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On The Eigenvalues Problem for Hemivariational Inequalities: Existence and Stability

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This paper is concerned with the eigenvalues problem for hemivariational inequalities. First we present our existence results by means of mixed version of Ky Fan's minimax inequality. Further results concerning the convergence of solutions are proved when the given data of the problem are perturbed. Some remarks and comments are given in the last section.

Keywords: Eigenvalues hemivariational inequality, Ky Fan's minimax inequality, qualitative stability, rate of convergence

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1. Introduction

Let V be a Hilbert space which is supposed imbedded in $L^p(\Omega)$ where Ω is an open subset of $\mathbb{R}^n$ and $p \geq 2$. The theory of hemivariational inequalities was first introduced by P.D. Panagiotopoulos in 1981 (see [24] and [19]), as : find $u \in V$ such that

$$\alpha\,(u, v-u) + \int_\Omega j^0\,(u\,(x)\,; v\,(x) - u\,(x))\,dx \geq 0 \quad \forall v \in V.$$

This problem can be considered as a nonconvex generalization of the classical variational inequalities of J. L. Lions and G. Stampacchia. For typical examples in connection with mechanics and engineering we refer to the books of Panagiotopoulos [20, 22] and [18]. The techniques used for resolution of hemivariational inequalities are subsequently based on arranging fixed point theorems, Galerkin methods and the convolution product regularization, see [15]-[17], [21], [22] and the bibliography therein.

In the last few years, much attention has been focused to the existence theory of such inequalities by means of the generalized Ky Fan minimax theorem [5, 4].

It is the aim of the present paper to investigate the variational-hemivariational inequality (VHI): find $u \in D$ and $\lambda \in \mathbb{R}$ such that $\forall v \in D$

$$\begin{aligned}\lambda\langle H(u), v-u\rangle \quad &\leq \quad \alpha\,(u, v-u) + \langle C\,(u)\,, v-u\rangle \\ &\quad + \int_\Omega j^0\,(u\,(x)\,; v\,(x) - u\,(x))\,dx + \Phi(v) - \Phi(u).\end{aligned}$$

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Here V is supposed a reflexive Banach space, $D \subset V$ is convex, H is the duality mapping (i.e. $H(x) := \{x^* \in V^* \;:\; \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\}$), $\alpha : V \times V \to I\!R$ is a continuous bilinear form, Φ is a proper convex lower semicontinuous function with domain $D = D(\Phi) := \{u \in V : \Phi(u) < +\infty\}$, $C : D \to V^*$ is a nonlinear operator and $j : I\!R \to I\!R$ is a locally Lipschitz function.

Notice that if $j = 0$, the problem (VHI) reduces to a generalized variational inequality, if $\lambda = 0$ and $\Phi = 0$ problem (VHI) becomes the hemivariational inequality considered by Naniewicz and Panagiotopoulos [18, 20, 22], and if $\lambda = 0, \Phi = 0$ and $C = 0$ we refer to [4, 8, 21]. In a Hilbert framework with $\Phi = 0$ Motreanu and Panagiotopoulos [14] provided the existence for (VHI) by using the critical points method.

It is the aim of our work to investigate existence and stability of solution for (VHI) by means of a mixed version of Ky Fan's minimax inequality (see [3], [2] and [6]).

After recalling some basic tools we need in the sequel, in Section 3 we present existence theorems under the $\delta-$positivity condition on the bilinear form α (Theorems 3.7 and 3.11) which generalizes and unifies some results obtained in [5], [11] and [14].

Section 4 is devoted to the qualitative convergence of solutions. More precisely, taking (for $i = 1, 2$) a solution u_i of (VHI_i), we estimate the value $\|u_1 - u_2\|$ in terms of "adequate distances" between the operators C_i, the functions j_i and the bilinear forms α_i, Theorems 4.1, 4.5. Afterwards, see Proposition 4.7 we cope with the convergence of solutions u_n when the sequences C_n , α_n and j_n are converging in adequate sense to C, α and j, respectively.

Finally, some remarks and comments are given in the last section.

2. Basic tools and preliminaries

We treat (VHI) problem by a variational method involving a generalized Ky Fan's minimax approach. Roughly speaking we need the following existence result of Blum and Oettli [3], [6].

Theorem 2.1. *Let X be a topological vector space, D a nonempty closed convex subset of X and f, g be two real functions defined on $D \times D$ such that :*

(i) *For each x in D, $f(x,x) = g(x,x) = 0$.*

(ii) *For each $y \in D$, $g(.,y)$ is upper hemicontinuous, i.e., $g(.,y)$ is upper semicontinuous on each line segment in D.*

(iii) *g is monotone, i.e., $g(x,y) + g(y,x) \leq 0$ for each $x, y \in D$.*

(iv) *For each x in D, $f(x,.)$ and $g\,(x,.)$ are convex.*

(v) *For each x in D, $g\,(x,.)$ is lower semicontinuous.*

(vi) *For each y in D, $f(.,y)$ is upper semicontinuous.*

(vii) *(Coercivity) There exists a nonempty convex compact $A \subset D$ such that $\forall x \in A \backslash core_D A$, $\exists y \in core_D A$ such that $f(x,y) + g(x,y) \leq 0$.*

Then, $f + g$ admits an equilibrium point $\overline{x} \in D$, i.e., $f(\overline{x}, y) + g(\overline{x}, y) \geq 0$ $\forall y \in D$.

Here the core of A relative to D, denoted by $core_D A$, is defined through

$$x \in core_D A \Leftrightarrow x \in A, \text{ and } A \cap (x,y] \neq \emptyset \forall y \in D \backslash A.$$

Note that $core_D D = D$.

Remark 2.2. [6] When X is a reflexive Banach space, endowed with the weak topology, a sufficient condition for the coercivity requirement (vii) in Theorem 2.1, which is obviously satisfied if D is bounded, is :

$\exists a \in D$ such that $f(x,a) \leq M\|x-a\| \forall x \in D$ with $\|x-a\| \geq c$, and

$g(x,a)/\|x-a\| \to -\infty$ if $\|x-a\| \to +\infty, x \in D$, for some positive constants M and c.

In this respect, we make use of the previous result by endowing the space V with the weak topology.

We shall henceforth make the following assumptions :

(H1) V be a reflexive[1] Banach space, and D a nonempty closed convex subset of V.

(H2) V is imbedded in $L^p(\Omega)$ where Ω is an open bounded subset of $\mathbb{R}^n$ and $p > 1$, and the imbedding is supposed to be compact. If we denote by $\|.\|$ the norm of V and by $\|.\|_p$ the norm of $L^p(\Omega)$, then $\forall u \in V$, $\|u\|_p \leq c_p \|u\|$ for some positive constant c_p.

