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Separation Theorems for Abstract Convex Structures

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Let S be a nonvoid set endowed with a "segment structure" which generalizes the notion of a segment in a linear space, and which allows to define affine functions. The problem is treated, whether a pair of sets $X, Y \subset S$ can be separated by some affine function $f : S \to \mathbb{R}$. Here separation means $f(y) \geq f(z), y \in Y, z \in Z$ in it's weakest and $\inf_{y \in Y} f(y) > \sup_{z \in Z} f(z)$ in it's strongest form. Several solutions of this problem are presented as a consequence of von Neumann's minimax theorem. As special cases we obtain all the classical separation theorems for linear spaces, linear topological spaces, locally convex spaces, normed spaces, etc., but also new results for convex metric spaces are derived.

Keywords: Segment structure, affine function, separation of sets, convex metric space

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1. Preliminaries

1.1. Notation

Let S be a nonvoid set and 2^S the power set of S. Then every nonvoid subset $\mathcal{P} \subset 2^S$ is called a *paving* in S and $(S, \mathcal{P})$ is a *paved space.* We write $\mathcal{P}(S)$ ($\mathcal{E}(S)$) for the paving of all nonvoid (finite) subsets of S. If S is a topological space, then we set $\mathcal{F}(S)$ ($\mathcal{K}(S)$) for the paving of all closed (compact) subsets of S.

A paving $\mathcal{P}$ in S is called *upward filtrating* iff $\forall A, B \in \mathcal{P}\ \exists C \in \mathcal{P} : C \supset A \cup B$, and $\mathcal{P}$ resp. the paved space $(S, \mathcal{P})$ is called *compact* iff every subpaving $\mathcal{Q} \subset \mathcal{P}$ with the finite intersection property $\bigcap\{P : P \in \mathcal{R}\} \neq \emptyset\ \forall\ \mathcal{R} \in \mathcal{E}(\mathcal{Q})$, has the global intersection property $\bigcap\{P : P \in \mathcal{Q}\} \neq \emptyset$, and a subset T of S is called *compact* (in $\mathcal{P}$) iff the *trace* $\mathcal{P}_T := \{C \cap T : C \in \mathcal{P}\}$ is compact.

A paving is said to be $\cap_f$*−closed* ($\cap_c$*−closed*, $\cap_a$*−closed*) iff it is closed under finite (countable, arbitrary) intersections and it is called $\cup_f$−closed iff it is closed under finite unions.

We shall use the extensions $\mathbb{R}^\bullet := \mathbb{R} \cup \{+\infty\}$, $\mathbb{R}_\bullet := \mathbb{R} \cup \{-\infty\}$, and $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ of the set $\mathbb{R}$ of reals, and we set $\mathbb{P}^n := \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n : \alpha_i \geq 0, i \leq n, \sum_{i=1}^n \alpha_i = 1\}$.

For a nonvoid subset $R \subset \overline{\mathbb{R}}$ let R^S denote the set of all functions $f : S \to R$, and let $\inf_T f := \inf_{t \in T} f(t), T \in \mathcal{P}(S)$. For $\mathsf{H} \subset \overline{\mathbb{R}}^S$ and $T \in \mathcal{P}(S)$ the family of all restrictions $h|T$, $h \in \mathsf{H}$, will be denoted by $\mathsf{H}|T$, and we set $\mathsf{H} + \mathbb{R} := \{h + \gamma : h \in \mathsf{H}, \gamma \in \mathbb{R}\}$. H is called $\vee_f-$ ($\vee_c-, \vee_a-$)closed iff H is closed w.r.t. formation of finite (countable, arbitrary) suprema.

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Now let $(S,\mathcal{P})$ be a paved space. Then a function $f \in \overline{\mathbb{R}}^S$ is called *lower semicontinuous* iff $\{f \leq \alpha\} := \{s \in S : f(s) \leq \alpha\} \in \mathcal{P}$ for all $\alpha \in \mathbb{R}$, and f is *continuous* iff f and $-f$ are lower semicontinuous. We set $LSC(S,\mathcal{P})$ for the family of all lower semicontinous functions and $C(S,\mathcal{P})$ for the family of all continuous *real-valued* functions.

Remark 1.1. Let $(S,\mathcal{P})$ be a paved space.

a) If $T \in \mathcal{P}(S)$ is compact in $\mathcal{P}$, then every $f|T,\ f \in LSC(S,\mathcal{P})$, attains its minimum.

b) (i) $\gamma f \in LSC(S,\mathcal{P})\ \forall \gamma > 0, f \in LSC(S,\mathcal{P})$.
$f + \gamma \in LSC(S,\mathcal{P})\ \forall f \in LSC(S,\mathcal{P}), \gamma \in \mathbb{R}$.

(ii) $S \in \mathcal{P}, f \in LSC(S,\mathcal{P}) \Longrightarrow f \wedge \gamma \in LSC(S,\mathcal{P})\ \forall\ \gamma \in \mathbb{R}$.
$\emptyset \in \mathcal{P}, f \in LSC(S,\mathcal{P}) \Longrightarrow f \vee \gamma \in LSC(S,\mathcal{P})\ \forall\ \gamma \in \mathbb{R}$.

(iii) If $\mathcal{P}$ is $\cap_f - (\cap_c -, \cap_a -)$ closed, then $LSC(S,\mathcal{P})$ is $\vee_f - (\vee_c -, \vee_a -)$ closed.

(iv) If $\mathcal{P}$ is $\cup_f - \cap_c -$closed, then $LSC(S,\mathcal{P}) \cap \mathbb{R}^{\bullet S}$ $(LSC(S,\mathcal{P}) \cap \mathbb{R}^S_{\bullet})$ is a convex subcone of $\mathbb{R}^{\bullet S}$ $(\mathbb{R}^S_{\bullet})$, and $C(S,\mathcal{P})$ is a linear sublattice of $\mathbb{R}^S$.

(v) $\{\emptyset, S\} \subset \mathcal{P} \iff \mathbb{R} \subset C(S,\mathcal{P}) \iff \mathbb{R} \subset LSC(S,\mathcal{P})$.

Proof. a) W.l.g. let $\inf_T f < \infty$. For $P_n := \{f \leq (\inf_T f) \vee (-n) + \frac{1}{n}\}$ we have $P_1 \supset \ldots \supset P_n \in \mathcal{P}$ and $P_n \cap T \neq \emptyset,\ n \in \mathbb{N}$, hence argmin $f|T = \bigcap_{n=1}^{\infty} P_n \cap T \neq \emptyset$.
b) Part (iv) follows from the identity $\{f + g \leq \alpha\} = \bigcap_{\rho \in \mathbb{Q}} \{f \leq \rho\} \cup \{g \leq \alpha - \rho\}$. The other assertions are obvious. □

1.2. Abstract segment spaces

Let S be a nonvoid set. Then a function $\langle\cdot,\cdot\rangle : S \times S \to 2^S$ with $\{s,t\} \subset \langle s,t\rangle,\ (s,t) \in S^2$, is called a *segment function* and the sets $\langle s,t\rangle$ are *segments* in S. (Compare [5], [19], [22], [34], [36], [41] for examples.) A subset $T \subset S$ is *convex* iff $\{s,t\} \subset T$ implies $\langle s,t\rangle \subset T$. The paving $\mathcal{C}$ of convex subsets of S contains $\emptyset$ and S, and it is closed with respect to arbitrary intersections and nested unions. Here $LSC(S,\mathcal{C})$ is the family of all *quasiconvex* or *lower convexity preserving* [41] functions $f : S \to \overline{\mathbb{R}}$.
We shall now consider special segment spaces with an additional structure which makes it possible to define *convex* functions forming a convex subcone of $\mathbb{R}^{\bullet S}$ or $\mathbb{R}^S_{\bullet}$ contained in $LSC(S,\mathcal{C})$:

A function $\mu : S \times S \times [0,1] \to S$ with $\mu(s,t,1) = s$ and $\mu(s,t,0) = t$ for $s,t \in S$ will be called a *segment structure* for S, and (S,μ) endowed with the segments

$$\langle s,t\rangle_\mu = \mu(s,t,[0,1]) := \{\mu(s,t,\lambda) : \lambda \in [0,1]\},\ s,t \in S$$

is a *structured segment space*. The convex subsets are also called $\mu-$*convex*, a function $f : S \to \overline{\mathbb{R}}$ is said to be $(\mu-)$*convex* iff $f \in \mathbb{R}^{\bullet S}$ or $f \in \mathbb{R}^S_{\bullet}$ and

$$f(\mu(s,t,\lambda)) \leq \lambda f(s) + (1-\lambda) f(t),\ s,t \in S,\ \lambda \in [0,1], \tag{1}$$

and f is $(\mu-)$*affine* iff relation (1) holds with equality.
We set $\mathcal{C}_\mu$ for the paving of all $\mu-$convex subsets of S, C_μ for the family of all $\mu-$convex functions $f : S \to \overline{\mathbb{R}}$, and $\mathbb{A}_\mu$ for the family of all *real-valued* $\mu-$affine functions.

A *paved structured segment space* is given by a triplet $\Sigma = (S,\mathcal{P},\mu)$, where $(S,\mathcal{P})$ is a paved space and μ is a segment structure for S. We set $\mathcal{C}^*(\Sigma) = \mathcal{C}_\mu \cap\ \mathcal{P}$ and $\mathbb{A}^*(\Sigma) = \mathbb{A}_\mu \cap C(S,\mathcal{P})$.

A segment structure μ is called

- *reflexive*, iff $\mu(s,s,\lambda) = s$ for all $s \in S, \lambda \in [0,1]$,
- *symmetric*, iff $\mu(x,y,\lambda) = \mu(y,x,(1-\lambda))$ for all $(x,y,\lambda) \in S \times S \times [0,1]$,
- *associative*, iff $\mu(x,\mu(y,z,\tau),\lambda) = \mu(\mu(x,y,\lambda[\lambda+(1-\lambda)\tau]^{-1}), z, \lambda+(1-\lambda)\tau)$ for all $(x,y,z,\lambda,\tau) \in S \times S \times S \times (0,1) \times (0,1)$,
- *cancellative* iff $\mu(x,y,\lambda) = \mu(x,z,\lambda)$ for some $(x,y,z,\lambda) \in S \times S \times S \times (0,1)$ implies $y = z$,
- *convexor* [35] iff it is reflexive, symmetric and associative.

The segment space (S,μ) is said to have the

- *Pasch Property* [41] iff for all $(s_0,s_1,s_2,\lambda_1,\lambda_2) \in S \times S \times S \times (0,1) \times (0,1)$ there exists a pair $(\tau_1,\tau_2) \in (0,1]^2$ with

$$\mu(s_2,\mu(s_1,s_0,\lambda_1),\tau_1) = \mu(s_1,\mu(s_2,s_0,\lambda_2),\tau_2) \tag{2}$$

  If one can choose $\tau_i = (1-\lambda_i)\lambda_{3-i}(1-\lambda_1\lambda_2)^{-1}, i \in \{1,2\}$, then we say that (S,μ) has the *Algebraic Pasch Property*.

Remark 1.2. Let (S,μ) be a structured segment space. Then the following properties hold:

a) The paving $\mathcal{C}_\mu$ contains the singletons iff μ is reflexive.
b) $\mathsf{C}_\mu \cap \mathbb{R}^{\bullet S}$ ($\mathsf{C}_\mu \cap \mathbb{R}^S_\bullet, \mathsf{C}_\mu \cap \mathbb{R}^S$) is a $\vee_a-$closed ($\vee_f-$closed) convex subcone of $\mathbb{R}^{\bullet S}$ ($\mathbb{R}^S_\bullet, \mathbb{R}^S$) and a subset of $LSC(S,\mathcal{C}_\mu)$.
c) $\mathbb{A}_\mu$ is a linear subspace of $\mathbb{R}^S$ and a subset of $C(S,\mathcal{C}_\mu)$.
d) A symmetric segment structure possesses the Algebraic Pasch Property iff it is associative.

Let (S,μ) be a structured segment space. Then for $T \subset S$ the ($\mu-$)*core* of T (in S) is defined according to

$$\mathrm{cor}_\mu T = \{t \in T : \forall s \in S \setminus \{t\}\ \exists \lambda \in (0,1] \text{ with } \mu(s,t,\lambda) \in T\}.$$

This definition, which is not standard in literature, turns out to be useful in the sequel. For convex subsets of linear spaces it coincides with the usual one (cf. Example 1.5 below).

