Journal of Convex Analysis
Volume 9 (2002), No. 2, 363–399

# Limits of Discrete Systems with Long-Range Interactions

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Received September 22, 2000

We prove that under some structure and decay hypotheses, limits of discrete systems with multi-neighbourhood interactions give rise to local energies with bulk and surface part, and that the limit energy densities can be recovered through a scaling and superposition procedure.

*Keywords:* Γ-limit, thin films, homogenization

*1991 Mathematics Subject Classification:* 35E99, 35M10, 49J45, 73C50, 73M25

## 1. Introduction

In this paper we consider the problem of the description of the continuum limit of discrete systems with long-range interactions defined on a cubic lattice as the lattice parameters tend to 0. Consider a domain $\Omega$ in $\mathbb{R}^N$, which will parameterize the limit continuum region, and the portion $Z_n$ of the lattice $\lambda_n\mathbb{Z}^N$ of step size $\lambda_n$ contained in $\Omega$. Let $u$ be a function defined in $Z_n$. If $u : Z_n \to \mathbb{R}^N$ then we may interpret $u(x)$ as the displacement of a particle parameterized by $x \in Z_n$. The interaction between each pair of points $x, y$ in $Z_n$ will be described by an energy $\Psi_n$ depending on $u(x)$ and $u(y)$, and on the mutual position of the points in the lattice. The total energy of the interactions among points of this discrete system described by $u$ is then given by the sum of these pairwise interactions, which we can write in the form

$$\mathcal{H}_n(u) = \sum_{x,y\in Z_n,\ x\neq y} \Psi_n(u(x) - u(y), x - y).$$

In order to describe the continuum limit of these energies we identify the discrete displacements $u$ with functions defined on $\Omega$ which are constant on each cube of side length $\lambda_n$ with vertices on the lattice $\lambda_n\mathbb{Z}^N$. We denote by $\mathcal{A}_n(\Omega)$ the set of these functions, and we regard the functional $\mathcal{H}_n$ to be defined as above on $\mathcal{A}_n(\Omega)$ interpreted as a subset of $L^1(\Omega;\mathbb{R}^N)$. We can thus apply the techniques of Γ-convergence for energies defined on $L^1(\Omega;\mathbb{R}^N)$. We recall that the Γ-convergence of a sequence of functionals is in a sense equivalent to the study of the convergence of all minimum problems involving these functionals and their continuous perturbations (see [16], [11] Part II).

ISSN 0944-6532 / $ 2.50 

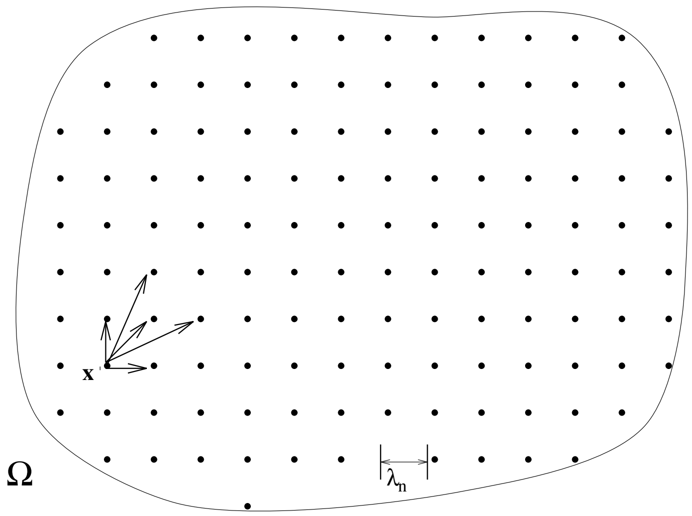


Figure 1.1: interactions on the lattice $Z_n$

The main result of this paper is showing that, under some qualitative hypotheses on the dependence of the energy densities $\Psi_n$ on $u(x) - u(y)$ and under some quantitative hypotheses on their dependence on $x-y$, the limit of the energies $\mathcal{H}_n$ gives a local energy $\mathcal{H}$ defined on functions which may have a discontinuity along a hypersurface. These energies contain a bulk term and a surface term accounting for fracture. We show that the domain of the functional $\mathcal{H}$ is the space $GSBV(\Omega)$ of generalized special functions of bounded variation, where it can be written in the form

$$\mathcal{H}(u) = \int_\Omega \mathcal{F}(\nabla u)\,dx + \int_{S(u)} \mathcal{G}([u], \nu_u) d\mathcal{H}^{N-1}.$$

The space $GSBV(\Omega)$ has been introduced by De Giorgi and Ambrosio [17] to give a variational framework for energies in fracture mechanics and computer vision (see the book by Ambrosio, Fusco and Pallara [5] for a complete introduction to $GSBV$ and free-discontinuity problems). We recall that the set $S(u)$ is the $(N-1)$-dimensional set of discontinuity points for $u$, with normal $\nu_u$, while $\nabla u$ denotes the (approximate) gradient of $u$, which is defined on $\Omega \setminus S(u)$ and hence a.e. on $\Omega$, and $[u]$ is the jump of $u$ across $S(u)$. We recall that in the terminology of fracture mechanics $S(u)$ can be interpreted as the fracture site, while $u$ describes the displacement on the uncracked region, so that $\mathcal{F}$ is a bulk energy density, while $\mathcal{G}$ is a fracture-initiation energy density (see [4]). Our work is connected to a previous use of $\Gamma$-convergence techniques to derive a continuous model for fracture and softening problems from a discrete approach by Braides, Dal Maso and Garroni [10], where only nearest-neighbour interactions were taken into account in the 1-dimensional case, and to the description of finite-difference approximation of free-discontinuity problems by Gobbino [20] (see also [21] and [14]), where only a special class of interaction potentials is taken into account. The idea of a passage from a discrete to a continuous setting using implicitly a variant of $\Gamma$-convergence is also present in the earlier work by Truskinovsky [24].

Under only suitable growth assumptions on $\Psi_n$ the description of the limit energy densities involves complex limit formulas (see [13], [12], [23] for the one-dimensional case), whose explicit computation is not easy. We consider further assumptions on $\Psi_n$, under which the form of $\mathcal{F}$ and $\mathcal{G}$ is explicit (for a particular case see [14]) and is derived from a *superposition principle*. This principle also applies with some variants to obtain an approximation of Griffith energy functionals in Fracture Mechanics (see [1]). Note that by comparison, our results describe the limit domain and give some bounds also for more general energies for which such principle does not apply (for an application to scaled Lennard-Jones potentials see [12]).

We will treat the case of scalar-valued $u$ only, as in this case the hypotheses on $\Psi_n$ are quite general. Some of our techniques carry on to vector-valued $u$; for example if $\Psi_n(z,w)$ depends on $z$ only through $|z|$. The typical shape of a function $z \mapsto \Psi_n(z,w)$ which satisfies our hypotheses is convex in an interval $[T_n^+(w), T_n^-(w)]$, and concave on the two remaining half-lines. Our hypotheses on the dependence of $\Psi_n$ on $w$ amount essentially

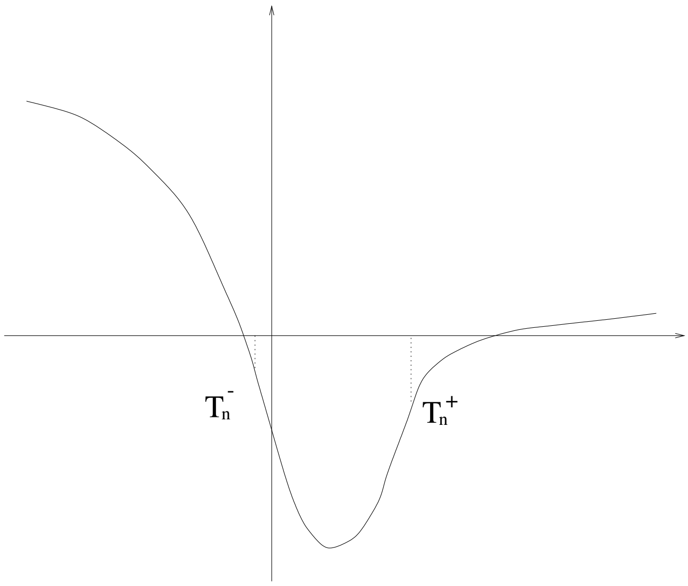


Figure 1.2: the typical shape of $\Psi_n(\cdot,w)$

to supposing that the effect of $\Psi_n(\cdot,w)$ decreases with $w$ in such a way as to avoid non-local effects on the limiting energy (see [7] for an example of a non-local limit), and are satisfied, for example, when we consider only a finite number of interactions. We will also assume the technical hypothesis that for all $w$ we have $T_n^+(w) \to +\infty$ and $T_n^-(w) \to -\infty$ as $n \to +\infty$, to ensure that no interaction occurs in $\mathcal{H}$ between the bulk and the surface parts, so as to avoid further complications in the description of $\mathcal{H}$. The effect of the interaction between bulk and surface energies in $\mathcal{H}$ in the 1-dimensional case when $T_n^+$ remains bounded can be found in [10].

In our hypotheses the integrands $\mathcal{F}$ and $\mathcal{G}$ can be recovered by examining the convex and concave parts of $\Psi_n$ separately. With fixed $w \in \mathbb{Z}^N \setminus \{0\}$, denote by $F_w$ the pointwise limit of $\Psi_n(\cdot,\lambda_n w)$ and with $G_w$ the limit of the scaled functions

$$G_{w,n}(z) = |w|\lambda_n \Psi_n\Big(\frac{z}{|w|\lambda_n}, \lambda_n w\Big).$$

Note that it is not restrictive by a compactness argument to suppose that both limits exist thanks to the convexity/concavity hypotheses on $\Psi_n$. The functions $F_w$ and $G_w$ describe the macroscopic effect of the convex and concave part of the interaction of discrete points in the lattice $\mathbb{Z}^N$ at distance $\lambda_n w$. The functions $\mathcal{F}$ and $\mathcal{G}$ are then obtained by summing all the contributions when $w$ varies in the lattice $\mathbb{Z}^N$, as

$$\mathcal{F}(z) = \sum_{w\in\mathbb{Z}^N\setminus\{0\}} k(w)F_w\Big(z\cdot\frac{w}{|w|}\Big)$$

$$\mathcal{G}(z,\nu) = \sum_{w\in\mathbb{Z}^N\setminus\{0\}} \frac{k(w)}{|w|}G_w(z\,\mathrm{sgn}\,(z\cdot w))|\nu\cdot w|,$$

where $k(w)\in\mathbb{N}$ denotes the ratio between $w$ and the minimal vector in $\mathbb{Z}^N$ with the same direction. Note that $\mathcal{F}$ may easily turn out to be isotropic (e.g., when $F_w$ is quadratic), while $\mathcal{G}$ in general is not. However, the explicit formula giving $\mathcal{G}$ allows easily to construct finite-difference schemes with interactions up to $M$-order-neighbours such that the anisotropy of $\mathcal{G}$ is arbitrarily reduced.

Boundary-value problems can also be treated in this scheme; we propose two ways of dealing with them. In the first one we consider discrete functions as defined on the whole $\lambda_n\mathbb{Z}^N$ and equal to a fixed function outside the domain $\Omega$; in this case the interactions 'across the boundary of $\Omega$' give rise to an additional boundary term in the limit energy. The second method consists in considering the functions as fixed only on $\partial\Omega$; in this case, the boundary term gives a different contribution, corresponding to a boundary-layer effect. We finally remark that our method easily extends to deal with lattices of different shapes, such as exagonal or slanted ones.

## 2. Notation and preliminaries

In the sequel $N$ will be a fixed positive integer denoting the dimension of the real euclidean space in which the discrete model is set (often omitted in the case $N=1$) and $S^{N-1}$ will be the set of unit-vectors in $\mathbb{R}^N$. If $x,y\in\mathbb{R}^N$, then $x\cdot y$ denotes their scalar product, $|x|,|y|$ their euclidean norms and $[x,y]$ the segment between $x$ and $y$. $B_\rho(x)$ denotes the ball of centre $x$ and radius $\rho$. For a set $A$ of $\mathbb{R}^N$ we denote $\mathrm{int}\,(A)$ the interior part of $A$. We write $\mathrm{sgn}\,t$ and $[t]$ to denote the sign of $t$ and the integer part of $t$, respectively. We write $\mathcal{L}_N(A)$ and $\mathcal{H}^k(A)$ to denote the Lebesgue ($N$-dimensional) measure and the $k$-Hausdorff measure of $A\subset\mathbb{R}^N$, respectively. We sometimes write $|A|=\mathcal{L}_N(A)$. We also write $\#A$ to denote the number of elements of $A$ ($+\infty$ is $A$ is not finite). We use standard notation for Sobolev and Lebesgue spaces. If $\mu$ is a Borel measure and $B$ is a Borel set, then the measure $\mu\llcorner B$ is defined as $\mu\llcorner B(A) := \mu(A\cap B)$. The notation $⨍_B f\,dx$ stands for the *mean value* of $f$ on $B$. The letter $c$ will denote a strictly positive constant whose value may vary from line to line.

### 2.1. Special functions of bounded variation

Let $\Omega$ be an open subset of $\mathbb{R}^N$, and let $\mathcal{M}(\Omega)$ be the set of all signed Radon measures on $\Omega$ with bounded total variation. We say that $u\in L^1(\Omega)$ is a *function of bounded variation*, and we write $u\in BV(\Omega)$, if all its distributional first derivatives $D_iu$ belong to $\mathcal{M}(\Omega)$. We denote by $Du$ the $\mathbb{R}^N$-valued measure whose entries are $D_iu$. The *jump set* $S(u)$ is the complement of the Lebesgue set of $u$; i.e., $x\notin S(u)$ if and only if

$$\lim_{\rho\to0^+}\rho^{-N}\int_{B_\rho(x)}|u(y)-z|\,dy = 0$$

for some $z \in \mathbb{R}$. If $u \in L^1(\Omega)$ then the set $S(u)$ is Lebesgue-negligible, and if, in addition, $u \in BV(\Omega)$ then the Hausdorff dimension of $S(u)$ is at most $N-1$. Precisely, $S(u)$ is *rectifiable* in that there is a countable sequence of $C^1$ hypersurfaces $\Gamma_i$ which covers $\mathcal{H}^{N-1}$-almost all of $S(u)$; i.e., $\mathcal{H}^{N-1}\big(S(u) \setminus \bigcup_{i=1}^{\infty} \Gamma_i\big) = 0$. Moreover, for $\mathcal{H}^{N-1}$-almost every $x \in S(u)$ it is possible to find $a$, $b \in \mathbb{R}$ and $\nu \in S^{N-1}$ such that

$$\lim_{\rho\to 0^+} \rho^{-N} \int_{B^{\nu}_{\rho}(x)} |u(y) - a|\,dy = 0, \qquad \lim_{\rho\to 0^+} \rho^{-N} \int_{B^{-\nu}_{\rho}(x)} |u(y) - b|\,dy = 0,$$

where $B^{\nu}_{\rho}(x) := \{y \in B_\rho(x) : (y-x)\cdot\nu > 0\}$. The triplet $(a, b, \nu)$ is uniquely determined up to a change of sign of $\nu$ and an interchange between $a$ and $b$, and in the sequel it will be denoted by $(u^+, u^-, \nu_u)$. We sometimes write $[u] = u^+ - u^-$ if no confusion may arise with the integer part of $u$. If $N = 1$ we choose $\nu_u = e_1$ on $S(u)$ so that we have $[u](t) = u(t+) - u(t-)$, where we denote $u(t\pm)$ the right and left limit in $t$ respectively.

The distributional derivative of a function $u \in BV(\Omega)$ admits the decomposition

$$Du = \nabla u\,\mathcal{L}_N + J_u + C_u,$$

where $\nabla u\,\mathcal{L}_N$ is the *Lebesgue part* of $Du$ (we use the notation $\dot u$ for $\nabla u$ if $N = 1$),

$$J_u := (u^+ - u^-)\nu_u\,\mathcal{H}^{N-1} \llcorner S(u)$$

is the *Hausdorff part* or *jump* part of $Du$, and $C_u$ is the *Cantor part* of $Du$. We recall that the measure $C_u$ is singular with respect to the Lebesgue measure and it is 'diffuse'; i.e., $C_u(S) = 0$ for every set $S$ with $\mathcal{H}^{N-1}(S) < +\infty$. For the general exposition of the theory of functions of bounded variation we refer the reader to Federer [19], Evans and Gariepy [18] and Ziemer [25].

We say that a function $u \in BV(\Omega)$ is a *special function of bounded variation* if $C_u \equiv 0$, or, equivalently, if

$$Du = \nabla u\,\mathcal{L}_N + (u^+ - u^-)\nu_u\,\mathcal{H}^{N-1} \llcorner S(u).$$

We denote the space of the special functions of bounded variation by $SBV(\Omega)$. The introduction of this space is due to De Giorgi and Ambrosio [17]. The space $GSBV(\Omega)$ of *generalized special function of bounded variation* is defined as the space of scalar functions in $L^1(\Omega)$ such that for all $T > 0$ the truncations $u_T := (-T) \wedge (u \vee T)$ belong to $SBV(\Omega)$. For the properties of functions in $SBV(\Omega)$ and $GSBV(\Omega)$ we refer to the book by Ambrosio, Fusco and Pallara [5].

Let $p > 1$. The spaces $SBV^p(\Omega)$ and $GSBV^p(\Omega)$ are defined as the subspaces of functions $u$ of $SBV(\Omega)$ and $GSBV(\Omega)$, respectively, such that

$$\mathcal{H}^{N-1}(S(u) \cap \Omega) < +\infty \qquad \text{and} \qquad \nabla u \in L^p(\Omega; \mathbb{R}^N).$$

These spaces are natural domains for the treatment of energies with bulk and surface contributions in the case where the bulk energy density grows superlinearly at infinity.