(H3) $\alpha : D \times D \to \mathbb{R}$ is a continuous bilinear form and δ-positive; i.e.

$$\delta := \inf_{u,v \in D, u \neq v} \frac{\alpha(u-v, u-v)}{\langle H(u) - H(v), u-v \rangle} \geq 0.$$

(H4) Φ is a proper convex lower semicontinuous function with domain $D \subset dom(\Phi) := \{u \in V : \Phi(u) < +\infty\}$.

(H5) $C : D \to V^*$ is a weakly-strongly continuous nonlinear operator (i.e., C is continuous from V endowed with the weak topology to V^* endowed with norm topology) and $C(D)$ is bounded.

(H6) $j : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function defined by $j(t) := \int_0^t \beta(s)ds$, where
(i) either $\beta \in L^\infty(\mathbb{R})$,
(ii) or $\exists \alpha_1 > 0, \alpha_2 > 0$ such that $|\beta(s)| \leq \alpha_1 + \alpha_2 |s|^{p-1} \forall s \in \mathbb{R}$.

Above j^0 denotes the Clarke's generalized derivative of j which is defined as follows

$$j^0(u;v) := \limsup_{\substack{x \to u \\ t \searrow 0}} \frac{1}{t}\left(j(x+tv) - j(x)\right)$$

and the generalized gradient of j is given by

$$\partial j(u) = \{\zeta \in X^* : \langle \zeta, v \rangle \leq j^0(u;v) \forall v \in X\}.$$

When we suppose that j is continuously differentiable, $\partial j(x)$ is reduced to $\{\nabla j(x)\}$. We end this section of preliminaries with useful properties of Clarke's generalized derivative.

Proposition 2.3. *[10, Prop. 2.1.1] Let ϕ be a real Lipschitz function of rank k near x. Then*

a) The function $v \to \phi^0(x;v)$ is positively homogeneous and subadditive (thus convex), continuous and Lipschitz of rank k on X,

[1]Without loss of generality we can assume that $0 \in D$ and the norms of V and V^* are strictly convex. Let us recall [25] that in that case H is one-to-one and strictly monotone.

b) For each v in X, one has $|\phi^0(x;v)| \leq k\|v\|$,

c) $\phi^0(x;v)$ is upper semicontinuous as a function of (x,v).

d) For every v in X, one has

$$\phi^0(x;v) = \max\{\langle \zeta, v\rangle : \zeta \in \partial\varphi(x)\}$$

Lemma 2.4. ([7], *see also* [*10, Example 2.2.5*])*Let* $\Phi \in L^\infty_{loc}(I\!R)$, *the function defined by* $\Psi(t) = \int_0^t \Phi(s)\,ds$ *is locally Lipschitz, and for* $\Phi^+(t) = \lim_{\delta\to 0} \operatorname*{ess\,sup}_{|s-t|\leq\delta} \Phi(s)$ *and* $\Phi_-(t) = \lim_{\delta\to 0} \operatorname*{ess\,inf}_{|s-t|\leq\delta} \Phi(s)$, *we have* $\Psi^0(t;z) \leq \Phi^+(t)z$ *if* $z > 0$ *and* $\Psi^0(t;z) \leq \Phi_-(t)z$ *if* $z < 0$, *and then* $\partial\Psi(t) \subset [\Phi_-(t), \Phi^+(t)]$.

3. The existence results

In this section we prove a basic existence result for the (VHI) problem. For this we introduce the following definition and collect some lemmata, which we need in the sequel.

Definition 3.1. We say that β is γ-lower essentially r-Hölder ($\gamma \in I\!R$), if

$$\beta^+(t_1) \leq \beta_-(t_2) + \gamma(t_2 - t_1)^r, \forall t_1 < t_2. \tag{1}$$

Remark 3.2. Note that the γ-lower essentially r-Hölder of β is incomparable to the following condition introduced by Panagiotopoulos in his existence results (see [11] and [23] and the bibliography therein): there exists $\delta > 0$ such that

$$\operatorname*{ess\,sup}_{(-\infty,-\delta)} \beta(s) \leq 0 \leq \operatorname*{ess\,inf}_{(\delta,+\infty)} \beta(s). \tag{2}$$

Lemma 3.3. *Suppose* β *is* γ*-lower essentially* r*-Hölder, then for every* $s,t \in I\!R$ *we have*

$$j^0(t;s-t) + j^0(s;t-s) \leq \gamma|s-t|^{r+1}$$

Proof. By Lemma 2.4, if $t \leq s$ we have,

$$\begin{aligned} j^0(t,s-t) + j^0(s,t-s) &\leq (s-t)\beta^+(t) + (t-s)\beta_-(s) \\ &= (s-t)\left(\beta^+(t) - \beta_-(s)\right) \\ &\leq \gamma(s-t)^{r+1} \end{aligned}$$

and if $t \geq s$ we have $j^0(t,s-t) + j^0(s,t-s) \leq (t-s)(\beta^+(s) - \beta_-(t)) \leq \gamma(t-s)^{r+1}$. □

Remark 3.4. From Lemma 3.3, it follows that if $\gamma < 0$, then ∂j is strongly monotone (i.e. $\forall s,t \in I\!R$, $\xi \in \partial j(s)$ and $\eta \in \partial j(t)$ imply $\langle \xi - \eta, s-t\rangle \geq -\gamma \mid s-t \mid^{r+1}$); and if $\gamma = 0$, ∂j is monotone. Thus when $\gamma \leq 0$ the (VHI) problem comes back to a variational inequality, since this case corresponds to convexity of j, whereas if $\gamma > 0$ the function j is not necessarily convex.

Remark 3.5. It is easy to check that j is Lipschitz near x of rank $R(x) := \operatorname*{ess\,sup}_{|s-x|\leq\tau} |\beta|(s)$ for some positive constant τ. Particularly, when β satisfies $(\mathbf{H}6)(i)$ we have $R(x) = R = \|\beta\|_\infty$ for all $x \in I\!R$.

Lemma 3.6. *Assume that* (**H**6) *is satisfied. Then, for all* v *in* V *the functional* $u \mapsto f(u,v) := \langle C(u), v-u\rangle + \int_\Omega j^0(u; v-u)dx$ *is weakly upper semicontinuous.*

Proof. Let $\{u_n\}$ be a weakly converging sequence to some u in V, we have to show that $\limsup f(u_n, v) \le f(u, v)$.