Lemma 1.3. *Let (S,μ) be a structured segment space, and let $T \in \mathcal{P}(S)$ with $\mathrm{cor}_\mu T \neq \emptyset$. Let $\alpha \in \mathbb{R}$ and let $f \in \overline{\mathbb{R}}^S$ be $\mu-$convex. Then the following holds:*

a) *$f|T \geq \alpha > \inf_S f$ implies $f|\ \mathrm{cor}_\mu T > \alpha$.*
b) *Let (S,μ) possess the Pasch Property, and let T be convex.*
  (i) For $s_0 \in \mathrm{cor}_\mu T, s_1 \in T$ and $\lambda \in [0,1)$ we have $z_1 := \mu(s_1,s_0,\lambda) \in \mathrm{cor}_\mu T$. In particular, $\mathrm{cor}_\mu T$ is a convex set.
  (ii) $f|\mathrm{cor}_\mu T \geq \alpha$ implies either $f|\mathrm{cor}_\mu T \equiv \infty$ or $f|T \geq \alpha$.

Proof. a) Let $t_0 \in \mathrm{cor}_\mu T$. Choose $s \in S$ with $f(s) < \alpha$ and $\lambda \in (0,1]$ with $x := \mu(s,t_0,\lambda) \in T$. Then $s \notin T$ implies $\lambda < 1$, and $\alpha \leq f(x) \leq \lambda f(s) + (1-\lambda)f(t_0) < \lambda\alpha + (1-\lambda)f(t_0)$ yields $f(t_0) > \alpha$.
b)(i) We may assume $\lambda_1 := \lambda \in (0,1)$. Let $s_2 \in S \setminus \{z_1\}$. Since $s_0 \in \mathrm{cor}_\mu T$ there exists a $\lambda_2 \in (0,1]$ with $z_2 := \mu(s_2,s_0,\lambda_2) \in T$. In case $\lambda_2 = 1$ we have $\mu(s_2,z_1,1) =$

$s_2 = z_2 \in T$. In case $\lambda_2 \in (0,1)$ we choose τ_1, τ_2 according to (2), and we obtain $\mu(s_2, z_1, \tau_1) = \mu(s_1, z_2, \tau_2) \in T$, since $s_1, z_2 \in T$.
(ii) Let $t \in T$ and $t_0 \in \mathrm{cor}_\mu T$ with $f(t_0) \neq \infty$. By (i) we have $z_n := \mu(t, t_0, 1 - \frac{1}{n}) \in \mathrm{cor}_\mu T$, and from $\alpha \leq f(z_n) \leq (1 - \frac{1}{n})f(t) + \frac{1}{n}f(t_0),\ n \in \mathbb{N}$, we infer $f(t) \geq \alpha$. □

Let (S, μ) and (T, ρ) be two structured segment spaces. Then a map $\varphi : S \to T$ is *affine* iff

$$\varphi(\mu(s_1, s_2, \lambda)) = \rho(\varphi(s_1), \varphi(s_2), \lambda),\ s_1, s_2 \in S, \lambda \in [0,1].$$

Especially, if $(T, \rho) = (\mathbb{R}, \nu)$ with the natural segment structure ν, then φ is affine iff $f \in \mathbb{A}_\mu$.

Remark 1.4. Let (S, μ) and (T, ρ) be two structured segment spaces and $\varphi : (S, \mu) \to (T, \rho)$ an affine bijection. Then the following holds:

a) φ^{-1} is affine, and μ is reflexive/symmetric/associative/cancellative iff ρ has these properties.
b) $\varphi(C) \in \mathcal{C}_\rho\ \forall C \in \mathcal{C}_\mu$.
c) $g \circ \varphi \in \mathbb{A}_\mu\ \forall g \in \mathbb{A}_\rho$.
d) $\varphi(\mathrm{cor}_\mu Y) = \mathrm{cor}_\rho \varphi(Y),\ Y \subset S$.

1.3. Some first examples

Example 1.5. Let E be a linear space over the reals and E' the (algebraic) dual space of all linear functionals $f : E \to \mathbb{R}$. Every nonvoid convex subset S of E can be equipped with the *natural segment structure*

$$\nu(s, t, \lambda) = \lambda s + (1 - \lambda)t,\ (s, t, \lambda) \in S \times S \times [0,1]$$

which is a cancellative convexor. It is well-known that this is – in essence – the *unique* example of a cancellative convexor. (Cf. Lemma 3.3 below.)
Now let $S = E$ and T a convex subset of E. Here $\mathbb{A}_\nu = E' + \mathbb{R}$ is point separating (take a Hamel base), and a $\nu-$affine function $f \in \mathbb{R}^{\bullet E}$ is real-valued iff $f \not\equiv \infty$. The $\nu-$core $\mathrm{cor}_\nu T$, the algebraic interior [17] cor T and the set of all internal points [7] coincide. In particular, by Lemma 1.3 b), cor T is convex.

In the sequel, if not otherwise stated, convex subsets of linear spaces will always be endowed with the natural segment structure.

Sometimes the following simple observation is helpful [15].

Example 1.6. Let S be an arbitrary set and C a convex subset of a real linear space. Suppose that there exists a bijection $\varphi : S \to C$. Then

$$\mu(s, t, \lambda) = \varphi^{-1}(\lambda\varphi(s) + (1 - \lambda)\varphi(t)),\ (s, t, \lambda) \in S \times S \times [0,1]$$

defines a segment structure on S such that $\varphi : S \to C$ is affine. Hence, Remark 1.4 applies. In particular, μ is a cancellative convexor.

Example 1.7. The set $\mathbb{R}^\bullet$ may be endowed with the "extended segment structure"

$$\nu^\bullet(s, t, \lambda) = \lambda s + (1 - \lambda)t,\ (s, t, \lambda) \in \mathbb{R}^\bullet \times \mathbb{R}^\bullet \times [0,1],$$

which is a noncancellative convexor. Here $\mathbb{A}_{\nu^\bullet} = \mathbb{R}$, i.e. every real-valued affine function is constant. The nonconstant affine functions $f : \mathbb{R}^\bullet \to \mathbb{R}^\bullet$ are of the form

$$f(x) = \begin{cases} \alpha x + \beta & : \quad x \in \mathbb{R} \\ \infty & : \quad x = \infty \end{cases}$$

with $\alpha, \beta \in \mathbb{R}$.
A nonvoid subset T of $\mathbb{R}^\bullet$ has nonvoid $\nu^\bullet-$core iff $\infty \in T$. In this case, $\mathrm{cor}_{\nu^\bullet} T = \mathrm{cor}(T \setminus \{\infty\}) \cup \{\infty\}$.

Example 1.8. (The circle) Let $\mathbb{S}^1 := \{e^{i\xi} : 0 \leq \xi < 2\pi\}$ be the unit circle in the complex plane $\mathbb{C}$. We define a segment structure on $\mathbb{S}^1$ according to

$$\sigma(e^{i\xi}, e^{i\eta}, \lambda) = \begin{cases} e^{i(\lambda\xi + (1-\lambda)\eta)} & : \quad |\xi - \eta| \leq \pi \\ e^{i(\lambda(2\pi+\xi)+(1-\lambda)\eta)} & : \quad \eta > \xi + \pi \\ e^{i(\lambda\xi+(1-\lambda)(2\pi+\eta))} & : \quad \xi > \eta + \pi \end{cases}$$

i.e., in case $|\xi - \eta| \neq \pi$, $\sigma(e^{i\xi}, e^{i\eta}, [0,1])$ is the shortest arc joining $e^{i\xi}$ and $e^{i\eta}$. This segment structure is reflexive, symmetric, and cancellative, but not associative.
Indeed, for $\lambda = \tau = \frac{1}{2}$ we have $\lambda + (1-\lambda)\tau = \frac{3}{4}$, $\lambda(\lambda + (1-\lambda)\tau)^{-1} = \frac{2}{3}$, but

$$\sigma(e^{i\frac{\pi}{4}}, \sigma(e^{i\pi}, e^{i\frac{7\pi}{4}}, \frac{1}{2}), \frac{1}{2}) = e^{i\frac{29\pi}{16}} \neq e^{i\frac{5\pi}{16}} = \sigma(\sigma(e^{i\frac{\pi}{4}}, e^{i\pi}, \frac{2}{3}), e^{i\frac{7\pi}{4}}, \frac{3}{4}).$$

Especially, for $f \in \mathbb{A}_\sigma$ we have $f(e^{i\frac{29\pi}{16}}) = f(e^{i\frac{5\pi}{16}}) =: \alpha$ and $\sigma(e^{i\frac{29\pi}{16}}, e^{i\frac{5\pi}{16}}, \frac{5}{8}) = e^{i0} = 1$ together with $\sigma(e^{i0}, e^{i2\xi}, \frac{1}{2}) = e^{i\xi}$, $\xi \in [0, \frac{\pi}{2}]$, implies $f \equiv \alpha = f(1)$, i.e., every affine function $f : \mathbb{S}^1 \to \mathbb{R}$ is constant.
It is easy to see, however, that the $\sigma-$affine functions $f : S \to \mathbb{R}$ are point separating for every nonvoid $\sigma-$convex proper subset S of $\mathbb{S}^1$.

Example 1.9. (Products of segment spaces) Let $(S_i, \mu_i)_{i\in I}$ be a family of structured segment spaces, let $S = \prod_{i\in I} S_i$ be the cartesian product and $\pi_i : S \to S_i$ the projections. Then (S, μ) with

$$\mu(s, t, \lambda) = (\mu_i(\pi_i(s), \pi_i(t), \lambda)_{i\in I}), \; s, t \in S, \; \lambda \in [0, 1]$$

is the *product space*.
Here μ is reflexive/symmetric/associative/cancellative iff every μ_i has these properties. Moreover, the projections are affine and $f_i \circ \pi_i \in \mathbb{A}_\mu$ for all $f_i \in \mathbb{A}_{\mu_i}$.

Example 1.10. (The cylinder) Let (S, μ) be the product of the circle $(\mathbb{S}^1, \sigma)$ with $(\mathbb{R}, \nu)$ according to Example 1.9. Then $\mu(s, t, [0, 1])$ is the geodesic segment joining s and t.
Let $f \in \mathbb{A}_\mu$. Then, by Example 1.8, we have $f(e^{i\xi}, r) = f(1, r)$ for all $\xi \in [0, 2\pi), r \in \mathbb{R}$. On the other hand $f(1, \cdot) : \mathbb{R} \to \mathbb{R}$ is a $\nu-$affine function. Therefore every $f \in \mathbb{A}_\mu$ is of the form

$$f(e^{i\xi}, r) = \alpha r + \beta, \quad \alpha, \beta \in \mathbb{R}.$$

Here two points can be separated by an affine function iff they do not lie on the same circle.

The following closely related example was given by Horvath to demonstrate that convex sets need not be contractible.

Example 1.11. ([18]) Let $S = \{re^{i\xi} : 0 \le \xi < 2\pi, r \ge 1\}$ be the complement of the open unit disc in the plane, endowed with the segment structure

$$\mu(r_1 e^{i\xi_1}, r_2 e^{i\xi_2}, \lambda) = (\lambda r_1 + (1-\lambda) r_2) e^{i(\lambda\xi_1 + (1-\lambda)\xi_2)}$$

Here the annuli

$$\{re^{i\xi} : 0 \le \xi < 2\pi, a \le r \le b\}, \quad 1 \le a \le b$$

are convex sets.
The map $\varphi : S \to [0, 2\pi) \times [1, \infty)$ with $\varphi(re^{i\xi}) = (\xi, r)$ is an affine bijection. Therefore every $f \in \mathbb{A}_\mu$ is of the form

$$f(re^{i\xi}) = \alpha\xi + \beta r + \gamma, \quad (\alpha, \beta, \gamma) \in \mathbb{R}^3.$$

Perhaps a more natural segment structure would be

$$\mu'(r_1 e^{i\xi_1}, r_2 e^{i\xi_2}, \lambda) = (\lambda r_1 + (1-\lambda) r_2)\sigma(e^{i\xi_1}, e^{i\xi_2}, \lambda)$$

with σ defined as in Example 1.8. Here, similar to Example 1.10, every $f \in \mathbb{A}_{\mu'}$ is of the form

$$f(re^{i\xi}) = \beta r + \gamma, \quad (\beta, \gamma) \in \mathbb{R}^2.$$

Example 1.12. (Hyperspaces) Let $(S, \mathcal{P}, \mu)$ be a paved structured segment space with $\emptyset \notin \mathcal{P}$. Then $\mathcal{P}$ can be endowed with a segment structure $\tilde{\mu}$ according to

$$\tilde{\mu}(A, B, \lambda) = \{\mu(a, b, \lambda) : a \in A, \ b \in B\}, \ A, B \in \mathcal{P}, \ \lambda \in [0, 1].$$

It is easy to see that $\tilde{\mu}$ is reflexive/symmetric/associative if μ has these properties, but cancellativity does not carry over, in general, from μ to $\tilde{\mu}$. For example, take $S = \mathbb{R}$ and $\mu = \nu$ the natural segment structure. If $\mathcal{P}$ is the paving of closed convex or of bounded convex subsets, then $\tilde{\nu}$ is easily seen to be noncancellative, but if one takes for $\mathcal{P}$ the paving of compact convex subsets, then $\tilde{\nu}$ is cancellative, which is a basic fact in interval arithmetics.
A more general result is due to Urbański [40]:
Let S be a (Hausdorff) linear topological space and $\mathcal{P}$ the paving of bounded, closed, convex subsets. Let $\bar{\mu}(A, B, \lambda)$ be the closure of $\tilde{\mu}(A, B, \lambda)$. Then $\bar{\mu}$ is a cancellative convexor.