### 2.2. Γ-convergence

We recall the definition of De Giorgi's Γ-*convergence* in a metric space space $(X, d)$: given a family of functionals $F_n : X \to [0, +\infty]$, $n \in \mathbb{N}$, for $u \in X$ we define

$$F'(u) = \Gamma(d)\text{-}\liminf_n F_n(u) := \inf\Big\{\liminf_n F_n(u_n) \;:\; \lim_n d(u_n, u) = 0\Big\},$$

and

$$F''(u) = \Gamma(d)\text{-}\limsup_n F_n(u) := \inf\left\{\limsup_n F_n(u_n) \ : \ \lim_n d(u_n, u) = 0\right\}.$$

If these two quantities coincide then their common value is called the $\Gamma$-*limit* of the sequence $(F_n)$ at $u$, and is denoted by $\Gamma\text{-}\lim_n F_n(u)$ or $\Gamma(d)\text{-}\lim_n F_n(u)$. Equivalently, $F(u) = \Gamma\text{-}\lim_n F_n(u)$ if and only if the two following conditions are satisfied:

(i) (*lower semicontinuity inequality*) for all sequences $(u_n)$ converging to $u$ in $X$ we have $F(u) \leq \liminf_n F_n(u_n)$;

(ii) (*existence of a recovery sequence*) there exists a sequence $(u_n)$ converging to $u$ in $X$ such that $F(u) \geq \limsup_n F_n(u_n)$.

For a comprehensive study of $\Gamma$-convergence we refer to the book of Dal Maso [16] (see also [11] Part II). Note that the functions $F'$ and $F''$ are lower semicontinuous. The reason for the introduction of this notion is explained by the following fundamental theorem.

**Theorem 2.1.** *Let $F = \Gamma\text{-}\lim_n F_n$, and let a compact set $K \subset X$ exist such that $\inf_X F_n = \inf_K F_n$ for all $n$. Then*

$$\exists \min_X F = \lim_n \inf_X F_n.$$

*Moreover, if $(u_n)$ is a converging sequence such that $\lim_n F_n(u_n) = \lim_n \inf_X F_n$ then its limit is a minimum point for $F$.*

## 2.3. Some Preliminary results

We first recall a density result in $SBV^p(\Omega)$ due to Cortesani and Toader [15] (see also [9]).

Let $\mathcal{W}(\Omega)$ be the space of all functions $w \in SBV(\Omega)$ satisfying the following properties:

(i) $\mathcal{H}^{N-1}(\overline{S(w)} \setminus S(w)) = 0$;

(ii) $\overline{S(w)}$ is the intersection of $\Omega$ with the union of a finite number of $(N-1)$-dimensional simplexes;

(iii) $w \in W^{k,\infty}(\Omega \setminus \overline{S(w)})$ for every $k \in \mathbb{N}$.

**Theorem 2.2.** *Let $\Omega$ be an open subset of $\mathbb{R}^N$ with Lipschitz boundary. Let $u \in SBV^p(\Omega) \cap L^\infty(\Omega)$. Then there exists a sequence $(w_j)$ in $\mathcal{W}(\Omega)$ such that*

$$w_j \to u \qquad \text{strongly in } L^1(\Omega), \tag{1}$$

$$\nabla w_j \to \nabla u \quad \text{strongly in } L^p(\Omega) \tag{2}$$

$$\limsup_{j\to+\infty} \|w_j\|_\infty \leq \|u\|_\infty \tag{3}$$

*and*

$$\limsup_{j\to+\infty} \int_{S_{w_j}} \phi(w_j^+, w_j^-, \nu_{w_j})\, d\mathcal{H}^{N-1} \leq \int_{S_u} \phi(u^+, u^-, \nu_u)\, d\mathcal{H}^{N-1} \tag{4}$$

*for every upper semicontinuous function $\phi : \mathbb{R}\times\mathbb{R}\times S^{N-1} \to [0, +\infty)$ such that $\phi(a, b, \nu) = \phi(b, a, -\nu)$ for every $a, b \in \mathbb{R}$ and $\nu \in S^{N-1}$.*

We will treat $N$-dimensional problems by reducing to 1-dimensional one. To this end we introduce some notation and a 'slicing' result. Let $\xi \in S^{N-1}$ and let $\Pi^\xi := \{y \in \mathbb{R}^N :$

$y \cdot \xi = 0\}$ be the linear hyperplane orthogonal to $\xi$ and $P^\xi : \mathbb{R}^N \to \Pi^\xi$ the orthogonal projection on $\Pi^\xi$. If $y \in \Pi^\xi$ and $E \subset \mathbb{R}^n$ we set

$$E^{\xi,y} := \{t \in \mathbb{R} : y + t\xi \in E\}. \tag{5}$$

Moreover, if $u : E \to \mathbb{R}$ we define the function $u^{\xi,y} : E^{\xi,y} \to \mathbb{R}$ by

$$u^{\xi,y}(t) := u(y + t\xi). \tag{6}$$

**Theorem 2.3.**

(a) *Let $u \in GSBV(\Omega)$. Then, for all $\xi \in S^{N-1}$ the function $u^{\xi,y}$ belongs to $GSBV(\Omega^{\xi,y})$ for $\mathcal{H}^{N-1}$-a.e. $y \in \Pi^\xi$. Moreover for such $y$ we have*

$$\dot u^{\xi,y}(t) = \nabla u(y + t\xi) \cdot \xi \quad \textit{for a.e. } t \in \Omega^{\xi,y},$$

$$S(u^{\xi,y}) = \{t \in \mathbb{R} : y + t\xi \in S(u)\},$$

$$u^{\xi,y}(t\pm) = u^{\pm}(y + t\xi) \quad \textit{or} \quad u^{\xi,y}(t\pm) = u^{\mp}(y + t\xi),$$

*according to the cases $\nu_u \cdot \xi > 0$ or $\nu_u \cdot \xi < 0$ (the case $\nu_u \cdot \xi = 0$ being negligible) and for all Borel functions $g$*

$$\int_{\Pi^\xi} \sum_{t \in S(u^{\xi,y})} g(t)\, d\mathcal{H}^{N-1}(y) = \int_{S(u)} g(x)|\nu_u \cdot \xi|\, d\mathcal{H}^{N-1}.$$

(b) *Conversely, if $u \in L^1(\Omega)$ and for all $\xi \in \{e_1, \ldots, e_N\}$ and for $\mathcal{H}^{N-1}$-a.e. $y \in \Pi^\xi$ $u^{\xi,y} \in SBV(\Omega^{\xi,y})$ and*

$$\int_{\Pi^\xi} \Big( \int_{\Omega^{\xi,y}} |\dot u^{\xi,y}|^p + \#(S(u^{\xi,y})) \Big)\, d\mathcal{H}^{N-1}(y) < +\infty,$$

*then $u \in GSBV^p(\Omega)$.*

We finally recall a simple 1-dimensional $\Gamma$-convergence result (for a direct proof see [12]).

**Proposition 2.4.** *For any $n \in \mathbb{N}$ let $h_n : \mathbb{R} \to [0, +\infty]$ be convex and lower semicontinuous functions such that the limit $\lim_n h_n(x) =: h(x)$ exists for all $x \in \mathbb{R}$. Assume in addition that $h$ is lower semicontinuous and* $\operatorname{int}(\{x : h(x) \neq +\infty\}) \neq \emptyset$. *Then for all $\phi_n, \phi \in L^p(a,b)$ such that $\phi_n \rightharpoonup \phi$ weakly in $L^p(a,b)$, we have*

$$\liminf_n \int_a^b h_n(\phi_n)\, dt \geq \int_a^b h(\phi)\, dt.$$

## 3. The one-dimensional case

We first recall some results in the case of nearest-neighbour interaction, and subsequently use this result to obtain the $\Gamma$-convergence theorem for long-range interactions.

Consider an open interval $(a,b)$ of $\mathbb{R}$ and two sequences $(\lambda_n)$, $(a_n)$ of positive real numbers with $a_n \in [a, a + \lambda_n)$ and $\lambda_n \to 0$. For $n \in \mathbb{N}$ let $a \leq x_n^1 < \ldots < x_n^{N_n} < b$ be the partition of $(a,b)$ induced by the intersection of $(a,b)$ with the set $a_n+\lambda_n\mathbb{Z}$. We define $\mathcal{A}_n(a,b)$ as the space of piecewise constant functions such that $c_n^0,\dots,c_n^{N_n}\in\mathbb{R}$ exist satisfying

$$u(x)=\begin{cases} c_n^0 & \text{if } a<x<x_n^1\\ c_n^i & \text{if } x_n^i\le x<x_n^{i+1}, i=1,\dots,N_n-1\\ c_n^{N_n} & \text{if } x_n^{N_n}\le x<b\end{cases} \tag{7}$$

that is, $\mathcal{A}_n(a,b)$ is the set of the restrictions to $(a,b)$ of functions constant on each $[a_n+k\lambda_n, a_n+(k+1)\lambda_n)$, $k\in\mathbb{Z}$.

### 3.1. Nearest-neighbourhood interaction

For $n\in\mathbb{N}$ let $\psi_n:\mathbb{R}\to[0,+\infty]$ be given, and let $E_n : L^1(a,b)\to[0,+\infty]$ be defined as

$$E_n(u)=\begin{cases} \displaystyle\sum_{i=1}^{N_n-1}\lambda_n\psi_n\left(\frac{u(x_n^{i+1})-u(x_n^i)}{\lambda_n}\right) & u\in\mathcal{A}_n(a,b)\\ +\infty & \text{otherwise in } L^1(a,b).\end{cases} \tag{8}$$

The following Γ-convergence is a particular case of the results in [12].

**Proposition 3.1.** *For all $n\in\mathbb{N}$ let $T_n^\pm\in\mathbb{R}$ exist with*

$$\lim_n T_n^\pm=\pm\infty, \qquad \lim_n \lambda_n T_n^\pm=0, \tag{9}$$

*and such that, if we define $F_n, G_n:\mathbb{R}\to[0,+\infty]$ as*

$$F_n(z)=\begin{cases}\psi_n(z) & T_n^-\le z\le T_n^+\\ +\infty & z\in\mathbb{R}\setminus[T_n^-,T_n^+]\end{cases} \tag{10}$$

$$G_n(z)=\begin{cases}\lambda_n\psi_n\left(\dfrac{z}{\lambda_n}+T_n^{\operatorname{sign} z}\right) & z\neq 0\\ 0 & \text{if } z=0\end{cases} \tag{11}$$

*the following conditions are satisfied: there exists $p>1$ such that*

$$F_n(z)\ge|z|^p \qquad \forall z\in\mathbb{R} \tag{12}$$

$$\sup_n \inf_{z\in\mathbb{R}} F_n(z)<+\infty \tag{13}$$

$$G_n(z)\ge c>0 \qquad \forall z\neq 0, \tag{14}$$

*$F_n$ is convex, and the restrictions of each $G_n$ to $(-\infty,0)$ and to $(0,+\infty)$ are concave, and, moreover, there exist $F,G:\mathbb{R}\to[0,+\infty)$ such that*

$$\lim_n F_n=F, \tag{15}$$

$$\lim_n G_n=G. \tag{16}$$

*Then, $(E_n)_n$ $\Gamma$-converges to $E$ with respect to the convergence in measure and with respect to the $L^1(a,b)$ convergence on $L^1(a,b)$, where*

$$E(u) = \begin{cases} \displaystyle\int_a^b F(\dot u)\,dt \;+\; \sum_{t\in S(u)} G([u](t)) & u \in SBV(a,b) \\ +\infty & \textit{otherwise in } L^1(a,b). \end{cases}$$

**Remark 3.2.** Note that the convergence hypotheses in (15) and (16) are not restrictive up to passing to a subsequence. Note moreover that $F$ is convex, $G$ is concave and that if all $\lambda_n\psi_n(\cdot/\lambda_n)$ are locally equibounded then we also have

$$G(x) = \lim_n \lambda_n \psi_n\Big(\frac{x}{\lambda_n}\Big)$$

by the uniform convergence of $G_n$ to $G$ on compact subsets of $\mathbb{R}\setminus\{0\}$.

By [12] we may drop the convexity and concavity hypotheses, in which case the description of $F$ and $G$ is more complex.

## 3.2. Multiple-neighbourhood interaction

With fixed $n \in \mathbb{N}$ and for $k = 1,\dots,N_n$, let $\psi_n^k : \mathbb{R} \to [0,+\infty)$ be given functions. We will investigate the limiting behaviour of the following energies defined on $L^1(a,b)$:

$$\mathcal{E}_n(u) = \begin{cases} \displaystyle\sum_{k=1}^{N_n}\sum_{i=1}^{k}\sum_{j=0}^{\left[\frac{N_n-i}{k}\right]-1} k\lambda_n\psi_n^k\left(\frac{u(x_n^{i+(j+1)k}) - u(x_n^{i+jk})}{k\lambda_n}\right) & \text{if } u \in \mathcal{A}_n(a,b) \\ +\infty & \text{otherwise.} \end{cases} \tag{17}$$

We begin by proving the following result.

**Proposition 3.3.** *Suppose that for every $n \in \mathbb{N}$ and $k \in \{1,\dots,N_n\}$ there exist points $T^k_{n,-}, T^k_{n,+}$ such that the following conditions are satisfied:*

$$\lim_n T^k_{n,-} = -\infty, \qquad \lim_n T^k_{n,+} = +\infty, \qquad \lim_n \lambda_n T^k_{n,\pm} = 0, \tag{18}$$

$$\psi_n^k\big|_{[T^k_{n,-},T^k_{n,+}]} \ \textit{is convex and lower semicontinuous,} \tag{19}$$

$$\psi_n^k\big|_{(-\infty,T^k_{n,-}]} \ \textit{and}\ \psi_n^k\big|_{[T^k_{n,+},+\infty)} \ \textit{are concave and lower semicontinuous,} \tag{20}$$

*for some $p > 1$*

$$\psi_n^1(x) \;\geq\; |x|^p \qquad \textit{if } T^1_{n,-} \leq x \leq T^1_{n,+}, \tag{21}$$

$$\lambda_n\psi_n^1(x) \;\geq\; c > 0 \qquad \textit{if } x \notin [T^1_{n,-}, T^1_{n,+}], \tag{22}$$

*there exist $F^k, G^k : \mathbb{R} \to [0,+\infty)$ such that*

$$\lim_n \psi_n^k(x) = F^k(x) \qquad \textit{for every } x \in \mathbb{R} \tag{23}$$

$$\lim_n k\lambda_n \psi_n^k \left(\frac{x}{k\lambda_n}\right) = G^k(x) \quad \textit{for every } x \in \mathbb{R}, \tag{24}$$

*and $G^k$ is superlinear in $0$, i.e., taking into account that $G^k(0) = 0$ by (24),*

$$\lim_{z\to 0} \frac{G^k(z)}{|z|} = +\infty. \tag{25}$$

*If $\mathcal{E}'(u)$ denotes the $\Gamma$-$\liminf_n \mathcal{E}_n(u)$ with respect to the convergence in measure then $\mathcal{E}'(u) \geq \mathcal{E}(u)$, where $\mathcal{E}(u)$ is defined by*

$$\mathcal{E}(u) = \begin{cases} \displaystyle\int_a^b \mathcal{F}(\dot{u})\,dt + \sum_{t\in S(u)} \mathcal{G}([u](t)) & \textit{if } u \in SBV(a,b) \\ +\infty & \textit{otherwise in } L^1(a,b), \end{cases} \tag{26}$$

*with*

$$\mathcal{F}(x) = \sum_{k=1}^{+\infty} kF^k(x) \quad \textit{and} \quad \mathcal{G}(x) = \sum_{k=1}^{+\infty} kG^k(x). \tag{27}$$

The proof of the proposition will make use of the following lemma.

**Lemma 3.4.** *Let $\psi_n : [0,+\infty) \to [0,+\infty)$ be non decreasing functions and let $T_n$ be positive real numbers such that*

$$\lim_n \lambda_n T_n = 0, \lim_n T_n = +\infty \tag{28}$$

$$\psi_n\big|_{(0,T_n)} \textit{ is convex} \tag{29}$$

$$\psi_n\big|_{(T_n,+\infty)} \textit{is concave.} \tag{30}$$

*Assume in addition that there exist $F, G : [0,+\infty) \to [0,+\infty)$ such that $G$ is superlinear in $0$ and*

$$F(x) = \lim_n \psi_n(x), \qquad G(x) = \lim_n \lambda_n \psi_n \left(\frac{x}{\lambda_n}\right) \tag{31}$$

*for every $x > 0$. Then, for every sequence $(T_n')$ such that $\lim_n \lambda_n T_n' = 0$ and $T_n' > T_n$ for all $n$ there exist non-decreasing $\phi_n : [0,+\infty) \to [0,+\infty)$ with $\phi_n \leq \psi_n$, satisfying*

$$\phi_n\big|_{(0,T_n')} \textit{ is convex} \tag{32}$$

$$\phi_n\big|_{(T_n',+\infty)} \textit{is concave,} \tag{33}$$

*and such that*

$$\lim_n \phi_n(x) = F(x) \quad \textit{and} \quad \lim_n \lambda_n \phi_n \left(\frac{x}{\lambda_n}\right) = G(x) \tag{34}$$

*for every $x > 0$.*

**Proof.** We denote $G_n(x) = \lambda_n\psi_n((x-\lambda_nT_n)/\lambda_n)$, which is a concave function on $(0,+\infty)$. The sequence $(G_n)$ converges uniformly to $G$ on all compact subsets of $(0,+\infty)$. Since $G$ is superlinear in 0, we claim that for all $M>0$ there exists $\varepsilon>0$ and $n_M\in\mathbb{N}$ such that $G_n(x)\geq Mx$ on $[0,\varepsilon]$ for all $n\geq n_M$. Suppose otherwise and choose $\varepsilon>0$ such that $G(\varepsilon)>M\varepsilon$; if, up to subsequences, we have $G_n(x_n)<Mx_n$ for some $x_n<\varepsilon$, then, as $G_n$ is positive and concave,we have also $G_n(\varepsilon)<M\varepsilon$, which gives a contradiction letting $n$ tend to $+\infty$.