Step 1. Suppose (**H**6)(i) is satisfied. Then j is Lipschitz of rank R, and by Proposition 2.3, the function Ψ_n given by

$$\Psi_n(x) := j^0(u_n(x); v(x) - u_n(x)) - R\,|u_n(x) - v(x)|$$

is nonpositive.
V is supposed compactly imbedded in $L^p(\Omega)$, we deduce for a subsequence also denoted by $\{u_k\}$ we have strong convergence to u in $L^p(\Omega)$. It follows for an other subsequence also denoted by $\{u_k\}$, that $u_k(x) \to u(x)$ almost everywhere on Ω.
Using Fatou's Lemma, we have

$$\limsup_k \int_\Omega \Psi_k(x)\,dx \le \int_\Omega \limsup_k \Psi_k(x)\,dx.$$

Taking into account the usual properties of $\limsup$ we have

$$\begin{aligned} &\limsup \int_\Omega j^0(u_k; v-u_k)dx \\ &\le \int_\Omega \limsup j^0(u_k; v-u_k)dx + R \lim \int_\Omega \left(|u_k(x) - v(x)| - |u(x) - v(x)|\right) dx \\ &\le \int_\Omega \limsup j^0(u_k; v-u_k)dx + R\ mes(\Omega)^{(p-1)/p} \lim \|u_k - u\|_p \\ &= \int_\Omega \limsup j^0(u_k; v-u_k)dx, \end{aligned}$$

since $u_k \to u$ *in* $L^p(\Omega)$. We deduce

$$\limsup \int_\Omega j^0(u_k; v-u_k)dx \le \int_\Omega \limsup j^0(u_k; v-u_k)dx.$$

As $j^0(.,.)$ is upper semicontinuous, we deduce

$$\limsup \int_\Omega j^0(u_k; v-u_k)dx \le \int_\Omega j^0(u; v-u)\,dx.$$

Step 2. Suppose (**H**6)(ii) is satisfied. Let us first prove for each $x, y \in \mathbb{R}$ that

$$j^0(x;y) \le (\alpha_1 + 2^{p-1}\alpha_2 |x|^{p-1})|y|. \tag{3}$$

To this end, take $\gamma > 0$ and fix s in $[x-\gamma, x+\gamma]$.
By $(H_4)(ii)$ it follows that

$$\begin{aligned} |\beta(s)| &\le \alpha_1 + \alpha_2(|x| + \gamma)^{p-1} \\ &\le \alpha_1 + 2^{p-2}\alpha_2(|x|^{p-1} + \gamma^{p-1}). \end{aligned}$$

(The last inequality is trivial for $p = 2$ and comes from convexity of $t \to x^{p-1}$ upon $\mathbb{R}^+$ for $p > 2$.)

Thus

$$| \operatorname*{ess\,sup}_{|s-t|\le\delta} \beta(s)| \le \alpha_1 + 2^{p-2}\alpha_2(|x|^{p-1} + \gamma^{p-1}).$$

Thus we conclude that

$$|\beta^+(x)| = |\lim_{\delta\to 0} \operatorname*{ess\,sup}_{|s-t|\le\delta} \beta(s)| \le \alpha_1 + 2^{p-2}\alpha_2|x|^{p-1}.$$

In the much same way, we can check that

$$|\beta_-(x)| \le \alpha_1 + 2^{p-2}\alpha_2|x|^{p-1}.$$

On the other hand, by Lemma 2.4 we have

$$\partial j(x) \subset [\beta^+(x), \beta_-(x)],$$

hence we deduce for all $\xi \in \partial j(x)$ that

$$|\xi| \le max(|\beta^+(x)|, |\beta_-(x)|) \le \alpha_1 + 2^{p-2}\alpha_2|x|^{p-1}.$$

It turns out from Proposition 2.3 (d) that

$$\begin{aligned} j^0(x;y) &= \max_{\xi\in\partial j(x)} \langle \xi, y\rangle \\ &\le (\alpha_1 + 2^{p-2}\alpha_2|x|^{p-1})|y|. \end{aligned}$$

Since $\{u_n\}$ is a weakly converging sequence to some $u \in V$ and the embedding of V in $L^p(\Omega)$ is compact, for a subsequence $\{u_k\}$, one has $u_k \to u$ (strongly) in L^p, and $u_k(x) \to u(x)$ almost everywhere on Ω.
From Egoroff's Theorem, since $mes(\Omega) < +\infty$, we get for each $\varepsilon > 0$, the existence of a measurable subset A_ε of Ω such that $mes\,(A_\varepsilon) \le \varepsilon$ and $u_k \to u$ uniformly on A_ε^c.
Now taking into account (3) one sees by setting $R_k(x) := \alpha_1 + 2^{p-1}\alpha_2|u_k(x)|^{p-1}$, that for each k and $x \in A_\varepsilon^c$,

$$j^0(u_k\,(x)\,; v\,(x) - u_k\,(x)) \le R_k(x)\,|v\,(x) - u_k\,(x)|\,.$$

So let us show that

$$\limsup_k \int_{A_\varepsilon^c} j^0(u_k; v - u_k)dx \le \int_{A_\varepsilon^c} \limsup_k j^0(u_k; v - u_k)dx. \tag{4}$$

To this end, consider

$$\Psi_k\,(x) := j^0\,(u_k\,(x)\,; v\,(x) - u_k\,(x)) - R_k(x)\,|u_k\,(x) - v\,(x)|$$

Since $\Psi_k \le 0$, Fatou's Lemma implies that

$$\limsup_k \int_{A_\varepsilon^c} \Psi_k\,(x)\,dx \le \int_{A_\varepsilon^c} \limsup_k \Psi_k\,(x)\,dx.$$

As $R_k(x)|v(x) - u_k(x)| \to R(x)|v(x) - u(x)|$ almost everywhere on A_ε^c, where $R(x) := \alpha_1 + 2^{p-1}\alpha_2|u(x)|^{p-1}$, and u_k is dominated independently on k for k large enough (because is u_k uniformly converging to u on A_ε^c) it follows that

$$\lim_k \int_{A_\varepsilon^c} R_k(x)|v(x)-u_k(x)|dx = \int_{A_\varepsilon^c} \lim_{k\to\infty} R_k(x)|v(x)-u_k(x)|dx = \int_{A_\varepsilon^c} R(x)|v(x)-u(x)|dx.$$

Therefore, from the previous Fatou's inequality we have

$$\begin{aligned}\limsup_k \int_{A_\varepsilon^c} j^0(u_k; v-u_k)dx + \lim_{k\to\infty}\int_{A_\varepsilon^c} R_k(x)|v(x)-u_k(x)|dx \leq \\ \leq \int_{A_\varepsilon^c} \limsup_k j^0(u_k; v-u_k)dx + \int_{A_\varepsilon^c} \lim_{k\to\infty} R_k(x)|v(x)-u_k(x)|dx\end{aligned}$$

