

Nonconvex Duality and Viscosity Solutions of the Hamilton-Jacobi-Bellman Equation in Optimal Control

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In this work, we characterize the solutions of a nonconvex optimal control problem, using the Klötzler-Vinter nonconvex duality approach, in terms of generalized solutions of the Hamilton-Jacobi-Bellman equation (HJB). The dual problem is to find the supremum of the viscosity subsolutions of the HJB equation. We prove, without convexity assumptions, a weak duality between the primal and dual problems by using the technique of convolution and mollification. This weak duality provides necessary and sufficient conditions of optimality and leads to an error estimate. We also establish strong duality under an additional convexity hypothesis.

Keywords: Optimal control, Hamilton-Jacobi-Bellman equation, nonconvex duality, convolution, viscosity subsolution

1. Introduction

The Hamilton-Jacobi verification technique provides sufficient conditions on minimizing arcs, for an optimal control problem, in terms of solutions of the Hamilton-Jacobi-Bellman (HJB) equation (or inequality), which are called verification functions. Classically such results have required the smoothness of the verification functions, however for general optimal control problems, smooth solutions may not exist. A number of remedies are available based on a variety of generalized solutions: Viscosity solutions [17], [18], [9], contingent solutions [21], [27], lower semicontinuous solutions [22], solutions in terms of the Clarke generalized gradient [13], [15],.... Another approach by means of duality has been developed by Klötzler [23] (see also [1], [2] and [29]), Fleming and Vermes [20] and Vinter [26], for which the dual problem is the upper hull of smooth subsolutions of the HJB equation. A restrictive feature of this duality approach is the smoothness properties of subsolutions and the fact that the necessary and sufficient conditions are proved with the strong duality. But without convexity hypothesis the duality gaps may occur and strong duality may fail. Our attention in this paper is focused upon nonconvex duality. The dual problem is to find the supremum of the viscosity subsolutions of the HJB equation.

Nonconvex duality with the HJB equation as it bears here, contains three distinguishing features: we establish weak duality without requiring any assumption of convexity on the cost function. This weak duality provides necessary and sufficient conditions and leads to an error estimate. Another feature is the mild nature of the hypotheses on the verification functions involved in the dual problem. These functions are required only to be continuous viscosity subsolutions of the HJB equation (not smooth subsolutions as in Vinter's and Klötzler's papers: [23], [26]). On the other hand we prove strong duality under a partially convexity assumption. This paper is expository in nature and the proofs are largely self-contained. In the second section we introduce the primal problem. In the third section we use the technique of mollification to construct a family of dual regularized problems involving the functions regularized by convolution and we establish weak duality between those problems and the primal problem. In the forth section we exploit the convergence properties of the mollifier sequence to prove weak duality between the primal problem and the "viscosity dual problem" involving the supremum of viscosity subsolutions of the HJB equation. The weak duality leads to an estimate error and provides necessary and sufficient conditions of optimality. The strong duality is proved under an additional hypothesis. We conclude by an example in which we show how the nonconvex duality may confirm the optimality of a suspected candidate.

2. Primal Problem

Among the problems in which optimal control plays a crucial role are those arising from economy and biology, particularly the bioeconomic problems of renewable resources management. In this class of problems (and others), the time dependence is usually expressed by an exponential term $e^{-\delta t}$, where δ denote the instantaneous discount rate, see [[16] chap. I, section 1.5 Discounting], (see also [12] and [5]). In this work we will use a general cost function for this type.

We consider the following differential inclusion formulation of the optimal control problem:

$$(P) \quad \begin{cases} V(0, x_0) = \inf_x J(x) := \int_0^T e^{-\delta t} l(x(t), \dot{x}(t)) dt + g(x(T)), \\ \dot{x}(t) \in F(t, x(t)) \text{ a.e. } t \in [0, T], \\ x(t) \in K, \forall t \in [0, T], \\ x(0) = x_0. \end{cases}$$

The infimum is taken on $x : [0, T] \rightarrow \mathbb{R}^n$ absolutely continuous ($x \in AC$), a class of functions we call arcs.

The functions $g : \mathbb{R}^n \rightarrow \mathbb{R}$, $l : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$, the set valued map $F : [0, T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ and the set $K \subset \mathbb{R}^n$ are the data of the problem. We recall that the function $x \in AC$ is a trajectory of F if $\dot{x}(t) \in F(t, x(t))$ a.e. $t \in [0, T]$.

Consider the following set

$$S_{[t, T]}(\xi) := \{x \text{ trajectory of } F \text{ on } [t, T] \text{ such that } x(t) \in K \forall t \in [t, T] \text{ and } x(T) = \xi\}.$$

We say that $x \in AC$ is admissible for (P) if $x \in S_{[0, T]}(x_0)$.

The value function V is given at $(t, \bar{x})$ as

$$V(t, \bar{x}) = \inf \left\{ \int_t^T e^{-\delta s} l(x(s), \dot{x}(s)) ds + g(x(T)) : x \in S_{[t, T]}(\bar{x}) \right\}.$$

Hypotheses.

H_1 – g is lower semicontinuous (l.s.c.), l is bounded, k –lipschitz.

H_2 – K is a compact subset of $\mathbb{R}^n$ such that $K = cl(int(K))$,

where $cl(int(K))$ denotes the closure of the interior of K .

H_3 – $\forall (t, x)$ the set $F(t, x)$ is nonempty, compact, convex and there exist γ and c such that, for all (t, x)

$$v \in F(t, x) \implies \|v\| \leq \gamma \|x\| + c. \quad (1)$$

H_4 – F is locally lipschitz i.e. for all $\varsigma > 0$, There exist $c_\varsigma > 0$ such that for all $(t_1 - t_2, x_1 - x_2) \in B_\varsigma(0)$, we have

$$F(t_1, x_1) \subset F(t_2, x_2) + c_\varsigma \|(t_1 - t_2, x_1 - x_2)\| B_1(0). \quad (2)$$

where $B_m(0)$ denotes the closed ball in $[0, T] \times \mathbb{R}^n$, with center 0 and radius m .

H_5 – There exists at least one admissible arc for the problem (P).

Our object in the following sections is to construct a dual problem to problem (P) involving the viscosity subsolutions of the HJB equation.

Let H be the Hamiltonian defined by

$$H(t, x, p) := \min_{v \in F(t, x)} \{v \cdot p + e^{-\delta t} l(x, v)\}. \quad (3)$$

and $\tilde{H}$ the augmented Hamiltonian

$$\tilde{H}(t, x, \theta, p) := \theta + H(t, x, p). \quad (4)$$

The hamiltonian H is locally lipschitz with respect the variables (t, x, p) (see [13]). In the following proposition we give a proof that the hamiltonian H is locally lipschitz with respect the variables t and x in order to find explicitly the lipschitz constants which we will use later.

Proposition 2.1. *Assume that the hypotheses $H_1 - H_4$ are satisfied.*

Then $H(., ., p)$ is locally lipschitz.

Proof. Let $v_1 \in F(r, x)$ the point where the minimum of $H(r, x, p)$, given by the relation (3), is achieved. Then we have

$$H(t, x, p) - H(r, x, p) \leq (v - v_1) \cdot p + e^{-\delta t} l(x, v) - e^{-\delta r} l(x, v_1) \quad \forall v \in F(t, x). \quad (5)$$

On the other hand, according to H_4 , F is locally lipschitz, then, since $v_1 \in F(r, x)$, there exist $\tilde{v} \in F(t, x)$ such that the inclusion (2) implies

$$\|\tilde{v} - v_1\| \leq c_T |r - t|,$$

this for all r such that $|r - t| \leq T$.