The motivation for the study of abstract segment structures arose from the foundation of a utility theory in game theory and mathematical economics [43] (compare also [14, 16, 23, 44]) but there are also other fields of application such as operational quantum mechanics [13, 14] and color vision [14]. Later on, inspired by a paper of Rusin [32] on the "nonlinear" blending behavior of the octane number of gasoline, Gudder and Schroeck [15] pointed out that the classical theory is too special to describe certain blending situations arising in color vision, threshold phenomena, and chemistry. They showed that these examples can only be described in an adequate way by segment structures where certain properties of a convexor are violated.

2. Separation Theorems

It is well–known that von Neumann's minimax theorem [42] can be derived from the classical separation theorem for Euclidean spaces. But there are also completely elementary proofs using only basic linear algebra [24, 30, 45]. In this paper we go the reverse way. We prove our separation theorems with the aid of the following lemma which is closely related to Ky Fan's minimax theorem [8]:

Lemma 2.1. *Let* (S, μ) *be a structured segment space,* T *a nonvoid convex subset of* S, $\mathsf{F} = \{f_1, \ldots, f_n\}$ *a finite subset of* $\mathbb{C}_\mu$*, and* $\{\gamma_1, \ldots, \gamma_n\} \subset \mathbb{R}$.

a) *If* $\mathsf{F} \subset \mathbb{R}^{\bullet S}$*, and if every* f_i *that attains the value* ∞ *on* T *is bounded from below on* T*, then the following are equivalent:*
(α) $\exists \epsilon > 0 \ \forall t \in T \ \exists i \le n : f_i(t) > \gamma_i + \epsilon$
(β) $\exists (\alpha_1, \ldots, \alpha_n) \in \mathbb{P}^n : \ \inf_T \sum_{i=1}^n \alpha_i f_i > \sum_{i=1}^n \alpha_i \gamma_i$

b) *If* $\mathsf{F} \subset \mathbb{R}_\bullet^S$*, then the following are equivalent:*
(γ) $\forall t \in T \ \exists i \le n : f_i(t) \ge \gamma_i$
(δ) $\exists (\alpha_1, \ldots, \alpha_n) \in \mathbb{P}^n : \ \inf_T \sum_{i=1}^n \alpha_i f_i \ge \sum_{i=1}^n \alpha_i \gamma_i$

Proof. W.l.g. we may assume $\gamma_i = 0$ and $T = S$.
a) Let (α) be satisfied. We set $Z := \{s \in S : f_i(s) < \infty \ \forall i \le n\}$, and we fix an $A = \{s_1, \ldots s_m\} \in \mathcal{E}(Z)$. By von Neumann's minimax theorem there exist vectors $(\alpha_1, \ldots, \alpha_n) \in \mathbb{P}^n$ and $(\beta_1, \ldots, \beta_m) \in \mathbb{P}^m$ with

$$\min_{k \le m} \sum_{i=1}^n \alpha_i f_i(s_k) = \max_{i \le n} \sum_{k=1}^m \beta_k f_i(s_k). \tag{3}$$

From $f_i \in \mathbb{C}_\mu$ it follows by induction that there exists an $s_0 \in S$ with

$$f_i(s_0) \le \sum_{k=1}^m \beta_k f_i(s_k) \ \forall i \le n. \tag{4}$$

By (3) and (4) together with (α) there exists an $\epsilon > 0$ with

$$\bigcap_{s \in A} \left\{ (\alpha_1, \ldots, \alpha_n) \in \mathbb{P}^n : \sum_{i=1}^n \alpha_i f_i(s) \ge \epsilon \right\} \ne \emptyset \quad \forall A \in \mathcal{E}(Z).$$

Since $\mathbb{P}^n$ is compact, we obtain $\inf_{s \in Z} \sum_{i=1}^n \alpha_i f_i(s) \ge \epsilon$ for some $(\alpha_1, \ldots, \alpha_n) \in \mathbb{P}^n$ and some $\epsilon > 0$.

If every f_i is real-valued, we are done. Otherwise we proceed as in [29]:
Let $J = \{j \in \{1, \ldots, n\} : f_j(s) = \infty$ for some $s \in S\}$, and let $r = \text{card}\, J \ge 1$. Choose an $M \in \mathbb{N}$ with $f_j(s) \ge -M$ for all $j \in J, s \in S$. Then for $N > \epsilon^{-1} M r$ we get

$$\sum_{i=1}^n \alpha_i f_i(s) + \frac{1}{N} \sum_{j \in J} f_j(s) \ge \epsilon - \frac{Mr}{N} > 0 \quad \forall s \in S,$$

and for

$$\alpha_i' = \begin{cases} (\alpha_i + \frac{1}{N})(1+\frac{r}{N})^{-1} & : \quad i \in J \\ \alpha_i(1+\frac{r}{N})^{-1} & : \quad i \in \{1,\dots,n\} \setminus J \end{cases}$$

we have $(\alpha_1', \dots, \alpha_n') \in \mathbb{P}^n$ and $\inf_{s \in S} \sum_{i=1}^n \alpha_i' f_i(s) > 0$.
The implication $(\beta) \Longrightarrow (\alpha)$ is obvious.
b) Let (γ) be satisfied. By a) the sets

$$Q_N := \bigcap_{s \in S} \left\{ (\alpha_1, \dots \alpha_n) \in \mathbb{P}^n : \sum_{i=1}^n \alpha_i \left(f_i(s) \vee (-N) + \frac{1}{N} \right) \geq 0 \right\}, \; N \in \mathbb{N}$$

are nonvoid, and every $(\alpha_1, \dots \alpha_n) \in \bigcap_{N=1}^\infty Q_N$ satisfies (δ). □

2.1. Separation of a point from a set

Theorem 2.2. *Let (S, μ) be a structured segment space, T a nonvoid convex subset of S, $\mathsf{F} = \{f_1, \dots, f_n\}$ a finite subset of $\mathbb{C}_\mu$, and $s \in S$.*

a) *If $\mathsf{F} \subset \mathbb{R}^{\bullet S}$ and if every f_i that attains the value ∞ on T is bounded from below on T, then the following are equivalent:*
(α) $\exists \epsilon > 0 \; \forall t \in T \; \exists i \leq n : f_i(t) > f_i(s) + \epsilon$.
(β) $\exists (\alpha_1, \dots, \alpha_n) \in \mathbb{P}^n : \; \inf_T \sum_{i=1}^n \alpha_i f_i > \sum_{i=1}^n \alpha_i f_i(s)$.

b) *If $\mathsf{F} \subset \mathbb{R}_\bullet^S$, then the following are equivalent:*
(γ) $\forall t \in T \; \exists i \leq n : f_i(t) \geq f_i(s)$
(δ) $\exists (\alpha_1, \dots, \alpha_n) \in \mathbb{P}^n : \; \inf_T \sum_{i=1}^n \alpha_i f_i \geq \sum_{i=1}^n \alpha_i f_i(s)$

Proof. a) Apply Lemma 2.1 a) to $\mathsf{F}' := \{f_i : f_i(s) \in \mathbb{R}\}$ and $\gamma_i = f_i(s)$.
b) Apply Lemma 2.1 b) with $\gamma_i = f_i(s)$. Note that condition (δ) is satisfied for $\alpha_j = \frac{1}{n}$ if $f_i(s) = -\infty$ for some $i \leq n$. □

Theorem 2.3. *Let $\Sigma = (S, \mathcal{P}, \mu)$ be a paved structured segment space, T a nonvoid convex subset of S which is compact in $\mathcal{C}^*(\Sigma)$, let F be a nonvoid convex subset of $LSC(S, \mathcal{P}) \cap \mathbb{C}_\mu$, and let $\theta : \mathsf{F} \to \overline{\mathbb{R}}$ be convex. Assume that either*

(i) $\mathsf{F} \subset \mathbb{R}_\bullet^S$ *and* $\theta : \mathsf{F} \to \mathbb{R}_\bullet$*, or*
(ii) $\mathsf{F} \subset \mathbb{R}^{\bullet S}$*, every $f \in \mathsf{F}$ that attains the value ∞ on T is bounded from below on T and $\theta : \mathsf{F} \to \mathbb{R}^\bullet$.*

Then the following are equivalent:

(a) $\forall t \in T \; \exists f \in \mathsf{F} : \; f(t) > \theta(f)$.
(b) $\exists f \in \mathsf{F} \; : \; \inf_T f > \theta(f)$.

Especially, for $s \in S \setminus T$ the following are equivalent:

(c) $\forall t \in T \; \exists f \in \mathsf{F} : f(t) > f(s)$.
(d) $\exists f \in \mathsf{F} : \inf_T f > f(s)$.

Proof. (a) $\Longrightarrow$ (b): For $t \in T$ choose $f_t \in \mathsf{F}$ and $\beta_t, \gamma_t \in \mathbb{R}$ with $f_t(t) > \beta_t > \gamma_t > \theta(f_t)$. By Remark 1.2 b) we have $\mathsf{F} \subset LSC(S, \mathcal{C}^*(\Sigma))$. Hence $\bigcap_{t \in T} \{f_t \leq \beta_t\} \cap T = \emptyset$ implies $\bigcap_{i=1}^n \{f_{t_i} \leq \beta_{t_i}\} \cap T = \emptyset$ for some finite set $\{t_1, \dots, t_n\} \subset T$. In particular, $\bigcap_{i=1}^n \{f_{t_i} \leq \gamma_{t_i} + \epsilon\} \cap T = \emptyset$ for $\epsilon = \min_{i \leq n}(\beta_{t_i} - \gamma_{t_i})$. By Lemma 2.1 there exists an $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{P}^n$

such that for $f = \sum_{i=1}^n \alpha_i f_{t_i}$ we have $\inf_T f \geq \sum_{i=1}^n \alpha_i \gamma_{t_i} > \sum_{i=1}^n \alpha_i \theta(f_{t_i}) \geq \theta(f)$.
(b) $\Longrightarrow$ (a) is obvious, and (c) $\iff$ (d) follows with $\theta(f) = f(s)$. □

2.2. Separation of two sets

Let (Y, Z) be a pair of nonvoid subsets of a nonvoid set S and $f \in \overline{\mathbb{R}}^S$. Then (Y, Z) is said to be

- *f-separated* iff there exists an $\alpha \in \mathbb{R}$ such that $f(y) \geq \alpha \geq f(z), y \in Y,\ z \in Z$, holds,
- *properly f-separated* iff (Y, Z) is f-separated and $\sup_Y f > \inf_Z f$ holds,
- *strictly f-separated* iff there exists an $\alpha \in \mathbb{R}$ such that $f(y) > \alpha > f(z),\ y \in Y,\ z \in Z$, holds, and
- *strongly $f-$separated* iff $\inf_Y f > \sup_Z f$ holds.

For $\mathsf{F} \subset \overline{\mathbb{R}}^S$ a pair (Y, Z) of nonvoid subsets of S is called

- *F-separated* iff there exists a nonconstant function $f \in \mathsf{F}$ such that (Y, Z) is $f-$separated,
- *properly/strictly/strongly F-separated* iff there exists a function $f \in \mathsf{F}$ such that (Y, Z) is properly/strictly/strongly $f-$separated, and
- *pointwise F-separated* iff for every pair $(y, z) \in (Y, Z)$ there exists an $f \in \mathsf{F}$ with $f(y) \neq f(z)$.

and F is called *point separating* iff for every pair $(y, z) \in S^2$ with $y \neq z$ there exists an $f \in \mathsf{F}$ with $f(y) \neq f(z)$.