Thus

$$\limsup_k \int_{A_\varepsilon^c} j^0(u_k; v-u_k)dx \leq \int_{A_\varepsilon^c} \limsup_k j^0(u_k; v-u_k)dx$$

On the other hand $\lim_{\varepsilon\to 0} mes\ (A_\varepsilon) = 0$ we get

$$\lim_{\varepsilon\to 0}\int_{A_\varepsilon} \limsup_{k\longrightarrow+\infty} j^0(u_k; v-u_k)dx = 0.$$

Then, thanks to a diagonalization Lemma [1, Corollary 1.16, p.33] it results that for some converging sequence $\{\varepsilon(k)\}$ to 0

$$\begin{aligned}\int_\Omega j^0(u, v-u)dx &\geq \limsup_{\varepsilon\to 0}\limsup_{k\to+\infty}\int_{A_\varepsilon^c} j^0(u_k; v-u_k)dx \\ &\geq \limsup_{k\to+\infty}\int_{A_{\varepsilon(k)}^c} j^0(u_k; v-u_k)dx. \qquad (5)\end{aligned}$$

Now, using (3), then $\forall k \in I\!N^*, \forall x \in A_{\varepsilon(k)}$,

$$j^0(u_k(x); v(x)-u_k(x)) \leq (\alpha_1 + \alpha_2|u_k(x)|^{p-1}|)v(x) - u_k(x)|.$$

Therefore

$$\begin{aligned}\int_{A_{\varepsilon(k)}} j^0(u_k; v-u_k)dx &\leq \int_{A_{\varepsilon(k)}} \left(\alpha_1 + \alpha_2\left|u_k(x)\right|^{p-1}\right)\left|v\left(x\right) - u_k(x)\right| dx \\ &\leq \left(\alpha_1 mes(A_{\varepsilon(k)})^{1/q} + \alpha_2\left(\int_{A_{\varepsilon(k)}} |u_k(x)|^p\right)^{1/q}\right)\left\|v-u_k\right\|_p, \qquad (6)\end{aligned}$$

where $q = (p-1)/p$. Letting, in (6), $mes(A_{\varepsilon(k)}) \to 0$ and $u_k \to u$ in $L^p(\Omega)$ when $k \to +\infty$, we obtain

$$\limsup_{k\to+\infty}\int_{A_{\varepsilon(k)}} j^0(u_k; v-u_k)dx \leq 0. \qquad (7)$$

Using (5) and (7), it turns out that

$$\begin{aligned}\int_{\Omega} j^0(u,v-u)dx &\geq \limsup_k \int_{A^c_{\varepsilon(k)}} j^0(u_k;v-u_k)dx + \limsup_k \int_{A_{\varepsilon(k)}} j^0(u_k;v-u_k)dx \\ &\geq \limsup_k \left[\int_{A^c_{\varepsilon(k)}} j^0(u_k;v-u_k)dx + \int_{A_{\varepsilon(k)}} j^0(u_k;v-u_k)dx\right] \\ &= \limsup_k \int_{\Omega} j^0(u_k;v-u_k)dx.\end{aligned}$$

Combining these two steps we confirm

$$\limsup \int_{\Omega} j^0(u_k;v-u_k)dx \leq \int_{\Omega} j^0\left(u;v-u\right)dx.$$

This is true for each sub-subsequence of $\{u_n\}$, it follows that the function $u \rightarrow \int_{\Omega} j^0(u,v-u)dx$ is upper semicontinuous.
On the other hand C is weakly-strongly continuous, thus $C(u_n) \longrightarrow C(u)$ as $n \rightarrow +\infty$. We conclude that $f(.,v)$ is weakly upper semicontinuous . □

Theorem 3.7. *Suppose that* $(\mathbf{H}1)-(\mathbf{H}5)$ *are satisfied. Assume either*
a)$(\mathbf{H}6)(i)$,
or
b)$(\mathbf{H}6)(ii)$ *for* $p \in (1,2)$,
or
c)$(\mathbf{H}6)(ii)$ *for* $p \geq 2$ *and* β *is* γ*-lower essentially* r*-Hölder with* $r<1$.
Then, for all $\lambda < \delta$, *the set of solutions to* (VHI) *is nonempty.*

Remark 3.8. In [6], the authors recalled a quite different coercivity condition ensuring the compactness of the set of solutions.

Remark 3.9. Note that the use of minimax approach is fruitful since it will lead to existence theorems without the structural decomposition assumption upon the operator C as in [14]. However, we demand the positivity of the bilinear form and either the boundedness or some Hölder type condition on the function β (see Definition 3.1), which will be useful for existence results and will also prove to play a crucial role for the stability ones.

Proof. The argument is based on Theorem 2.1 with V endowed with the weak topology $\sigma\left(V,V^*\right)$, and for each $u,v \in D$

$g\left(u,v\right) := \alpha\left(u,v-u\right) - \lambda\langle H(u),v-u\rangle + \Phi(v) - \Phi(u)$,

$f\left(u,v\right) := \langle C(u),v-u\rangle + \int_{\Omega} j^0\left(u;v-u\right)dx$.
The assumptions i), iv) and v) are immediately satisfied, and vi) follows from Lemma 3.6.
Assumption ii) follows from the demicontinuity of the duality mapping H.
Assumption iii) is also satisfied since using $(\mathbf{H}3)$ and monotonicity of H, we have for every $u,v \in D$

$$\begin{aligned}g(u,v)+g(v,u) &= -\alpha(v-u,v-u) + \lambda\langle H(v)-H(u),v-u\rangle \\ &\leq (\lambda-\delta)\langle H(v)-H(u),v-u\rangle \leq 0.\end{aligned}$$

We have only to prove (vii). Assume contradiction that (vii) is not true, then for each $n \in I\!N^*$ there exists some $u_n \in D$ with $\|u_n\| = n$ and $f(u_n, 0) + g(u_n, 0) > 0$. Then

$$\begin{aligned} 0 &\leq -\alpha(u_n, u_n) + \lambda\langle H(u_n), u_n\rangle + \Phi(0) - \Phi(u_n) - \langle C(u_n), u_n\rangle + \int_\Omega j^0(u_n; -u_n)\, dx \\ &\leq n^2 \left((\lambda - \delta) + \frac{1}{n^2}(\Phi(0) - \Phi(u_n)) - \langle C(u_n), \frac{u_n}{n^2}\rangle + \frac{1}{n^2}\int_\Omega j^0(u_n; -u_n)\, dx \right). \end{aligned}$$

Using convexity of Φ, it is readily shown that

$$\begin{aligned} 0 < \delta - \lambda &\leq \limsup_{n\to+\infty} (\Phi(0) - \Phi(\frac{u_n}{n^2})) + \limsup_{n\to+\infty} \langle C(u_n), \frac{-u_n}{n^2}\rangle + \\ &\qquad + \limsup_{n\to+\infty} \frac{1}{n^2}\int_\Omega j^0(u_n; -u_n)\, dx. \end{aligned}$$

Since Φ is lower semicontinuous and $C(D)$ is bounded we deduce

$$0 < \delta - \lambda \leq \limsup_{n\to+\infty} \frac{1}{n^2}\int_\Omega j^0(u_n; -u_n)\, dx.$$

We shall prove that $\limsup_{n\to+\infty} \frac{1}{n^2}\int_\Omega j^0(u_n; -u_n)\, dx \leq 0$; once this has been done we end to a contradiction.