Let

$$x_n = \max\Big\{x\in[0,T_n]:\ \psi_n'(x-)\leq \frac{\psi_n(T_n')-\psi_n(x)}{T_n'-x}\Big\}$$

($x_n=0$ if the set on the right hand side is empty) We set

$$\phi_n(x)=\begin{cases}\psi_n(x) & \text{if } x<x_n\\ \psi(x_n)+\dfrac{\psi_n(T_n')-\psi_n(x_n)}{T_n'-x_n}(x-x_n) & \text{if } x_n\leq x\leq T_n'\\ \psi_n(x) & \text{if } x>T_n'.\end{cases}$$

Clearly $\phi_n$ is convex on $(0,T_n')$ and concave on $(T_n',+\infty)$. Moreover, it can be immediately checked that $\phi_n\leq\psi_n$ and that $\phi_n$ is non-decreasing. The only thing left to prove is that $\lim_n x_n=+\infty$. To check this, note that

$$\frac{\psi_n(T_n')-\psi_n(x)}{T_n'-x}=\frac{G_n((T_n'-T_n)\lambda_n)-\lambda_n\psi_n(x)}{\lambda_n(T_n'-x)}.$$

For what proven above and since $\lambda_n(T_n'-T_n)\to 0$ for all fixed $x$ we have

$$\lim_n \frac{\psi_n(T_n')-\psi_n(x)}{T_n'-x}=+\infty.$$

On the other hand for all fixed $x>0$ we have $\limsup_n \phi_n'(x-)<+\infty$, so that $x<x_n$ for $n$ large enough. □

**Proof of Proposition 3.3.** First, we rewrite our functionals as follows

$$\mathcal{E}_n(u)=\sum_{k=1}^{N_n}\sum_{i=1}^{k}E_n^{k,i}(u), \tag{35}$$

where $E_n^{k,i}$ is defined by

$$E_n^{k,i}(u)=\sum_{j=0}^{\left[\frac{N_n-i}{k}\right]-1} k\lambda_n\psi_n^k\left(\frac{u(x_n^{i+(j+1)k})-u(x_n^{i+jk})}{k\lambda_n}\right). \tag{36}$$

Let $u_n,u\in L^1(a,b)$ be such that $u_n\to u$ in measure and suppose that $\liminf_n\mathcal{E}_n(u_n)<+\infty$. Up to subsequences, we also can assume that $u_n\to u$ pointwise and $\liminf_n\mathcal{E}_n(u_n)=$

$\lim_n \mathcal{E}_n(u_n)$. By (21) and (22) we can apply Proposition 3.1 to $E_n^{1,1}$. As $\liminf_n E_n^{1,1}(u_n) \leq \lim_n \mathcal{E}_n(u_n) < +\infty$, we get that $u \in SBV^p(a,b)$ and

$$\liminf_n E_n^{1,1}(u_n) \geq \int_a^b F^1(\dot u)\,dt + \sum_{S(u)} G^1([u]). \tag{37}$$

Fix an integer $k \geq 2$ and $i \in \{1,\dots,k\}$. We will prove that although $\psi_n^k$ satisfies weaker hypotheses than $\psi_n^1$, we still have

$$\liminf_n E_n^{k,i}(u_n) \geq \int_a^b F^k(\dot u)\,dt + \sum_{S(u)} G^k([u]). \tag{38}$$

Once this inequality is proved, the thesis follows immediately: with fixed $m \in \mathbb{N}$ we have from Fatou's Lemma

$$\liminf_n \mathcal{E}_n(u_n) \geq \liminf_n \sum_{k=1}^m \sum_{i=1}^k E_n^{k,i}(u_n) \geq \sum_{k=1}^m k\Big(\int_a^b F^k(\dot u)\,dt + \sum_{S(u)} G^k([u])\Big),$$

so that we have $\mathcal{E}'(u) \geq \mathcal{E}(u)$ by letting $m \to +\infty$.

The proof of (38) is divided into two steps. For the sake of simplicity we assume $T_{n,+}^k = -T_{n,-}^k$, $T_{n,+}^1 = -T_{n,-}^1$ and we denote these points by $T_n^k$ and $T_n^1$, respectively. The necessary changes for the general case will be clear from the proof.

*Step* 1 Assume that $T_n^1 \leq T_n^k$ for every $n \in \mathbb{N}$. Define

$$I_n^{k,i} = \left\{ j \in \left\{0,\dots,\left[\frac{N_n - i}{k}\right]\right\} : |u_n(x_n^{i+(j+1)k}) - u_n(x_n^{i+jk})| > T_n^k k \lambda_n \right\}.$$

Note that $\# I_n^{1,1}$ is equibounded by (22). Moreover, it can be easily checked that if $j \in I_n^{k,i}$ then there exists $l \in \{i+jk,\dots,i+(j+1)k-1\}$ such that $l \in I_n^{1,1}$. Hence, we get that $\# I_n^{k,i} \leq \# I_n^{1,1} \leq c$.

Define $u_n^{k,i}$ on $[x_n^{i+jk}, x_n^{i+(j+1)k})$ as follows

$$u_n^{k,i}(x) = \begin{cases} u_n(x_n^{i+jk}) & \text{if } j \in I_n^{k,i} \\ \dfrac{u_n(x_n^{i+(j+1)k}) - u_n(x_n^{i+jk})}{k\lambda_n (x - x_n^{i+jk}) + u_n(x_n^{i+jk})} & \text{if } j \notin I_n^{k,i}. \end{cases}$$

We have that for every $k$ and $i$ fixed, $(u_n^{k,i})_n$ converges to $u$ in measure or, equivalently, $u_n^{k,i} - u_n$ converges to 0 in measure. Indeed, fixed $\varepsilon > 0$, let $s$ be an index such that

$$[x_n^s, x_n^{s+1}) \cap \{x : |u_n^{k,i}(x) - u_n(x)| > \varepsilon\} \neq \emptyset \tag{39}$$

and let $j$ be such that $s \in \{i+jk,\dots,i+(j+1)k-1\}$. If $x \in [x_n^s, x_n^{s+1}) \cap \{x : |u_n^{k,i}(x) - u_n(x)| > \varepsilon\}$, since $u_n^{k,i}(x)$ is a convex combination of the values $u_n^{k,i}(x_n^{i+jk})$,

$u_n^{k,i}(x_n^{i+(j+1)k})$, we have

$$\begin{aligned}\varepsilon &< |u_n^{k,i}(x) - u_n(x)| \\ &\leq \lambda |u_n^{k,i}(x_n^{i+jk}) - u_n(x_n^s)| + (1-\lambda)|u_n^{k,i}(x_n^{i+(j+1)k}) - u_n(x_n^s)| \\ &\leq \sum_{l=i+jk}^{i+(j+1)k-1} |u_n(x_n^{l+1}) - u_n(x_n^l)| \\ &\leq \max_{l=i+jk,\dots,i+(j+1)k-1} |u_n(x_n^{l+1}) - u_n(x_n^l)| k.\end{aligned}$$

Then, for each index $s$ such that (39) holds, we can find an index $s'$ with $|s-s'|<k$ and $|u_n(x_n^{s'+1}) - u_n(x_n^{s'})| > \frac{\varepsilon}{k}$. Note that $s' \in I_n^{1,1}$ for $n$ large, so that

$$|\{x : |u_n^{k,i}(x) - u_n(x)| > \varepsilon\}| \leq \# I_n^{1,1} k \lambda_n.$$

We claim that $\dot{u}_n^{k,i}$ is equi-integrable. Indeed, fixed an index $j \notin I_n^{k,i}$, if $\{i+jk, \dots, i+(j+1)k-1\} \cap I_n^{1,1} = \emptyset$, a simple convexity argument shows that

$$\int_{x_n^{i+jk}}^{x_n^{i+(j+1)k}} |\dot{u}_n^{k,i}|^p \, dt \leq \int_{x_n^{i+jk}}^{x_n^{i+(j+1)k}} |\dot{u}_n^{1,1}|^p \, dt.$$

Since there are at most $\# I_n^{1,1}$ indices such that $\{i+jk, \dots, i+(j+1)k-1\} \cap I_n^{1,1} \neq \emptyset$, for every measurable set $A$ we have

$$\int_A |\dot{u}_n^{k,i}| \, dt \leq |A|^{\frac{1}{q}} \left( \int_a^b |\dot{u}_n^{1,1}|^p \, dt \right)^{\frac{1}{p}} + ck\lambda_n T_n^k,$$

which proves that the sequence is equi-integrable.

Now the lower semicontinuity and compactness theorem (see e.g. [6] Theorem 2.3) in $SBV(a,b)$ gives

$$\dot{u}_n^{k,i} \rightharpoonup \dot{u} \qquad \text{weakly in } L^1(a,b), \qquad \sum_{S(u_n^{k,i}) \cap (t-\varepsilon, t+\varepsilon)} [u_n^{k,i}] \to [u](t) \tag{40}$$

where $t$ is a point in $S(u)$ and $\varepsilon > 0$ is any small-enough real number. By using the subadditivity of $\psi_n^k$ on $(-\infty, -T_n^k) \cup (T_n^k, +\infty)$, we have

$$E_n^{k,i}(u_n) \geq \int_a^b \psi_n^k(\dot{u}_n^{k,i}) \, dt + \sum_{t \in S(u)} k\lambda_n \psi_n^k \left( \frac{\sum_{S(u_n^{k,i}) \cap (t-\varepsilon, t+\varepsilon)} [u_n^{k,i}]}{k\lambda_n} \right).$$

Using (19), (40), (23) and (24), we get

$$\liminf_n E_n^{k,i}(u_n) \geq \int_a^b F^k(\dot{u}) \, dt + \sum_{S(u)} G^k([u])$$

by Proposition 2.4.

*Step* 2 Suppose that $T_n^1 > T_n^k$ for infinitely many $n$. For the sake of simplicity suppose that it holds for all $n$. Note that hypotheses (18), (19), (20), (22) and the finiteness of $F_k$ imply that there exists a minimum point $c_n^k \in [-T_n^k, T_n^k]$ for $\psi_n^k$. Suppose first that $c_n^k \in (-T_n^k, T_n^k)$. We can apply Lemma 3.4 twice, to the functions $\psi_n(x) = \psi_{n,\pm}^k(x) = \psi_n^k(c_n^k \pm x)$, with $T_n = T_n^k \pm c_n^k$ and $T_n' = T_n^1 \pm c_n^k$, to obtain functions $\phi_{n,\pm}^k$. We define then

$$\phi_n^k(x) = \begin{cases} \phi_{n,-}^k(c_n^k - x) & \text{if } x \le c_n^k \\ \phi_{n,+}^k(x - c_n^k) & \text{if } x \ge c_n^k \end{cases}$$

If $c_n^k = T_n^k$ then we just choose as $\phi_n^k$ on $[T_n^k, T_n^1]$ the affine function satisfying $\phi_n^k(T_n^k) = \psi_n^k(T_n^k)$ and $\phi_n^k(T_n^1) = \psi_n^k(T_n^1)$. Similarly, we deal the case $c_n^k = -T_n^k$.

The new sequence $\phi_n^k$ satisfies all the hypotheses of Proposition 3.3 relative to the index $k$. In addition, each $\phi_n^k$ is convex in $[-T_n^1, T_n^1]$ and $\phi_n^k \le \psi_n^k$. We can apply Step 1 to the functionals $\tilde{E}_n^{k,i}$ defined as in (36) with $\psi_n^k$ replaced by $\phi_n^k$, noting that by Lemma 3.4 the limit functions $F^k$ and $G^k$ remain unchanged. We then obtain (38) since $E_n^{k,i}(u_n) \ge \tilde{E}_n^{k,i}(u_n)$ for all $n$. □

In the following proposition we deal with the upper inequality for the $\Gamma$-limit.

**Proposition 3.5.** *Let $\psi_n^k : \mathbb{R} \to [0, +\infty]$ satisfy hypotheses* (18)–(24) *of Proposition* 3.3 *and assume in addition that there exist $\mathcal{F}, \mathcal{G} : \mathbb{R} \to [0, +\infty]$ given by* (27) *such that*

$$\mathcal{F}(x) = \lim_n \sum_{k=1}^{N_n} k\psi_n^k(x) \qquad \textit{for every } x \in \mathbb{R} \tag{41}$$

$$\mathcal{G}(x) = \lim_n \sum_{k=1}^{N_n} k^2 \lambda_n \psi_n^k\left(\frac{x}{k\lambda_n}\right) \qquad \textit{for every } x \in \mathbb{R}. \tag{42}$$

*Then, for every $u \in L^1(a,b)$, $\Gamma\text{-}\limsup_n \mathcal{E}_n(u) \le \mathcal{E}(u)$ where the $\Gamma$-limsup is taken with respect to the strong convergence in $L^1(a,b)$.*

**Proof.** The case $\mathcal{F} \equiv +\infty$ is trivial. We therefore assume that $\{x : \mathcal{F}(x) \ne +\infty\} \ne \emptyset$, and consider $u \in L^1(a,b)$ such that $\mathcal{E}(u) < +\infty$. Note that $u \in SBV^p(a,b)$ by (21) and (22), since $\mathcal{F} \ge F^1$ and $\mathcal{G} \ge G^1$.

We will first prove the $\Gamma$-limsup inequality assuming in addition that $\dot{u} \in L^\infty(a,b)$ and $\text{ess-inf}\, \dot{u}, \text{ess-sup}\, \dot{u} \in \{x : \mathcal{F}(x) \ne +\infty\}$. We claim that a recovery sequence for such a function is simply the piecewise-constant interpolation function $u_n \in \mathcal{A}_n(a,b)$ of $u$ defined to be identically $u(x_n^i+)$ on the interval $[x_n^i, x_n^{i+1})$, for all $i \in \{1, \dots, N_n - 1\}$ and equal to $u(a+)$ and $u(b-)$ on $(a, x_n^1)$ and $[x_n^{N_n}, b)$, respectively. It can be easily checked that $u_n \to u$ strongly in $L^1(a,b)$. Indeed, for any interval $[x_n^i, x_n^{i+1})$, we have

$$\begin{aligned} \int_{x_n^i}^{x_n^{i+1}} |u_n(x) - u(x)|\, dx &\le \int_{x_n^i}^{x_n^{i+1}} \Big( \int_{x_n^i}^{x} |\dot{u}(t)|\, dt + \Big| \sum_{S(u)\cap(x_n^i, x_n^{i+1}]} [u] \Big| \Big)\, dx \\ &\le \int_{x_n^i}^{x_n^{i+1}} \lambda_n \Big( \|\dot{u}\|_\infty + \sum_{S(u)} |[u]| \Big)\, dx. \end{aligned}$$

Summing over $i$ we get

$$\int_a^b |u_n - u|\,dx \le (b-a)\Big(\|\dot u\|_\infty + \sum_{S(u)} |[u]|\Big)\lambda_n.$$

For any $\varepsilon > 0$ fixed, we claim that there exists $m = m(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{k=m}^{N_n} \sum_{i=1}^{k} E_n^{k,i}(u_n) \le \varepsilon \tag{43}$$

for $n$ large enough, from which we deduce immediately that for such $m$

$$\limsup_n \mathcal{E}_n(u_n) \le \sum_{k=1}^{m} \sum_{i=1}^{k} \limsup_n E_n^{k,i}(u_n) + \varepsilon. \tag{44}$$

We divide the proof of (43) in three steps.