A paving $\mathcal{P}$ in S is called *(properly,...) F-separated* iff every pair (Y, Z) of nonvoid disjoint sets $Y, Z \in \mathcal{P}$ is (properly,...) F-separated.

Theorem 2.4. *Let $\Sigma = (S, \mathcal{P}, \mu)$ be a paved structured segment space and Y, Z nonvoid convex subsets of S which are compact in $\mathcal{C}^*(\Sigma)$. Let F be a nonvoid absolutely convex subset of $\mathbb{A}^*(\Sigma)$. Then the following are equivalent:*

(a) The pair (Y, Z) is pointwise $\mathsf{F}-$separated.
(b) The pair (Y, Z) is strongly $\mathsf{F}-$separated.

Proof. (b) $\Longrightarrow$ (a) is obvious.
(a) $\Longrightarrow$ (b): We fix a $y \in Y$. By (a) and $\mathsf{F} = -\mathsf{F}$, for every $z \in Z$ there exists a $g \in \mathsf{F}$ with $g(z) > g(y)$. Hence, by Theorem 2.3 there exists an $f \in \mathsf{F}$ with $f(y) > \sup_Z f$. Now with Theorem 2.3 applied to $T = Y$ and $\theta(f) = \sup_Z f, f \in \mathsf{F}$, the assertion follows. □

Corollary 2.5. *Let a paved structured segment space $\Sigma = (S, \mathcal{P}, \mu)$ and a nonvoid absolutely convex family $\mathsf{F} \subset \mathbb{A}^*(\Sigma)$ be given. Let $\mathcal{K}$ be a compact subpaving of $\mathcal{C}_\mu$ with*

$$C \cap K \in \mathcal{K} \cup \{\emptyset\} \quad \forall C \in \mathcal{C}^*(\Sigma),\ K \in \mathcal{K}.$$

Then a pair of nonvoid sets $(Y, Z) \in \mathcal{K} \times \mathcal{K}$ is strongly $\mathsf{F}-$separated iff it is pointwise $\mathsf{F}-$separated.

Theorem 2.6. *Let $\Sigma = (S, \mathcal{P}, \mu)$ be a paved structured segment space and F a nonvoid $\vee_f-$closed convex subset of $C(S, \mathcal{P}) \cap LSC(S, \mathcal{C}_\mu)$ with $\mathsf{F} = \mathsf{F} + \mathbb{R}$. Let $Y, Z \in \mathcal{P}(S)$ be subsets with Z convex and Y, Z compact in $\mathcal{C}^*(\Sigma)$. Assume that*

$$\forall (y, z) \in Y \times Z\ \exists f \in \mathsf{F} \cap \mathbb{A}_\mu :\ f(y) > f(z).$$

Then the pair (Y, Z) is strongly $\mathsf{F}-$separated.

Proof. We first fix a $y \in Y$. By Theorem 2.3, applied to $T = Z$ and F replaced by $-\mathsf{F} \cap \mathbb{A}_\mu$, there exists an $f_y \in \mathsf{F} \cap \mathbb{A}_\mu$ and an $\alpha_y \in \mathbb{R}$ with $\sup_Z f_y < \alpha_y < f_y(y)$. W.l.g. we may assume $\alpha_y = 0$ since $\mathbb{A}_\mu = \mathbb{A}_\mu + \mathbb{R}$.
From $\bigcap_{y \in Y} \{f_y \leq 0\} \cap Y = \emptyset$ it follows that $\bigcap_{y \in B} \{f_y \leq 0\} \cap Y = \emptyset$ for some $B \in \mathcal{E}(Y)$, since $\{f_y \leq 0\} \in \mathcal{C}^*(\Sigma)$ and Y is compact in $\mathcal{C}^*(\Sigma)$. Now for $f = \bigvee_{t \in B} f_t$ we have $\sup_Z f < 0 < f(y),\ y \in Y$. □

Corollary 2.7. *Let $\Sigma = (S, \mathcal{P}, \mu)$ be a paved structured segment space with $\cup_f - \cap_c -$closed $\mathcal{P}$ and $\{\emptyset, S\} \subset \mathcal{P}$. Let Y and Z be nonvoid subsets of S with Z convex and Y, Z compact in $\mathcal{C}^*(\Sigma)$ such that (Y, Z) is pointwise $\mathbb{A}^*(\Sigma)-$separated. Then (Y, Z) is strongly $C(S, \mathcal{P}) \cap \mathbb{C}_\mu-$separated.*

Proof. By Remarks 1.1 b) and 1.2 b), $\mathsf{F} = C(S, \mathcal{P}) \cap \mathsf{C}_\mu$ is a $\vee_f-$closed convex subcone of $\mathbb{R}^S$ with $\mathbb{R} \subset \mathsf{F} \subset LSC(S, \mathcal{C}_\mu)$. Hence Theorem 2.6 applies. □

Theorem 2.8. *Let (S, μ) be a structured segment space, and let two upward filtrating pavings $\mathcal{Y}, \mathcal{Z} \subset \mathcal{P}(S)$ be given such that every pair $(Y, Z) \in \mathcal{Y} \times \mathcal{Z}$ is properly $\mathbb{A}_\mu-$separated. Suppose that every nonnegative $f \in \mathbb{A}_\mu$ is constant. Let $Y_0 = \bigcup\{Y : Y \in \mathcal{Y}\}$ and $Z_0 = \bigcup\{Z : Z \in \mathcal{Z}\}$. If $\mathrm{cor}_\mu Y \neq \emptyset$ for some $Y \in \mathcal{Y}$, then there exists a $\mu-$affine function $f \in \mathbb{R}^{\bullet S}$ separating the pair (Y_0, Z_0) properly.*

Proof. Let $Y' \in \mathcal{Y}$ with $\mathrm{cor}_\mu Y' \neq \emptyset$, and let $y_0 \in \mathrm{cor}_\mu Y'$. W.l.g. we may assume $Y' \subset Y\ \forall\ Y \in \mathcal{Y}$. For every $s \in S \setminus \{y_0\}$ we choose an $\alpha_s \in (0, 1]$ with $y_s := \mu(s, y_0, \alpha_s) \in Y'$. Since $\mathbb{A}_\mu = \mathbb{A}_\mu + \mathbb{R}$, for every pair $(Y, Z) \in \mathcal{Y} \times \mathcal{Z}$ there exists a $g = g_{Y,Z} \in \mathbb{A}_\mu$ with $\inf_Y g \geq 0 \geq \sup_Z g$ and $\sup_Y g > \inf_Z g$. Now $\alpha_s g(s) + (1 - \alpha_s) g(y_0) = g(y_s) \geq 0$ yields $g(y_0) > 0$, since otherwise $g(s) \geq 0, s \in S$, would imply that g is constant. W.l.g we may assume $g(y_0) = 1$, and we arrive at $g(s) \geq 1 - {\alpha_s}^{-1}$. Now take a subnet of $(g_{Y,Z})$ converging pointwise to some $f \in \overline{\mathbb{R}}^S$. Then f is $\mathbb{R}^\bullet-$valued and $\mu-$affine with $\inf_{Y_0} f \geq 0 \geq \sup_{Z_0} f$ and $f(y_0) = 1$. □

Theorem 2.9. *Let (S, μ) be a structured segment space, and let two upward filtrating pavings $\mathcal{Y}, \mathcal{Z} \subset \mathcal{P}(S)$ be given such that every pair $(Y, Z) \in \mathcal{Y} \times \mathcal{Z}$ is strictly $\mathbb{A}_\mu-$separated. Let $Y_0 = \bigcup\{Y : Y \in \mathcal{Y}\}$ and $Z_0 = \bigcup\{Z : Z \in \mathcal{Z}\}$.*

- *a) For every $y_0 \in \mathrm{cor}_\mu Y_0$ and $z_0 \in Z_0$ there exists a $\mu-$affine function $f \in \mathbb{R}^{\bullet S}$ with $\inf_{Y_0} f \geq \sup_{Z_0} f$ and $f(y_0) = 1 = -f(z_0)$.*
- *b) If $\mathrm{cor}_\mu Y_0 \neq \emptyset$ and $\mathrm{cor}_\mu Z_0 \neq \emptyset$, then the pair (Y_0, Z_0) is properly separated by some $f \in \mathbb{A}_\mu$, and every such f separates $(\mathrm{cor}_\mu Y_0, \mathrm{cor}_\mu Z_0)$ strictly.*

Proof. a) W.l.g. we may assume $y_0 \in \bigcap\{Y : Y \in \mathcal{Y}\}$ and $z_0 \in \bigcap\{Z : Z \in \mathcal{Z}\}$. For every $s \in S \setminus \{y_0\}$ we choose an $\alpha_s \in (0, 1]$ with $y_s := \mu(s, y_0, \alpha_s) \in Y_0$. By assumption, for every pair $(Y, Z) \in \mathcal{Y} \times \mathcal{Z}$ there exists a $g = g_{Y,Z} \in \mathbb{A}_\mu$ with $g(y) > g(z)\ \forall y \in Y, z \in Z$. Since $\mathbb{A}_\mu$ is a linear space containing the constant functions, we may assume $g(y_0) = 1 = -g(z_0)$. Then we have $\alpha_s g(s) + (1 - \alpha_s) g(y_0) = g(y_s) \geq g(z_0)$, and therefore $g(s) \geq 1 - 2{\alpha_s}^{-1}$ for all $g = g_{Y,Z}$ with $y_s \in Y$. Now take a subnet of $(g_{Y,Z})$ converging pointwise to some $f \in \overline{\mathbb{R}}^S$. Then f is $\mathbb{R}^\bullet-$valued and $\mu-$affine with $\inf_{Y_0} f \geq \sup_{Z_0} f$ and $f(y_0) = 1 = -f(z_0)$.
b) For $y_0 \in \mathrm{cor}_\mu Y_0$ and $z_0 \in \mathrm{cor}_\mu Z_0$ choose f as in a). Then there exist numbers $\beta_s \in$

$(0,1], s \in S \setminus \{z_0\}$, with $\mu(s, z_0, \beta_s) \in Z_0$, and as above it follows that $f(s) \le 2{\beta_s}^{-1} - 1$. In particular, we have $f \in \mathbb{R}^S$ and therefore $f \in \mathbb{A}_\mu$.
Now let $f \in \mathbb{A}_\mu$ and $\alpha \in \mathbb{R}$ with $f|Y_0 \ge \alpha \ge f|Z_0$ and $\sup_{Y_0} f > \inf_{Z_0} f$. Suppose that $f|Z_0 \equiv \alpha$. Then Lemma 1.3 a) implies $f|\mathrm{cor}_\mu Z_0 < \alpha$, a contradiction. But, by Lemma 1.3 a), $\inf_{Z_0} f < \alpha \le f|Y_0$ implies $f|\mathrm{cor}_\mu Y_0 > \alpha$ and similarly, $\sup_{Y_0} f > \alpha \ge f|Z_0$ yields $f|\mathrm{cor}_\mu Z_0 < \alpha$. □

3. Applications

We shall now present some applications of our abstract separation theorems. First we shall show how the classical separation theorems for linear and linear topological spaces fit in our concept. Then metric spaces are studied which are convex in the sense of Menger or Takahashi.

3.1. Linear spaces and cancellative convexors

We first recall some facts concerning the finite topology:

Remark 3.1. Let E be a linear space equipped with the natural segment structure ν as in Example 1.5. For $v = (s_1, \ldots, s_n) \in E^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{P}^n, n \in \mathbb{N}$, we put $\sigma_v(\alpha) = \sum_{i=1}^n \alpha_i s_i$. Let $\mathbb{P}^n$ be endowed with the Euclidean topology. Then the *finite topology* on E is the finest topology for which the maps $\sigma_v, v \in \bigcup_{n=1}^\infty E^n$, are continuous. Here a map f from E into a topological space Z is continuous iff the maps $f \circ \sigma_v : \mathbb{P}^n \to Z$ are continuous for all $v \in E^n, n \in \mathbb{N}$ (cf. [4], §2.4). In particular, every $f \in \mathbb{A}_\nu$ is continuous, since $f \circ \sigma_{(s_1,\ldots,s_n)}(\alpha) = \sum_{i=1}^n \alpha_i f(s_i)$ is continuous, and every polytope $[A]_\nu := \{\sum_{i=1}^n \alpha_i s_i : (\alpha_1, \ldots, \alpha_n) \in \mathbb{P}^n\} = \sigma_{(s_1,\ldots,s_n)}(\mathbb{P}^n),\ A = \{s_1, \ldots, s_n\} \in \mathcal{E}(E)$ is compact.