First, suppose that $\beta \in L^\infty(I\!R)$. Then $j^0(u_n; -u_n) \leq \|\beta\|_\infty |u_n|$, and by Hölder inequality it follows that

$$\begin{aligned} \limsup_{n\to+\infty} \frac{1}{n^2}\int_\Omega j^0(u_n; -u_n)\, dx &\leq \|\beta\|_\infty \limsup_{n\to+\infty} \frac{1}{n^2}\int_\Omega |u_n| dx \\ &\leq \|\beta\|_\infty \limsup_{n\to+\infty} \frac{c_p}{n^2} mes(\Omega)^{1/q}\|u_n\| \\ &= 0. \end{aligned}$$

Suppose now $(\mathbf{H}6)(ii)$ is satisfied with $p \in (1, 2)$, then

$$\begin{aligned} \limsup_{n\to+\infty} \frac{1}{n^2}\int_\Omega j^0(u_n; -u_n)\, dx &\leq \limsup_{n\to+\infty} \frac{1}{n^2}\int_\Omega (\alpha_1 + \alpha_2|u_n|^{p-1})|u_n| dx \\ &\leq \limsup_{n\to+\infty} \frac{1}{n^2}\left(\alpha_1 c_p mes(\Omega)^{1/q}\|u_n\| + \alpha_2 c_p^p \|u_n\|^p\right) \\ &\leq \limsup_{n\to+\infty} \left(\alpha_1 c_p mes(\Omega)^{1/q} n^{-1} + \alpha_2 c_p^p n^{p-2}\right). \end{aligned}$$

So that since $p \in (1, 2)$ we have $\limsup_{n\to+\infty} \frac{1}{n^2}\int_\Omega j^0(u_n; -u_n)\, dx \leq 0$.

We are going to treat the case where β is γ-lower essentially r-Hölder with $r < 1$ and $p \geq 2$.

Remarking that $p \geq r + 1$, by means of Lemma 3.3, we have

$$\begin{aligned} \frac{1}{n^2}\int_\Omega j^0(u_n, -u_n) dx &\leq \frac{1}{n^2}\left(-\int_\Omega j^0(0, u_n) dx + \gamma\int_\Omega |u_n|^{r+1} dx\right) \\ &\leq \frac{1}{n^2}\left(R(0)\int_\Omega |u_n| dx + \gamma\int_\Omega |u_n|^{r+1} dx\right) \\ &\leq \left(R(0) c_p mes(\Omega)^{1/q} n^{-1} + \gamma c_p^{r+1} mes(\Omega)^{1-(r+1)/p} n^{r-1}\right). \end{aligned}$$

Thus $\limsup_{n\to+\infty} \frac{1}{n^2}\int_\Omega j^0(u_n;-u_n)\,dx \le 0$.
The proof is therefore complete. $\square$

Remark 3.10. Theorem 3.7 remains valid if, instead of (**H**5), the operator C is weakly-strongly continuous and for each sequence (u_n) with $\|u_n\| = n$ one has

$$\liminf_{n\to\infty}\langle C(u_n), \frac{u_n}{n^2}\rangle > \lambda - \delta,$$

which is a more general condition.

As it was remarked by one of the referees, the minimax technique may be used to establish the existence of solution to (VHI) under more general conditions. He suggested, instead of the γ-lower essential r-Hölder, the more larger condition:
(**H**6*) $\lim_{R\to\infty}$ ess $\inf_{|s|>R}\frac{\beta(s)}{s} \ge 0$. Following the arguments of the referee, we have the following result.

Theorem 3.11. *Assume that* (**H**1) – (**H**5), (**H**6)(ii) *and* (**H**6*) *are satisfied. Then, for every $\lambda < \delta$, the set of solutions to (VHI) is nonempty.*

Proof. As (**H**6)(ii) is fulfilled, Lemma 3.6 implies that condition (vi) from the minimax inequality is satisfied. All we have to prove is condition (vii), which, follows if relation

$$\limsup_{n\to+\infty}\frac{1}{n^2}\int_\Omega j^0(u_n;-u_n)\,dx \le 0$$

is established every time when $\|u_n\| = n$.
To prove this, let $\epsilon > 0$; from (**H**6*) it follows that there is $R > 0$ such that $\frac{\beta(s)}{s} \ge -\epsilon$ for almost every s such that $|s| > R$. Then $\beta(s) \ge -\epsilon s$ for almost every $s > R$, and $\beta(s) \le -\epsilon s$ for almost every $s < -R$, so

$$\beta_-(s) \ge -\epsilon s\,\forall s > R,\ \beta^+(s) \le -\epsilon s\ \forall s < -R.$$

On the other hand, from condition (**H**6)(ii) we read that

$$|\beta^+(s)| \le \alpha_1 + \alpha_2 R^{p-1},\ |\beta_-(s)| \le \alpha_1 + \alpha_2 R^{p-1}\,\forall |s| \le R$$

and that

$$\beta_-(s) \ge -\alpha_1 - \alpha_2 s^{p-1},\ \beta^+(s) \le \alpha_1 + \alpha_2|s|^{p-1}\,\forall s < -R.$$

Accordingly,

$$j^0(u_n(x);-u_n(x)) \le min(\epsilon|u_n|^2, \alpha_1|u_n| + \alpha_2|u_n|^p),$$

for every $x \in \Omega$ such that $|u_n| > R$, and

$$j^0(u_n(x);-u_n(x)) \le \alpha_1 R + \alpha_2 R^p,$$

for every $x \in \Omega$ such that $|u_n| \le R$.
Thus

$$\begin{aligned}\frac{1}{n^2}\int_\Omega j^0(u_n;-u_n)dx \ &\le\ \frac{1}{n^2}mes(\Omega)(\alpha_1 R + \alpha_2 R^p)\\ &\quad +min(\frac{1}{n^2}\int_\Omega \epsilon|u_n|^2dx, \frac{1}{n^2}\int_\Omega(\alpha_1|u_n| + \alpha_2|u_n|^p)dx)\end{aligned}$$