*Step* 1. As remarked in the proof of Proposition 3.3 each function $\psi_n^k$ is non-increasing up to a point $c_n^k$ and non-decreasing afterwards. From this monotonicity property we get, for all $m \in \mathbb{N}$,

$$\begin{aligned}\sum_{k=m}^{N_n} \sum_{i=1}^{k} E_n^{k,i}(u_n) \;\le\; & \sum_{k=m}^{N_n} \sum_{i=1}^{k} \Big( (b-a)(\psi_n^k(\text{ess-inf }\dot u) + \psi_n^k(\text{ess-sup }\dot u)) \\ & + \sum_{j \in J_n^{k,i}} G_n^k\Big(\int_{y_n^j}^{y_n^{j+1}} \dot u\,dt + \sum_{S(u)\cap(y_n^j, y_n^{j+1}]} [u]\Big)\Big),\end{aligned}$$

where we set

$$G_n^k(x) = k\lambda_n \psi_n^k\Big(\frac{x}{k\lambda_n}\Big), \tag{45}$$

$$y_n^j = x_n^{i+jk}, \text{ for } j = 0, \dots, M_n^{k,i} \text{ and}$$
$$J_n^{k,i} = \{j \in \{0, \dots, M_n^{k,i}\} \,:\, S(u) \cap (y_n^j, y_n^{j+1}] \ne \emptyset\}.$$

Since, for every $m \in \mathbb{N}$,

$$\lim_n \sum_{k=m}^{N_n} k(\psi_n^k(\text{ess-inf }\dot u) + \psi_n^k(\text{ess-sup }\dot u)) = \sum_{k=m}^{+\infty} k(\psi^k(\text{ess-inf }\dot u) + \psi^k(\text{ess-sup }\dot u))$$

and, by our assumptions,

$$\sum_{k=1}^{+\infty} k(\psi^k(\text{ess-inf }\dot u) + \psi^k(\text{ess-sup }\dot u)) < +\infty,$$

with fixed $\varepsilon > 0$ there exists $m = m(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{k=m}^{+\infty} k(\psi^k(\text{ess-inf }\dot u) + \psi^k(\text{ess-sup }\dot u)) < \varepsilon,$$

and there exists $n_0 = n_0(\varepsilon, m) \in \mathbb{N}$ such that, for any $n \geq n_0$,

$$\sum_{k=m}^{N_n} k(\psi_n^k(\text{ess-inf } \dot{u}) + \psi_n^k(\text{ess-sup } \dot{u})) \leq \sum_{k=m}^{+\infty} k(\psi^k(\text{ess-inf } \dot{u}) + \psi^k(\text{ess-sup } \dot{u})) + \varepsilon.$$

*Step* 2. It remains to estimate

$$\sum_{k=m}^{N_n} \sum_{i=1}^{k} \sum_{j \in J_n^{k,i}} G_n^k\Big(\int_{y_n^j}^{y_n^{j+1}} \dot{u}\,dt + \sum_{S(u)\cap(y_n^j, y_n^{j+1}]} [u]\Big).$$

If $\mathcal{G} \equiv +\infty$ on $\mathbb{R} \setminus \{0\}$ there is nothing to prove. Assume that $\mathcal{G}$ is everywhere finite. It then suffices to notice that

$$\begin{aligned} \text{ess-inf } \dot{u}(b-a) - \sum_{S(u)} |u^+ - u^-| \ &\leq \ \int_{y_n^j}^{y_n^{j+1}} \dot{u}\,dt + \sum_{S(u)\cap(y_n^j, y_n^{j+1}]} [u] \\ &\leq \ \text{ess-sup } \dot{u}(b-a) + \sum_{S(u)} |u^+ - u^-| \end{aligned}$$

and, once again by the monotonicity properties of $\psi_n^k$, which translate into analogous properties of $G_n^k$,

$$\begin{aligned} &G_n^k\Big(\int_{y_n^j}^{y_n^{j+1}} \dot{u}\,dt + \sum_{S(u)\cap(y_n^j, y_n^{j+1}]} [u]\Big) \\ &\leq G_n^k\Big(\text{ess-inf } \dot{u}(b-a) - \sum_{S(u)} |u^+ - u^-|\Big) \\ &\quad + G_n^k\Big(\text{ess-sup } \dot{u}(b-a) + \sum_{S(u)} |u^+ - u^-|\Big). \end{aligned}$$

Since $\# J_n^{k,i} \leq \# S(u)$, we can repeat the reasoning of Step 1, applied to

$$\sum_{k=m}^{N_n} k G_n^k(\text{ess-inf } \dot{u}(b-a) - \sum_{S(u)} |u^+ - u^-|) + k G_n^k(\text{ess-sup } \dot{u}(b-a) + \sum_{S(u)} |u^+ - u^-|),$$

to find $m', n' \in \mathbb{N}$ such that for $n \geq n'$, we have

$$\sum_{k=m'}^{N_n} \sum_{i=1}^{k} \sum_{j \in J_n^{k,i}} G_n^k\Big(\int_{y_n^j}^{y_n^{j+1}} \dot{u}\,dt + \sum_{S(u)\cap(y_n^j, y_n^{j+1}]} [u]\Big) \leq \varepsilon. \tag{46}$$

*Step* 3. There is only left the case $\mathcal{G} \equiv +\infty$ on a half-line. Assume for instance that $\mathcal{G} \equiv +\infty$ on $(-\infty, 0)$, and $\mathcal{G}$ is finite on $[0, +\infty)$. This assumption implies that $[u](t) > 0$ for every $t \in S(u)$. Hence,

$$(\text{ess-inf } \dot{u}) k \lambda_n \leq \int_{y_n^j}^{y_n^{j+1}} \dot{u}\,dt + \sum_{S(u)\cap(y_n^j, y_n^{j+1}]} [u] \leq \text{ess-sup } \dot{u}(b-a) + \sum_{S(u)} |u^+ - u^-|.$$

So, for any $m \in \mathbb{N}$, we get

$$\sum_{k=m}^{N_n} \sum_{i=1}^{k} \sum_{j \in J_n^{k,i}} G_n^k\Big(\int_{y_n^j}^{y_n^{j+1}} \dot u \, dt + \sum_{S(u)\cap(y_n^j, y_n^{j+1}]} [u]\Big)$$

$$\leq \sum_{k=m}^{N_n} k\Big(\#S(u) G_n^k\Big(\text{ess-sup } \dot u(b-a) + \sum_{S(u)} |u^+ - u^-|\Big)$$

$$+(b-a)\psi_n^k(\text{ess-inf } \dot u)\Big).$$

Since

$$\lim_n \sum_{k=1}^{N_n} k G_n^k(\text{ess-sup } \dot u(b-a) + \sum_{S(u)} |u^+ - u^-|) < +\infty,$$

we can proceed as in Step 1 to obtain inequality (46) for some $m', n' \in \mathbb{N}$.

We conclude the proof of the proposition in the following additional three steps.

*Step* 4. We now check that for any $k$ and $i$

$$\limsup_n E_n^{k,i}(u_n) \leq \int_a^b F^k(\dot u)\, dt + \sum_{S(u)} G^k([u]). \tag{47}$$

We have

$$E_n^{k,i}(u_n) \leq \sum_{j \notin J_n^{k,i}} k\lambda_n \psi_n^k\Big(⨍_{y_n^j}^{y_n^{j+1}} \dot u\, dt\Big) + \sum_{j \in J_n^{k,i}} G_n^k\Big(\int_{y_n^j}^{y_n^{j+1}} \dot u\, dt + \sum_{S(u)\cap(y_n^j, y_n^{j+1}]} [u]\Big).$$

Let $n$ be large enough so that $T_{n,-}^k < \text{ess-inf } \dot u \leq \text{ess-sup } \dot u < T_{n,+}^k$. For every $t \in S(u)$ there exists $j_n^t \in \{0, \ldots, M_n^{k,i}\}$ such that $S(u) \cap (y_n^{j_n^t}, y_n^{j_n^t+1}] = \{t\}$. Then, by convexity,

$$E_n^{k,i}(u_n) \leq \int_a^b \psi_n^k(\dot u)\, dt + \sum_{j=j_n^t,\ t\in S(u)} G_n^k\Big(\int_{y_n^j}^{y_n^{j+1}} \dot u\, dt + [u](t)\Big).$$

Since $F_n^k, G_n^k$ tend to $F^k, G^k$, respectively, uniformly on compact sets of $\mathbb{R}$ and $\mathbb{R} \setminus \{0\}$, respectively, and, for every $j$, $\lim_n \int_{y_n^j}^{y_n^{j+1}} \dot u\, dt = 0$, passing to the limsup we get (47).

*Step* 5. By substituting (47) in (44) and letting $\varepsilon \to 0+$, we get the $\Gamma$-limsup inequality.

*Step* 6. We finally extend the result to a general $u$ such that $\mathcal{E}(u) < +\infty$.

Let $c_1 = \inf\{x \in \mathbb{R} : \mathcal{F}(x) < +\infty\}$, $c_2 = \sup\{x \in \mathbb{R} : \mathcal{F}(x) < +\infty\}$. We may assume $c_1 \neq c_2$ otherwise there is nothing left to prove. For $k \in \mathbb{N}$, define

$$m_k = \begin{cases} c_1 + \dfrac{1}{k} & \text{if } c_1 \in \mathbb{R} \\ -k & \text{if } c_1 = -\infty \end{cases} \qquad M_k = \begin{cases} c_2 - \dfrac{1}{k} & \text{if } c_2 \in \mathbb{R} \\ k & \text{if } c_2 = +\infty. \end{cases}$$

If $u \in SBV^p(a,b)$ is such that $\mathcal{E}(u) < +\infty$, $u_k$ is defined as

$$u_k(x) = u(a+) + \int_a^x (\dot u \vee m_k) \wedge M_k \, dt + \sum_{y \in S(u),\ y<x} [u](y).$$

It is easily checked that $u_k \to u$ in $L^p(a,b)$ and $\lim_k \mathcal{E}(u_k) = \mathcal{E}(u)$. We get $\Gamma\text{-}\limsup_n \mathcal{E}_n(u) \leq \mathcal{E}(u)$ by using the lower semicontinuity of the $\Gamma$-limsup, □

## 4. The N-dimensional case

In this section we extend the results obtained in Propositions 3.3 and 3.5 to the general $N$-dimensional case. We will describe the continuum limit of energies

$$\mathcal{H}_n(u) = \sum_{\substack{x,y \in Z_n \\ x \neq y}} \Psi_n(u(x) - u(y), x - y). \tag{48}$$

where $Z_n$ is the portion of a lattice of step size $\lambda_n$ contained in a fixed open set $\Omega$ (see the Introduction). A key point will be the reduction to the 1-dimensional case by considering 1-dimensional fibers, where we can apply the previous results. In order to simplify the description of the limit we will suppose that $\Omega$ is convex, so that these fibers are always intervals. In the general case it is necessary to neglect the interactions between points $x, y$ such that the interval with endpoints $x$ and $y$ does not lie inside $\Omega$. We will give a description of the limit in terms which are equivalent to, but differ a little from, those in the Introduction, by gruping the interactions first by their direction (indexed by a 'rational direction' $\nu$) and then by relative length (indexed by a positive integer $k$).

Let $\Omega$ be a bounded, convex, smooth open set of $\mathbb{R}^N$ with $\partial\Omega$ of class $C^1$ and let $e_1, \dots, e_N$ denote a fixed orthonormal basis of $\mathbb{R}^N$. In order to rewrite functional $\mathcal{H}_n$ as defined on a subset of $L^1(\Omega)$ we identify the functions defined in $Z_n = \lambda_n \mathbb{Z}^N \cap \Omega$ as the set $A_n(\Omega)$ of functions which are constant on each cube $\alpha + (0, \lambda_n)^N$ with $\alpha \in \lambda_n \mathbb{Z}^N$. For such $\alpha$ the value $u(\alpha)$ is defined as the constant value taken by $u$ on $\alpha + (0,\lambda_n)^N$ a.e. Let $D \subset S^{N-1}$ be the set of 'rational directions' in $\mathbb{R}^N$, defined as

$$D = \{\xi/|\xi| : \xi \in \mathbb{Z}^N \setminus \{0\}\}.$$

If $\nu \in D$ we denote

$$\xi(\nu) := \min\{|\xi| : \nu = \xi/|\xi|,\ \xi \in \mathbb{Z}^N \setminus \{0\}\}.$$

We will also write $D_M$ to denote the set of directions $\nu \in D$ such that $\xi(\nu) \leq M$. For any $\nu \in D$, $n \in \mathbb{N}$ and $k = 1, \dots, N_n(\nu)$, let $\psi_n^{k,\nu} : \mathbb{R} \to [0, +\infty)$ be continuous functions. We define $\mathcal{H}_n : L^1(\Omega) \to [0, +\infty]$ as

$$\mathcal{H}_n(u) := \begin{cases} \displaystyle\sum_{\nu \in D} \sum_{k=1}^{N_n(\nu)} \sum_{\alpha \in R_n^{k,\nu}} k\lambda_n^N \xi(\nu) \psi_n^{k,\nu}\left(\frac{u(\alpha + k\lambda_n\xi(\nu)\nu) - u(\alpha)}{k\lambda_n\xi(\nu)}\right) & \\ & \text{if } u \in A_n(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega) \end{cases}$$

where $N_n(\nu) := \sup_y \operatorname{diam}(\Omega^{\nu,y})(\lambda_n\xi(\nu))^{-1}$ and

$$R_n^{k,\nu} := \{\alpha \in \lambda_n\mathbb{Z}^N : \alpha, \alpha + k\lambda_n\xi(\nu)\nu \in \Omega\}.$$

In this way we have rewritten the functional $\mathcal{H}_n$ considered in (48) if we take

$$\psi_n^{k,\nu}(z) = \frac{1}{k\lambda_n^N\xi(\nu)}\Psi_n(k\lambda_n\xi(\nu)z, k\lambda_n\xi(\nu)\nu).$$

The following theorem gives a convergence result for the functionals $\mathcal{H}_n$ as $n \to +\infty$.

**Theorem 4.1.** *Assume that real numbers $T_{n,\pm}^{k,\nu}$ and $p > 1$ exist such that the following conditions are satisfied:*

(1) (conditions on the lattice parameters) *for all $\nu \in D$, $k \in \mathbb{N}$*

$$\lim_n \lambda_n T_{n,\pm}^{k,\nu} = 0, \quad \lim_n T_{n,\pm}^{k,\nu} = \pm\infty; \tag{49}$$

(2) (structure conditions on $\psi_n^{k,\nu}$)

$$\begin{aligned} &\psi_n^{k,\nu} \textit{ is convex on } [T_{n,-}^{k,\nu}, T_{n,+}^{k,\nu}] \\ &\psi_n^{k,\nu} \textit{ is concave on } (-\infty, T_{n,-}^{k,\nu}] \\ &\psi_n^{k,\nu} \textit{ is concave on } [T_{n,+}^{k,\nu}, +\infty); \end{aligned} \tag{50}$$

(3) (growth conditions on nearest-neighbour interactions) *if $\nu \in \{e_1, \dots, e_N\}$ then*

$$\begin{aligned} &\psi_n^{1,\nu}(x) \geq |x|^p \textit{ if } x \in [T_{n,-}^{1,\nu}, T_{n,+}^{1,\nu}] \\ &\lambda_n\, \psi_n^{1,\nu}(x) \geq c > 0 \textit{ if } x < T_{n,-}^{1,\nu} \textit{ or } x > T_{n,+}^{1,\nu}; \end{aligned} \tag{51}$$

(4) (existence of single-interaction limit energy densities) *for all $\nu \in D$, $k \in \mathbb{N}$ there exist $F^{k,\nu}$, $G^{k,\nu} : \mathbb{R} \to [0, +\infty)$ such that $G^{k,\nu}$ is superlinear in $0$ and*

$$F^{k,\nu}(x) = \lim_n \psi_n^{k,\nu}(x), \qquad G^{k,\nu}(x) = \lim_n k\lambda_n\xi(\nu)\, \psi_n^{k,\nu}\Big(\frac{x}{k\lambda_n\xi(\nu)}\Big) \tag{52}$$

*for all $x \in \mathbb{R}$;*

(5) (existence of limit energy densities) *if $\mathcal{F}^\nu, \mathcal{G}^\nu : \mathbb{R} \to [0, +\infty)$ are defined by*

$$\mathcal{F}^\nu = \sum_{k=1}^\infty kF^{k,\nu} \qquad \textit{and} \qquad \mathcal{G}^\nu = \sum_{k=1}^\infty kG^{k,\nu}$$

*then*

$$\begin{aligned}
\mathcal{F}^\nu(x) &= \lim_n \sum_{k=1}^{N_n(\nu)} k\psi_n^{k,\nu}(x), \\
\mathcal{G}^\nu(x) &= \lim_n \sum_{k=1}^{N_n(\nu)} k^2\lambda_n\xi(\nu)\,\psi_n^{k,\nu}\Big(\frac{x}{k\lambda_n\xi(\nu)}\Big), \qquad (53)\\
\sum_{\nu\in D}\xi(\nu)\mathcal{F}^\nu(x) &= \lim_n \sum_{\nu\in D}\xi(\nu)\sum_{k=1}^{N_n(\nu)} k\psi_n^{k,\nu}(x), \qquad (54)\\
\sum_{\nu\in D}\xi(\nu)\mathcal{G}^\nu(x) &= \lim_n \sum_{\nu\in D}\xi(\nu)\sum_{k=1}^{N_n(\nu)} k^2\lambda_n\xi(\nu)^2\,\psi_n^{k,\nu}\Big(\frac{x}{k\lambda_n\xi(\nu)}\Big) \qquad (55)
\end{aligned}$$

*for all* $x \in \mathbb{R}$*;*

(6) (growth condition on the limit bulk energy density) *we have*

$$\sum_{\nu\in Q}\xi(\nu)\,\mathcal{F}^\nu(x) \le c(1+|x|^p) \qquad (56)$$

*for all* $x\in\mathbb{R}$*.*

*Then* $\mathcal{H}_n(u)$ $\Gamma$*-converges to the functional* $\mathcal{H} : L^1(\Omega)\to[0,+\infty]$ *defined as*

$$\mathcal{H}(u) = \begin{cases} \displaystyle\int_\Omega \mathcal{F}(\nabla u(x))\,dx + \int_{S(u)} \mathcal{G}(u^+-u^-,\nu_u)\,d\mathcal{H}^{N-1} & \text{if } u\in GSBV(\Omega)\\ +\infty & \text{otherwise in } L^1(\Omega)\end{cases}$$

*with respect both to the* $L^1(\Omega)$*-convergence and to the convergence in measure, where* $\mathcal{F}:\mathbb{R}\to[0,+\infty)$*,* $\mathcal{G}:\mathbb{R}\times S^{N-1}\to[0,+\infty)$ *are defined as*

$$\begin{aligned}
\mathcal{F}(x) &:= \sum_{\nu\in D}\xi(\nu)\mathcal{F}^\nu(x\cdot\nu)\\
\mathcal{G}(z,\eta) &:= \sum_{\nu\in D}\xi(\nu)\mathcal{G}^\nu(z\,\mathrm{sgn}\,(\eta\cdot\nu))|\eta\cdot\nu|.
\end{aligned}$$

Before proving this result it is worth commenting hypotheses (1)–(6).