Example 3.2. For a linear space E and its algebraic dual E' the following holds:

a) The paving $\mathcal{P}_\nu$ of polytopes is strongly $E'-$separated.
b) Let Y and Z be nonvoid convex subsets of E with cor $Y \ne \emptyset$ and $Z\cap$ cor $Y = \emptyset$. Then there exists a non-zero $f \in E'$ and an $\alpha \in \mathbb{R}$ with $f|Y \ge \alpha \ge f|Z$. For every such f we have $f|\mathrm{cor}Y > \alpha$.
c) The paving of algebraically open convex subsets is strictly $E'-$separated.

Proof. a) Let E be endowed with the finite topology and the natural segment structure ν. Let $\mathcal{K}$ denote the paving of all convex compact subsets of E. Then for $\Sigma = (E, \mathcal{P}, \nu)$ with $\mathcal{P} = \{C \in \mathcal{C}_\nu : C \cap K \in \mathcal{K} \cup \{\emptyset\}\ \forall K \in \mathcal{K}\}$ we have $\mathcal{C}^*(\Sigma) = \mathcal{P}$. By Remark 3.1 we have $\mathcal{P}_\nu \subset \mathcal{K}$ and $\mathbb{A}^*(\Sigma) = \mathbb{A}_\nu \cap C(E, \mathcal{P}) = \mathbb{A}_\nu = E' + \mathbb{R}$ is point separating. Now by Corollary 2.5 the paving $\mathcal{K}$ is strongly $E'-$separated.
b) Let $\mathcal{Y}$ and $\mathcal{Z}$ be the pavings of all polytopes contained in cor Y resp. in Z. Then $\mathcal{Y}$ and $\mathcal{Z}$ are upward filtrating, since cor Y and Z are convex. By Theorem 2.9 a) together with a) and Example 1.5 there exists an $f \in E' \setminus \{0\}$ separating the pair $(\mathrm{cor}Y, Z)$. Finally let $f \in E' \setminus \{0\}$ and $\alpha \in \mathbb{R}$ with $f|Y \ge \alpha$. Then $\inf_E f = -\infty$ yields $f|\mathrm{cor}Y > \alpha$ according to Lemma 1.3 a).
c) This follows from b). □

The above separation theorem for linear spaces carries over to abstract cancellative convexors by means of the following well-known characterization:

Lemma 3.3. *([13, 14, 16, 23, 27, 37]) For a structured segment space (S,μ) the following are equivalent:*

(a) *(S,μ) is isomorphic to a convex set, i.e., there exists a linear space E, a convex subset C of E and an affine bijection $\varphi : (S,\mu) \to (C,\nu)$, where ν denotes the natural segment structure.*

(b) *$\mathbb{A}_\mu$ is point separating.*

(c) *μ is a cancellative convexor.*

Example 3.4. (Flood [9]) Let (S,μ) be a structured segment space with a cancellative convexor μ. Let Y, Z be two nonvoid convex subsets of S with $\mathrm{cor}_\mu Y \neq \emptyset$ and $Z \cap \mathrm{cor}_\mu Y = \emptyset$. Then there exists an $f \in \mathbb{A}_\mu$ and an $\alpha \in \mathbb{R}$ with $f|Y \geq \alpha \geq f|Z$ and $f|\mathrm{cor}_\mu Y > \alpha$.

Proof. By Lemma 3.3 there exists a linear space E, a convex subset C of E and an affine bijection $\varphi : (S,\mu) \to (C,\nu)$. By Example 1.5 and Remark 1.4 the sets $Y' = \varphi(Y)$ and $Z' = \varphi(Z)$ are convex, $\mathrm{cor}\, Y' = \varphi(\mathrm{cor}_\mu Y) \neq \emptyset$ and $Z' \cap \mathrm{cor} Y' = \emptyset$. By Example 3.2 b) there exists a $g \in E'$ and an $\alpha \in \mathbb{R}$ with $g|Y' \geq \alpha \geq g|Z'$ and $g|\mathrm{cor} Y' > \alpha$. Now, by Remark 1.4 c), $f = g \circ \varphi$ has the desired properties. □

3.2. Linear topological spaces

We now consider linear topological spaces E over the reals. We denote by E^* the (topological) dual space of all linear continuous functionals $f : E \to \mathbb{R}$. Linear topological spaces are always assumed to be Hausdorff.
Recall that $\mathrm{int}\, T = \mathrm{cor}\, T$ for every convex subset T of E with nonvoid interior $\mathrm{int}\, T$ ([17], §11).

Lemma 3.5. *Let E be a linear topological space, T a nonvoid open subset of E, and F a nonvoid subset of E' such that $\alpha := \sup\{f(t - t_0) : t \in T, f \in \mathsf{F}\} < \infty$ for some $t_0 \in T$. Then F is equicontinuous [7]. In particular, every $f \in E'$ with $\sup_T f < \infty$ is continuous.*

Proof. Choose a balanced 0−neighborhood $V \subset T - t_0$. Then for $U = (\alpha+1)^{-1}\epsilon V$ we have $\sup_{f\in\mathsf{F}} \sup_{u\in U} |f(s) - f(s+u)| < \epsilon,\ s \in S$. □

Example 3.6. Let E be a linear topological space, and let Y and Z be nonvoid convex subsets such that the topological interior $\mathrm{int}\, Y$ is nonvoid and $Z \cap \mathrm{int}\, Y = \emptyset$. Then there exists a non-zero $f \in E^*$ and an $\alpha \in \mathbb{R}$ with $f|Y \geq \alpha \geq f|Z$. For every such f we have $f|\,\mathrm{int}\, Y > \alpha$. In particular, $f|Y > \alpha > f|Z$ if Y and Z are open.

Proof. By Example 3.2 b) there exists a non-zero $f \in E'$ and an $\alpha \in \mathbb{R}$ with $f|Y \geq \alpha \geq f|Z$, and for every such f we have $f|\mathrm{int} Y = f|\mathrm{cor} Y > \alpha$. From Lemma 3.5 it follows that f is continuous. □

Example 3.7. Let E be a locally convex linear topological space, and let $Y, Z \in \mathcal{P}(E)$ be two disjoint convex subsets with Y compact and Z closed. Then the pair (Y,Z) is strongly E^*−separated. In particular, E^* is point separating.

Proof. By [31], Theorem 1.10, there exists a convex 0−neigborhood V such that $(Y + V) \cap (Z + V) = \emptyset$. Since $Y + V$ and $Z + V$ are open and convex, there exists an $f \in E^*$ and an $\alpha \in \mathbb{R}$ with $f|(Y+V) > \alpha > f|(Z+V)$ according to Example 3.6

and therefore $f|Y > \alpha > f|Z$. Since f is continuous and Y is compact, we arrive at $\min_Y f > \alpha \geq \sup_Z f$, so f strongly separates the pair (Y, Z). □

Example 3.8. Let E be a Euclidean space. Then the following holds:

a) $E' = E^*$, i.e., every linear function $f \in \mathbb{R}^E$ is continuous.
b) The paving of all compact convex subsets of E is strongly $E'-$separated.
c) The paving of all convex subsets of E is $E'-$separated.

Proof. a) This is obvious.
b) This follows from Example 3.7.
c) Let Y and Z be two nonvoid disjoint convex subsets of E. Then (Y, Z) is $E'-$separated iff $(Y - Z, \{0\})$ is $E'-$separated, where $Y - Z$ denotes the algebraic difference. Hence, w.l.g. we may assume $Z = \{0\}$. If 0 is contained in the closure clY, then $0 \in H := \text{lin } Y$, the linear hull of Y. In H we have cor $Y \neq \emptyset$ ([17], §2), and the assertion follows from Example 3.2 b), since every $g \in H'$ can be extended to an $f \in E'$. Otherwise, by Example 3.7, the pair (clY, $\{0\}$) is even strongly separated. □

Example 3.9. (Dragomirescu [6]). Let E be a linear topological space, and let $\mathcal{Y}$ and $\mathcal{Z}$ be two upward filtrating subpavings of $\mathcal{P}(E)$ such that one of the sets $Y \in \mathcal{Y}$ or $Z \in \mathcal{Z}$ has nonempty topological interior. Then the following are equivalent:

(a) Every pair of sets $(Y, Z) \in \mathcal{Y} \times \mathcal{Z}$ is E^*-separated.
(b) The sets $Y_0 := \bigcup\{Y : Y \in \mathcal{Y}\}$ and $Z_0 := \bigcup\{Z : Z \in \mathcal{Y}\}$ are E^*-separated.

Proof. (b) $\Longrightarrow$ (a) is obvious.
(a) $\Longrightarrow$ (b): W.l.g. we may assume that $\text{cor}_\nu Y = \text{int} Y$ is nonvoid for every $Y \in \mathcal{Y}$. As in the proof of Example 3.2 b) it follows that every pair $(Y, Z) \in \mathcal{Y} \times \mathcal{Z}$ is properly E^*-separated. By Theorem 2.8 there exists a non-zero $f \in E'$ and an $\alpha \in \mathbb{R}$ with $f|Y_0 \geq \alpha \geq f|Z_0$. By Lemma 3.5 f is continuous. □

Lemma 3.10. *Let S be a topological space and F a nonvoid equicontinuous subset of $C(S, \mathcal{F}(S))$. Then every real-valued function $g = \bigvee_{i \in I}(f_i - \alpha_i), f_i \in \mathsf{F}, \alpha_i \in \mathbb{R}, i \in I$, I an index set, is continuous.*

Proof. For $s \in S$ and $\epsilon > 0$ choose a neigborhood V of s such that

$$\sup_{f \in \mathsf{F}} \sup_{v \in V} |f(s) - f(v)| \leq \epsilon.$$

For fixed $v \in V$ (and for s) choose $i, j \in I$ with $g(s) \leq f_i(s) - \alpha_i + \epsilon$ and $g(v) \leq f_j(v) - \alpha_j + \epsilon$. Then we have

$$|g(v) - g(s)| \leq (f_j(v) - f_j(s)) \vee (f_i(s) - f_i(v)) + \epsilon \leq 2\epsilon.$$

□

Example 3.11. (Nehse [28]). Let E be a linear topological space and $Y, Z \in \mathcal{P}(E)$ such that Y is convex with nonvoid interior and $Z \cap$ int $Y = \emptyset$. Then there exists a continuous convex function $f : E \to \mathbb{R}$ with $f|Z \geq 0 \geq f|Y$ and $f|$ int $Y < 0$. For every such f we have $f|\text{int}Z > 0$.

Proof. Choose $y_0 \in \text{int } Y$. By Example 3.6 there exist $f_z \in E^*$ with $f_z(z) \geq \sup_Y f_z$ and $f_z(z) > f_z(y),\ y \in \text{int } Y,\ z \in Z$, and w.l.g. we may assume $f_z(z) - f_z(y_0) = 1,\ z \in Z$. Then we have $f_z(y - y_0) = f_z(y) - f_z(z) + 1 < 1,\ y \in \text{int } Y,\ z \in Z$. By Lemma 3.5 the family $\mathsf{F} = \{f_z : z \in Z\}$ is equicontinuous. Now take $f = \bigvee_{z \in Z}(f_z - f_z(z))$. For arbitrary $s \in S$ there exists a $\lambda \in (0, 1]$ with $\lambda s + (1 - \lambda)y_0 =: y \in Y$, and from $\lambda f_z(s) + (1 - \lambda)f_z(y_0) = f_z(y) \leq f_z(z)$ we infer $f(s) \leq \frac{1}{\lambda} - 1 < \infty$. By Lemma 3.10, f is continuous (and convex) and satisfies $f|Z \geq 0 \geq f|Y$ and $f(y_0) = -1$. Now let $y_1 \in \text{int}Y \setminus \{y_0\}$. Then there exists a $\lambda \in (0, 1)$ and a $y_2 \in Y$ with $y_1 = \lambda y_0 + (1 - \lambda)y_2$, which implies $f_z(y_1) - f_z(z) = -\lambda + (1 - \lambda)(f_z(y_2) - f_z(z)) \leq -\lambda, z \in Z$, and therefore $f(y_1) \leq -\lambda < 0$ by Lemma 1.3 a). □

Example 3.12. (Bălaj [2]). Let E be a linear topological space and $Y, Z \in \mathcal{P}(E)$ such that Y is open and convex and $\text{clY} \cap \text{Z} = \emptyset$. Then there exists a continuous convex function $f : E \to \mathbb{R}$ with $f|Z > 0 > f|Y$. If Z is compact, then $\inf_Z f > 0 > \sup_Y f$ can be acheived.