But

$$\int_\Omega |u_n|dx \leq (mes(\Omega))^{\frac{p-1}{p}} \|u_n\|_p \leq c_p(mes(\Omega))^{\frac{p-1}{p}} n, \int_\Omega |u_n|^p dx = \|u_n\|_p^p \leq c_p^p n^p,$$

and

$$(p \geq 2) \Rightarrow (\int_\Omega |u_n|^2 \leq (mes(\Omega))^{\frac{p-2}{p}} \|u_n\|_p^2 \leq c_p^2(mes(\Omega))^{\frac{p-2}{p}} n^2.$$

Consequently,

$$min(\frac{1}{n^2}\int_\Omega \epsilon|u_n|^2 dx, \frac{1}{n^2}\int_\Omega (\alpha_1|u_n| + \alpha_2|u_n|^p)dx) \leq \alpha_1 c_p mes(\Omega))^{\frac{p-1}{p}} n^{-1} + \alpha_2 c_p^{p-2} n^{p-2},$$

and if $p \geq 2$ then

$$min(\frac{1}{n^2}\int_\Omega \epsilon|u_n|^2 dx, \frac{1}{n^2}\int_\Omega (\alpha_1|u_n| + \alpha_2|u_n|^p)dx) \leq c_p^2(mes(\Omega))^{\frac{p-2}{p}} \epsilon.$$

When $p < 2$, the sequence $\alpha_1 c_p mes(\Omega))^{\frac{p-1}{p}} n^{-1} + \alpha_2 c_p^{p-2} n^{p-2}$ converges to 0, so

$$\limsup_{n\to\infty} min(\frac{1}{n^2}\int_\Omega \epsilon|u_n|^2 dx, \frac{1}{n^2}\int_\Omega (\alpha_1|u_n| + \alpha_2|u_n|^p)dx) \leq c_p^2(mes(\Omega))^{\frac{p-2}{p}} \epsilon,$$

and as

$$\lim_{n\to\infty} \frac{1}{n^2} mes(\Omega)(\alpha_1 R + \alpha_2 R^p) = 0,$$

we deduce that

$$\limsup_{n\to+\infty} \frac{1}{n^2}\int_\Omega j^0(u_n; -u_n)dx \leq c_p^2(mes(\Omega))^{\frac{p-2}{p}} \epsilon.$$

Since the previous relation holds for every $\epsilon > 0$, it follows that

$$\limsup_{n\to+\infty} \frac{1}{n^2}\int_\Omega j^0(u_n; -u_n)dx \leq 0,$$

and the proof is completed. □

Remark 3.12. Remark that, if $p < 2$ then $(\mathbf{H}6)(ii) \Rightarrow (\mathbf{H}6)^*$, and if β is γ-lower essential r-Hölder with $r < 1$, then $(\mathbf{H}6^*)$ is fulfilled.

Condition $(\mathbf{H}6^*)$ is much larger than the γ-lower essential r-Hölder for some $r < 1$, as functions like $\sqrt{|s|}$ and $\frac{-s}{ln(s)}$, which are not γ-lower essentially r-Hölder for any $r < 1$, fulfills $(\mathbf{H}6^*)$ which seems to be more natural. In addition, $(\mathbf{H}6^*)$ is a condition "at the infinity", therefore this condition may be regarded as γ-lower essential r-Hölder "at the infinity".

4. Stability results

In this section, we are interested to the qualitative stability results for solutions of (VHI) when the data are perturbed. Throughout this section we suppose that V is a Hilbert space, and $\delta - \lambda > 0$.

4.1.

We first treat the case where only the operator C is perturbed. Let us consider C_i for $i = 1, 2$, then by Theorem 3.7 the solution-set of the corresponding problem (VHI_i) is nonempty.

Theorem 4.1. *Assume that α is δ-positive and let $\beta \in L^\infty(I\!R)$ be a γ-lower essentially r-Hölder function, and $p \geq 2$. If u_i is a solution of (VHI_i) and $0 < \delta - \lambda$, then*

$$(\delta - \lambda)\|u_1 - u_2\| \leq \|C_1(u_1) - C_2(u_2)\| + \gamma \int_\Omega |u_2 - u_1|^{r+1} dx \tag{8}$$

Proof. Since u_i is a solution of (VHI_i), we have $\forall v \in V$

$$\lambda\langle u_i, v - u_i\rangle \leq \alpha(u_i, v - u_i) + \langle C_i(u_i), v - u_i\rangle + \int_\Omega j^0\left(u_i; v - u_i\right) dx + \Phi(v) - \Phi(u_i). \tag{9}$$

By setting $v = u_j$ in (VHI_i) for $j \neq i$, and adding the two relations, we obtain

$$\begin{aligned} -\lambda\left\|u_1 - u_2\right\|^2 \;\; \leq \;\; & -\alpha\left(u_1 - u_2, u_1 - u_2\right) + \left\langle C_1\left(u_1\right) - C_2\left(u_2\right), u_2 - u_1\right\rangle \\ & + \int_\Omega \left(j^0\left(u_1; u_2 - u_1\right) + j^0\left(u_2; u_1 - u_2\right)\right) dx \end{aligned} \tag{10}$$

Using the fact that α is δ-positive and Lemma 3.3, it follows that

$$(\delta - \lambda)\|u_1 - u_2\| \leq \|C_1(u_1) - C_2(u_2)\| + \gamma \int_\Omega |u_2 - u_1|^{r+1} dx$$

□

Remark 4.2. If $r = 1$, $K := \gamma {c_p}^2 (mes(\Omega))^{(p-2)/p}$ and C_2 is c-monotone, i.e. $\langle C_2\left(u_1\right) - C_2\left(u_2\right), u_1 - u_2\rangle \geq c\|u_1 - u_2\|^2$, with $c + \delta - K - \lambda > 0$ we obtain from the evaluation (10) the following estimate

$$(\delta - \lambda + c)\left\|u_1 - u_2\right\| \leq \left\|C_1\left(u_1\right) - C_2\left(u_1\right)\right\| + K\left\|u_1 - u_2\right\|.$$

In fact, from (10) we can write

$$\begin{aligned} -\lambda\left\|u_1 - u_2\right\|^2 \;\; \leq \;\; & -\delta\|u_1 - u_2\|^2 + \left\langle C_1\left(u_1\right) - C_2\left(u_1\right), u_2 - u_1\right\rangle \\ & + \gamma \int_\Omega |u_2 - u_1|^2 dx \\ & + \left\langle C_2\left(u_1\right) - C_2\left(u_2\right), u_2 - u_1\right\rangle \end{aligned}$$

which leads to

$$\begin{aligned} -\lambda\left\|u_1 - u_2\right\|^2 \;\; \leq \;\; & -\delta\|u_1 - u_2\|^2 + \left\|C_1\left(u_1\right) - C_2\left(u_1\right)\right\|\|u_2 - u_1\| \\ & + \int_\Omega |u_2 - u_1|^2 dx \\ & - c\|u_2 - u_1\|^2. \end{aligned}$$