By choosing, in the inequality (5), $v = \tilde{v}$ and by adding and subtracting the term $e^{-\delta t} l(x, v_1)$, we obtain

$$\begin{aligned} H(t, x, p) - H(r, x, p) &\leq c_T \cdot \|p\| \cdot |r - t| + e^{-\delta t} (l(x, v) - l(x, v_1)) \\ &\quad + (e^{-\delta t} - e^{-\delta r}) l(x, v_1). \end{aligned}$$

But the facts that the function $\exp(\cdot)$ is locally lipschitz and l is bounded k -lipschitz implies that there exist $q := q_1 \cdot q_2$, where q_1 is the local lipschitz constant of the exponential function (on $t : |r - t| \leq T$) and q_2 is the constant by which l is bounded, such that

$$H(t, x, p) - H(r, x, p) \leq c_T \cdot \|p\| \cdot |r - t| + k \cdot c_T \cdot |r - t| e^{-\delta t} + q |r - t|. \quad (6)$$

From the fact that $e^{-\delta t} \leq 1$ for all $t \in [0, T]$, it follows that

$$H(t, x, p) - H(r, x, p) \leq [c_T \cdot \|p\| + k \cdot c_T + q] |r - t|.$$

Since r and t are arbitrary, we conclude that

$$|H(t, x, p) - H(r, x, p)| \leq [c_T \cdot \|p\| + k \cdot c_T + q] |r - t|. \quad (7)$$

Likewise, let $v_2 \in F(t, y)$ the point where the minimum of $H(t, y, p)$ is achieved, we have

$$\begin{aligned} H(t, x, p) - H(t, y, p) &\leq p \cdot (v - v_2) + e^{-\delta t} (l(x, v) - l(y, v_2)) \\ &\leq \|p\| \cdot \|v - v_2\| + e^{-\delta t} k \cdot [\|x - y\| + \|v - v_2\|]. \end{aligned}$$

But according to H_4 we have, for $a > 0$, if $\|x - y\| \leq a$ then there exist c_a such that

$$\|v - v_2\| \leq c_a \|x - y\|.$$

Therefore

$$\begin{aligned} H(t, x, p) - H(t, y, p) &\leq c_a \|p\| \|x - y\| + e^{-\delta t} k \cdot [\|x - y\| + c_a \|x - y\|] \\ &\leq [c_a \|p\| + e^{-\delta t} k \cdot (1 + c_a)] \|x - y\| \\ &\leq [c_a \|p\| + k \cdot (1 + c_a)] \|x - y\|. \end{aligned}$$

We permute x and y , this leads to the following

$$|H(t, x, p) - H(t, y, p)| \leq [c_a \|p\| + k \cdot (1 + c_a)] \|x - y\|. \quad (8)$$

By combining the inequality (7) and (8) we obtain

$$|H(t, x, p) - H(r, y, p)| \leq b[|r - t| + \|x - y\|], \quad (9)$$

where $b := \sup\{[c_T \cdot \|p\| + k \cdot c_T + q], [c_a \|p\| + k \cdot (1 + c_a)]\}$.

Which completes the proof of theorem. $\square$

3. Regularized Dual Problems

Our interest here is to construct a family of dual problems of the problem (P) by using convolution and mollification techniques. We will exploit the smoothness properties of regularized functions obtained by convolution.

3.1. Regularization by Convolution

Let $\varphi : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a bounded l.s.c. function, $\varepsilon > 0$ small and $h > 0$ arbitrary.

By the convolution technique, (see for instance [3], [8] and [24]), we regularize φ in order

to obtain a function ψ_ε locally lipschitz and then a function $\varphi_\varepsilon \in C^1(\mathbb{R} \times \mathbb{R}^n)$.
Let, for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$

$$\psi_\varepsilon(t, x) := \sup_{(s, y) \in \mathbb{R} \times \mathbb{R}^n} \left\{ \varphi(s, y) - e^{-ht} \frac{\|x - y\|^2}{\varepsilon^2} - \frac{|t - s|^2}{\varepsilon^4} \right\}. \quad (10)$$

Note that the supremum in ψ_ε is achieved for $(\bar{s}, \bar{y})$ satisfying

$$e^{-ht} \frac{\|x - \bar{y}\|^2}{\varepsilon^2} + \frac{|t - \bar{s}|^2}{\varepsilon^4} \leq M^2, \quad (11)$$

where $M = \sqrt{2\|\varphi\|_\infty}$, (see [8]).

Consider now the following mollifier sequence $(\rho_\varepsilon)_\varepsilon$

$$\rho_\varepsilon(t, x) := \varepsilon^{-(n+1)} \rho\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right),$$

where $\rho \in C^\infty(\mathbb{R}^{n+1})$, nonnegative, $\text{supp}(\rho) \subset B_1(0)$ and $\iint_{\mathbb{R}^{n+1}} \rho(\tau, \xi) d\tau d\xi = 1$.

It is easy to see that

$$\iint_{\mathbb{R}^{n+1}} \rho_\varepsilon(\tau, \xi) d\tau d\xi = 1.$$

We now define φ_ε as

$$\varphi_\varepsilon(t, x) = \iint_{\mathbb{R}^{n+1}} \psi_\varepsilon(t + \tau, x + \xi) \rho_\varepsilon(\tau, \xi) d\tau d\xi. \quad (12)$$

Proposition 3.1. ψ_ε is locally lipschitz and $\varphi_\varepsilon \in C^1(\mathbb{R} \times \mathbb{R}^n)$.

The proof is similar to the proof of the Lemma 5.5 in [[8], p.136].

Proposition 3.2. Assume that φ is bounded continuous ($\in CB(\mathbb{R} \times \mathbb{R}^n)$).

Then the sequences $\{\psi_\varepsilon\}$ and $\{\varphi_\varepsilon\}$ converge pointwise towards φ as $\varepsilon \rightarrow 0$.

Proof. First we prove that ψ_ε converges pointwise towards φ .

Fixe (t, x) in $\mathbb{R} \times \mathbb{R}^n$ and let $\Lambda > 0$, since $\psi_\varepsilon(t, x) \geq \varphi(t, x)$, we have

$$\begin{aligned} |\varphi(t, x) - \psi_\varepsilon(t, x)| &= \psi_\varepsilon(t, x) - \varphi(t, x) \\ &= \sup_{(s, y) \in \mathbb{R} \times \mathbb{R}^n} \left\{ \varphi(s, y) - e^{-ht} \frac{\|x - y\|^2}{\varepsilon^2} - \frac{|t - s|^2}{\varepsilon^4} \right\} - \varphi(t, x). \end{aligned}$$

So

$$|\varphi(t, x) - \psi_\varepsilon(t, x)| = \varphi(\bar{s}, \bar{y}) - e^{-ht} \frac{\|x - \bar{y}\|^2}{\varepsilon^2} - \frac{|t - \bar{s}|^2}{\varepsilon^4} - \varphi(t, x),$$

where $(\bar{s}, \bar{y})$ is the point where the supremum is achieved.

The term $-e^{-ht} \frac{\|x - \bar{y}\|^2}{\varepsilon^2} - \frac{|t - \bar{s}|^2}{\varepsilon^4}$ is negative, then we have

$$|\varphi(t, x) - \psi_\varepsilon(t, x)| \leq \varphi(\bar{s}, \bar{y}) - \varphi(t, x).$$

The function φ is continuous, so, there exist $\eta(t, x) > 0$ such that, for all (s, y) satisfying $\|x - y\| + |t - s| \leq \eta$, we have

$$|\varphi(t, x) - \varphi(s, y)| \leq \Lambda.$$

But we know from (11) that

$$e^{-ht} \frac{\|x - \bar{y}\|^2}{\varepsilon^2} + \frac{|t - \bar{s}|^2}{\varepsilon^4} \leq M^2.$$

It follows that

$$\|x - \bar{y}\| + |t - \bar{s}| \leq \varepsilon M(e^{\frac{h}{2}t} + \varepsilon).$$

For small ε we have

$$\|x - \bar{y}\| + |t - \bar{s}| \leq \eta(t, x).$$

This implies that

$$|\varphi(t, x) - \psi_\varepsilon(t, x)| \leq \Lambda,$$

as required.

We turn now to the convergence of the sequence $\{\varphi_\varepsilon\}$:

$$\begin{aligned} |\varphi(t, x) - \varphi_\varepsilon(t, x)| &= \left| \varphi(t, x) - \iint_{\mathbb{R}^{n+1}} \psi_\varepsilon(t + \tau, x + \xi) \rho_\varepsilon(\tau, \xi) d\tau d\xi \right| \\ &= \left| \iint_{\mathbb{R}^{n+1}} [\varphi(t, x) - \psi_\varepsilon(t + \tau, x + \xi)] \rho_\varepsilon(\tau, \xi) d\tau d\xi \right|. \end{aligned}$$

Thus

$$\begin{aligned} |\varphi(t, x) - \varphi_\varepsilon(t, x)| &\leq \iint_{\mathbb{R}^{n+1}} |\varphi(t, x) - \psi_\varepsilon(t + \tau, x + \xi)| \rho_\varepsilon(\tau, \xi) d\tau d\xi \\ &\leq \iint_{\mathbb{R}^{n+1}} \{|\varphi(t, x) - \varphi(t + \tau, x + \xi)| + \\ &\quad + |\varphi(t + \tau, x + \xi) - \psi_\varepsilon(t + \tau, x + \xi)|\} \rho_\varepsilon(\tau, \xi) d\tau d\xi. \end{aligned}$$

But, we know that $\lim_{\varepsilon \rightarrow 0} \psi_\varepsilon(t + \tau, x + \xi) = \varphi(t + \tau, x + \xi)$, then, for $\Lambda > 0$, there exist $\varepsilon_0(t, x, \tau, \xi) > 0$ such that, for all $\varepsilon < \varepsilon_0(t, x, \tau, \xi)$, we have

$$|\varphi(t + \tau, x + \xi) - \psi_\varepsilon(t + \tau, x + \xi)| \leq \frac{\Lambda}{2}. \quad (13)$$

Since the support of ρ_ε is in $B_\varepsilon(0)$, we can choose $\varepsilon_0 = \inf_{\tau, \xi} \varepsilon_0(t, x, \tau, \xi)$, so, inequality (13) remains true for all $(\tau, \xi) \in B_\varepsilon(0)$.