Proof. By Example 3.11 there exists a continuous convex function $f \in \mathbb{R}^E$ with $f|Y < 0 < f|(E \setminus \text{clY})$. If Z is compact, take $g = f - \frac{1}{2}\min_Z f$ instead. □

Example 3.13. (Bălaj [2]). Let E be a locally convex linear topological space and Y, Z two nonvoid disjoint subsets of E such that Y is closed and convex and Z is compact. Then there exists a continuous convex function $f : E \to \mathbb{R}$ with $\inf_Z f > 0 > \sup_Y f$.

Proof. As in the proof of Example 3.7 there exist disjoint open sets G, H with G convex, $G \supset Y$ and $H \supset Z$. By Example 3.12 there exists a continuous convex function $f : E \to \mathbb{R}$ with $\inf_Z f > 0 > \sup_G f\ (\geq \sup_Y f)$. □

3.3. Metric segment spaces

A triplet $(S, d, \langle\cdot,\cdot\rangle)$, where (S, d) is a metric space and $\langle\cdot,\cdot\rangle$ is a segment function for S, will be called a *metric segment space*.

A triplet (S, d, μ), where (S, d) is a metric space and μ is a segment structure for S, will be called a *structured metric segment space*. Here $(S, d, \langle\cdot,\cdot\rangle_\mu)$ is a metric segment space. We set $\mathbb{A}^*_\mu = \mathbb{A}_\mu \cap C(S, \mathcal{F}(S))$ for the linear space of all $\mu-$affine continuous functions $f : S \to \mathbb{R}$.

Example 3.14. In a structured metric segment space (S, d, μ) a pair (Y, Z) of nonvoid convex compact sets is strongly $\mathbb{A}^*_\mu-$separated iff it is pointwise $\mathbb{A}^*_\mu-$separated.

Proof. Apply Corollary 2.5 to $\mathcal{P} = \mathcal{F}(S), \mathcal{K} = \mathcal{K}(S) \cap \mathcal{C}_\mu$, and $\mathsf{F} = \mathbb{A}^*_\mu$. □

Remark 3.15. For a structured metric segment space (S, d, μ) the following are equivalent:

(i) The open balls $B^\circ(s, \rho) := \{t \in S : d(s, t) < \rho\}, s \in S, \rho > 0$ are convex.
(ii) The closed balls $B(s, \rho) := \{t \in S : d(s, t) \leq \rho\}, s \in S, \rho > 0$ are convex.
(iii) The metric d is separately quasiconvex, i.e., $d(s, \cdot) \in LSC(S, \mathcal{C}_\mu), s \in S$.
(iv) $d(z, \mu(s, t, \lambda)) \leq \max\{d(z, s), d(z, t)\}\ \forall(s, t, z) \in S^3, \lambda \in [0, 1]$.

In this case, we say that (S, d, μ) *has convex balls*.

A structured metric segment space (S, d, μ) will be said to have *Property UC* iff for any $z \in S$ and all sequences $(s_n), (t_n)$ in S with

$$\lim_{n\to\infty} d(z, \mu(s_n, t_n, \frac{1}{2})) = \lim_{n\to\infty} d(z, s_n) = \lim_{n\to\infty} d(z, t_n)$$

it follows that $\lim_{n\to\infty} d(s_n, t_n) = 0$.

The following lemma generalizes a result from [11], where it was proved for uniformly convex Banach spaces. (Cf. Example 3.29 below.)

Lemma 3.16. *Let (S, d, μ) be a complete structured metric segment space with convex balls and with Property UC. Then for the paving $\mathcal{K}$ of all closed, bounded and convex subsets of S the following holds.*

a) For each decreasing sequence (C_n) in $\mathcal{K} \setminus \{\emptyset\}$ and for each $z \in S$ with $\gamma := \sup_{n\in\mathbb{N}} d(z, C_n) < \infty$ there exists a unique point $\hat{x} \in \bigcap_{n\in\mathbb{N}} C_n$ with $d(z, \hat{x}) = \gamma$.

b) $\mathcal{K}$ is compact.

Proof. We adapt the proof from [11]:
a) Let $P_n := C_n \cap B(z, \gamma + \frac{1}{n})$. Then (P_n) is a decreasing sequence of nonempty closed sets. We show that $\lim_{n\to\infty} \delta(P_n) = 0$, where $\delta(P_n)$ denotes the diameter of P_n. Let $s_n, t_n \in P_n$ with $d(s_n, t_n) \geq \delta(P_n) - \frac{1}{n}$. Then $\mu(s_n, t_n, \frac{1}{2}) \in P_n$, since P_n is convex. Hence, $\{d(z, s_n), d(z, t_n), d(z, \mu(s_n, t_n, \frac{1}{2})\} \subset [d(z, C_n), \gamma + \frac{1}{n}]$ implies $\lim_{n\to\infty} d(s_n, t_n) = 0$, because (S, d) has Property UC. By Cantor's theorem there exists a point $\hat{x}$ with $\bigcap_{n\in\mathbb{N}} P_n = \{\hat{x}\}$, and from $d(z, C_n) \leq d(z, \hat{x}) \leq \gamma$, $n \in \mathbb{N}$, we infer $d(z, \hat{x}) = \gamma$. Conversely, for $\tilde{x} \in \bigcap_{n\in\mathbb{N}} C_n$ with $d(z, \tilde{x}) = \gamma$ we have $\tilde{x} \in \bigcap_{n\in\mathbb{N}} P_n = \{\hat{x}\}$.
b) Let $\mathcal{C}$ be a subpaving of $\mathcal{K}$ with the finite intersection property. We fix a $C_0 \in \mathcal{C}$ and a $z \in C_0$, and we set $C_{\mathcal{R}} := \bigcap_{R\in\mathcal{R}} R$, $\mathcal{R} \in \mathcal{E}_0 := \{\mathcal{Q} \in \mathcal{E}(\mathcal{C}) : C_0 \in \mathcal{Q}\}$. Since $z \in C_0$ and C_0 is bounded we have $\gamma := \sup_{\mathcal{R}\in\mathcal{E}_0} d(z, C_{\mathcal{R}}) \leq \sup_{s\in C_0} d(z, s) < \infty$. We choose an increasing sequence $\mathcal{R}_n$ in $\mathcal{E}_0$ with $d(z, C_{\mathcal{R}_n}) + \frac{1}{n} \geq \gamma$, $n \in \mathbb{N}$. By a) there is a unique point $\hat{x} \in \bigcap_{n\in\mathbb{N}} C_{\mathcal{R}_n}$ with $d(z, \hat{x}) = \gamma$.

Now let $\mathcal{R} \in \mathcal{E}_0$ be arbitrary, and let $C_n := C_{\mathcal{R}} \cap C_{\mathcal{R}_n}$. Then again (C_n) is a decreasing sequence in $\mathcal{K} \setminus \{\emptyset\}$ with $\sup_{n\in\mathbb{N}} d(z, C_n) = \gamma$, and by a) there exits an $\tilde{x} \in \bigcap_{n\in\mathbb{N}} C_n = C_{\mathcal{R}} \cap \bigcap_{n\in\mathbb{N}} C_{\mathcal{R}_n}$ with $d(z, \tilde{x}) = \gamma$. Hence, $\hat{x} = \tilde{x} \in \bigcap\{C_{\mathcal{R}} : \mathcal{R} \in \mathcal{E}_0\} = \bigcap\{C : C \in \mathcal{C}\}$. □

Example 3.17. Let (S, d, μ) be a complete structured metric segment space with convex balls and Property UC. Then a pair (Y, Z) of closed, bounded, and convex subsets of S is strongly $\mathbb{A}^*_\mu$−separated iff it is pointwise $\mathbb{A}^*_\mu$− separated.

Proof. Apply Corollary 2.5 together with Lemma 3.16. □

In [12] various examples in hyperbolic geometry can be found satisfying the assumptions of the above example. Classical examples are the Poincaré disc and the uniformly convex Banach spaces (cf. Examples 3.24 and 3.29 below).

3.3.1. Metrically convex metric spaces

Every metric space (S, d) can be endowed with the *geodesic segments*

$$\langle x, y\rangle_d = \{s \in S : d(x, s) + d(s, y) = d(x, y)\},\ x, y \in S.$$

Here $(S, d, \langle\cdot,\cdot\rangle_d)$ is a metric segment space. A subset $T \subset S$ is called *d−convex* iff $\langle x, y\rangle_d \subset T$ for all $\{x, y\} \subset T$.

Let $\mathcal{C}_d$ denote the paving of all $d-$convex subsets of S and C_d the $\vee_a-$closed convex cone of all functions $f : S \to \mathbb{R}^\bullet$ which are *d–convex*, i.e., with

$$f(s_0)d(s_1, s_2) \leq f(s_1)d(s_0, s_2) + f(s_2)d(s_0, s_1) \quad \text{for all } s_0, s_1, s_2 \in S \text{ with } s_0 \in \langle s_1, s_2\rangle_d. \tag{5}$$

We set A_d for the linear space of all functions $f : S \to \mathbb{R}$ which are *d–affine*, i.e. where relation (5) holds with equality, and $\mathbb{A}_d^* := \mathbb{A}_d \cap C(S, \mathcal{F}(S))$ for the linear space of all continuous $d-$affine functions $f : S \to \mathbb{R}$.

A metric space (S, d) is called

- *Menger–convex* iff $\langle x, y\rangle_d \setminus \{x, y\} \neq \emptyset$ for all $x, y \in S$ with $x \neq y$.
- *(strictly) metrically convex* iff it can be endowed with a (unique) segment structure μ such that

$$d(s, \mu(s, t, \lambda)) = (1 - \lambda)d(s, t) \text{ and } d(t, \mu(s, t, \lambda)) = \lambda d(s, t) \quad \text{for all } (s, t, \lambda) \in S \times S \times [0, 1]. \tag{6}$$

- *strongly metrically convex* iff it can be endowed with a segment structure μ such that

$$d(\mu(s, t, \alpha), \mu(s, t, \beta)) = |\alpha - \beta|d(s, t), \ s, t \in S, \ \alpha, \beta \in [0, 1]. \tag{7}$$

A *(strictly) convex structured metric segment space* is a structured metric segment space (S, d, μ) such that μ (and no other segment structure) satisfies relation (6).

A *strongly convex structured metric segment space* is a structured metric segment space (S, d, μ) such that μ satisfies relation (7).

Remark 3.18. Let (S, d, μ) be a convex structured metric segment space. Then

a) μ is reflexive, and μ is symmetric if (S, d) is strictly metrically convex,

b) (i) $\langle s, t\rangle_\mu \subset \langle s, t\rangle_d, \ (s, t) \in S^2$, hence $\mathcal{C}_d \subset \mathcal{C}_\mu$,
and
(ii) $\langle s, t\rangle_\mu = \langle s, t\rangle_d, \ (s, t) \in S^2$, iff (S, d) is strictly metrically convex. Especially, $\mathcal{C}_d = \mathcal{C}_\mu$ in this case.

c) (i) Every $d-$convex function $f : S \to \mathbb{R}^\bullet$ is $\mu-$convex,
and
(ii) $\mathbb{C}_d = \mathbb{C}_\mu \cap \mathbb{R}^{\bullet S}$ if (S, d) is strictly metrically convex.

d) If $\mathbb{A}_d$ is point separating, then (S, d) is strictly metrically convex.