Corollary 4.3. *Suppose that β is γ-lower essentially 1-Hölder, α is δ-positive and C_2 is Lipschitz of rank k; then for λ such that $K + k < \delta - \lambda$, the solution u_2 of (VHI_2) is unique and the solution-set S_1 of (VHI_1) is included in the ball of center u_2 and radius*

$$r := \sup_{v \in D} \|C_1(v) - C_2(v)\| / (\delta - \lambda - K - k).$$

We are going now to consider the distance d between two strongly continuous operators C_1 and C_2 defined by

$$d(C_1, C_2) := \max_{u \in D} \|C_1(u) - C_2(u)\|.$$

Let $\{C_n, n = 1, 2, ...\}$ be a sequence of nonlinear compact and Lipschitz operators (with the same rank k) corresponding to problems (VHI_n) for $n = 1, 2, ...$
From the Corollary 4.3, for each $n = 1, 2, ...$ there exists u_n the unique solution of (VHI_n) such that

$$e(S, u_n) := \sup_{v \in D} \|v - u_n\| \leq \sup_{v \in D} \|v - u_n\| \leq \frac{1}{\delta - \lambda - K - k} d(C_n, C).$$

Here S denotes the solution-set of (VHI).

Corollary 4.4. *Suppose that α is δ-positive, β is γ-essential 1-Hölder and C_n are converging to C relatively to d. Then, for $\lambda < \delta - K - k$, the sequence $\{u_n\}$ converges to a unique solution $\overline{u}$ of (VHI).*

Proof. Since $e(S, u_n) \leq d(C_n, C)/(\delta - \lambda - K - k)$ and $\lim d(C_n, C) = 0$, we deduce that $\{u_n\}$ converges to some $\overline{u}$ which must be the unique solution of (VHI). □

4.2.

Let us consider now the problems (VHI_i) $i = 1, 2$ where the corresponding positive bilinear forms α_i, nonlinear operators C_i and functions j_i given by $j_i(t) = \int_0^t \beta_i(s)ds$ (where $\beta_i \in L^\infty(I\!R)$) are perturbed.
Between j_1 and j_2 we consider the distance

$$D(j_1, j_2) := \|\beta_1 - \beta_2\|_\infty.$$

Theorem 4.5. *Suppose that α_2 is δ-positive and $\beta_1, \beta_2 \in L^\infty(I\!R)$ such that β_2 is γ-lower essentially 1-Hölder, and $p \geq 2$. If u_i of (VHI_i), for $i = 1, 2$, and $\lambda < \delta - K$, then*

$$\|u_1 - u_2\| \leq \frac{1}{\delta - K - \lambda}(\|C_1(u_1) - C_2(u_2)\| + \tau\|\alpha_1 - \alpha_2\| + \sigma D(j_1, j_2))$$

where $\sigma = c_p(mes(\Omega))^{(p-1)/p}$ and $\tau \geq \|u_1\|$.

Proof. By definition of u_i, setting $v = u_j$ in (VHI_i) for $j \neq i$, and adding these two relations we obtain

$$\begin{aligned} -\lambda\|u_1 - u_2\|^2 \leq\ & \alpha_1(u_1, u_2 - u_1) + \alpha_2(u_2, u_1 - u_2) + \langle C_1(u_1) - C_2(u_2), u_2 - u_1\rangle \\ & + \int_\Omega \left(j_1^0(u_1; u_2 - u_1) + j_2^0(u_2; u_1 - u_2)\right) dx. \end{aligned}$$

As β_2 satisfies (1) for $r = 1$, it follows from Lemma 3.3 that

$$\begin{aligned}\int_\Omega \left(j_1^0(u_1; u_2 - u_1) + j_2^0(u_2; u_1 - u_2)\right) dx \quad &\leq \\ &\leq \int_\Omega \left(j_1^0(u_1; u_2 - u_1) - j_2^0(u_1; u_2 - u_1)\right) dx + K\|u_1 - u_2\|^2 \\ &\leq \int_\Omega (j_1 - j_2)^0(u_1; u_2 - u_1)dx + K\|u_1 - u_2\|^2 \\ &\leq \|\beta_1 - \beta_2\|_\infty \int_\Omega |u_1(x) - u_2(x)|\, dx + K\|u_1 - u_2\|^2 \\ &\leq \sigma \|\beta_1 - \beta_2\|_\infty \|u_1 - u_2\| + K\|u_1 - u_2\|^2.\end{aligned}$$

Since α_2 is positive, we obtain

$$\begin{aligned}-\lambda \|u_1 - u_2\|^2 \;\leq\; & (\alpha_1 - \alpha_2)(u_1, u_2 - u_1) - \delta \|u_1 - u_2\|^2 + K\|u_1 - u_2\|^2 \\ & + \|C_1(u_1) - C_2(u_2)\| \|u_1 - u_2\| + \sigma D(j_1, j_2) \|u_1 - u_2\|.\end{aligned}$$

Thus

$$\|u_1 - u_2\| \leq \frac{1}{\delta - \lambda - K} \left(\tau\|\alpha_1 - \alpha_2\| + \|C_1(u_1) - C_2(u_2)\| + \sigma D(j_{1,}j_2)\right).$$

□

Remark 4.6. If moreover C_2 is Lipschitz of rank k such that $K + k < \delta - \lambda$, then for $M := 1/(\delta - \lambda - K - k)$ we obtain

$$\|u_1 - u_2\| \leq M\left[\tau\|\alpha_1 - \alpha_2\| + d(C_1, C_2) + \sigma D(j_{1,}j_2)\right].$$

When C_2 is c-monotone, the same result holds by taking $M = 1/(\delta - \lambda - K + c)$.

Let us consider $\{C_n; n = 1, 2, ...\}$ be a sequence of weakly- strongly continuous and Lipschitz operators of the same rank k with $K + k < \delta - \lambda$,$j_n(t) = \int_0^t \beta_n(s)ds$ (where $\beta, \beta_n \in L^\infty(I\!R)$), and α_n, $n = 1, 2, ...$ Let u_n $(n = 1, 2, ...)$ be the unique solution of the associated problem (VHI_n).
When we suppose that all these sequences converge, then Remark 4.6 ensures that $\{u_n\}$ is a Cauchy sequence and moreover if u is limit of $\{u_n\}$

$$\|u - u_n\| \leq \frac{1}{\delta - \lambda - K - k}(\tau\|\alpha - \alpha_n\| + d(C, C_n) + \sigma D(j, j_n)).$$

Here τ is taken so as : $\tau \geq \|u\|$.