We obtain therefore

$$|\varphi(t, x) - \varphi_\varepsilon(t, x)| \leq \iint_{\mathbb{R}^{n+1}} |\varphi(t, x) - \varphi(t + \tau, x + \xi)| \rho_\varepsilon(\tau, \xi) d\tau d\xi + \frac{\Lambda}{2}.$$

On the other hand, φ is continuous, i.e., there exist $\eta(t, x) > 0$ such that, for all (s, y) satisfying $\|y - x\| + |t - s| \leq \eta(t, x)$, we have

$$|\varphi(t, x) - \varphi(s, y)| \leq \frac{\Lambda}{2}.$$

But the support of ρ_ε is in $B_\varepsilon(0)$; i.e., $\|\xi\| + |\tau| = \|x + \xi - x\| + |t + \tau - t| \leq \varepsilon$, then for small values of ε , (i.e. $\varepsilon \leq \eta(t, x)$), we have

$$|\varphi(t, x) - \varphi(t + \tau, x + \xi)| \leq \frac{\Lambda}{2}.$$

Hence,

$$|\varphi(t, x) - \varphi_\varepsilon(t, x)| \leq \frac{\Lambda}{2} + \frac{\Lambda}{2} = \Lambda.$$

This completes the proof of the proposition. $\square$

3.2. Regularized Dual Formulation

Our goal now is to construct a family of dual problems to (P) based on the regularized functions defined above.

Consider the following dual problem

$$(D_\varepsilon) \quad \begin{cases} \beta_\varepsilon = \sup_{\varphi} \left\{ \int_0^T \inf_{x \in K} \tilde{H}(t, x, \frac{\partial \varphi_\varepsilon}{\partial t}(t, x), \frac{\partial \varphi_\varepsilon}{\partial x}(t, x)) dt + \delta_\varepsilon + \varphi_\varepsilon(0, x_0) \right\}, \\ \varphi(T, \cdot) \leq g(\cdot) \text{ on } K. \end{cases}$$

The supremum is taken on $\varphi : [0, T] \times K \rightarrow \mathbb{R}$ that are continuous functions, $\tilde{H}$ is the augmented hamiltonian given by (4), φ_ε is obtained by convolution from $\tilde{\varphi}$ as described above, where $\tilde{\varphi}$ is the continuous extension of φ to $\mathbb{R} \times \mathbb{R}^n$ as follows:

$$\tilde{\varphi}(t, y) := \begin{cases} G(t, y) & \text{for all } y \in K, \\ G(t, \tilde{y}) & \text{for all } y \text{ outside } K. \end{cases} \quad (14)$$

Where $\tilde{y}$ is an intersection of the line segment $[y_0, y]$, (connecting y at an arbitrary fixed element $y_0 \in \text{int}K$), and the boundary of the set K , and where G is defined on $\mathbb{R} \times K$ by:

$$G(t, \cdot) := \begin{cases} \varphi(0, \cdot) & \text{for all } t \leq 0, \\ \varphi(t, \cdot) & \text{for all } 0 \leq t \leq T, \\ \varphi(T, \cdot) + T - t & \text{for all } t > T. \end{cases} \quad (15)$$

The parameter $\delta_\varepsilon := \inf_{x \in K} (g(x) - g_\varepsilon(x))$ where $g_\varepsilon(y) := \iint_{\mathbb{R}^{1+n}} g(y + \xi) \rho_\varepsilon(\tau, \xi) d\tau d\xi$.

In the following theorem we establish a weak duality between (P) and (D_ε) .

Theorem 3.3. *For small values of ε , we have*

$$\alpha \geq \beta_\varepsilon.$$

Proof. The proof of theorem uses the following lemma.

Lemma 3.4. *If φ is admissible for (D_ε) then*

$$\varphi_\varepsilon(T, y) \leq g_\varepsilon(y), \quad \forall y \in K.$$

Proof. The fact $\varphi(T, \cdot) \leq g(\cdot)$ on K implies, according to expressions (14) and (15), that

$$\tilde{\varphi}(T + \tau, y) < g(y), \quad \forall \tau > 0, \quad \forall y \in K.$$

We now prove that $\psi_\varepsilon(T + \tau, y) \leq g(y)$ on K , for small values of ε , where ψ_ε is the regularized function obtained from $\tilde{\varphi}$ by expression (10). Indeed,

let $\lambda := g(y) - \tilde{\varphi}(T + \tau, y) > 0$.

The fact that $\psi_\varepsilon(T + \tau, y)$ converges to $\tilde{\varphi}(T + \tau, y)$, says that there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon < \varepsilon_0$, we have

$$|\psi_\varepsilon(T + \tau, y) - \tilde{\varphi}(T + \tau, y)| \leq \lambda.$$

Since $|\psi_\varepsilon(T + \tau, y) - \tilde{\varphi}(T + \tau, y)| = \psi_\varepsilon(T + \tau, y) - \tilde{\varphi}(T + \tau, y)$ and $\lambda := g(y) - \tilde{\varphi}(T + \tau, y)$, we obtain

$$\psi_\varepsilon(T + \tau, y) \leq g(y). \quad (16)$$

Hence,

$$\begin{aligned} \varphi_\varepsilon(T, y) &:= \iint_{\mathbb{R}^{n+1}} \psi_\varepsilon(T + \tau, y + \xi) \rho_\varepsilon(\tau, \xi) d\tau d\xi \\ &\leq \iint_{\mathbb{R}^{n+1}} g(y + \xi) \rho_\varepsilon(\tau, \xi) d\tau d\xi. \end{aligned}$$

By setting $g_\varepsilon(y) := \iint_{\mathbb{R}^{n+1}} g(y + \xi) \rho_\varepsilon(\tau, \xi) d\tau d\xi$, we obtain

$$\varphi_\varepsilon(T, y) \leq g_\varepsilon(y), \quad \forall y \in K,$$

as required □

Turn now to the proof of the theorem.

First note that the function g_ε satisfies

$$g_\varepsilon(y) \rightarrow g(y) \text{ as } \varepsilon \rightarrow 0, \text{ uniformly on } K. \quad (17)$$

Let now x be a (P) admissible arc, we have

$$\begin{aligned} \int_0^T e^{-\delta t} l(x(t), \dot{x}(t)) dt + g(x(T)) &= \int_0^T e^{-\delta t} l(x(t), \dot{x}(t)) dt + g(x(T)) \\ &+ \int_0^T \frac{d\varphi_\varepsilon}{dt}(t, x(t)) dt - \int_0^T \frac{d\varphi_\varepsilon}{dt}(t, x(t)) dt \\ &= \int_0^T e^{-\delta t} l(x(t), \dot{x}(t)) dt + g(x(T)) + \\ &+ \int_0^T \left[\frac{\partial \varphi_\varepsilon}{\partial t}(t, x(t)) + \frac{\partial \varphi_\varepsilon}{\partial x}(t, x(t)) \cdot \dot{x}(t) \right] dt \\ &- \varphi_\varepsilon(T, x(T)) + \varphi_\varepsilon(0, x_0). \end{aligned}$$

By adding and subtracting the term $g_\varepsilon(x(T))$ we obtain

$$\begin{aligned} \int_0^T e^{-\delta t} l(x(t), \dot{x}(t)) dt + g(x(T)) &= \int_0^T \left[\frac{\partial \varphi_\varepsilon}{\partial t}(t, x(t)) + \frac{\partial \varphi_\varepsilon}{\partial x}(t, x(t)) \cdot \dot{x}(t) \right. \\ &+ \left. e^{-\delta t} l(x(t), \dot{x}(t)) \right] dt + g(x(T)) - g_\varepsilon(x(T)) \\ &+ g_\varepsilon(x(T)) - \varphi_\varepsilon(T, x(T)) + \varphi_\varepsilon(0, x_0). \end{aligned}$$

But according to Lemma (3.4) we have

$$\varphi_\varepsilon(T, x(T)) \leq g_\varepsilon(x(T)).$$

Hence, by using this inequality and the facts

$$\frac{\partial \varphi_\varepsilon}{\partial x}(t, x(t)) \cdot \dot{x}(t) + e^{-\delta t} l(x(t), \dot{x}(t)) \geq \min_{v \in F(t, x(t))} \left\{ \frac{\partial \varphi_\varepsilon}{\partial x}(t, x(t)) \cdot v + e^{-\delta t} l(x(t), v) \right\}$$

and $g(x(T)) - g_\varepsilon(x(T)) \geq \inf_{x \in K} (g(x) - g_\varepsilon(x))$, we obtain

$$\begin{aligned} \int_0^T e^{-\delta t} l(x(t), \dot{x}(t)) dt + g(x(T)) &\geq \int_0^T \left[\frac{\partial \varphi_\varepsilon}{\partial t}(t, x(t)) + \min_{v \in F(t, x(t))} \left\{ \frac{\partial \varphi_\varepsilon}{\partial x}(t, x(t)) \cdot v \right. \right. \\ &+ \left. \left. e^{-\delta t} l(x(t), v) \right\} \right] dt + \inf_{x \in K} (g(x) - g_\varepsilon(x)) + \varphi_\varepsilon(0, x_0). \end{aligned}$$

Let $\delta_\varepsilon := \inf_{x \in K} (g(x) - g_\varepsilon(x))$, it follows from the limit (17) that

$$\delta_\varepsilon \rightarrow 0 \text{ as } \varepsilon \rightarrow 0,$$

So,

$$\begin{aligned} \int_0^T e^{-\delta t} l(x(t), \dot{x}(t)) dt + g(x(T)) &\geq \int_0^T \inf_{x \in K} \tilde{H}(t, x, \frac{\partial \varphi_\varepsilon}{\partial t}(t, x), \frac{\partial \varphi_\varepsilon}{\partial x}(t, x)) dt + \delta_\varepsilon \\ &+ \varphi_\varepsilon(0, x_0). \end{aligned}$$

this for all (P) admissible arc x and all (D_ε) admissible function φ .