Proof. a), b) (i), and c) (i) are obvious.
b) (ii): Let (S, d) be strictly metrically convex. For $(s, t) \in S^2$ with $s \neq t$ and $x \in \langle s, t\rangle_d \setminus \{s, t\}$ set $\lambda = d(t, x)/d(s, t)$. Then, by (6), we have $d(t, \mu(s, t, \lambda)) = \lambda d(s, t) = d(t, x)$ and $d(s, \mu(s, t, \lambda)) = (1 - \lambda)d(s, t) = d(s, x)$, which implies $x = \mu(s, t, \lambda) \in \langle s, t\rangle_\mu$. Together with b (i) we obtain $\langle\cdot,\cdot\rangle_\mu = \langle\cdot,\cdot\rangle_d$.
Conversely, let $\langle\cdot,\cdot\rangle_\mu = \langle\cdot,\cdot\rangle_d$ be satisfied, and let $\tilde{\mu}$ be another segment structure satisfying relation (6). Let $s, t \in S, \ \lambda \in [0, 1]$ and $\tilde{x} = \tilde{\mu}(s, t, \lambda)$. Then $\tilde{x} \in \langle s, t\rangle_{\tilde{\mu}} \subset \langle s, t\rangle_d = \langle s, t\rangle_\mu$ implies $\tilde{x} = \mu(s, t, \hat{\lambda})$ for some $\hat{\lambda} \in [0, 1]$. Now $d(t, \tilde{x}) = \lambda d(s, t) = \hat{\lambda} d(s, t)$ implies $\hat{\lambda} = \lambda$,

i.e. $\mu(s,t,\lambda) = \tilde{\mu}(s,t,\lambda)$ in case $s \neq t$. Together with a) we obtain $\tilde{\mu} = \mu$.
c) (ii): Let f be $\mu-$convex. For s,t,x and λ as in the proof of b) (ii) we have $f(x)d(s,t) = f(\mu(s,t,\lambda))d(s,t) \leq (\lambda f(s) + (1-\lambda)f(t))d(s,t) = f(s)d(t,x) + f(t)d(s,x)$, i.e., f is $d-$convex.
d) Let μ' and μ'' be segment structures satisfying relation (6). Then for $f \in \mathbb{A}_d$ and $(s,t,\lambda) \in S \times S \times [0,1]$ with $s \neq t$ we have

$$f(\mu'(s,t,\lambda))d(s,t) = f(s)d(\mu'(s,t,\lambda),t) + f(t)d(\mu'(s,t,\lambda),s) =$$

$$f(s)\lambda d(s,t) + f(t)(1-\lambda)d(s,t) = f(\mu''(s,t,\lambda))d(s,t).$$

Since $\mathbb{A}_d$ separates points, we obtain $\mu'(s,t,\lambda) = \mu''(s,t,\lambda)$. Hence, $\mu' = \mu''$ by a). □

Lemma 3.19. *For a metric space (S,d) we have (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) for the conditions:*

(a) (S,d) *is strongly metrically convex.*
(b) (S,d) *is metrically convex.*
(c) (S,d) *is Menger–convex.*

If (S,d) is complete, then the three conditions are equivalent.

Proof. (a) $\Longrightarrow$ (b): Apply (7) with $\alpha \in \{0,1\}$ and $\beta = \lambda$.
(b) $\Longrightarrow$ (c) is obvious.
(c) $\Longrightarrow$ (a): Now let (S,d) be complete. By a theorem of Menger ([26], p. 89) for every pair $(s,t) \in S^2$ with $s \neq t$ there exists a map $\varphi = \varphi_{s,t} : [0, d(s,t)] \to S$ with $\varphi(0) = t$, $\varphi(d(s,t)) = s$ and

$$d(\varphi(\alpha d(s,t)), \varphi(\beta d(s,t))) = |\alpha - \beta| d(s,t), \ (\alpha,\beta) \in [0,1]^2.$$

Hence, $\mu(s,t,\lambda) = \varphi_{s,t}(\lambda d(s,t))$ has property (7). □

Example 3.20. In a complete Menger–convex metric space (S,d) a pair (Y,Z) of nonvoid $d-$convex compact subsets of S is strongly $\mathbb{A}_d^*-$separated iff it is pointwise $\mathbb{A}_d^*-$separated.

Proof. According to Lemma 3.19 (S,d) can be endowed with a metric segment structure μ satisfying relation (7). By Remark 3.18 we have $\mathcal{C}_d \subset \mathcal{C}_\mu$ and $\mathbb{A}_d^* \subset \mathbb{A}_\mu^*$. Now the assertion follows from Corollary 2.5 applied to $\mathcal{P} = \mathcal{F}(S), \mathcal{K} = \mathcal{C}_\mu \cap \mathcal{K}(S)$ and $\mathsf{F} = \mathbb{A}_d^*$. □

3.3.2. Takahashi–convex metric spaces

A metric space (S,d) is called *(strictly) Takahashi–convex* iff there exists a (unique) function $\mu : S \times S \times [0,1] \to S$ such that

$$d(z, \mu(x,y,\lambda)) \leq \lambda d(z,x) + (1-\lambda)d(z,y), \ (x,y,z) \in S^3, \ \lambda \in [0,1]. \tag{8}$$

In this case, μ is called a *Takahashi convex structure* [38, 39].

A *(strictly) Takahashi-convex structured metric segment space* is a structured metric segment space (S,d,μ) such that μ (and no other Takahashi convex structure) satisfies relation (8).

Remark 3.21. Every Takahashi convex structure is a reflexive segment structure. Hence, a metric space (S,d) is (strictly) Takahashi–convex iff it can be endowed with a (unique) segment structure μ such that the metric d is separately $\mu-$convex in both variables. In particular, every Takahashi-convex structured metric segment space has convex balls.

Example 3.22. Let (S, d, μ) be a complete Takahashi–convex structured metric segment space with Property UC. Then a pair (Y, Z) of closed, bounded and convex subsets of S is strongly $\mathbb{A}^*_\mu$–separated ($\mathbb{A}^*_d$–separated) iff it is pointwise $\mathbb{A}^*_\mu$–separated ($\mathbb{A}^*_d$–separated).

Proof. Apply Corollary 2.5 together with Lemma 3.16 and Remarks 3.18 c) and 3.21. □

Remark 3.23. a) (Takahashi [38]) Every Takahashi–convex structured metric segment space (S, d, μ) is a convex structured metric segment space, i.e., μ satisfies relation (6).

b) (Talman [39]) Every strictly Takahashi–convex structured metric segment space (S, d, μ) is a strongly convex structured metric segment space i.e., μ satisfies relation (7).

c) A structured metric segment space (S, d, μ) is strictly Takahashi–convex if it is Takahashi–convex and (S, d) is strictly metrically convex. The converse is not true, in general. (Compare Example 3.31 below).

Example 3.24. (The Poincaré disc) The open disc $D = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane with the Poincaré metric

$$d(z, w) = \tanh^{-1}\left|\frac{z - w}{1 - z\bar{w}}\right|$$

is a complete, strictly metrically convex ([12]; 2.4), and Takahashi convex ([12]; Lemma 6.8) metric space with Property UC ([12]; 2.5).

Example 3.25. A metric space (S, d) is said to be *Ptolemaic* provided that for each quadruple $(w, x, y, z) \in S^4$ the inequality

$$d(z, w)d(x, y) \leq d(z, x)d(y, w) + d(z, y)d(x, w) \tag{9}$$

holds.

In a Ptolemaic metric space the metric d is separately d–convex in both variables, hence the family $\mathbb{C}^*_d$ of continuous d–convex functions $f : S \to \mathbb{R}$ is point separating, and the converse of Remark 3.23 a) is also true:

If a Ptolemaic metric space (S, d) is endowed with a segment structure μ, then μ is a Takahashi convex structure if (and only if) (S, d, μ) is a convex structured metric segment space. (Set $w = \mu(x, y, \lambda)$ in (9)).

Examples of Ptolemaic metric spaces are Hilbert spaces (cf. Example 3.28 below) and hyperbolic spaces [21].

3.4. Normed linear spaces

In the sequel, $(E, \|\cdot\|)$ will be be a normed linear space endowed with the induced metric $d(s, t) = \|s - t\|$ and the natural segment structure $\nu(s, t, \lambda) = \lambda s + (1 - \lambda)t$.

Remark 3.26. (Bilyeu ([3]) In a normed linear space $(E, \|\cdot\|)$ the natural segment structure ν is the unique Takahashi convex structure.

Example 3.27. For a normed linear space $(E, \|\cdot\|)$ the following are equivalent:

(a) $(E, \|\cdot\|)$ is strictly normed, i.e.,

$$\|s+t\| = \|s\| + \|t\| \text{ and } t \neq 0 \text{ implies } s = \alpha t \text{ for some } \alpha \geq 0.$$

(b) (E, d) is strictly metrically convex.
(c) $(E, \|\cdot\|)$ is tense, i.e., $\langle\cdot,\cdot\rangle_\nu = \langle\cdot,\cdot\rangle_d$.
(d) A subset of E is convex iff it is $d-$convex.
(e) A function $f : E \to \mathbb{R}^\bullet$ is convex iff it is $d-$convex.
(f) $\mathbb{A}_d = E' + \mathbb{R}$.
(g) $\mathbb{A}_d$ is point separating.
(h) $\mathbb{A}_d^* = E^* + \mathbb{R}$.
(i) $\mathbb{A}_d^*$ is point separating.
(j) The paving of all compact convex subsets of E is strongly $\mathbb{A}_d^*-$separated.

The equivalence (a) $\iff$ (c) $\iff$ (d) is Theorem 11.2 in [36]. Compare also [10], Theorem 1.

Proof. (a) $\Longrightarrow$ (b): Of course, $\mu = \nu$ satisfies condition (6). Conversely, let $s, t \in S$ with $s \neq t$ and let μ satisfy (6). Then for $x = \mu(s,t,\lambda), \lambda \in (0,1)$ with $x \neq t$, say, we have $\|s - x\| + \|x - t\| = \|(s-x) + (x-t)\|$, hence $s - x = \alpha(x - t)$ for some $\alpha \geq 0$. In case $\lambda = 1$ we have $x = s = \nu(s,t,1)$. Otherwise, from $\|s - t\| = (1-\lambda)^{-1}\|s - x\| = (1-\lambda)^{-1}\alpha\|x - t\| = (1-\lambda)^{-1}\alpha\lambda\|s-t\|$ we obtain $x = \nu(s,t,\lambda)$, i.e., $\mu = \nu$.
(b) $\Longrightarrow$ (c), (e) and (g) $\Longrightarrow$ (b) follows from Remark 3.18.
(e) $\Longrightarrow$ (f) $\Longrightarrow$ (g) follows with Example 1.5.
(d) $\Longrightarrow$ (c): $\langle s,t\rangle_\nu$ is $\nu-$convex and therefore $d-$convex. Hence $\{s,t\} \subset \langle s,t\rangle_\nu$ implies $\langle s,t\rangle_d \subset \langle s,t\rangle_\nu$. The converse inclusion holds by Remark 3.18.
(c) $\Longrightarrow$ (a): From $\|s+t\| = \|s\| + \|t\|$ we infer $s \in \langle 0, s+t\rangle_d$. Hence, by (c) there exists a $\lambda \in [0,1]$ with $s = \lambda \cdot 0 + (1-\lambda)(s+t)$.
(h) $\Longrightarrow$ (i) follows with Example 3.7.
(i) $\Longrightarrow$ (j): Apply Corollary 2.5 to $\mathcal{P} = \mathcal{F}(\mathcal{S}), \mathcal{K} = \mathcal{K}(\mathcal{S}) \cap \mathcal{C}_\nu$, and $\mathsf{F} = \mathbb{A}_d^*$.
(c) $\Longrightarrow$ (d), (f) $\Longrightarrow$ (h) and (j) $\Longrightarrow$ (i) $\Longrightarrow$ (g) are obvious. □

Example 3.28. ([20, 21, 33]) For a normed linear space $(E, \|\cdot\|)$ the following are equivalent:

(a) (E, d) is Ptolemaic.
(b) $(E, \|\cdot\|)$ is symmetric, i.e., $\|\lambda x - y\| = \|x - \lambda y\|$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and all $\lambda \in \mathbb{R}$.
(c) $(E, \|\cdot\|)$ satisfies the parallelogram law

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

(d) There exists an inner product $\langle\cdot,\cdot\rangle$ on E with $\|x\| = \sqrt{\langle x, x\rangle},\ x \in E$.