**Remark 4.2.** The first hypothesis in (1) ensures that the concave parts of $\psi_n^{k,\nu}$ are meaningful in the description of the limit surface energy density. Indeed, if $\limsup_n \lambda_n T_{n,\pm}^{k,\nu} > 0$ then the corresponding $G^{k,\nu}$ may not give a contribution to the energy density $\mathcal{G}$, which should then be modified accordingly. If $\liminf_n T_{n,\pm}^{k,\nu} < +\infty$, on the other hand, then the description of $\mathcal{F}$ must be modified by taking a suitable convex modification of $\psi_n^{k,\nu}$ into account (in the case of 1-dimensional nearest-neighbour interaction a precise description of this procedure can be found in [10]).

Condition (2) may be weakened in view of the results in [12], but in general the $\Gamma$-limits with respect to the $L^1(\Omega)$ convergence and to the convergence in measure may be different.

Condition (3) ensures that the limit domain is contained in $GSBV(\Omega)$. Both conditions may be slightly modified by taking the coerciveness conditions for functionals defined on $GSBV(\Omega)$ into account (see [5]).

The superlinearity condition on $G^{k,\nu}$ in (4) may be dropped if we assume some monotonicity conditions on the points $T^{k,\nu}_{n\pm}$; e.g., that $T^{k,\nu}_{n,-} \leq T^{1,\nu}_{n,-} \leq T^{1,\nu}_{n,+} \leq T^{k,\nu}_{n,+}$ (see Step 1 in the proof of Proposition 3.3). Moreover, if only a finite number of interactions are considered then this condition may be dropped on those not taken into account (see the theorem below).

The existence of the functions $F^{k,\nu}$ and $G^{k,\nu}$ in (4) is not restrictive, upon extracting a subsequence, by the convexity and concavity conditions on $\psi^{k,\nu}_n$. Note that $F^{k,\nu}$ is convex and $G^{k,\nu}$ is concave on $(-\infty, 0]$ and on $[0,\infty)$.

Conditions (5) ensure that there is no contribution to $\mathcal{F}$ and $\mathcal{G}$ which cannot be captured by considering $\mathcal{F}^\nu$ and $\mathcal{G}^\nu$ only; i.e., there is no big contribution by $F^{k,\nu}$ and $G^{k,\nu}$ if $k\xi(\nu)$ is large. It can easily be seen that if this condition is not satisfied then the $\Gamma$-limit may not be local.

Condition (6) is technical, and is related to the general difficulty of representing bulk functionals which satisfy different growth estimate from above and below.

We can simplify Theorem 4.1 in the case of a finite set of interactions. The proof is the same, up to ignoring the contribution which are not present.

**Theorem 4.3.** *Let $\mathbb{D}$ be a finite set in $D$ containing $e_1, \ldots, e_N$, and for all $\nu \in \mathbb{D}$ let $I(\nu) \subset \mathbb{N}$ be a finite set. We suppose that $1 \in I(e_j)$ for all $j = 1, \ldots, N$, and we denote*

$$\Delta = \{(\nu, k) : \ \nu \in \mathbb{D}, \ k \in I(\nu)\}.$$

*Assume that real numbers $(T^{k,\nu}_{n,\pm})$ and $p > 1$ exist such that the conditions (1)–(4) of Theorem 4.1 are satisfied for $(\nu, k) \in \Delta$. Let $\mathcal{H}_n : L^1(\Omega) \to [0, +\infty]$ be defined by*

$$\mathcal{H}_n(u) := \begin{cases} \displaystyle\sum_{(\nu,k)\in\Delta} \sum_{\alpha \in R^{k,\nu}_n} k\lambda_n^N \xi(\nu) \psi^{k,\nu}_n \left( \frac{u(\alpha + k\lambda_n \xi(\nu)\nu) - u(\alpha)}{k\lambda_n \xi(\nu)} \right) & \\ & \text{if } u \in A_n(\Omega) \\ +\infty & \text{otherwise,} \end{cases}$$

*let*

$$\begin{aligned} \mathcal{F}(x) &:= \lim_n \sum_{(\nu,k)\in\Delta} \xi(\nu) k \psi^{k,\nu}_n(x \cdot \nu) \\ \mathcal{G}(z, \eta) &:= \lim_n \sum_{(\nu,k)\in\Delta} \xi(\nu) k^2 \lambda_n \xi(\nu)\, \psi^{k,\nu}_n \Big( \frac{z \operatorname{sgn}(\eta \cdot \nu)}{k\lambda_n \xi(\nu)} \Big) |\eta \cdot \nu|. \end{aligned}$$

*and suppose that $\mathcal{F}(x) \leq c(1 + |x|^p)$ for $x \in \mathbb{R}$. Then $\mathcal{H}_n(u)$ $\Gamma$-converges to the functional $\mathcal{H} : L^1(\Omega) \to [0, +\infty]$ defined as in Theorem 4.1.*

In order to simplify the proof of Theorem 4.1 we introduce some notation and state some preliminary remarks. We define $F_n^{k,\nu}, G_n^{k,\nu} : \mathbb{R} \to [0,+\infty)$ as follows

$$F_n^{k,\nu}(x) := \begin{cases} \psi_n^{k,\nu}(x) & \text{if } T_{n,-}^{k,\nu} \le |x| \le T_{n,+}^{k,\nu} \\ +\infty & \text{otherwise} \end{cases}$$

$$G_n^{k,\nu}(x) := \begin{cases} k\lambda_n\xi(\nu)\psi_n^{k,\nu}\Big(\dfrac{x}{k\lambda_n\xi(\nu)} + T_{n,\operatorname{sgn}(x)}^{k,\nu}\Big) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Note that $F_n^{k,\nu}$ is convex and $G_n^{k,\nu}$ is concave on $(-\infty, 0]$ and on $[0,+\infty)$, and that we still have $F^{k,\nu} = \lim_n F_n^{k,\nu}$ and $G^{k,\nu} = \lim_n G_n^{k,\nu}$.

We recall that if $\xi \in S^{N-1}$ we denote by $\Pi^\xi$ the linear hyperplane orthogonal to $\xi$ (which will be identified with $\mathbb{R}^{N-1}$ when needed) and $P^\xi : \mathbb{R}^N \to \Pi^\xi$ the orthogonal projection on $\Pi^\xi$. If $y \in \Pi^\xi$ and $E \subset \mathbb{R}^N$ we set $E^{\xi,y} := \{t \in \mathbb{R} : y + t\xi \in E\}$. Moreover, if $u : E \to \mathbb{R}$ we define the function $u^{\xi,y} : E^{\xi,y} \to \mathbb{R}$ by $u^{\xi,y}(t) := u(y + t\xi)$. We set

$$\Omega^\xi = \{y \in \Pi^\xi : \ \Omega^{\xi,y} \neq \emptyset\}.$$

**Remark 4.4.** If $y_1, y_2 \in \Omega^\xi$, let $(a_1, b_1) = \Omega^{\xi,y_1}$ and $(a_2, b_2) = \Omega^{\xi,y_2}$. Then, for any fixed $\eta > 0$, there exist two constants $\varrho, M > 0$, depending only on $\eta$ and $\xi$, such that $|a_1 - a_2| + |b_1 - b_2| \le M|y_1 - y_2|$ whenever $|y_1 - y_2| < \varrho$ for all $y_1, y_2 \in \Omega^\xi_\eta := \{y \in \Omega^\xi : \operatorname{dist}(y, \partial\Omega^\xi) > \eta\}$.

**Remark 4.5.** If $S = S_1 \cup \ldots \cup S_M$ is a finite union of $(N-1)$-simplexes and we denote

$$n(y) := \#\{t \in \Omega^{\xi,y} : y + t\xi \in S\}$$

for $y \in \Omega^\xi$, then there exists a closed set $B \subset \Pi^\xi$ with $\mathcal{H}^{N-1}(B) = 0$ such that $n(y) : \Omega^\xi \setminus B \to \mathbb{N}$ is locally constant.

**Remark 4.6.** By applying the result above we get that $\Omega^\xi \setminus B$ is a finite union of open disjoint sets $U$ such that $n(y)$ is constant on $U$. Moreover, if $y, z \in U$ and for any $i = 1, \ldots, M$ we denote $t^i_y, t^i_z$ the points in $\Omega^{\xi,y}$,$\Omega^{\xi,z}$, respectively, such that $y + t^i_y\xi, z + t^i_z\xi \in S_i$, then $|t^i_y - t^i_z| \le c|z - y|$ with $c = c(\xi_1, \ldots, \xi_M)$.

**Proof of Theorem 4.1.** In order to simplify the notation, we suppose that $\psi_n^{k,\nu}$ is even for all $k, \nu$ and $n$, the proof in the general case following easily. We begin by rewriting the functional $\mathcal{H}_n$ as a sum of 'nearest-neighbour type' functionals based on sub-lattices of $\lambda_n\mathbb{Z}^N$. First of all, note that

$$\mathcal{H}_n(u) = \sum_{\nu \in D} \mathcal{H}_n^\nu(u),$$

where

$$\mathcal{H}_n^\nu(u) := \sum_{k=1}^{N_n(\nu)} \sum_{\alpha \in R_n^{k,\nu}} k\lambda_n^N \xi(\nu)\psi_n^{k,\nu}\left(\frac{u(\alpha + k\lambda_n\xi(\nu)\nu) - u(\alpha)}{k\lambda_n\xi(\nu)}\right).$$

We will proceed by analizing the limiting behaviour of $\mathcal{H}_n^\nu$ first. To this end, with fixed $\xi_1 := \xi(\nu)\nu$, let $\xi_2, \ldots, \xi_N \in \mathbb{Z}^N \cap \Pi^\nu$ be such that $\xi_i \cdot \xi_j = 0$ for $i \neq j$. Denote

$\mathcal{M}(\nu) := |\det(\xi_1, \dots, \xi_N)|$ and $L(\nu) := \mathcal{M}(\nu)/|\xi_1|$. Note that $\mathcal{M}(\nu) \in \mathbb{Z}$. Let $z_i$ be the points in $\Pi^\nu$ such that

$$\{z_i : \ i = 1, \dots, \mathcal{M}(\nu)\} := \{z \in \mathbb{Z}^N : 0 \leq (z \cdot \xi_j) < |\xi_j|, \ j = 1, \dots, N\}$$

and let $R^\nu := \{m_1\xi_1 + \ldots + m_N\xi_N : m_i \in \mathbb{Z}\}$. Then, we can split $\mathbb{Z}^N$ into an union of disjoint copies of $R^\nu$ as

$$\mathbb{Z}^N = \bigcup_{i=1}^{\mathcal{M}(\nu)} (z_i + R^\nu).$$

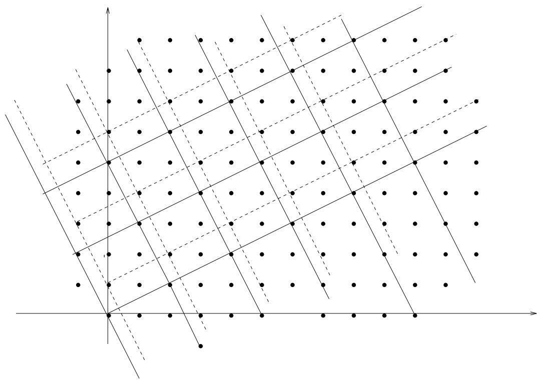

Figure 4.1: the lattices $z_i + R^\nu$

For $n \in \mathbb{N}$, $i = 1, \dots, \mathcal{M}(\nu)$ we write

$$R^{k,\nu}_{n,i} := \{\alpha \in R^{k,\nu}_n : {\lambda_n}^{-1}\alpha \in (z_i + R^\nu)\},$$

so that, $\mathcal{H}^\nu_n(u) = \sum_i \mathcal{H}^{\nu,z_i}_n(u)$, with

$$\mathcal{H}^{\nu,z_i}_n(u) := \sum_{k=1}^{N_n(\nu)} \sum_{\alpha \in R^{k,\nu}_{n,i}} k\lambda_n^N \xi(\nu) \psi^{k,\nu}_n \left( \frac{u(\alpha + k\lambda_n\xi(\nu)\nu) - u(\alpha)}{k\lambda_n\xi(\nu)} \right).$$

We now prove the Γ-liminf and Γ-limsup inequality separately.

*Proof of the* Γ*-liminf inequality.* We will use the 1-dimensional results of the previous section to provide an estimate of the functionals $\mathcal{H}^{\nu,z_i}_n$ in order to recover the desired inequality by a slicing technique. Consider $u_n, u \in L^1(\Omega)$ such that $u_n \to u$ in measure and $\sup_n \mathcal{H}_n(u_n) < +\infty$ and fix a direction $\nu \in Q$ and $z_i$ as above; if $\alpha \in R^{k,\nu}_{n,i}$ we will denote

$$Q^\nu_{\alpha,n} := \{x \in \mathbb{R}^N : 0 \leq (x - \alpha \cdot \xi_j) < \lambda_n|\xi_j| \ j = 1, \dots, N\},$$

and, for $\beta = P^\nu(\alpha)$, we also denote $Q_{\beta,n} := P^\nu(Q^\nu_{\alpha,n})$. Then, with fixed $\eta > 0$, for any function $u$ that is constant on each $Q^\nu_{\alpha,n}$, for n sufficiently large

$$\mathcal{H}^{\nu,z_i}_n(u) \geq \sum_{\beta \in \mathcal{I}^{\nu,\eta}_{n,z_i}} \lambda_n^{N-1} \mathcal{E}^{\nu,z_i}_n(u^{\nu,\beta}, \Omega^{\nu,\beta}) \tag{57}$$

holds, where $\mathcal{I}^{\nu,\eta}_{n,z_i} := \{\beta \in P^\nu(R^{1,\nu}_{n,i}) : Q_{\beta,n} \cap \Omega^\nu_\eta \neq \emptyset\}$ and $\mathcal{E}^{\nu,z_i}_n(\,\cdot\,, \Omega^{\nu,\beta})$ is the localized version of the functional defined in (17), obtained by replacing $\lambda_n$, $\psi^k_n$, $T^k_{n,\pm}$, $(a,b)$ and $\{x^i_n\}$ by $\lambda_n\xi(\nu)$, $\psi^{k,\nu}_n$, $T^{k,\nu}_{n,\pm}$, $\Omega^{\nu,\beta}$ and $\{i \in \lambda_n\xi(\nu)\mathbb{Z} : \beta + i\nu \in \Omega\}$, respectively.

Let $\mathcal{A}_n(\Omega, z_i + R^\nu)$ denote the restrictions to $\Omega$ of functions constant on each $Q^\nu_{\alpha,n}$ We want to define a sequence $v_n$ in $\mathcal{A}_n(\Omega, z_i + R^\nu)$ which coincides with $u_n$ on the edges of 'almost' each poly-rectangle $Q^\nu_{\alpha,n}$. With fixed $\beta \in \mathcal{I}^{\nu,\eta}_n$, denote

$$\underline{i} = \underline{i}(\beta) := \min\Big\{i \in \lambda_n\xi(\nu)\mathbb{Z} : Q^\nu_{\alpha,n} \subset \Omega \text{ where } \alpha = \beta + i\nu\Big\}$$
$$\overline{i} = \overline{i}(\beta) := \max\Big\{i \in \lambda_n\xi(\nu)\mathbb{Z} : Q^\nu_{\alpha,n} \subset \Omega \text{ where } \alpha = \beta + i\nu\Big\};$$

if $\alpha = \beta + i\nu$, we define $v_n(x)$ on $Q^\nu_{\alpha,n}$ as

$$v_n(x) = \begin{cases} u_n(\alpha) & \text{if } \underline{i} \le i \le \overline{i} \\ u_n(\beta + \underline{i}\nu) & \text{if } i < \underline{i} \\ u_n(\beta + \overline{i}\nu) & \text{if } i > \overline{i}. \end{cases} \tag{58}$$

We claim that

$$\liminf_n \mathcal{E}^{\nu,z_i}_n(v^{\nu,y}_n, \Omega^{\nu,y}_\eta) \ge \int_{\Omega^{\nu,y}_\eta} \mathcal{F}^\nu(\dot{u}^{\nu,y})\,dt + \sum_{S(u^{\nu,y})\cap\Omega^{\nu,y}_\eta} \mathcal{G}^\nu([u^{\nu,y}]), \tag{59}$$

where, for the sake of simplicity, we have set $\Omega_\eta := \{x \in \Omega : \operatorname{dist}(P^\nu(x), \partial\Omega^\nu) > \eta\}$.

