. Here for $r \in \mathbb{R}$, $s, t \in \overline{\mathbb{R}}$ we write $r \geq s - t$ if $r + t \geq s$; if s or t is finite the inequality $r \geq s - t$ has its usual meaning. We write $r \geq \sup_{i \in I}(s_i - t)$ if $r \geq s_i - t$ for each $i \in I$.

Lemma 2.1. *For any nonempty subsets C, D of X, one has*

$$e(C, D) \geq \sup\{h_C(x^*) - h_D(x^*) : x^* \in U\}.$$

If D is convex and bounded equality holds. If C and D are convex and bounded

$$d(C, D) = \sup\{|h_C(x^*) - h_D(x^*)| : x^* \in U\}.$$

Let us observe that there is an analogy between the biconjugate transform $f \mapsto f^{**}$ and the mapping $C \mapsto \overline{\text{co}}(C)$ which assigns to any subset C of X its closed convex hull. In fact, denoting by ι_C the indicator function of C (defined by $\iota_C(x) = 0$ if $x \in C$, $+\infty$ else) we have $\iota_C^{**} = \iota_{\overline{\text{co}}(C)}$. However, since ι_C is not positively homogeneous and definite positive in general, we cannot apply the result of the preceding section. However, one has the following elementary result which completes Proposition 1.2.

Proposition 2.2. *For any nonempty subsets C, D of a reflexive Banach space X one has $e(\overline{\text{co}}(C), \overline{\text{co}}(D)) \leq e(C, D)$ and $d(\overline{\text{co}}(C), \overline{\text{co}}(D)) \leq d(C, D)$.*

Proof. It suffices to prove the first inequality and to assume that $e(C, D) < \infty$. For any $r > e(C, D)$ we have $C \subset D + rU$. It follows from the convexity of U that $\text{co}(C) \subset \text{co}(D) + rU$. Since $\overline{co}(D) + rU$ is weakly closed, as easily deduced from the weak compactness of U, we get $\overline{co}(C) \subset \overline{co}(D) + rU$, hence $e(\overline{co}(C), \overline{co}(D)) \leq r$. The result follows by taking the infimum on $r > e(C, D)$. □

As an immediate consequence, we get the Lipschitz property of the bipolar operation. Recall that the polar C^0 of a subset C of X is given by

$$C^0 := \{x^* \in X^* : \forall x \in C \ \langle x^*, x\rangle \le 1\}$$

and that the bipolar C^{00} of C is $(C^0)^0 := \overline{co}(C \cup \{0\})$, by the bipolar theorem.

Corollary 2.3. *For any nonempty subsets C, D of a reflexive Banach space X one has $e(C^{00}, D^{00}) \le e(C, D)$ and $d(C^{00}, D^{00}) \le d(C, D)$.*

Proof. It suffices to apply the preceding proposition and to observe that $e(C, D) \ge e(C \cup \{0\}, D \cup \{0\})$, and $d(C, D) \ge d(C \cup \{0\}, D \cup \{0\})$. □

More generally, we have

Corollary 2.4. *For any nonempty subsets A, B, C, D of a reflexive Banach space X one has $e(\overline{co}(A \cup B), \overline{co}(C \cup D)) \le \max(e(A, C), e(B, D)$, and $d(\overline{co}(A \cup B), \overline{co}(C \cup D)) \le \max(d(A, C), d(B, D))$.*

Now let us turn to the study of the Minkowski difference of sets. Let us recall that the *Minkowski difference* of two subsets C, D of X is the set

$$C \overset{*}{-} D := \{x \in X : D + x \subset C\}.$$

It plays a role in differential games, in nonsmooth analysis, infinitesimal geometry and in d.c. optimization (see for instance [14], [4], [8], [11], [10] respectively). When C is convex, $C \overset{*}{-} D$ is convex; when C is closed, then $C \overset{*}{-} D$ is closed. We need the following observation noted in [3] under a slightly less precise formulation.