We conclude that

$$\begin{aligned} \inf_x \left\{ \int_0^T e^{-\delta t} l(x(t), \dot{x}(t)) dt + g(x(T)) \right\} &\geq \sup_\varphi \left\{ \int_0^T \inf_{x \in K} \tilde{H}(t, x, \frac{\partial \varphi_\varepsilon}{\partial t}(t, x), \frac{\partial \varphi_\varepsilon}{\partial x}(t, x)) dt \right. \\ &+ \left. \delta_\varepsilon + \varphi_\varepsilon(0, x_0) \right\}. \end{aligned}$$

On other word,

$$\alpha \geq \beta_\varepsilon,$$

for small values of ε , which complete the proof of theorem. $\square$

The result of the theorem can be interpreted as a first ε -error estimation of the α by β_ε in the case where the functions φ involved in the regularized dual problems are not required to be subsolutions of the HJB equation.

4. Viscosity Dual Problem

4.1. Weak Duality

The goal of this section is to establish a weak duality between the problem (P) and a dual problem (D) involving the supremum of viscosity subsolutions of the HJB equation and give some necessary and sufficient conditions for optimality by using this weak duality. We will adopt the following definition of the viscosity subsolutions (see [9] and [17], [18]). Let φ be a continuous function on $[0, T] \times K$. We say that φ is a viscosity subsolution of the following HJB equation

$$\tilde{H}(t, x, \frac{\partial \varphi}{\partial t}(t, x), \frac{\partial \varphi}{\partial x}(t, x)) = 0 \text{ on } [0, T] \times K,$$

if the following assertion is satisfied :

If, for $(t_0, x_0) \in [0, T] \times K$, there exists a function $\phi \in C^1(\mathbb{R}^{n+1})$ such that the function $\varphi - \phi : [0, T] \times K \rightarrow \mathbb{R}$, achieves a strict maximum at (t_0, x_0) , then we have

$$\tilde{H}(t_0, x_0, \frac{\partial \phi}{\partial t}(t_0, x_0), \frac{\partial \phi}{\partial x}(t_0, x_0)) \geq 0.$$

Consider now the dual problem (D)

$$(D) \quad \begin{cases} \beta = \sup_{\varphi} (\varphi(0, x_0)), \\ \text{the supremum is taken on } \varphi \in C([0, T] \times K) \text{ which are} \\ \text{viscosity subsolutions of the HJB equation and satisfying} \\ \varphi(T, \cdot) \leq g(\cdot) \text{ on } K. \end{cases}$$

The dual problem (D) involve the continuous viscosity subsolutions of HJB equation on $[0, T] \times K$, the existence of such subsolutions is assured under hypotheses $H_1 - H_4$, see [[7], Chapter III].

Theorem 4.1. *Under the hypotheses $H_1 - H_5$, we have*

$$\alpha \geq \beta. \tag{18}$$

We establish this weak duality without requiring, in the cost function of the primal problem, any hypothesis of convexity, which play a crucial role to establish a strong duality as we will show later, in this case the duality gaps may occur.

Before proceeding to the proof of the theorem, let us show how this weak duality gives some interesting results. Indeed, the weak duality provides a necessary and sufficient conditions as we will show below. On the other hand, this weak duality allows to a second estimate error of α by a (P) admissible arc, as follows.

Corollary 4.2. *For an arc x admissible for (P), we have*

$$|\alpha - J(x)| \leq \inf_{\varphi} \{J(x) - \varphi(0, x_0)\}. \quad (19)$$

The infimum is over the functions φ admissible for the dual problem (D).

Proof. Since $-\alpha \leq -\sup_{\varphi} \{\varphi(0, x_0)\} = \inf_{\varphi} \{-\varphi(0, x_0)\}$,

we have

$$|\alpha - J(x)| = J(x) - \alpha \leq \inf_{\varphi} \{J(x) - \varphi(0, x_0)\}. \quad \square$$

To compute this error estimate, the idea is to subdivide the set $[0, T] \times K$ in finite elements and for a piecewise smooth arc $\hat{x}$ constructed by respecting the nodes and vertices, we construct φ by numerical methods. By taking the infimum in (19) on φ so constructed, we obtain an error estimate of α by $J(\hat{x})$.

We pause now to show that this weak duality provides a necessary and sufficient conditions of optimality for the problem (P).

4.2. Necessary and Sufficient Conditions

Theorem 4.3. *Suppose that the hypotheses $H_1 - H_5$ are satisfied.*

Let $\hat{x}$ be a (P) admissible arc

- Necessary and sufficient conditions: $\hat{x}$ is a minimum for (P) iff

there exist a sequence $(\varphi_n)_n$ of a continuous viscosity subsolutions of the HJB equation on $[0, T] \times K$, satisfying

$$\varphi_n(T, \cdot) \leq g(\cdot) \text{ on } K, \quad (20)$$

and

$$\lim_{n \rightarrow +\infty} |J(\hat{x}) - \varphi_n(0, x_0)| \leq |J(x) - \varphi(0, x_0)|, \forall \varphi \in \text{adm}(D) \quad (21)$$

$$\forall x \in \text{adm}(P).$$

Where $\text{adm}(S) := \{\varphi \text{ admissible for } (S)\}$.

Proof.

- Necessary conditions: Assume that $\hat{x}$ is a minimum for (P).

Let $(\varphi_n)_n$ be a maximizing sequence for (D), then we have

$$\lim_{n \rightarrow +\infty} \varphi_n(0, x_0) = \sup \varphi(0, x_0),$$

the supremum is taken on φ admissible for (D).

According to the Theorem (4.1) we have

$$\sup \varphi(0, x_0) \leq \alpha = J(\hat{x}).$$

Since $(\varphi_n)_n$ is a maximizing sequence then the functions φ_n are continuous viscosity subsolutions of the HJB equation on $[0, T] \times K$ and the condition (20) is satisfied, it remains to prove the condition (21). We have

$$J(\hat{x}) \leq J(x), \forall x \text{ admissible for (P)}.$$

Moreover

$$\lim_{n \rightarrow +\infty} \varphi_n(0, x_0) = \sup \varphi(0, x_0) \geq \varphi(0, x_0), \forall \varphi \in \text{adm}(D).$$

This implies that

$$J(\hat{x}) - \lim_{n \rightarrow +\infty} \varphi_n(0, x_0) \leq J(x) - \varphi(0, x_0).$$

Since $J(\hat{x}) \geq \varphi_n(0, x_0) \forall n$ and $J(x) \geq \varphi(0, x_0)$, we have

$$\lim_{n \rightarrow +\infty} |J(\hat{x}) - \varphi_n(0, x_0)| \leq |J(x) - \varphi(0, x_0)|.$$

- Sufficient conditions: Suppose that the conditions (20 - 21) are satisfied. Then the condition (21) implies that

$$\begin{aligned} J(\hat{x}) - \lim_{n \rightarrow +\infty} \varphi_n(0, x_0) &\leq J(x) - \varphi(0, x_0), \forall x \in \text{adm}(P), \\ &\forall \varphi \in \text{adm}(D). \end{aligned}$$

So,

$$\begin{aligned} J(\hat{x}) &\leq J(x) - \varphi(0, x_0) + \lim_{n \rightarrow +\infty} \varphi_n(0, x_0) \\ &\leq J(x) - \sup_{\varphi} \varphi(0, x_0) + \lim_{n \rightarrow +\infty} \varphi_n(0, x_0). \end{aligned}$$

But, $\lim_{n \rightarrow +\infty} \varphi_n(0, x_0) \leq \sup_{\varphi} \varphi(0, x_0)$, therefore

$$J(\hat{x}) \leq J(x),$$

this for all x admissible for (P).