We first prove (59) in the case $\nu \in \{e_1, \ldots, e_N\}$; subsequently, we will infer the same inequality for every $\nu \in Q^\nu_{\alpha,n}$. Then, let $\nu = e_j$ and $v_n$ be as above; in this case we have to consider a single lattice, determined by $z_i = 0$. Note that $v_n \to u$ in measure in $L^1(\Omega_\eta)$; indeed,

$$\{x \in \Omega_\eta : v_n(x) \neq u_n(x)\} = \bigcup_{\beta \in \mathcal{I}^{e_j,\eta}_n} \Big\{\alpha + [0,\lambda_n)^N : \alpha = \beta + ie_j, i > \overline{i} \text{ or } i < \underline{i}\Big\}.$$

By Remark 4.4, we get that, for $n$ sufficiently large as to have $N\lambda_n < \rho$, for any $\beta \in \mathcal{I}^{e_j,\eta}_n$

$$\lambda_n \#\{i \in \lambda_n\mathbb{Z} : i > \overline{i} \text{ or } i < \underline{i}\} \le \lambda_n M \tag{60}$$

with $M = M(\frac{\eta}{2})$ and $\rho = \rho(\frac{\eta}{2})$ in Remark 4.4. Since $\#\mathcal{I}^{e_j,\eta}_n \le |\Omega^{e_j}|\lambda^{1-N}_n$, we obtain

$$\lim_n |\{x \in \Omega_\eta : v_n(x) \neq u_n(x)\}| \le \lim_n c\lambda_n = 0.$$

Hence, $v_n \to u$ in measure on $\Omega^{e_j,y}_\eta$ and, by construction, we have

$$\sum_{\beta \in \mathcal{I}^{e_j,\eta}_n} \lambda^{N-1}_n \mathcal{E}^{e_j,0}_n(v^{e_j,\beta}_n, \Omega^{e_j,\beta}_\eta) \ge \int_{\Omega^{e_j}_\eta} \mathcal{E}^{e_j,0}_n(v^{e_j,y}_n, \Omega^{e_j,y}_\eta)\,d\mathcal{H}^{N-1}(y) - c\mathcal{F}^{e_j}_n(0)\lambda_n. \tag{61}$$

Since $\mathcal{E}_n^{e_j,0}(\,\cdot\,,\Omega_\eta^{e_j,y})$ satisfies the hypotheses of Proposition 3.3, by taking the equiboundedness of $\mathcal{H}_n^\nu(u_n)$ and the convergence in measure of $v_n^{e_j,y}$ to $u^{e_j,y}$ into account, we get that $u^{e_j,y}\in SBV(\Omega_\eta^{e_j,y})$ and

$$\liminf_n \mathcal{E}_n^{e_j,0}(v_n^{e_j,y},\Omega_\eta^{e_j,y}) \geq \int_{\Omega_\eta^{e_j,y}} \mathcal{F}^{e_j}(\dot u^{e_j,y})\,dt + \sum_{S(u^{e_j,y})\cap\Omega_\eta^{e_j,y}} \mathcal{G}^{e_j}([u^{e_j,y}]).$$

Again by the uniform bound on $\mathcal{H}_n^\nu(u_n)$ with respect to $\nu$ and $\eta$, we deduce that $u\in GSBV(\Omega)$ by the slicing result Theorem 2.3(b).

We now turn our attention to $\nu\in D\setminus\{e_1,\dots,e_N\}$; it is easy to check that (60) still holds and, taking into account that $\mathcal{H}^{N-1}(Q_{\beta,n})=L(\nu)(\lambda_n)^{N-1}$, we can rewrite (61) as

$$\mathcal{H}_n^{\nu,z_i}(u_n) \geq \int_{\Omega_\eta^\nu} L(\nu)^{-1}\mathcal{E}_n^{\nu,z_i}(v_n^{\nu,y},\Omega_\eta^{\nu,y})\,d\mathcal{H}^{N-1}(y) - c\mathcal{F}_n^\nu(0)\lambda_n. \tag{62}$$

Note that, since $\psi_n^{1,\nu}$ does not satisfy in general hypothesis (51), we cannot use Proposition 3.3 directly. However, we can repeat the proof of Proposition 3.3, by defining the sets $I_n^{k,i}(\nu,z_i)$ and the piecewise affine functions $u_n^{k,i}(\nu,z_i)(\,\cdot\,)$ in the same way as the sets $I_n^{k,i}$ and the functions $u_n^{k,i}$ are defined there, and noticing that, if $v_n^1,\dots,v_n^N$ are the functions defined in (58) with respect to $e_1,\dots,e_N$, respectively, then we can estimate $u_n(x)-v_n(x)$, $\dot u_n^{k,i}(\nu,z_i)$ in terms of $u_n(x)-v_n^j(x)$, $\dot v_n^j$. Thus we get that $v_n$ still converges to $u$ in measure and (59) holds.

We can now take the liminf as $n$ goes to $+\infty$ using (59) and Fatou's Lemma to get

$$\liminf_n \mathcal{H}_n^{\nu,z_i}(u_n) \geq \int_{\Omega_\eta^\nu} \frac{1}{L(\nu)}\Big(\int_{\Omega_\eta^{\nu,y}} \mathcal{F}^\nu(\dot u^{\nu,y})dt + \sum_{S(u^{\nu,y})\cap\Omega_\eta^{\nu,y}} \mathcal{G}^\nu([u^{\nu,y}])\Big)d\mathcal{H}^{N-1}(y). \tag{63}$$

Letting $\eta$ tend to $0+$ and summing over $i$ we obtain

$$\begin{aligned}\liminf_n \mathcal{H}_n^\nu(u_n) \;\geq\; & \Big(\int_\Omega \xi(\nu)\mathcal{F}^\nu(\nabla u\cdot\nu)\,dx \\ & + \int_{S(u)} \xi(\nu)\mathcal{G}^\nu([u]\,\mathrm{sgn}\,(\nu_u\cdot\nu))|\nu_u\cdot\nu|\,d\mathcal{H}^{N-1}\Big).\end{aligned}$$

With fixed a positive number $M$ we then obtain

$$\begin{aligned}\liminf_n \mathcal{H}_n(u_n) \;\geq\; & \sum_{\nu\in D_M}\Big(\int_\Omega \xi(\nu)\mathcal{F}^\nu\left(\nabla u(x)\cdot\nu\right)\,dx \\ & + \int_{S(u)} \xi(\nu)\mathcal{G}^\nu([u]\,\mathrm{sgn}\,(\nu_u\cdot\nu))|\nu_u\cdot\nu|\,d\mathcal{H}^{N-1}\Big).\end{aligned}$$

Eventually, we obtain the desired inequality by letting $M\to+\infty$.

*Proof of the* $\Gamma$*-limsup inequality.* To prove the $\Gamma$-limsup inequality with respect to the $L^1$-strong convergence we first deal with functions in $\mathcal{W}(\mathbb{R}^N)$ (see Section 2.3). As in the 1-dimensional case, a recovery sequence will be given by the interpolates of $u$ on the lattice $\lambda_n\mathbb{Z}^N$. The technical difficulty derives in the fact that the 1-dimensional sections of these interpolations are not themselves interpolations.

Let $u \in \mathcal{W}(\mathbb{R}^N)$ be such that $\mathcal{H}(u) < +\infty$. Up to considering in the sequel the lattice $\lambda_n\mathbb{Z}^N + \xi_n$, for suitable $\xi_n \to 0$, we can assume that $\overline{S}(u) \cap \lambda_n\mathbb{Z}^N = \emptyset$ for all $n$. Then, we define $u_n \in \mathcal{A}_n(\Omega)$ by setting $u_n(x) := u(\alpha)$ on $\alpha + [0, \lambda_n)^N$. We have that $u_n \to u$ in $L^1(\Omega)$. Indeed, with fixed $\alpha \in \lambda_n\mathbb{Z}^N$ and $x \in \alpha + [0, \lambda_n)^N$, we have

$$\begin{aligned} |u_n(x) - u(x)| &= |u(\alpha) - u(x)| \\ &\leq \Big| \int_0^1 \frac{d}{dt} u(t\alpha + (1-t)x)\, dt \Big| + \sum_{z \in [\alpha, x] \cap S(u)} |u^+ - u^-|(z) \\ &\leq \|\nabla u\|_\infty \sqrt{N}\lambda_n + 2\|u\|_\infty M' \chi_{A_n}(x), \end{aligned}$$

where $M'$ is the number of $(N-1)$-simplexes contained in $\overline{S}(u)$ and $A_n$ is the set of those cubes whose intersection with $S(u)$ is non-empty. Since $A_n \subset \{x : \mathrm{dist}\,(x, \overline{S}(u)) \leq \sqrt{N}\lambda_n\}$, it is easy to compute that $|A_n| \leq c\lambda_n$. Hence, we get

$$\lim_n \|u_n - u\|_{L^1(\Omega)} \leq \lim_n c\Big(\|\nabla u\|_\infty |\Omega| + 2\|u\|_\infty \mathcal{H}^{N-1}(S(u))\Big)\lambda_n = 0$$

by integrating on $\alpha + [0, \lambda_n)^N$ and summing over $\alpha$.

We will proceed as follows: first we will prove that for every direction $\nu \in D$

$$\begin{aligned} &\limsup_n \mathcal{H}_n^\nu(u_n) \\ &\leq \int_\Omega \xi(\nu)\mathcal{F}^\nu(\nabla u(x) \cdot \nu) dx \\ &\qquad + \int_{S(u)} \xi(\nu)\mathcal{G}^\nu([u]\,\mathrm{sgn}\,(\nu_u \cdot \nu))|\nu_u \cdot \nu|\, d\mathcal{H}^{N-1}; \end{aligned} \tag{64}$$

subsequently, we prove that for every $\varepsilon > 0$ there exists $M > 0$ such that

$$\limsup_n \sum_{\nu \in D \setminus D_M} \mathcal{H}_n^\nu(u_n) \leq \varepsilon. \tag{65}$$

With fixed $\nu \in D$, we prove (64). For the rest of the proof it is useful to estimate the value of the functionals $\mathcal{H}_n^\nu$ with respect to sets of the form $P^{\nu-1}(C) \cap \Omega$ with $C \subset \Omega^\nu$. For $\xi \in \mathbb{R}^N$ let $B = B^\xi$ be the set of Remark 4.5 and, for $\varepsilon > 0$, denote

$$\begin{aligned} B_\varepsilon^\xi &:= \{y \in \Pi^\xi : \mathrm{dist}\,(y, B^\xi) < \varepsilon\}, \\ {Q_{\alpha,n}^\nu}^k &:= \{x \in \mathbb{R}^N : 0 \leq (x - \alpha \cdot \xi_1) < k\lambda_n|\xi_1|, \\ &\qquad\qquad 0 \leq (x - \alpha \cdot \xi_j) < \lambda_n|\xi_j|\, j = 2, \ldots, N\}, \\ A_{n,i}^{k,\nu} &:= \{\alpha \in R_{n,i}^{k,\nu} : {Q_{\alpha,n}^\nu}^k \cap \overline{S}(u) \neq \emptyset\}. \end{aligned}$$

Then, for $\alpha \in R_{n,i}^{k,\nu}$, according to the different cases we have the following estimates:

$$k\lambda_n^N \xi(\nu)\psi_n^{k,\nu}\left(\frac{u(\alpha + k\lambda_n\xi_1) - u(\alpha)}{k\lambda_n\xi(\nu)}\right) \leq \begin{cases} k\lambda_n^N\xi(\nu)F_n^{k,\nu}(\|\nabla u\|_\infty) \\ \qquad\qquad \text{if } \alpha \notin A_{n,i}^{k,\nu} \\ \lambda_n^{N-1}G_n^{k,\nu}(2M'\|u\|_\infty + c\|\nabla u\|_\infty) \\ \qquad\qquad \text{otherwise,} \end{cases} \tag{66}$$

with $M'$ the number of simplexes of $\overline{S(u)}$ and $c := \operatorname{diam}\Omega$. Hence, we get

$$\sum_{\alpha\in C^{\xi_1}_{\varepsilon,i}}\sum_{k=1}^{N_n(\nu)} k\lambda_n^N\xi(\nu)\psi_n^{k,\nu}\left(\frac{u(\alpha+k\lambda_n\xi_1)-u(\alpha)}{k\lambda_n\xi(\nu)}\right)\le c_n(\nu,\varepsilon),$$

where $C^{\xi}_{\varepsilon,i} := \bigcup_k\{\alpha\in R^{k,\nu}_{n,i} : P^{\xi}(Q^{\nu}_{\alpha,n}{}^k)\cap B^{\xi}_{\varepsilon}\neq\emptyset\}$ and

$$\begin{aligned} c_n(\nu,\varepsilon) \;&:=\; L(\nu)^{-1}\Big(\sum_{k=1}^{N_n(\nu)} k(F_n^{k,\nu}(\|\nabla u\|_\infty)+G_n^{k,\nu}(2M'\|u\|_\infty+c\|\nabla u\|_\infty))\Big)\\ &\quad\times(\mathcal{H}^{N-1}(B^{\nu}_{2\varepsilon})+|P^{\nu-1}(B^{\nu}_{2\varepsilon})|). \end{aligned} \tag{67}$$

Thanks to this bound, in the following we will confine our analysis to estimate the value of the functionals on poly-rectangles whose projection does not intersect the set $B^{\nu}_{\varepsilon}$. For such poly-rectangles, the function $n(y)$ defined in Remark 4.5 is constant along the set. Let $y\in\Omega^{\nu}$, then, for any $n\in\mathbb{N}$, there exists a unique $\beta\in\lambda_n\mathbb{Z}^N\cap\Pi^{\nu}$ such that $y\in Q^{\nu}_{\beta}$; we will denote this point (depending also on $n$) by $\beta(y)$. Note that $\#\overline{S}(u)^{\nu,y} = \#\overline{S}(u)^{\nu,\beta(y)}$ for $y\in\Omega^{\nu}\setminus B^{\nu}_{\varepsilon}$. We have that

$$\begin{aligned} \mathcal{H}^{\nu}_n(u_n) \;&\le\; \int_{\Omega^{\nu}\setminus B^{\nu}_{\varepsilon}} L(\nu)^{-1}\sum_{z_i}\mathcal{E}^{\nu,z_i}_n(u_n^{\nu,\beta(y)},\Omega^{\nu,y})\,d\mathcal{H}^{N-1}\\ &\qquad +\sum_{\beta}\sup_{y:\beta=\beta(y)}\#(I^{\beta}_n\setminus I^{y}_n)c\lambda_n^N+M(\nu)c_n(\nu,\varepsilon), \end{aligned} \tag{68}$$

where $I^y_n := \{i\in\lambda_n\xi(\nu)\mathbb{Z} : y+i\nu, y+(i+\lambda_n)\nu\in\Omega\}$. We claim that, for every $y\in\Omega^{e_j}$

$$\limsup_n \mathcal{E}^{\nu,z_i}_n(u_n^{\nu,\beta(y)},\Omega^{\nu,y}) \le \int_{\Omega^{\nu,y}_{\eta}}\mathcal{F}^{\nu}(\dot u^{\nu,y})\,dt+\sum_{S(u^{\nu,y})\cap\Omega^{\nu,y}_{\eta}}\mathcal{G}^{\nu}([u^{\nu,y}]); \tag{69}$$

i.e., $u_n^{\nu,\beta(y)}$ is a recovery sequence for $u^{\nu,\beta(y)}$ although it does not coincide in general with its piecewise-constant interpolation on $\lambda_n\xi(\nu)\mathbb{Z}\cap\Omega^{\nu}_y$. By reasoning as in the proof of Proposition 3.3, it is not difficult to see that it suffices to prove that $u_n^{\nu,\beta(y)}$ is a recovery sequence for the functionals $E^{k,i}_n$ defined in (36). For the sake of simplicity we will prove this result for $k=1$ and $\nu=e_j$, as the treatment of the general case amounts only to a more complex notation.

Starting from the value of $u$ at points of the lattice $\lambda_n\mathbb{Z}^N$, for any $n\in\mathbb{N}$, we provide a function $v_n$ which is piecewise affine along the direction $e_j$. More precisely, fixed $y\in\Omega^{e_j}\setminus B^{e_j}_{\varepsilon}$ and $i\in I^y_n$ we define $v^{j,y}_n$ for $t\in[i,i+\lambda_n)$ as

$$v^{j,y}_n(t) := \begin{cases} \dfrac{u^{e_j,\beta(y)}(i+\lambda_n)-u^{e_j,\beta(y)}(i)}{\lambda_n}(t-i)+u^{e_j,\beta(y)}(i) & i\in I^y_n\setminus S_{\beta(y)}\\ u^{e_j,\beta(y)}(i) & i\in S_{\beta(y)}, \end{cases} \tag{70}$$

where $S_{\beta(y)} := \{i : (i,i+\lambda_n)\cap(\overline{S}(u))^{e_j,\beta(y)}\neq\emptyset\}$. If $y\in\Omega^{e_j}\setminus B^{e_j}_{\varepsilon}$ we have that

$$\dot v^{j,y}_n(t)\to\dot u^{e_j,y}(t)\quad\text{a.e. in }\Omega^{e_j,y} \tag{71}$$

and, since $\#\overline{S}(u)^{e_j,y} = \#\overline{S}(u)^{e_j,\beta(y)}$, by taking Remark 4.6 into account, for all $s \in \overline{S}(u)^{e_j,y} =: S_{j,y}$ there exists unique $i_n(s)$ such that $i_n(s) - \lambda_n \in S_{\beta(y)}$, $\lim_n i_n(s) = s$ and

$$[v_n^{j,y}](i_n(s)) \to [u^{e_j,y}](s) \text{ uniformly with respect to } y. \tag{72}$$

Indeed, if $t \in \Omega^{e_j,y}$, for $n$ large we have

$$\begin{aligned} |\dot{v}_n^{j,y}(t) - \dot{u}^{e_j,y}(t)| &= \Big| \fint_i^{i+\lambda_n} \nabla u(\beta(y) + se_j) \cdot e_j \, ds - \nabla u(y + te_j) \cdot e_j \Big| \\ &\leq \fint_i^{i+\lambda_n} \|H(u)\|_\infty (|\beta(y) - y| + |s - t|) \, ds \leq c\lambda_n. \end{aligned}$$

To prove (72), with fixed $s \in \overline{S}(u)^{e_j,y}$ and $i_n(s) \in S_{\beta(y)}$, we may assume that $s > i_n(s)$. Hence, by Remark (4.6),

$$\begin{aligned} |v_n^{j,y}(i_n(s)) - u^{e_j,y}(s+)| &\leq |u(\beta(y) + i_n(s)e_j) - u(\beta(y) + se_j)| \\ &\qquad + |u(\beta(y) + se_j) - u(y + se_j+)| \\ &\leq \|\nabla u\|_\infty (|\beta(y) - y| + |i_n(s) - s|) \\ &\leq c|\beta(y) - y| \leq c\lambda_n. \end{aligned}$$

An analogous computation shows that $|v_n^{j,y}(i_n(s) - \lambda_n) - u^{e_j,y}(s-)| \leq c\lambda_n$. Since $c$ is independent of $y$ and $n$ we have that the convergence is uniform.