The implication "(b) $\Longrightarrow$ (d)" in the following example was proved by Granas and Lassonde [11]:

Example 3.29. Let $(E, \|\cdot\|)$ be a Banach space and $\mathcal{K}$ the paving of all bounded, closed, convex subsets of E. Then the implications (a) $\Longrightarrow$ (b) $\iff$ (c) $\Longrightarrow$ (d) $\Longrightarrow$ (e) hold for the conditions:

(a) $(E, \|\cdot\|)$ satisfies the parallelogram law.
(b) $(E, \|\cdot\|)$ is uniformly convex, i.e., for all sequences $(x_n), (y_n)$ in E with $\|x_n\| \to 1$, $\|y_n\| \to 1$ and $\|x_n + y_n\| \to 2$ one has $\|x_n - y_n\| \to 0$.
(c) (E, d, ν) has Property UC.
(d) $\mathcal{K}$ is compact.
(e) $\mathcal{K}$ is strongly E^*−separated.

Proof. (a) $\Longrightarrow$ (b) $\Longleftrightarrow$ (c) is obvious.
(c) $\Longrightarrow$ (d): This follows from Lemma 3.16.
(d) $\Longrightarrow$ (e): By Example 3.7, E^* is point separating. Now apply Corollary 2.5 with $\mathcal{P} = \mathcal{F}(S)$ and $\mathsf{F} = E^*$. □

As a consequence we obtain the following well-known result.

Example 3.30. Let H be a Hilbert space. Then the paving of all bounded, closed, convex subsets of H is strongly H^*−separated.

Example 3.31. We endow $\mathbb{R}^n$ with the metric d_n induced by the sum-norm $\|(s_1, \ldots, s_n)\|_n = \sum_{i=1}^n |s_i|$. Let ν_n denote the natural segment structure on $\mathbb{R}^n$. Here the metric segments are the boxes

$$\langle s, t\rangle_{d_n} = \langle s_1, t_1\rangle_{\nu_1} \times \ldots \times \langle s_n, t_n\rangle_{\nu_1}, \ s = (s_1, \ldots, s_n), t = (t_1, \ldots .t_n).$$

By Remark 3.26, $(\mathbb{R}^n, d_n)$ is strictly Takahashi convex with (unique) Takahashi convex structure $\mu = \nu_n$, and therefore strongly metrically convex according to Remark 3.23 b).

Joó and Stachó proposed in [19] to endow $\mathbb{R}^n$ with the segment function $\langle\cdot,\cdot\rangle_n$, inductively defined as follows:
For $n = 1$ and $s_1, t_1 \in \mathbb{R}$ let $\langle s_1, t_1\rangle_1 = \langle s_1, t_1\rangle_{\nu_1}$.
Let $\langle\cdot,\cdot\rangle_n$ be defined. For $s = (s_1, \ldots, s_n, s_{n+1})$ and $t = (t_1, \ldots, t_n, t_{n+1}) \in \mathbb{R}^{n+1}$ we set $s^n = (s_1, \ldots, s_n)$ and $t^n = (t_1, \ldots, t_n)$, and we define

$$\langle s, t\rangle_{n+1} = (\{s^n\} \times \langle s_{n+1}, t_{n+1}\rangle_1) \cup (\langle s^n, t^n\rangle_n \times \{t_{n+1}\})$$

in case $s_{n+1} \le t_{n+1}$. In case $s_{n+1} > t_{n+1}$ we set $\langle s, t\rangle_{n+1} = \langle t, s\rangle_{n+1}$ defined as above.

Similarly we define a segment structure μ_n for $\mathbb{R}^n$, $n \in \mathbb{N}$, as follows:
For $n = 1$ we set $\mu_1 = \nu_1$, hence $\langle\cdot,\cdot\rangle_1 = \langle\cdot,\cdot\rangle_{\nu_1} = \langle\cdot,\cdot\rangle_{\mu_1}$.
Let μ_n be defined. Let $s = (s_1, \ldots, s_n, s_{n+1}), t = (t_1, \ldots, t_n, t_{n+1}) \in \mathbb{R}^{n+1}$ and $\lambda \in [0, 1]$.
In case $s \ne t$ and $t_{n+1} \ge s_{n+1}$ we put $\alpha = \|s - t\|_{n+1}$ and $\beta = \|s^n - t^n\|_n$, and we set $\mu_{n+1}(s, t, \lambda) = \begin{cases} (s^n, s_{n+1} + (1-\lambda)\alpha) & : \ \lambda \ge \beta\alpha^{-1} \\ (\mu_n(s^n, t^n, \lambda\alpha\beta^{-1}), t_{n+1}) & : \ \lambda < \beta\alpha^{-1}. \end{cases}$ In case $t_{n+1} < s_{n+1}$ we set $\mu_{n+1}(s, t, \lambda) = \mu_{n+1}(t, s, 1 - \lambda)$.

Suppose that $\langle\cdot,\cdot\rangle_n = \langle\cdot,\cdot\rangle_{\mu_n}$ holds.

Then in case $t_{n+1} \ge s_{n+1}$ we have

$$\mu_{n+1}(s, t, [0, \beta\alpha^{-1}]) = \langle s^n, t^n\rangle_{\mu_n} \times \{t_{n+1}\},$$

and

$$\mu_{n+1}(s, t, [\beta\alpha^{-1}, 1]) = \{s^n\} \times \langle s_{n+1}, t_{n+1}\rangle_1$$

which implies $\langle s,t\rangle_{n+1} = \langle s,t\rangle_{\mu_{n+1}}$.

In case $t_{n+1} < s_{n+1}$ we have $\langle s,t\rangle_{n+1} = \langle t,s\rangle_{n+1} = \langle t,s\rangle_{\mu_{n+1}} = \langle s,t\rangle_{\mu_{n+1}}$.

Hence, $\langle\cdot,\cdot\rangle_n = \langle\cdot,\cdot\rangle_{\mu_n}, n \in \mathbb{N}$.

A short calculation shows that relation (7) and therefore relation (6) is satisfied for $\mu = \mu_n$. Thus, $(\mathbb{R}^n, d_n)$ is not strictly metrically convex for $n \geq 2$.

Of course, μ_n is a reflexive and symmetric segment structure, but, for $n \geq 2$, μ_n is neither cancellative nor associative.

To see this, take $x = 0, y = e^1 + e^2$ and $z = -e^1 + e^2$, where e^i denotes the $i-$th unit vector in $\mathbb{R}^n$. Then we have

$$\mu_n(x,y,\frac{1}{2}) = (\mu_2((0,0),(1,1),\frac{1}{2}),0,\ldots,0) = e^2 = \mu_n(x,z,\frac{1}{2}),$$

and for $\lambda = \frac{1}{3}$ and $\tau = \frac{1}{2}$, say, we obtain

$$\mu_n(\mu_n(x,y,\lambda[\lambda+(1-\lambda)\tau]^{-1}),z,\lambda+(1-\lambda)\tau) = \mu_n(e^2,z,\frac{2}{3}) =$$

$$-\frac{1}{3}e^1 + e^2 \neq \frac{2}{3}e^2 = \mu_n(x,e^2,\frac{1}{3}) = \mu_n(x,\mu_n(y,z,\tau),\lambda).$$

Moreover, $\mu_n(x,\mu_n(y,z,\frac{1}{2}),\tau_1) = \mu_n(y,\mu_n(x,z,\frac{1}{2}),\tau_2)$ implies $\tau_1 = \tau_2 = 0$, i.e., the Pasch Property is violated.

Now we show that every real-valued μ_n-affine function is constant.
First let $f \in \mathbb{A}_{\mu_2}$. Then f is separately (ν_1-)affine in both variables, i.e.,

$$f(s_1,s_2) = as_1s_2 + bs_1 + cs_2 + d$$

with $(a,b,c,d) \in S^4$. From

$$2f(0,\xi) = 2f(\mu_2((\xi,\xi),(0,0)\frac{1}{2})) = f(\xi,\xi) + f(0,0),\ \xi > 0$$

and

$$2f(1,2) = f(1,1) + f(1,-1)$$

we infer $a = b = c = 0$, i.e., $\mathbb{A}_{\mu_2} = \mathbb{R}$.
Now $\mathbb{A}_{\mu_n} = \mathbb{R}$ follows easily by induction. In particular, by Remark 3.18 c), every d_n-affine function $f : \mathbb{R}^n \to \mathbb{R}$ is constant.
On the other hand, there are μ_n-convex subsets S of $\mathbb{R}^n$ admitting nontrivial μ_n-affine functions $f : S \to \mathbb{R}$.

Similar to Lemma 3.3 above, there exists an imbedding theorem for structured metric segment spaces into normed spaces.

A structured metric segment space (S,d,μ) will be said to satisfy the *Theorem of Proportional Segments* iff

$$d(\mu(x,z,\alpha),\mu(y,z,\alpha)) = \alpha d(x,y),\ x,y,z \in S, \alpha \in [0,1] \tag{10}$$

holds. Obviously, (10) implies that μ is cancellative.

Lemma 3.32. *(Machado [25], cf. also Andalafte and Blumenthal [1]) For a structured metric segment space (S, d, μ) the following are equivalent:*

(a) *There exists a normed linear space $(E, \|\cdot\|)$, a convex subset C of E and an affine isometry $\varphi : (S, \mu) \to (C, \nu)$, where ν denotes the natural segment structure.*
(b) *The segment structure μ is a symmetric and associative Takahashi convex structure, and (S, d, μ) satisfies the Theorem of Proportional Segments.*
(c) *The segment structure μ is symmetric and associative, and (S, d, μ) has convex balls and satisfies the Theorem of Proportional Segments.*
(d) *The segment structure μ is a convexor, and (S, d, μ) satisfies the Theorem of Proportional Segments.*
(e) *$\mathbb{A}_\mu$ is point separating, and (S, d, μ) satisfies the Theorem of Proportional Segments.*
(f) *$\mathbb{A}^*_\mu$ is point separating, and (S, d, μ) satisfies the Theorem of Proportional Segments.*

Proof. (a) $\Longrightarrow$ (f): Obviously $(C, \|\cdot\|, \nu)$ satisfies the Theorem of Proportional Segments. This carries over to (S, d, μ). By Example 3.7, E^* and therefore $\{g \circ \varphi : g \in E^*\} \subset \mathbb{A}^*_\mu$ are point separating.
(f) $\Longrightarrow$ (e) is obvious.
(e) $\Longrightarrow$ (d) follows from Lemma 3.3.
(b) $\Longrightarrow$ (c) follows from Remark 3.21.
(c) $\Longrightarrow$ (d): From condition (iv) in Remark 3.15 it follows that μ is reflexive.
(d) $\Longrightarrow$ (b) $\Longrightarrow$ (a): This follows from [25] (Theorem 1 and p. 319 f), since every symmetric associative segment function satisfies Machado's Property (B) [25]. □

Example 3.33. Let (S, d, μ) be a structured metric segment space satisfying the Theorem of Proportional Segments, and let μ be a convexor. Let Y, Z be disjoint nonvoid convex and closed subsets of S with Y compact. Then the pair (Y, Z) is strongly A^*_μ−separated.

Proof. Choose an affine isometry $\varphi : (S, \mu) \to (C, \nu)$ according to Lemma 3.32 (a). Then the sets $\varphi(Y)$ and $\varphi(Z)$ are convex by Remark 1.4 b), $\varphi(Y)$ is compact and $\varphi(Z)$ is closed. By Example 1.9 there exists a $g \in E^*$ separating $(\varphi(Y), \varphi(Z))$ strongly, and therefore, $f = g \circ \varphi \in \mathbb{A}^*_\mu$ (by Remark 1.4 c)) separates (Y, Z) strongly □

Example 3.34. Let (S, d, μ) be a structured metric segment space satisfying the Theorem of Proportional Segments. Let μ be a convexor, and let (S, d) be complete with Property UC. Then the paving of all convex, closed, and bounded subsets of S is strongly A^*_μ−separated.

Proof. Apply Lemma 3.32 "(d) $\Longrightarrow$ (c), (f)" together with Example 3.17. □

Example 3.35. Let (S, d, μ) be a strictly convex structured metric segment space satisfying the Theorem of Proportional Segments. If μ is associative, then the paving of all convex compact subsets of S is strongly A^*_d−separated. If, moreover, (S, d) is complete and has Property UC, then the paving of all closed, bounded, and convex subsets of S is strongly A^*_d−separated.

Proof. By Remark 3.18 a) and c) μ is a convexor and $A^*_d = A^*_\mu$. Hence the assertion follows with Examples 3.33 and 3.34, respectively. □

4. Concluding remark

As in the classical case our abstract separation theorems have a broad range of applicability. They can be used to derive hull theorems such as bipolar theorems or Krein–Milman type theorems, abstract versions of Bauer's minimum principle, Helly and Klee type intersection theorems, minimax theorems and many other results. It is intended to treat these topics elsewhere.

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