We conclude that $\hat{x}$ is a minimum of (P), as required. $\square$

Moreover, with the above theorem, we recover Vinter's sufficient conditions for optimality [26], so that,

Corollary 4.4. *Let $\hat{x}$ be a (P) admissible arc.*

If there exists a sequence of continuous functions $(\varphi_n)_n$ viscosity subsolutions of HJB equation on $[0, T] \times K$ and satisfying

$$\varphi_n(T, \cdot) \leq g(\cdot) \text{ on } K, \tag{22}$$

$$\lim_{n \rightarrow +\infty} \varphi_n(0, x_0) = J(\hat{x}), \tag{23}$$

then $\hat{x}$ is a minimum for (P)

Proof. The proof is evident since the relation (23) implies the relation (21). $\square$

4.3. Proof of Theorem 4.1

Let φ be admissible for the dual problem (D). Extend φ on $\mathbb{R} \times \mathbb{R}^n$ to $\tilde{\varphi}$ as given in (14) and (15). We construct locally lipschitz functions ψ_ε from $\tilde{\varphi}$ by convolution as given by (10). Our goal is to prove that ψ_ε is a viscosity subsolution of a perturbed HJB equation on $O_\varepsilon \times K_\varepsilon$, where

$$O_\varepsilon := (M\varepsilon^2, T - M\varepsilon^2) \quad \text{and} \quad K_\varepsilon := \{x \in K : B(x, Me^{\frac{h}{2}T}\varepsilon) \subset K\}.$$

(h and M are given by the relations (10) and (11)).

Before proceeding to proof, let us prove the following lemma which we will use later.

Lemma 4.5. *The families $(O_\varepsilon)_\varepsilon$ and $(K_\varepsilon)_\varepsilon$ satisfies the following assertions:*

1. *We have, for $\varepsilon_1 \leq \varepsilon_2$*

$$O_{\varepsilon_2} \subset O_{\varepsilon_1} \quad \text{and} \quad K_{\varepsilon_2} \subset K_{\varepsilon_1}.$$

2. *We have*

$$\bigcup_{\varepsilon>0} O_\varepsilon = (0, T) \quad \text{and} \quad \bigcup_{\varepsilon>0} K_\varepsilon = \text{int}K.$$

3. *For $d > 0$ arbitrary chosen and a open neighborhood $U(\partial K, d)$ of boundary ∂K of K , with diameter d , there exist $(\varepsilon_i)_{1 \leq i \leq n}$ such that*

$$(0, T) \subset \bigcup_{i=1}^n O_{\varepsilon_i} \cup]0, d[\cup]T - d, T[,$$

and

$$\text{int}K \subset \bigcup_{i=1}^n K_{\varepsilon_i} \cup (U(\partial K, d) \cap \text{int}K).$$

Proof. The first assertion is obvious. For the second assertion, we only prove that

$$\bigcup_{\varepsilon>0} K_\varepsilon = \text{int}K.$$

It's easy to see that

$$K_\varepsilon \subset \text{int}K.$$

Then,

$$\bigcup_{\varepsilon>0} K_\varepsilon \subset \text{int}K.$$

Conversely, for all $x \in \text{int}K$, there exist $r(x) > 0$ such that $B(x, r(x)) \subset K$ which implies that

$$x \in K_\varepsilon.$$

where $\varepsilon = \frac{r(x)}{Me^{\frac{hT}{2}}}$.

It follows that

$$\text{int}K \subset \bigcup_{\varepsilon>0} K_\varepsilon.$$

Hence,

$$\bigcup_{\varepsilon>0} K_\varepsilon = \text{int}K,$$

as required.

Turn now to assertion three. Since $[0, T] = (0, T) \cup \{0\} \cup \{T\}$ and $K = \text{int}K \cup \partial K$, then, for d arbitrary chosen, we have, according to second assertion of this lemma, that

$$[0, T] \subset \bigcup_{\varepsilon>0} O_\varepsilon \cup]-d, d[\cup]T-d, T+d[,$$

and

$$K \subset \bigcup_{\varepsilon>0} K_\varepsilon \cup U(\partial K, d),$$

where $U(\partial K, d)$ denotes a open neighborhood of the boundary of K , ∂K , with diameter d .

Hence, we have covering $[0, T]$ and K by a families of open sets. But K and $[0, T]$ are compact sets, then we can extract from these coverings a finites coverings such that

$$[0, T] \subset \bigcup_{i=1}^n O_{\varepsilon_i} \cup]-d, d[\cup]T-d, T+d[,$$

and

$$K \subset \bigcup_{i=1}^n K_{\varepsilon_i} \cup U(\partial K, d).$$

Observe that we have use the same value n in both inclusions. To see this, it suffices to choose $n := \sup(n_1, n_2)$, if the first inclusion is true with n_1 and the second is true with n_2 , then both inclusions are true for n .

We deduce that

$$(0, T) \subset \bigcup_{i=1}^n O_{\varepsilon_i} \cup]0, d[\cup]T-d, T[,$$

and

$$\text{int}K \subset \bigcup_{i=1}^n K_{\varepsilon_i} \cup (U(\partial K, d) \cap \text{int}K).$$

Which complete the proof of the lemma. $\square$

Return now to proof of the theorem. Let $\varepsilon_0 := \inf\{\varepsilon_i : i := 1, \dots, n\}$. For $\varepsilon \leq \varepsilon_0$, let $\phi \in C^1(\mathbb{R}^{n+1})$ be such that $(\bar{t}, \bar{x}) \in O_\varepsilon \times K_\varepsilon$ is a strict maximum of $\psi_\varepsilon - \phi$ on $O_\varepsilon \times K_\varepsilon$ and let $(\bar{s}, \bar{y})$ be the point where the supremum of the function $\psi_\varepsilon(\bar{t}, \bar{x})$ is achieved.

So,

$$\psi_\varepsilon(\bar{t}, \bar{x}) = \tilde{\varphi}(\bar{s}, \bar{y}) - e^{-h\bar{t}} \frac{\|\bar{x} - \bar{y}\|^2}{\varepsilon^2} - \frac{|\bar{t} - \bar{s}|^2}{\varepsilon^4}.$$

But according to (11) it follows that $|\bar{t} - \bar{s}| \leq M\varepsilon^2$, so, according to fact that $\bar{t} \in O_\varepsilon$ we have

$$0 \leq \bar{s} \leq T.$$

Similarly it follows from (11) that $\|\bar{x} - \bar{y}\| \leq Me^{\frac{h}{2}T}\varepsilon$, so, from the fact that $\bar{x} \in K_\varepsilon$ we have

$$\bar{y} \in K.$$

Then

$$\psi_\varepsilon(\bar{t}, \bar{x}) = \varphi(\bar{s}, \bar{y}) - e^{-h\bar{t}} \frac{\|\bar{x} - \bar{y}\|^2}{\varepsilon^2} - \frac{|\bar{t} - \bar{s}|^2}{\varepsilon^4}.$$

Consider the following function θ :

$$\theta : (t, x, s, y) \rightarrow \varphi(s, y) - e^{-ht} \frac{\|x - y\|^2}{\varepsilon^2} - \frac{|t - s|^2}{\varepsilon^4} - \phi(t, x).$$

The function θ achieves a strict maximum on $O_\varepsilon \times K_\varepsilon \times [0, T] \times K$ at $(\bar{t}, \bar{x}, \bar{s}, \bar{y})$.

Let $(t, x) := (\bar{t}, \bar{x})$ and consider the function $\theta(\bar{t}, \bar{x}, s, y)$, we have

$$\theta(\bar{t}, \bar{x}, s, y) = \varphi(s, y) - [e^{-h\bar{t}} \frac{\|\bar{x} - y\|^2}{\varepsilon^2} + \frac{|\bar{t} - s|^2}{\varepsilon^4} + \phi(\bar{t}, \bar{x})].$$

The function $(s, y) \mapsto e^{-h\bar{t}} \frac{\|\bar{x} - y\|^2}{\varepsilon^2} + \frac{|\bar{t} - s|^2}{\varepsilon^4} + \phi(\bar{t}, \bar{x})$ is $C^1(\mathbb{R}^{n+1})$ and $\theta(\bar{t}, \bar{x}, \cdot, \cdot) : [0, T] \times K \mapsto \mathbb{R}$ achieves a strict maximum at $(\bar{s}, \bar{y}) \in [0, T] \times K$. Then according to the fact that φ is a viscosity subsolution to the HJB equation, we conclude from above that

$$(s, y) \rightarrow e^{-h\bar{t}} \frac{\|\bar{x} - y\|^2}{\varepsilon^2} + \frac{|\bar{t} - s|^2}{\varepsilon^4} + \phi(\bar{t}, \bar{x}),$$

satisfies the HJB inequality at $(\bar{s}, \bar{y})$, i.e.