As a review of this convergence's rate, we introduce the following local convergence of j_n to j:

$$D_{\tau,u}(j_n, j) := \|k_\tau(\beta, \beta_n)(u(.))\|_{L^q(\Omega)} \text{ where } k_\tau(\beta, \beta_n)(t) := \operatorname*{ess\,sup}_{|s-t|\leq\tau} |(\beta - \beta_n)(s)|.$$

Proposition 4.7. *Let C_n as stated in Remark 4.6 and suppose that, for $n = 1, 2, ..,$ α_n is δ-positive, β_n is γ-lower essential 1-Hölder,* (**H**6)(ii) *is verified with α_1 and α_2*

independent of n *and* β_n *converges to* β *almost uniformly on each bounded line segment of* $I\!R$. *Then*

$$\|u-u_n\| \leq \frac{1}{\delta-\lambda-k-c}[\eta\|\alpha-\alpha_n\|+d_\eta(C,C_n)+c_p D_{\tau,u}(j,j_n)] \tag{11}$$

where $\eta>0$ *is such that* $\|u\|\leq\eta$ *and*

$$d_\eta(C,C_n):=\max_{\|u\|\leq\eta}\|C_1(u)-C_n(u)\|.$$

Proof. By using the same argument as in the proof of Theorem 4.5, we have

$$\begin{aligned}(\delta-\lambda-K-c)\left\|u-u_n\right\|^2 &\leq (\alpha-\alpha_n)(u,u_n-u)+\left\|C_n(u)-C(u)\right\|\left\|u_n-u\right\| \\ &\quad+\int_\Omega (j-j_n)^0(u;u_n-u)dx.\end{aligned}$$

Note, for some $\tau>0$, we have from Remark 2.2 and Proposition 2.3

$$\int_\Omega (j-j_n)^0(u;u_n-u)dx \leq \int_\Omega k_\tau(\beta,\beta_n)(u(x))\left|u_n(x)-u(x)\right|dx. \tag{12}$$

Since β and β_n satisfy $(\mathbf{H}6)(ii)$, it follows that $k_\tau(\beta,\beta_n)(u(.))\in L^q(\Omega)$ and

$$\begin{aligned}\textstyle\int_\Omega (j-j_n)^0(u;u_n-u)dx &\leq \|k_\tau(\beta,\beta_n)(u(.))\|_{L^q(\Omega)}\|u_n-u\|_p \\ &\leq c_p D_{\tau,u}(j_n,j)\|u_n-u\|\end{aligned}$$

where $D_{\tau,u}(j_n,j):=\|k_\tau(\beta,\beta_n)(u(.))\|_{L^q(\Omega)}$. Thus (11) is satisfied. Now, thanks to $(\mathbf{H}6)$ (ii), the function $k_{\tau,u}(\beta_n,\beta)$ is dominated independently on n in $L^q(\Omega)$. Using Lebesgue's Theorem, we deduce that $D_{\tau,u}(j_n,j)$ converges to 0 in $L^q(\Omega)$ and thus $u_n\to u$. □

5. Comments and Remarks

5.1.

By setting $D=V$, $\alpha(u,v)=\langle Au,v\rangle$, $\Phi=0$ and $J(u)=\int_\Omega j(u(x))$, then under suitable conditions the problem (VHI) can be expressed by

$$\langle Au,v\rangle+\langle Cu,v\rangle+J^0(u;v)\geq\lambda\langle H(u),v\rangle \text{ for each } v\in V$$

which means

$$J^0(u,v)\geq<\lambda H(u)-Au-Cu,v\rangle \text{ for each } v\in V.$$

Using Clarke's subdifferential we obtain $\lambda H(u)-Au-Cu\in\partial J(u)$, i.e. $\lambda H(u)\in(A+C+\partial J)(u)$. This justifies the eigenvalue nomenclature.

5.2.

In [14], the solution to (VHI) has been established for all λ belonging to the resolvent set of the bilinear form α.
In Theorem 3.7, the condition on λ implies that λ is contained in the resolvent set of the bilinear form α.

Indeed, the spectrum $\sigma(\alpha)$ of α is included in the closure of numerical range of α (see [12, p. 171]) that is $\sigma(\alpha)\subset\overline{\{\alpha(u,u):\|u\|=1\}}$.
As α is δ-positive, we have $\sigma(\alpha)\subset[\delta,+\infty[$ which join the assumption of [14] upon λ.

5.3.

Neither existence nor stability results are affected if we consider the following hemivariational inequality

$$(\mathbf{P}_1)\quad \begin{array}{l} \text{find } u \in X \text{ such that } \forall v \in X, \\ \alpha(u, v-u) + \langle C(u), v-u\rangle + \int_\Omega j^0(u; v-u)dx \geq \langle l, v-u\rangle \end{array}$$

where X is a reflexive Banach space and $l \in X^*$. In fact, this inequality would require only minor changes. Note that, if we take $C = 0$ in $(\mathbf{P}_1)$, we find again the result obtained in a recent paper by O. Chadli, Z. Chbani and H. Riahi, by weaking their assumption (h_4), see [5] for more details.

5.4.

As we have mentioned before, on a Hilbert space, with $\Phi = 0$, the (VHI) reduces to the problem introduced and studied in [14]. Here, our approach imposes only the positivity on the bilinear form α, but not the symmetry. However, we do not involve any structural decomposition upon the operator C (assumption $(\mathbf{H}_1)$ there). Also, in our case, the space V is a reflexive Banach and not necessarily dense in $L^p(\Omega)$. Moreover, using Assumption (1), we obtain the convergence of solutions with an estimate for the rate of convergence.

5.5.

If we take $\lambda = 0$ and $C = -h$ where h is a derivative of a Gâteaux-Differentiable function G, the (VHI) contains as a particular case the problem considered and treated in [11], namely : find $u \in D$ such that $\forall v \in V$

$$(\mathbf{P}_2)\qquad \alpha(u, v-u) + \int_\Omega j^0(u; v-u)dx + \Phi(v) - \Phi(u) \geq \langle h(u), v-u\rangle,$$

where h is a derivative of a Gâteaux-Differentiable G.
With just the positivity but not necessarily coercivity on the bilinear form α, our Theorem 3.7 extends the result of [11], by weakening assumption A_4) there, to the case where V is imbedded in $L^p(\Omega)$ for p such that $2 \leq p \leq 3$.

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