Now, we get

$$\begin{aligned} E_n^{e_j}(u_n^{e_j,\beta(y)}, \Omega^{e_j,y}) &\leq \int_{\Omega^{e_j,y}} F_n^{e_j}(\dot{v}_n^{j,y}) \, dt + \sum_{s \in S_{j,y}} G_n^{e_j}([v_n^{j,y}](i_n(s))) \\ &=: (I)_n + (II)_n. \end{aligned} \tag{73}$$

Hence, $\limsup_n E_n^{e_j}(u_n^{e_j,\beta(y)}, \Omega^{e_j,y}) \leq \limsup_n (I)_n + \limsup_n (II)_n$. We now compute each of these quantities. Since $F_n^{e_j} \to F^{e_j}$ uniformly on compact sets, by property (1) of Theorem 2.2 and by (71), we get

$$\limsup_n \, (I)_n \leq \int_{\Omega^{e_j,y}} F^{e_j}(\dot{u}^{e_j,y}) \, dt$$

by using the Dominated Convergence Theorem. It remains to estimate the last term. Consider for $k \in \mathbb{N}$, the set $S_{j,y}^k := \{x \in S(u)^{e_j,y} : |[u^{e_j,y}]| > \frac{1}{k}\}$. Then

$$(II)_n \leq c\#(S(u^{e_j,y}) \setminus S_{j,y}^k) + \sum_{s \in S_{j,y}^k} G_n^{e_j}([v_n^{j,y}](i_n(s))).$$

Since, by (72), $[v_n^{j,y}](i_n(s)) \to [u^{e_j,y}](s)$ uniformly as $n \to +\infty$, by taking (52) into account, we have

$$\begin{aligned} \limsup_n (II)_n &\leq c\#(S(u)^{e_j,y} \setminus S_{j,y}^k) + \sum_{s \in S_{j,y}^k} G^{e_j}([u^{e_j,y}](s)) \\ &\leq c\#(S(u)^{e_j,y} \setminus S_{j,y}^k) + \sum_{s \in S_{j,y}} G^{e_j}([u^{e_j,y}](s)). \end{aligned}$$

Since $\lim_k \#(S(u)^{e_j,y} \setminus S^k_{j,y}) = 0$, we get

$$\limsup_n (II)_n \leq \sum_{s\in S_{j,y}} G^{e_j}([u^{e_j,y}](s)).$$

We now prove that

$$\limsup_n \sum_\beta \sup_{y:\beta=\beta(y)} \#(I^\beta_n \setminus I^y_n)c\lambda^N_n = 0. \tag{74}$$

With fixed $\eta > 0$, by Remark 4.4, we have

$$\sum_\beta \sup_{\{y:\beta=\beta(y)\}} \#(I^\beta_n \setminus I^y_n)c\lambda^N_n \leq M(\frac{\eta}{2})\lambda_n \mathcal{H}^{N-1}(\Omega^\nu_\eta) + \mathcal{H}^{N-1}(\Omega^\nu \setminus \Omega^\nu_\eta) \sup_{y\in\Omega^\nu} \mathcal{H}^1(\Omega^{\nu,y}).$$

Hence,

$$\limsup_n \sum_\beta \sup_{\{y:\beta=\ \beta(y)\}} \#(I^\beta_n \setminus I^y_n)c\lambda^N_n \leq c\mathcal{H}^{N-1}(\Omega^\nu \setminus \Omega^\nu_\eta).$$

Since $\mathcal{H}^{N-1}(\Omega^\nu \setminus \Omega^\nu_\eta) \to 0$ as $\eta \to 0+$, we get (74).

By (67) and (53), it can be easily seen that $\limsup_n \mathcal{M}(\nu)c(\nu,\varepsilon) \leq c\varepsilon$. Then, it suffices to pass to the lim sup as $n \to +\infty$ in (68), use (69) and let $\varepsilon$ tend to 0.

It remains to prove (65). Let $M$ be a fixed positive real number. Then taking (66) into account, it can be easily seen that

$$\sum_{\nu\in D\setminus D_M} \mathcal{H}^\nu_n(u_n) \leq c \sum_{\nu\in D\setminus D_M} (\xi(\nu)\mathcal{F}^\nu_n(\|\nabla u\|_\infty)|\Omega| + \xi(\nu)\mathcal{G}^\nu_n(T(u))\mathcal{H}^{N-1}(\Omega^\nu)),$$

where we denote $T(u) := 2M'\|u\|_\infty + \operatorname{diam} \|\nabla u\|_\infty$. Passing to the limsup as $n \to +\infty$ and using (54), (55), we get

$$\limsup_n \sum_{\nu\in D\setminus D_M} \mathcal{H}^\nu_n(u_n) \leq c \sum_{\nu\in D\setminus D_M} (\xi(\nu)\mathcal{F}^\nu(\|\nabla u\|_\infty) + \xi(\nu)\mathcal{G}^\nu(T(u))).$$

Since by the finiteness of $\mathcal{F}$ and $\mathcal{G}$

$$\lim_{M\to+\infty} \sum_{\nu\notin D_M} \xi(\nu)\Big(\mathcal{F}^\nu(\|\nabla u\|_\infty) + \mathcal{G}^\nu(T(u))\Big) = 0$$

we get the thesis.

Finally, let $u \in L^\infty(\Omega)$ be such that $\mathcal{H}(u) < +\infty$. Then, by Theorem 2.2, we can find $u_n \in \mathcal{W}(\mathbb{R}^N)$ such that $u_n \to u$ strongly in $L^1(\Omega)$ and $\lim_n \mathcal{H}(u_n) = \mathcal{H}(u)$. The inequality follows by the lower semicontinuity of the Γ-limsup. The hypothesis that $u \in L^\infty(\Omega)$ can be easily removed by a truncation argument, by taking hypothesis (56) into account. □

## 5. Examples and applications

A convergence theorem for discrete functionals with non-cubic underlying lattices can be obtained from Theorem 4.1 by a superposition argument.

**Example 5.1.** (*General lattices*) Let $\mathcal{P} := \{p_1, \dots, p_N\}$ be linearly independent vectors in $\mathbb{R}^N$ and let $\mathcal{R} := \{m_1 p_1 + \dots + m_N p_N : m_i \in \mathbb{Z} \text{ for } i = 1, \dots, N\}$ be the integer lattice associated to $\mathcal{P}$. Set

$$D^{\mathcal{P}} := \{\frac{\xi}{|\xi|} : \xi \in \mathcal{R} \setminus 0\}, \qquad \xi(\nu_{\mathcal{P}}) := \min\{r > 0 : r\nu_{\mathcal{P}} \in \mathcal{R}\} \text{ if } \nu_{\mathcal{P}} \in D^{\mathcal{P}}.$$

With fixed $\lambda_n > 0$ we define $\mathcal{A}_n^{\mathcal{P}}(\Omega)$ as the set of restrictions to $\Omega$ of functions $u$ constant on $\{x \in \mathbb{R}^N : 0 \le (x - \lambda_n \gamma \cdot p_i) < \lambda_n |p_i| \text{ for } i = 1, \dots, N\}$ for each $\gamma \in \mathcal{R}$, which correspond to the set of functions defined on $\lambda_n \mathcal{R} \cap \Omega$.

Given functions $\psi_{n,\mathcal{P}}^{k,\nu_{\mathcal{P}}} : \mathbb{R} \to [0, +\infty)$ for al $k, n \in \mathbb{N}$ and $\nu_{\mathcal{P}} \in D^{\mathcal{P}}$, we define $\mathcal{H}_n^{\mathcal{P}} : L^1(\Omega) \to [0, +\infty]$ as

$$\mathcal{H}_n^{\mathcal{P}}(u) := \sum_{\nu_{\mathcal{P}} \in D^{\mathcal{P}}} \sum_{k=1}^{N_n(\nu_{\mathcal{P}})} \sum_{\alpha \in R_{n,\mathcal{P}}^{k,\nu_{\mathcal{P}}}} k\lambda_n^N \xi(\nu_{\mathcal{P}}) \psi_{n,\mathcal{P}}^{k,\nu_{\mathcal{P}}} \left( \frac{u(\alpha + k\lambda_n \xi(\nu_{\mathcal{P}})\nu_{\mathcal{P}}) - u(\alpha)}{k\lambda_n \xi(\nu_{\mathcal{P}})} \right)$$

if $u \in A_n^{\mathcal{P}}(\Omega)$, and $\mathcal{H}_n^{\mathcal{P}} = +\infty$ otherwise in $L^1(\Omega)$, where

$$R_{n,\mathcal{P}}^{k,\nu} := \{\alpha \in \lambda_n \mathcal{R} \,:\, \alpha, \alpha + k\lambda_n \xi(\nu_{\mathcal{P}})\nu_{\mathcal{P}} \in \Omega\}$$

and $N_n(\nu_{\mathcal{P}}) := \sup_y \operatorname{diam}(\Omega^{\nu_{\mathcal{P}},y})(\lambda_n \xi(\nu_{\mathcal{P}}))^{-1}$.

If $(\psi_{n,\mathcal{P}}^{k,\nu_{\mathcal{P}}})$ satisfies hypotheses (49)-(56) (where we replace $e_i$ by $p_i$), then $\mathcal{H}_n^{\mathcal{P}}$ $\Gamma$-converges with respect to the convergence in $L^1(\Omega)$ and the convergence in measure. Moreover, if

$$F_{\mathcal{P}}^{k,\nu_{\mathcal{P}}}(z) = \lim_n \psi_{n,\mathcal{P}}^{k,\nu_{\mathcal{P}}}(z), \qquad G_{\mathcal{P}}^{k,\nu_{\mathcal{P}}}(z) = \lim_n k\lambda_n \xi(\nu_{\mathcal{P}}) \psi_{n,\mathcal{P}}^{k,\nu_{\mathcal{P}}} \left( \frac{z}{k\lambda_n \xi(\nu_{\mathcal{P}})} \right),$$

and $A : \mathbb{R}^N \to \mathbb{R}^N$ is the linear operator such that $A(e_i) = p_i$, then the limit functional $\mathcal{H}^{\mathcal{P}}$ is given by

$$\mathcal{H}^{\mathcal{P}}(u) = \int_\Omega \sum_{\nu_{\mathcal{P}} \in D^{\mathcal{P}}} \xi(\nu_{\mathcal{P}}) \sum_{k=i}^{+\infty} k F_{\mathcal{P}}^{k,\nu_{\mathcal{P}}} (\nabla u(x) \cdot \nu_{\mathcal{P}}) \, |\mathrm{det} A|^{-1} \, dx \tag{75}$$

$$+ \int_{S(u)} \sum_{\nu_{\mathcal{P}} \in D^{\mathcal{P}}} \xi(\nu_{\mathcal{P}}) \sum_{k=i}^{+\infty} k G_{\mathcal{P}}^{k,\nu_{\mathcal{P}}} \Big( [u] \operatorname{sgn}(\nu_u \cdot \nu_{\mathcal{P}}) \Big) |\nu_u \cdot \nu_{\mathcal{P}}| |\mathrm{det} A|^{-1} \, d\mathcal{H}^{N-1} \tag{76}$$

$$\tag{77}$$

if $u \in GSBV(\Omega)$, and $\mathcal{H}^{\mathcal{P}}(u) = +\infty$ otherwise in $L^1(\Omega)$. This result can be easily obtained by applying Theorem 4.1 with $\psi_n^{k,\nu}(x) := |A\nu| \psi_{n,\mathcal{P}}^{k,\nu_{\mathcal{P}}} \left( \frac{x}{|A\nu|} \right)$ and $\nu_{\mathcal{P}} = \frac{A\nu}{|A\nu|}$ and noticing that $\mathcal{H}_n^{\mathcal{P}}(u) = \mathcal{H}_n(u \circ A)$ for every $u \in \mathcal{A}_n^{\mathcal{P}}(\Omega)$.

As a particular case of the previous example, we can also treat nearest-neighbour interactions on hexagonal lattices by considering them as second-neighbour interactions on a slanted lattice.

**Example 5.2.** (*Hexagonal lattice*) Let $N = 2$ and $p_1 = e_1$, $p_2 = -\frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_2$. Fix $\lambda_n := n^{-2}$ and assume that $\psi_{n,\mathcal{P}}^{k,\nu_{\mathcal{P}}} \neq 0$ if and only if $k = 1$ and $\nu_{\mathcal{P}} \in \{p_1, p_2, p_1 + p_2\}$, i.e., every point in the lattice $\mathcal{R}$ is supposed to have interaction only with the vertices of a regular hexagon of center the point itself.

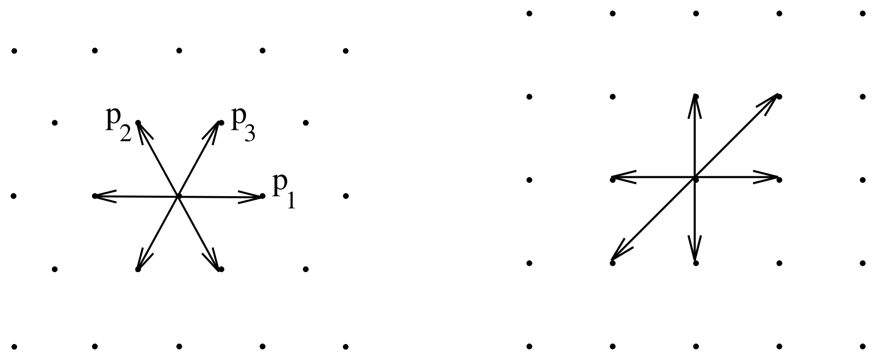


Figure 5.1: first-order interaction in the hexagonal lattice and the corresponding second-order anisotropic interaction in the standard lattice

Consider for example

$$\psi_n^{1,p_i}(z) = \frac{a_i}{\lambda_n}\Big((\lambda_n z^2) \wedge c_i^2\Big),$$

with $a_i, c_i \in \mathbb{R}^+$ and $p_3 := p_1 + p_2$, then, by using formula (75), we get

$$\mathcal{H}^{\mathcal{P}}(u) = \int_\Omega \sum_{i=1}^{3} a_i |\nabla u \cdot p_i|^2 \frac{2}{\sqrt{3}}\, dx + \int_{S(u)} \sum_{i=1}^{3} a_i c_i^2 |\nu \cdot p_i| \frac{2}{\sqrt{3}}\, d\mathcal{H}^1.$$

In particular, if we choose $a_1 = a_2 = a_3 = \sqrt{3}/2$ and $c_1 = c_2 = c_3 = 1/\sqrt{2}$, we have that

$$\mathcal{H}^{\mathcal{P}}(u) = \int_\Omega |\nabla u|^2\, dx + \int_{S(u)} |\Phi(\nu_u)|\, d\mathcal{H}^1,$$

where $\Phi(x) = \frac{1}{2}\sum_{i=1}^{3} |\frac{x}{|x|} \cdot p_i| x$ is the deformation of $\mathbb{R}^2$ into itself that applies the unitary ball into the hexagon of vertices $\pm p_1, \pm p_2, \pm p_3$ and is positively homogeneous of degree 1.

**Example 5.3.** (*Energy with a fixed range of interactions*) According to the 'local-type' interactions of many mechanical models, we confine our attention to the case in which the potentials $\psi_n^{k,\nu}$ are null if $k\xi(\nu) > R$, for $R > 0$ fixed. In this case we deal with $n(R)$ non-negligible interactions. If $N = 2$ and $R > 1$, it is easy to see that $n(R)$ is a multiple of 8. Indeed, the set of directions $D$ is invariant under the action of the linear trasformations below:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

If we assume that the interaction relative to the direction $(\nu_1, \nu_2)$ is equal to the one relative to $(-\nu_1, \nu_2)$, then it can be shown that, for potential of the form $\psi_n^{k,\nu}(z) = c_{k,\nu} \min\{z^2, d_{k,\nu}\}$, the limit energy is isotropic in the volume part. The surface part will retain some of the symmetries of the polygon identified by the directions in $D_R$. It is easy to find suitable $c_{k,\nu}$, $d_{k,\nu}$ in a way that the surface part can be written as the euclidean norm of the deformation of $\mathbb{R}^2$ positively homogeneous of degree 1 that maps the unitary ball in the polygon.

For example, let $R = \sqrt{2}$, $\lambda_n = n^{-2}$ and let

$$\begin{aligned} \psi_n^{1,\nu}(z) &= (\sqrt{2}-1)\frac{1}{2\lambda_n}\Big((\lambda_n z^2) \wedge 1\Big) && \text{if } \nu \in \{\pm e_1, \pm e_2\} \\ \psi_n^{1,\nu}(z) &= (\sqrt{2}-1)\frac{1}{2\lambda_n}\Big((\lambda_n z^2) \wedge \frac{1}{\sqrt{2}}\Big) && \text{if } \nu \in \{e_1 \pm e_2, -e_1 \pm e_2\}. \end{aligned}$$

Then, the limit energy is

$$\mathcal{H}(u) = \int_\Omega |\nabla u|^2 \, dx + \int_{S(u)} |\Phi(\nu_u)| \, d\mathcal{H}^1,$$

where $\Phi$ is the deformation of $\mathbb{R}^2$ relative to the regular octagon with center 0 and one vertex in $e_1$.