$$-\frac{2(\bar{t} - \bar{s})}{\varepsilon^4} + H(\bar{s}, \bar{y}, -2e^{-h\bar{t}} \frac{\bar{x} - \bar{y}}{\varepsilon^2}) \geq 0. \quad (24)$$

On the other hand the function $\theta(t, x, \bar{s}, \bar{y})$ achieves a strict maximum at $(\bar{t}, \bar{x})$, therefore, the t and x partial derivatives vanish at $\bar{t}$ and $\bar{x}$ respectively, i.e.

$$-\frac{\partial \phi}{\partial t}(\bar{t}, \bar{x}) - \frac{2(\bar{t} - \bar{s})}{\varepsilon^4} + he^{-h\bar{t}} \frac{\|\bar{x} - \bar{y}\|^2}{\varepsilon^2} = 0,$$

and

$$-\frac{\partial \phi}{\partial x}(\bar{t}, \bar{x}) - 2e^{-h\bar{t}} \frac{\bar{x} - \bar{y}}{\varepsilon^2} = 0. \quad (25)$$

It follows according to relation (24) that

$$\frac{\partial \phi}{\partial t}(\bar{t}, \bar{x}) + H(\bar{s}, \bar{y}, \frac{\partial \phi}{\partial x}(\bar{t}, \bar{x})) \geq he^{-h\bar{t}} \frac{\|\bar{x} - \bar{y}\|^2}{\varepsilon^2}. \quad (26)$$

Now, according to inequality (7) in Proposition 2.1, it follows that

$$H(\bar{s}, \bar{y}, \frac{\partial \phi}{\partial x}(\bar{t}, \bar{x})) - H(\bar{t}, \bar{y}, \frac{\partial \phi}{\partial x}(\bar{t}, \bar{x})) \leq [c_T(\left\| \frac{\partial \phi}{\partial x}(\bar{t}, \bar{x}) \right\| + k) + q] |\bar{t} - \bar{s}|,$$

and from inequality (8) it follows that

$$H(\bar{t}, \bar{y}, \frac{\partial \phi}{\partial x}(\bar{t}, \bar{x})) - H(\bar{t}, \bar{x}, \frac{\partial \phi}{\partial x}(\bar{t}, \bar{x})) \leq [c_a \left\| \frac{\partial \phi}{\partial x}(\bar{t}, \bar{x}) \right\| + k(1 + c_a)] \|\bar{x} - \bar{y}\|,$$

where a is such that $\|\bar{x} - \bar{y}\| \leq a$.

We obtain

$$\begin{aligned} H(\bar{s}, \bar{y}, \frac{\partial \phi}{\partial x}(\bar{t}, \bar{x})) - H(\bar{t}, \bar{x}, \frac{\partial \phi}{\partial x}(\bar{t}, \bar{x})) &\leq [c_T(\left\| \frac{\partial \phi}{\partial x}(\bar{t}, \bar{x}) \right\| + k) + q] |\bar{t} - \bar{s}| \\ &\quad + [c_a \left\| \frac{\partial \phi}{\partial x}(\bar{t}, \bar{x}) \right\| + k(1 + c_a)] \|\bar{x} - \bar{y}\|. \end{aligned}$$

By adding the term $\frac{\partial \phi}{\partial t}(\bar{t}, \bar{x})$ we obtain

$$\begin{aligned} \frac{\partial \phi}{\partial t}(\bar{t}, \bar{x}) + H(\bar{t}, \bar{x}, \frac{\partial \phi}{\partial x}(\bar{t}, \bar{x})) &\geq \frac{\partial \phi}{\partial t}(\bar{t}, \bar{x}) + H(\bar{s}, \bar{y}, \frac{\partial \phi}{\partial x}(\bar{t}, \bar{x})) - [c_T(\left\| \frac{\partial \phi}{\partial x}(\bar{t}, \bar{x}) \right\| + k) \\ &\quad + q] |\bar{t} - \bar{s}| - [c_a \left\| \frac{\partial \phi}{\partial x}(\bar{t}, \bar{x}) \right\| + k(1 + c_a)] \|\bar{x} - \bar{y}\|. \end{aligned}$$

Since $\left\| \frac{\partial \phi}{\partial x}(\bar{t}, \bar{x}) \right\| = 2e^{-h\bar{t}} \frac{\|\bar{x} - \bar{y}\|}{\varepsilon^2}$ according to (25), it follows from this fact and inequality (26), that

$$\begin{aligned} \frac{\partial \phi}{\partial t}(\bar{t}, \bar{x}) + H(\bar{t}, \bar{x}, \frac{\partial \phi}{\partial x}(\bar{t}, \bar{x})) &\geq he^{-h\bar{t}} \frac{\|\bar{x} - \bar{y}\|^2}{\varepsilon^2} - [c_T(2e^{-h\bar{t}} \frac{\|\bar{x} - \bar{y}\|}{\varepsilon^2} + k) + q] |\bar{t} - \bar{s}| \\ &\quad - [c_a 2e^{-h\bar{t}} \frac{\|\bar{x} - \bar{y}\|}{\varepsilon^2} + k(1 + c_a)] \|\bar{x} - \bar{y}\|. \end{aligned}$$

But we have from (11) that $e^{-h\bar{t}} \frac{\|\bar{x} - \bar{y}\|}{\varepsilon^2} \leq \frac{M}{\varepsilon} e^{-\frac{h}{2}\bar{t}}$ and $|\bar{t} - \bar{s}| \leq M\varepsilon^2$, therefore we obtain

$$\begin{aligned} \frac{\partial \phi}{\partial t}(\bar{t}, \bar{x}) + H(\bar{t}, \bar{x}, \frac{\partial \phi}{\partial x}(\bar{t}, \bar{x})) &\geq (h - 2c_a)e^{-h\bar{t}} \frac{\|\bar{x} - \bar{y}\|^2}{\varepsilon^2} - [2\frac{M}{\varepsilon} e^{-\frac{h}{2}\bar{t}} c_T + kc_T + q] M\varepsilon^2 \\ &\quad - k(1 + c_a) M\varepsilon e^{\frac{h}{2}\bar{t}}. \end{aligned}$$

It suffices to choose h , which is arbitrary, such that $h \geq 2c_a$ to have

$$\begin{aligned} \frac{\partial \phi}{\partial t}(\bar{t}, \bar{x}) + H(\bar{t}, \bar{x}, \frac{\partial \phi}{\partial x}(\bar{t}, \bar{x})) &\geq -2M^2 e^{-\frac{h}{2}\bar{t}} c_T \varepsilon - (kc_T + q) M\varepsilon^2 - k(1 + c_a) M e^{\frac{h}{2}\bar{t}} \varepsilon \\ &\geq -2M^2 \varepsilon c_T - (kc_T + q) M\varepsilon^2 - k(1 + c_a) M e^{\frac{h}{2}T} \varepsilon. \end{aligned}$$

In other words

$$\frac{\partial \phi}{\partial t}(\bar{t}, \bar{x}) + H(\bar{t}, \bar{x}, \frac{\partial \phi}{\partial x}(\bar{t}, \bar{x})) \geq -\mu_\varepsilon,$$

where $\mu_\varepsilon := 2M^2 \varepsilon c_T + (kc_T + q) M\varepsilon^2 + k(1 + c_a) M e^{\frac{h}{2}T} \varepsilon$, and where μ_ε converge to 0 as $\varepsilon \rightarrow 0$.

We deduce that ψ_ε is a viscosity subsolution of the following HJB equation

$$\frac{\partial \phi}{\partial t}(t, x) + H_\varepsilon(t, x, \frac{\partial \phi}{\partial x}(t, x)) \geq 0, \text{ on } O_\varepsilon \times K_\varepsilon. \quad (27)$$

Where $H_\varepsilon(t, x, p) = H(t, x, p) + \mu_\varepsilon$.

Since $\varepsilon \leq \varepsilon_i$ for all $i = 1, \dots, n$, we have, according to first assertion of previous lemma, that, for all $1 \leq i \leq n$

$$O_{\varepsilon_i} \subset O_\varepsilon \text{ and } K_{\varepsilon_i} \subset K_\varepsilon.$$

It follows that

$$\bigcup_{i=1}^n O_{\varepsilon_i} \subset O_\varepsilon \text{ and } \bigcup_{i=1}^n K_{\varepsilon_i} \subset K_\varepsilon.$$

Then, according to assertion three of previous lemma, we have that

$$(0, T) \subset O_\varepsilon \cup ([0, d[\cup]T - d, T]) \text{ and } \text{int}K \subset K_\varepsilon \cup (\text{int}K \cap U(\partial K, d)).$$

In other word,

$$(0, T) \setminus ([0, d[\cup]T - d, T]) \subset O_\varepsilon \text{ and } \text{int}K \setminus (\text{int}K \cap U(\partial K, d)) \subset K_\varepsilon.$$

Hence ψ_ε is a viscosity subsolution of the HJB equation (27) on $(0, T) \setminus ([0, d[\cup]T - d, T]) \times \text{int}K \setminus (\text{int}K \cap U(\partial K, d))$. But d is arbitrary chosen, then by tending d towards 0, we deduce that ψ_ε is a viscosity subsolution of the HJB equation (27) on $(0, T) \times \text{int}K$.