Lemma 2.5. *Let C and D be nonempty subsets of X, D being bounded. Let h_C and h_D denote the support functions of C and D respectively. Then the support function $h_{C \overset{*}{-} D}$ of $C \overset{*}{-} D$ satisfies $h_{C \overset{*}{-} D} \le (h_C - h_D)^{**}$. If C is closed convex equality holds.*

Proof. The inequality is trivial when $C \overset{*}{-} D$ is empty. For any $x \in C \overset{*}{-} D$ and any $x^* \in X^*$ we have $h_C(x^*) \ge h_D(x^*) + \langle x^*, x\rangle$, hence, taking the supremum over $x \in C \overset{*}{-} D$ and observing that h_D is finite-valued, $h_{C \overset{*}{-} D} \le h_C - h_D$. Since $h_{C \overset{*}{-} D}$ is closed proper convex, we get $h_{C \overset{*}{-} D} \le (h_C - h_D)^{**}$ and $(h_C - h_D)^{**}(0) = 0$. Conversely, suppose $(h_C - h_D)^{**}(0) = 0$ and let $x \in \partial(h_C - h_D)^{**}(0) = \partial(h_C - h_D)(0)$, i.e. $\langle \cdot, x\rangle \le h_C - h_D$ or $\langle \cdot, x\rangle + h_D \le h_C$; such an x exists since $(h_C - h_D)^{**}$ is the supremum of the continuous linear forms in $\partial(h_C - h_D)^{**}(0)$. Then, when C is closed convex, the Hahn-Banach theorem ensures that $x + D \subset C$, so that $x \in C \overset{*}{-} D$. Then, for each $x^* \in X^*$, we have

$$(h_C - h_D)^{**}(x^*) = \sup\{\langle x^*, x\rangle : x \in \partial(h_C - h_D)^{**}(0)\} \le h_{C \overset{*}{-} D}(x^*).$$

It follows that $(h_C - h_D)^{**}(0) = 0$ iff $C \overset{*}{-} D$ is nonempty; in the contrary case we have $(h_C - h_D)^{**}(0) = -\infty$ and the equality $h_{C \overset{*}{-} D} = (h_C - h_D)^{**}$ still holds. □

Proposition 2.6. *Let A and B be two nonempty bounded subsets of X, A being closed convex. Suppose there exist $z \in X$ and $r > 0$ such that A contains the r-enlargement $U(B,r) := B + z + rU$ of $B + z$. Then, for each $\rho \in]0, r/2[$, there exists $k > 0$ such that for any closed convex subsets C, C' and any convex subsets D, D' satisfying*

$$\max(d(C,A), d(C',A), d(D,B), d(D',B)) < \rho,$$

one has

$$d(C \overset{*}{-} D, C' \overset{*}{-} D') \leq \lambda(d(C,C') + d(D,D')).$$

Proof. Since $A \supset B + (z + rU)$, the preceding lemma ensures that

$$h_A(x^*) - h_B(x^*) \geq (h_A - h_B)^{**}(x^*) \geq h_{A \overset{*}{-} B}(x^*) \geq h_{z+rU}(x^*) = \langle x^*, z\rangle + r\,\|x^*\|\,.$$

Let $\lambda = (d+1)(r-2\rho)^{-1}(\alpha + \beta + \rho)$ where $A \subset \alpha U$, $B \subset \beta U$. Let $\rho \in]0, r/2[$ and let closed convex subsets C, C' and convex subsets D, D' satisfy

$$\max(d(C,A), d(C',A), d(D,B), d(D',B)) < \rho.$$

Lemma 2.1 yields

$$\begin{aligned} \|(h_A - h_B) - (h_C - h_D)\| &< 2\rho, \\ \|(h_A - h_B) - (h_{C'} - h_{D'})\| &< 2\rho. \end{aligned}$$

Since $\|h_A - h_B\| \leq \alpha + \beta$, Lemmas 2.1, 2.5 and Proposition 1.2 ensure that

$$\begin{aligned} d(C \overset{*}{-} D, C' \overset{*}{-} D') &= \left\| h_{C \overset{*}{-} D} - h_{C' \overset{*}{-} D'} \right\| \\ &= \|(h_C - h_D)^{**} - (h_{C'} - h_{D'})^{**}\| \\ &\leq \lambda\,\|(h_C - h_D) - (h_{C'} - h_{D'})\| \\ &\leq \lambda\,\|h_C - h_D\| + \lambda\,\|h_{C'} - h_{D'}\| \\ &\leq \lambda d(C,D) + \lambda d(C',D'). \end{aligned}$$

Remark. In the special case of $D = D'$, the estimate of λ given in the preceding proof can be improved. Let us note in particular that the estimate of [17, Lemma 2.23] for the special case $D = D' = r\alpha U$ is more precise. This fact is an incentive to improve Proposition 1.2 in such special cases. Tighter estimates of the Lipschitz rate of the Minkowski difference are provided in [12].

Acknowledgements. We are grateful to C. Zălinescu for his remarks about the manuscript; in particular we are indebted to him of a reformulation of Lemma 1.3. We also acknowledge the suggestion of an anonymous referee to extend the results of the present paper to nontraditional dualities as in [5]-[7]. This program will not be carried out here.

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