**Example 5.4.** (*Potentials with a separate dependence on the reference position*) Consider the case in which the potential $\Psi_n$, in the notation of the Introduction, are of the form

$$\Psi_n(z, w) = \rho\Big(\frac{w}{\lambda_n}\Big)\psi_n(z),$$

$\rho$ and $\psi_n$ assigned. In particular, we can deal with

$$\rho(w) = e^{-\delta|w|^\beta}, \qquad \text{and} \qquad \rho(w) = |w|^{-\alpha},$$

with $\delta, \beta > 0$, and $\alpha > 4$, respectively. If $w \in \mathbb{Z}^N \setminus \{0\}$, $\nu = \frac{w}{|w|}$, and $k = \frac{|w|}{\xi(\nu)}$, then we can consider $\psi_n^{k,\nu}(x) = \Psi_n(\lambda_n |w| z, \lambda_n w)$. Under the hypotheses of Theorem 4.1, the sequence of the relative energies $\Gamma$-converges and the limit energy can be expressed in terms of $\rho$ and of the limits

$$F(x) = \lim_n \psi_n(\lambda_n x), \qquad G(x) = \lim_n \lambda_n \psi_n(x).$$

In particular, if $\psi_n(x) = \lambda_n^{-1}\psi(x^2\lambda_n)$ with $\psi(z) = z \wedge 1$, then the limit energy can be written as

$$\mathcal{H}(u) = c(\rho) \int_\Omega |\nabla u|^2 \, dx + \int_{S(u)} \sum_{w \in \mathbb{Z}^N \setminus \{0\}} |\nu_u \cdot w||w|\rho(w) \, d\mathcal{H}^{N-1}$$

with $c(\rho) = \frac{1}{N} \sum_{w \in \mathbb{Z}^N \setminus \{0\}} |w|^3 \rho(w)$. For a deeper analysis of this case we refer to [14].

## 6. Boundary value problems for the discrete energies

### 6.1. Boundary values as interactions through the boundary

We give a notion of boundary value problem for a discrete system on $\Omega$ by defining the boundary datum $\varphi$ on a neighbourhood of $\partial\Omega$, and considering all functions as equal to $\varphi$ outside $\Omega$. We then separate 'interior interactions' from those 'crossing the boundary'; the latter give rise to a boundary term. For the sake of simplicity we consider the case of a finite number of interactions only.

In order to consider a suitable notion of boundary value, let $\varphi \in SBV^p_{\rm loc}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ be fixed and such that $\mathcal{H}^{N-1}(S(\varphi) \cap \partial\Omega) = \emptyset$ and let $\Delta$ be defined as in Theorem 4.3. For $u \in \mathcal{A}_n(\Omega)$, let $\mathcal{B}_n(u) := \mathcal{H}_n(u) + \mathcal{H}^\varphi_n(u)$ where

$$\mathcal{H}^\varphi_n(u) := \sum_{(\nu,k)\in\Delta} \sum_{\alpha\in R^{k,\nu}_n(\partial\Omega)} k\lambda_n\xi(\nu)\psi^{k,\nu}_n\Big(\frac{\varphi(\alpha + k\lambda_n\xi(\nu)\nu) - u(\alpha)}{\lambda_n}\Big),$$

with $R^{k,\nu}_n(\partial\Omega) := \{\alpha \in \lambda_n\mathbb{Z}^N : \alpha \in \Omega, \alpha + k\lambda_n\xi(\nu)\nu \notin \Omega\}$, i.e., we consider separately the interactions crossing the boundary of $\Omega$, and where we set the value

$$\varphi(\alpha) := \limsup_{\rho\to 0+} \frac{1}{\rho^N} \int_{\alpha+[0,\rho)^N} \varphi(y)\, dy.$$

**Theorem 6.1.** *Under the hypotheses of Theorem 4.3 we have that $\mathcal{B}_n(u)$ $\Gamma$-converges with respect to the $L^1(\Omega)$-strong topology to the functional $\mathcal{B}$ defined in $L^1(\Omega)$ as*

$$\mathcal{B}(u) = \begin{cases} \mathcal{H}(u) + \displaystyle\int_{\partial\Omega} \mathcal{G}(\gamma(u) - \varphi, \nu_{\partial\Omega})\, d\mathcal{H}^{N-1} & \text{if } u \in GSBV(\Omega) \\ +\infty & \text{otherwise,} \end{cases}$$

*where $\gamma(u)$ is the inner trace of $u$ with respect to $\partial\Omega$ (i.e.,*

$$\gamma(u)(x) = \lim_{\rho\to 0+} \fint_{B(x,\rho)\cap\Omega} u(y)\, dy),$$

*the value of $\varphi$ on $\partial\Omega$ is in the sense of traces of functions in $SBV$ and $\nu_{\partial\Omega}$ is the inner normal to $\partial\Omega$.*

**Proof.** In the sequel it will be useful to extend functions in $L^1(\Omega)$ and in $\mathcal{A}_n(\Omega)$ to functions belonging to $L^1_{\rm loc}(\mathbb{R}^N)$ and $\mathcal{A}_n(\mathbb{R}^N)$ that take into account the value of $\varphi$ outside $\Omega$. Thus, $T_\varphi : L^1(\Omega) \to L^1_{\rm loc}(\mathbb{R}^N)$, $T^n_\varphi : \mathcal{A}_n(\Omega) \to \mathcal{A}_n(\mathbb{R}^N)$ will be defined as follows:

$$T_\varphi(u) = \begin{cases} u & \text{in } \Omega \\ \varphi & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \qquad T^n_\varphi(u) = \begin{cases} u(\alpha) & \text{if } \alpha \in \Omega \\ \varphi(\alpha) & \text{if } \alpha \notin \Omega. \end{cases}$$

If $u_n \in \mathcal{A}_n(\Omega)$ and $u_n \to u$ in $L^1(\Omega)$, then it can be easily seen that $T^n_\varphi(u_n) \to T_\varphi(u)$ in $L^1_{\rm loc}(\mathbb{R}^N)$. In the sequel it will be useful to define, in the notation of the introduction, $\mathcal{H}_n(u, B) := \sum_{x,y\in B} \Psi_n(u(x)-u(y), x-y)$ for a general set $B$. Let $\eta > 0$ and set $\Omega_\eta = \{x \in$

$\mathbb{R}^N : \operatorname{dist}(x,\Omega) < \eta\}$. Then, for $n$ large enough $\mathcal{B}_n(u) \geq \mathcal{H}_n(T^n_\varphi(u),\Omega_\eta) - \mathcal{H}_n(\varphi,\Omega_\eta\setminus\Omega)$. It can be seen that $\mathcal{H}_n(\varphi,\Omega_\eta\setminus\Omega) \leq \omega(\eta)$ with $\lim_{\eta\to 0}\omega(\eta) = 0$. Then, if $u_n \to u$ in $L^1(\Omega)$, using the $\Gamma$-convergence of $\mathcal{H}_n(\cdot,\Omega_\eta)$ to $\mathcal{H}(\cdot,\Omega_\eta)$ and the estimate above, we get

$$\begin{aligned}&\liminf_n \mathcal{B}_n(u_n) \geq \mathcal{H}(T_\varphi(u),\Omega_\eta) - w(\eta)\\ =\;& \int_{\Omega_\eta} \mathcal{F}(\nabla(T_\varphi(u))\,dx + \int_{S(T_\varphi(u))\cap\Omega_\eta} \mathcal{G}([T_\varphi(u)],\nu_{T_\varphi(u)})\,d\mathcal{H}^{N-1} - w(\eta).\end{aligned}$$

The $\Gamma$-liminf inequality follows by letting $\eta$ tend to 0.

Conversely, let $u$ be such that $\mathcal{B}(u) < +\infty$ and define $u_n \in \mathcal{A}_n(\Omega)$ to be the piecewise-constant interpolation of $u$ on the points of the lattice $\lambda_n\mathbb{Z}^N$. Then $T^n_\varphi(u_n)$ is the piecewise constant interpolation of $T_\varphi(u)$, and, by the proof of Theorem 4.1, it is also a recovering sequence for $\mathcal{H}(T_\varphi(u),\Omega_\eta)$ for any $\eta$. Hence

$$\limsup_n \mathcal{B}_n(u_n) \leq \limsup_n \mathcal{H}_n(T^n_\varphi(u_n),\Omega_\eta) \leq \mathcal{H}(T_\varphi(u),\Omega_\eta)$$

and the thesis follows by letting $\eta \to 0$. □

### 6.2. Convergence of boundary value problems

Thanks to Theorem 6.1 we can state a convergence result for boundary value problems as follows.

**Theorem 6.2.** *Let the hypotheses of Theorem* 6.1 *be satisfied with* $\varphi \in L^\infty(\mathbb{R}^N)$. *Then the minimum values*

$$\min\Big\{\mathcal{B}_n(u) : \ u \in \mathcal{A}_n(\Omega)\Big\} \tag{78}$$

*converge to the minimum value*

$$\min\Big\{\mathcal{H}(u) + \int_{\partial\Omega} \mathcal{G}(\gamma(u)-\varphi,\nu_{\partial\Omega})\,d\mathcal{H}^{N-1} : \ u \in SBV(\Omega)\Big\}. \tag{79}$$

*Moreover, if* $(u_n)$ *is a sequence of minimizers for* (78) *which is bounded in* $L^\infty(\Omega)$ *then it admits a subsequence converging to a minimizer of* (79).

**Proof.** By a truncation argument, we can find a sequence $(u_n)$ of minimizers for (78) with $\|u_n\|_\infty \leq \|\varphi\|_\infty$. We then obtain that the sequence $(v_n)$ constructed in the proof of Theorem 4.1 is precompact in $L^1(\Omega)$, so that also $(u_n)$ is precompact in $L^1(\Omega)$. By the uniform bound the limit is in $SBV(\Omega)$. We can then apply Theorem 2.1. □

**Remark 6.3.** In the same way we can deal with the convergence of minimum problems with Neumann boundary values of the form

$$\min\Big\{\mathcal{B}_n(u) - \int_\Omega hu\,dx : \ u \in \mathcal{A}_n(\Omega),\ u_n \in K\Big\}, \tag{80}$$

where $K$ is a compact set of $\mathbb{R}$ and $h \in L^1(\Omega)$, or with mixed boundary conditions.

### 6.3. Boundary values as a condition on the boundary: boundary layers in the 1-dimensional case

Let $I = [0,\ell]$ and $\lambda_n = \ell n^{-1}$. We can identify the discrete system $\{i\lambda_n\}_{i=0,\dots,n}$ with the reference configuration of $n+1$ particles disposed on a bar of lenght $\ell$ and interacting pairwise with interaction-energy given by potentials $\psi_n^k$. In order to study the convergence of minimum points for the discrete energies with prescribed displacements in 0 and $\ell$, we study the $\Gamma$-convergence of functionals that take into account the boundary conditions.

With fixed a positive real number $d$, let $\mathcal{A}_n^d(I) := \{u \in \mathcal{A}_n(I) : u(0) = 0, u(\ell) = d\}$ and let $\mathcal{E}_n^d$ be defined as

$$\mathcal{E}_n^d(u) = \begin{cases} \displaystyle\sum_{k=1}^{n+1}\sum_{i=0}^{k-1}\sum_{j=0}^{\left[\frac{n-i}{k}\right]} k\lambda_n\psi_n^k\Big(\frac{u(i+(j+1)k)\lambda_n) - u(i+jk\lambda_n)}{k\lambda_n}\Big) & \text{if } u \in \mathcal{A}_n^d(I) \\ +\infty & \text{otherwise.} \end{cases}$$

We have the following result.

**Theorem 6.4.** *Under the hypotheses of Proposition* 3.3, $\mathcal{E}_n^d$ $\Gamma$*-converges with respect to the strong topology in* $L^1(I)$ *to the functional* $\mathcal{E}^d$ *defined as*

$$\mathcal{E}^d(u) = \begin{cases} \mathcal{E}(u) + \displaystyle\sum_{k=1}^{+\infty}\big(G^k(u(a+)) + G^k(d - u(b-))\big) & u \in GSBV(I), \\ +\infty & \textit{otherwise} \end{cases}$$

*in* $L^1(I)$

**Proof.** With fixed $u_n \in \mathcal{A}_n^d(I)$ converging to $u$, we deal with the $\Gamma$-liminf inequality first. It suffices to study the limit behaviour of $\sum_{i=0}^{k-1} E_n^{k,i}(u_n)$ where

$$E_n^{k,i}(u) := \sum_{x,y\in[0,\ell],|x-y|=k\lambda_n} k\lambda_n\psi_n^k\Big(\frac{u(x)-u(y)}{k\lambda_n}\Big).$$

For $k$ fixed let $i(k) := n - \left[\frac{n}{k}\right]k$. Note that $i(k)$ is the unique value in $\{0,\dots,k-1\}$ such that $i \equiv n$ modulo $k$. Let $\alpha < 0 < \ell < \beta$; for $i \in \{0,\dots,k-1\}$ define $v_n^i \in \mathcal{A}_n(\alpha,\beta)$ as

$$v_n^i((i+jk)\lambda_n) = \begin{cases} u_n((i+jk)\lambda_n) & \text{if } j = i,\dots,j_{max} \\ u_n(i\lambda_n) & \text{on } (\alpha, i\lambda_n) \\ u_n((i+j_{max}k)\lambda_n) & \text{on } ((i+j_{max}k)\lambda_n, \beta), \end{cases}$$

if $i \neq 0, i(k)$, with $j_{max} := \max\{j \in \mathbb{N} : i + jk < \ell\}$,

$$v_n^i(x) = \begin{cases} u_n(x) & \text{on } (0,\ell) \\ 0 & \text{on } (\alpha, 0) \\ d & \text{on } (\ell, \beta), \end{cases}$$

if $i = 0, i(k)$. In particular, $E_n^{k,i}(u_n) \geq E_n^{k,i}(v_n^i) - (|\alpha| + |\beta - \ell|)\psi_n^k(0)$, and $v_n^i$ converges in $L^1(\alpha,\beta)$ to $v^i$, defined as

$$v^i(x) = \begin{cases} u(x) & \text{on } (0,\ell) \\ u(0+) & \text{on } (\alpha,0) \\ u(\ell-) & \text{on } (\ell,\beta) \end{cases}$$

if $i \neq 0, i(k)$,

$$v^i(x) = \begin{cases} u(x) & \text{on } (0,\ell) \\ 0 & \text{on } (\alpha,0) \\ d & \text{on } (\ell,\beta). \end{cases}$$

if $i = 0, i(k)$. Hence, we have

$$\liminf_n E_n^{k,i}(u_n) \geq \int_\alpha^\beta F^k(\dot v^i)\,dt + \sum_{S(v^i)} G^k([v^i]) - (|\alpha| + |\beta-\ell|)F^k(0).$$

Letting $\alpha \to 0, \beta \to \ell$ we get the inequality for $k$ fixed. It remains to sum over $k$ and proceed with standard arguments.

Now let $u$ be such that $\mathcal{E}^d(u) < +\infty$. Assume that $S(u) \cap (\lambda_n\mathbb{Z}) = \emptyset$ and define $u_n \in \mathcal{A}_n^d(I)$ to be the piecewise-constant interpolation of $u$ on the points $\{\frac{\ell}{n}, \dots, \ell - \frac{\ell}{n}\}$. Then, if $\mathcal{E}_n(\cdot,(0,\ell))$ is the functional relative to the partition $\{\frac{\ell}{n}, \dots, \ell - \frac{\ell}{n}\}$, we have

$$\mathcal{E}_n^d(u_n) \leq \mathcal{E}_n(u_n,(0,\ell)) + \sum_{k=1}^{n+1} \Big( G_n^k\Big(u\Big(\frac{k\ell}{n}\Big)\Big) + G_n^k\Big(d - u\Big(\ell - \frac{\ell}{n}\Big)\Big)\Big). \tag{81}$$

Since $u_n$ restricted to $(0,\ell)$ is the piecewise-constant interpolation of $u$ in $(0,\ell)$, we have that $\limsup \mathcal{E}_n(u_n,(0,\ell) \leq \mathcal{E}(u)$. By the boundedness of $u$, reasoning as in Step 1 of the proof of Theorem 4.3, we can neglect $\sum_{k=m}^{n+1}\big(G_n^k(u(\frac{k\ell}{n})) + G_n^k(d - u(\ell - \frac{\ell}{n}))\big)$, for $m$ large. Thus, it remains to show that $\limsup_n (G_n^k(u(\frac{k\ell}{n})) + G_n^k(d - u(\ell - \frac{\ell}{n}))) = G^k(u(a+)) + G^k(d - u(b-))$. This can be easily seen in the case $u(a+) \neq 0$, $u(b-) \neq d$, by using the uniform convergence of $G_n^k$ to $G^k$ on compact sets of $\mathbb{R} \setminus \{0\}$. In the other case it suffices to notice that the inequality (81) can be refined, for example for $u(a+) = 0$, as

$$\mathcal{E}_n^d(u_n) \leq \mathcal{E}_n(u_n,[0,\ell)) + \sum_{k=1}^{n+1} G_n^k\Big(d - u\Big(\ell - \frac{\ell}{n}\Big)\Big),$$

where $\mathcal{E}_n(\cdot,[0,\ell))$ is the functional relative to the partition $\{0, \dots, \ell - \frac{\ell}{n}\}$ and for which the interpolation of $u$ on the lattice $\{0, \dots, \ell - \frac{\ell}{n}\}$ is still a recovery sequence fo $\mathcal{E}(u)$. □

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