On the other hand, ψ_ε is locally lipschitz, therefore it's almost everywhere differentiable. Hence, we have

$$\frac{\partial \psi_\varepsilon}{\partial t}(t, x) + H(t, x, \frac{\partial \psi_\varepsilon}{\partial x}(t, x)) \geq -\mu_\varepsilon \text{ a.e. } (t, x) \in [0, T] \times \text{int}K. \quad (28)$$

Lemma 4.6. *We have*

$$\frac{\partial \varphi_\varepsilon}{\partial t}(t, x) + H(t, x, \frac{\partial \varphi_\varepsilon}{\partial x}(t, x)) \geq -\sigma_\varepsilon, \quad \forall (t, x) \in [0, T] \times \text{int}K,$$

where φ_ε is defined by relation (12) and $\sigma_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Proof. According to relation (28) we have $\frac{\partial \psi_\varepsilon}{\partial t}(t, x) + \frac{\partial \psi_\varepsilon}{\partial x}(t, x).v + e^{-\delta t}l(x, v) \geq -\mu_\varepsilon$ a.e. $(t, x) \in [0, T] \times \text{int}K, \forall v \in F(t, x)$.

By convoluting both sides of the inequality we obtain

$$\begin{aligned} \frac{\partial \varphi_\varepsilon}{\partial t}(t, x) + \frac{\partial \varphi_\varepsilon}{\partial x}(t, x).v + \iint_{\mathbb{R}^{n+1}} e^{-\delta(t+\tau)} l(x + \xi, v) \rho_\varepsilon(\tau, \xi) d\tau d\xi \geq -\mu_\varepsilon, \\ \text{a.e. } (t, x) \in [0, T] \times \text{int}K, \end{aligned} \quad (29)$$

where φ_ε is defined by relation (12). The convolution is taken on variables (t, x) . Consider

$$l_\varepsilon(t, x, v) := \iint_{\mathbb{R}^{n+1}} e^{-\delta(t+\tau)} l(x + \xi, v) \rho_\varepsilon(\tau, \xi) d\tau d\xi.$$

Then, we have

$$l_\varepsilon(t, x, v) \xrightarrow{\varepsilon \rightarrow 0} e^{-\delta t} l(x, v),$$

uniformly with respect to (t, x) on $[0, T] \times K$ for fixed v .

Therefore inequality (29) becomes

$$\frac{\partial \varphi_\varepsilon}{\partial t}(t, x) + \frac{\partial \varphi_\varepsilon}{\partial x}(t, x).v + l_\varepsilon(t, x, v) \geq -\mu_\varepsilon \quad \text{a.e. } (t, x) \in [0, T] \times \text{int}K, \forall v \in F(t, x).$$

By adding and subtracting the term $e^{-\delta t}l(x, v)$ we have

$$\frac{\partial \varphi_\varepsilon}{\partial t}(t, x) + \frac{\partial \varphi_\varepsilon}{\partial x}(t, x).v + e^{-\delta t}l(x, v) + l_\varepsilon(t, x, v) - e^{-\delta t}l(x, v) \geq -\mu_\varepsilon,$$

$$\text{a.e. } (t, x) \in [0, T] \times \text{int}K, \forall v \in F(t, x).$$

It follows that

$$\frac{\partial \varphi_\varepsilon}{\partial t}(t, x) + \frac{\partial \varphi_\varepsilon}{\partial x}(t, x).v + e^{-\delta t}l(x, v) + \sup_{(t, x) \in [0, T] \times K} \left\{ \sup_{v \in F(t, x)} (l_\varepsilon(t, x, v) - e^{-\delta t}l(x, v)) \right\} \geq -\mu_\varepsilon,$$

$$\text{a.e. } (t, x) \in [0, T] \times \text{int}K, \forall v \in F(t, x).$$

Then we have

$$\frac{\partial \varphi_\varepsilon}{\partial t}(t, x) + \frac{\partial \varphi_\varepsilon}{\partial x}(t, x).v + e^{-\delta t}l(x, v) \geq -\mu_\varepsilon - \gamma_\varepsilon,$$

$$\text{a.e. } (t, x) \in [0, T] \times \text{int}K, \forall v \in F(t, x),$$

where $\gamma_\varepsilon := \sup_{(t, x) \in [0, T] \times K} \left\{ \sup_{v \in F(t, x)} (l_\varepsilon(t, x, v) - e^{-\delta t}l(x, v)) \right\}$. The sequence $\gamma_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$

because the supremum on v is achieved since l_ε and l are continuous with respect v and $F(t, x)$ is compact, and because $l_\varepsilon(., ., v) \rightarrow e^{-\delta t}l(., v)$ as $\varepsilon \rightarrow 0$ uniformly on compact sets.

We deduce that

$$\frac{\partial \varphi_\varepsilon}{\partial t}(t, x) + \frac{\partial \varphi_\varepsilon}{\partial x}(t, x).v + e^{-\delta t}l(x, v) \geq -\sigma_\varepsilon \quad \text{a.e. } (t, x) \in [0, T] \times \text{int}K, \forall v \in F(t, x),$$

where $\sigma_\varepsilon = \mu_\varepsilon + \gamma_\varepsilon$ and $\sigma_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$.

This can be written as

$$\frac{\partial \varphi_\varepsilon}{\partial t}(t, x) + H(t, x, \frac{\partial \varphi_\varepsilon}{\partial x}(t, x)) \geq -\sigma_\varepsilon \quad \text{a.e. } (t, x) \in [0, T] \times \text{int}K.$$

On other hand, since the Hamiltonian H is locally lipschitz and φ_ε is continuously differentiable, the above inequality remains true for all elements of $[0, T] \times \text{int}K$.

This completes the proof of the lemma. $\square$

We turn now to the proof of Theorem 4.1.

According to Theorem (3.3) we have

$$\alpha = \inf(P) \geq \beta_\varepsilon = \int_0^T \inf_{x \in K} \tilde{H}(t, x, \frac{\partial \varphi_\varepsilon}{\partial x}(t, x), \frac{\partial \varphi_\varepsilon}{\partial t}(t, x)) dt + \delta_\varepsilon + \varphi_\varepsilon(0, x_0).$$

But according to Lemma (4.6) we have

$$\begin{aligned} \tilde{H}(t, x, \frac{\partial \varphi_\varepsilon}{\partial t}(t, x), \frac{\partial \varphi_\varepsilon}{\partial x}(t, x)) &:= \frac{\partial \varphi_\varepsilon}{\partial t}(t, x) + H(t, x, \frac{\partial \varphi_\varepsilon}{\partial x}(t, x)) \geq -\sigma_\varepsilon, \\ \forall (t, x) &\in [0, T] \times \text{int}K. \end{aligned}$$

On other hand, since the compact set K is such that $K = \text{cl}(\text{int}(K))$ and $\tilde{H}$ is locally lipschitz and φ_ε is continuously differentiable we have

$$\inf_{x \in \text{int}K} \tilde{H}(t, x, \frac{\partial \varphi_\varepsilon}{\partial t}(t, x), \frac{\partial \varphi_\varepsilon}{\partial x}(t, x)) = \inf_{x \in K} \tilde{H}(t, x, \frac{\partial \varphi_\varepsilon}{\partial t}(t, x), \frac{\partial \varphi_\varepsilon}{\partial x}(t, x)).$$

Hence,

$$\alpha = \inf(P) \geq \int_0^T -\sigma_\varepsilon dt + \delta_\varepsilon + \varphi_\varepsilon(0, x_0).$$

So,

$$\alpha \geq -T\sigma_\varepsilon + \delta_\varepsilon + \varphi_\varepsilon(0, x_0).$$

By letting ε tend towards 0, we obtain

$$\alpha \geq \varphi(0, x_0),$$

this for all φ viscosity subsolution of the HJB equation and satisfying

$$\varphi(T, \cdot) \leq g(\cdot) \quad \text{on } K.$$

Therefore

$$\alpha \geq \sup_{\varphi} \varphi(0, x_0),$$

where φ is any function admissible for the dual problem (D).

This completes the proof of the theorem. □

4.4. Strong Duality

Assume that the following additional hypothesis is satisfied.

H_6 — The restriction of $l(x, \cdot)$ on $F(t, x)$ is convex for all (t, x) fixed.

Under this additional hypothesis the primal problem (P) has a minimizer. Indeed ;

Proposition 4.7. *Under the hypotheses $H_1 - H_6$ the infimum of the problem (P) is achieved and the value function V is l.s.c.*

The proof is based on a classic arguments, see [13] (see also [4] and [6]).

We now prove that under the additional hypothesis H_6 , no duality gaps occur.

Theorem 4.8. *Under the hypotheses $H_1 - H_6$, we have*

$$\alpha = \beta. \tag{30}$$

Proof. The Theorem (4.1) gives the first inequality $\alpha \geq \beta$.

Conversely to prove the inverse inequality it suffices to prove that the hypotheses of Vinter's theorem [Th. 2.1, [26]] are satisfied, in this case we have that

$$\alpha = \sup \psi(0, x_0) \tag{31}$$

where the supremum is taken over the functions $\psi \in C^1(\mathbb{R}^{n+1})$ which are smooth subsolutions of HJB equation. But each $\psi \in C^1(\mathbb{R}^{n+1})$ which is smooth subsolution of HJB equation is in particular a viscosity subsolution on $[0, T] \times K$.

Then

$$\beta = \sup \varphi(0, x_0) \geq \sup \psi(0, x_0)$$

where the supremum on the left is taken over the functions $\varphi \in C([0, T] \times K)$ which are viscosity subsolutions of HJB equation and the supremum on the right is taken over the functions $\psi \in C^1(\mathbb{R}^{n+1})$ which are smooth subsolutions of HJB equation.

So, it follows from the relation (31) that

$$\beta \geq \alpha.$$

Then, we conclude that

$$\beta = \alpha.$$

Let us now verify the hypotheses of Vinter's theorem.

To be in the context of Vinter [26], it suffices to have $H_1 - H_3$, H_5 and H_6 and we must prove that the set

$$A := \{(v, (t, x)) \in \mathbb{R}^n \times ([0, T] \times \mathbb{R}^n) : v \in F(t, x), (t, x) \in [0, T] \times K\}$$

is compact which is the consequence of the hypotheses $H_3 - H_4$, indeed;

Let $(v, (t, x)) \in A$, we have

$$\|(v, (t, x))\| \leq \|v\| + T + \|x\|.$$

But it follows from the hypothesis H_3 that

$$\|v\| \leq \gamma \|x\| + c.$$

So,

$$\|(v, (t, x))\| \leq (\gamma + 1) \sup_K \|x\| + c + T.$$

It follows that $\|(v, (t, x))\|$ is bounded.

On the other hand, the set A is closed since $\text{graph} F$ is closed (because F is locally lipschitz, hypothesis H_4 , with compact values, see [4]) and K is compact. $\square$

Obviously the necessary and sufficient conditions of the theorem 4.3 remain true. We now prove that, with the strong duality, the sufficient conditions of the Corollary 4.4 become necessary and sufficient conditions which extend the Vinter's theorem [Th. 2.2, [26]] of necessary and sufficient conditions.

Theorem 4.9. *Under the hypotheses $H_1 - H_6$.*

Let $\hat{x}$ be an admissible arc for (P) , then

$\hat{x}$ is a minimum for (P) iff there exists a sequence of continuous functions $(\varphi_n)_n$ viscosity subsolutions of HJB equation on $[0, T] \times K$ and satisfying

$$\varphi_n(T, \cdot) \leq g(\cdot) \text{ on } K, \tag{32}$$

$$\lim_{n \rightarrow +\infty} \varphi_n(0, x_0) = J(\hat{x}). \quad (33)$$

Proof. The Corollary (4.4) guarantees that the conditions of the theorem are sufficient. We now prove that these conditions are necessary.

Suppose that $\hat{x}$ is a minimum for (P).

Then it follows from the Theorem (4.3) that there exist a sequence of continuous functions $\{\varphi_n\}_n$ viscosity subsolutions of HJB equation on $[0, T] \times K$ satisfying (32) as well as the following inequality

$$\begin{aligned} \lim_{n \rightarrow +\infty} |J(\hat{x}) - \varphi_n(0, x_0)| &\leq |J(x) - \varphi(0, x_0)|, \forall \varphi \in \text{adm}(D), \\ &\forall x \in \text{adm}(P). \end{aligned}$$

It implies that

$$\lim_{n \rightarrow +\infty} \varphi_n(0, x_0) \geq \varphi(0, x_0), \forall \varphi \in \text{adm}(D).$$

Then

$$\lim_{n \rightarrow +\infty} \varphi_n(0, x_0) \geq \sup_{\varphi \in \text{adm}(D)} \varphi(0, x_0) = \beta.$$

But $\alpha = \beta$, then

$$\lim_{n \rightarrow +\infty} \varphi_n(0, x_0) \geq \alpha.$$

On the other hand, we have $\forall n \varphi_n(0, x_0) \leq \alpha$, then

$$\lim_{n \rightarrow +\infty} \varphi_n(0, x_0) \leq \alpha \leq \lim_{n \rightarrow +\infty} \varphi_n(0, x_0).$$

Hence

$$\lim_{n \rightarrow +\infty} \varphi_n(0, x_0) = \alpha = J(\hat{x}).$$

This completes the proof. □

4.5. Example

Consider the following problem

$$(P_e) \quad \begin{cases} \inf_x \sqrt{|x(2)|}, \\ \dot{x}(t) \in [-x(t), x(t)] \quad t \in [0, 2], \\ x(0) = 0, \\ -1 \leq x(t) \leq 1. \end{cases}$$

$g(x) = \sqrt{|x|}$, $l \equiv 0$, The multifunction F is autonomous and given by $F(x) = [-x, x]$.

It's easy to prove that $\hat{x}(t) \equiv 0$ is an optimal solution, but our object here is to show that the duality may in some cases confirm the optimality of a suspected arc.

The function g is quasiconvex. We will use the results of Barron [10] to prove that the continuous viscosity solution of the following HJB equation

$$\begin{cases} \frac{\partial \varphi}{\partial t}(t, x) + H(\frac{\partial \varphi}{\partial x}(t, x)) = 0, \\ \varphi(T, \cdot) = g(\cdot) \text{ on } [-1, 1], \end{cases} \quad (34)$$

where

$$H(p) := \min_{-1 \leq x \leq 1} \langle x, p \rangle, \quad (35)$$

$$H(p) = -|p|,$$

is given by

$$\varphi(t, x) = \inf\{\gamma \in R : \sup_{p \in R} \{px - g^\sharp(\gamma, p) + (2-t)H(p)\} \leq 0\}, \quad \forall (t, x) \in [0, 2] \times [-1, 1], \quad (36)$$

where $g^\sharp$ denote the quasiconvex conjugate function of g defined by

$$g^\sharp(\gamma, p) := \sup\{p \cdot y : y \in \mathbb{R}^n \text{ and } g(y) \leq \gamma\}.$$

First, we note that H given by (35) satisfies the following hypotheses of Barron [10].

Lemma 4.10. *We have*

- (i) $H(\lambda p) = \lambda H(p)$ for all $\lambda \geq 0$.
- (ii) $|H(p) - H(p')| \leq k_H |p - p'|$.

Proof. The proof is obvious.

Hence, according to Barron [10] we prove that φ given by the relation (36) is a continuous viscosity solution of the equation (34).

Let us now compute $\varphi(t, x)$:

We have

$$g^\sharp(\gamma, p) = \begin{cases} |p| \gamma^2 & \text{if } \gamma \geq 0, \\ -\infty & \text{otherwise.} \end{cases}$$

Therefore

$$\begin{aligned} \varphi(t, x) &= \inf\{\gamma \in R : \sup_{p \in R} \{px - g^\sharp(\gamma, p) + (2-t)H(p)\} \leq 0\} \\ &= \inf\{\gamma \geq 0 : \sup_{p \in R} \{px - |p| \gamma^2 + (2-t)H(p)\} \leq 0\}. \end{aligned}$$

By using the expression of H we obtain the expression

$$\sup_{p \in R} \{px - |p| \gamma^2 + (2-t)H(p)\} = \begin{cases} 0 & \text{if } \gamma^2 \geq x - (2-t), \\ +\infty & \text{if } \gamma^2 \leq x - (2-t). \end{cases}$$

So,

$$\begin{aligned} \varphi(t, x) &= \inf\{\gamma \geq 0 : \gamma^2 \geq x - (2-t)\}, \quad \forall (t, x) \in [0, 2] \times [-1, 1] \\ &= \begin{cases} \sqrt{x - (2-t)} & \text{if } x \geq (2-t), \\ 0 & \text{if } x \leq (2-t). \end{cases} \end{aligned}$$

For $t = 0$ and $x = 0$, we have

$$\varphi(0, 0) = 0.$$

According to Corollary 4.2 we have

$$|\alpha - J(\hat{x})| = |\alpha| \leq 0 - \varphi(0, 0) = 0.$$

We deduce that $\hat{x}(t) = 0$ is an optimal trajectory.

5. Conclusion

In this work our attention was focused on nonconvex duality and we have established a weak duality without convexity assumptions in the cost function, together with necessary and sufficient conditions for optimality and an error estimate for α . On the other hand we established strong duality under mild additional convexity assumption.

Our next work will be devoted to studying the numerical aspects and computing the error estimate, and we hope apply our study to certain problems arising in economic applications.

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