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An Extension of the Serrin's Lower Semicontinuity Theorem

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In this paper we present a new extension of the celebrated Serrin's lower semicontinuity theorem. We consider an integral of the calculus of variation $\int_\Omega f(x,u,Du)\,dx$ and we prove its lower semicontinuity in $W^{1,1}_{loc}(\Omega)$ with respect to the strong L^1_{loc} norm topology, under the usual *continuity* and *convexity* property of the integrand $f(x,s,\xi)$, only assuming a mild (more precisely, *local*) condition on the independent variable $x \in \mathbb{R}^n$, say *local Lipschitz continuity*, which - we show with a specific counterexample - cannot be replaced, in general, by local *Hölder continuity*.

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1. Introduction

The aim of this paper is to determinate some new sufficient conditions for lower semicontinuity with respect to the strong convergence in L^1_{loc} for functionals of integral type

$$F(u,\Omega) = \int_\Omega f(x,u(x),Du(x))\,dx\,, \tag{1}$$

where Ω is an open set of $\mathbb{R}^n$, u varies in the Sobolev class $W^{1,1}_{loc}(\Omega)$, Du denotes the gradient of u, and the function $f = f(x,s,\xi)$, for $x \in \Omega$, $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, satisfies the conditions

$$\begin{cases} f \textit{ is continuous in } \Omega \times \mathbb{R} \times \mathbb{R}^n, \\ f \textit{ is nonnegative in } \Omega \times \mathbb{R} \times \mathbb{R}^n, \\ f(x,s,\xi) \textit{ is convex in } \xi \in \mathbb{R}^n \textit{ for all } (x,s) \in \Omega \times \mathbb{R}. \end{cases} \tag{2}$$

The integral functional F is *lower semicontinuous in* $W^{1,1}_{loc}(\Omega)$ *with respect to the strong convergence in* L^1_{loc} if, for every $u_h, u \in W^{1,1}_{loc}(\Omega)$ such that $u_h \to u$ in $L^1_{loc}(\Omega)$, then

$$\liminf_{h\to+\infty} F(u_h,\Omega) \geq F(u,\Omega)\,.$$

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Since the example given in 1941 by Aronszajn (see Pauc [15]; in particular page 54), it is known that condition (2) alone is not sufficient for strong lower semicontinuity of the integral F in (1). Serrin published in 1961 an article [16] proposing, in addition to (2), some *sufficient conditions* for strong lower semicontinuity. One of the most known and celebrated Serrin's theorem on this subject is the following one (see Theorem 12 in [16]).

Theorem 1.1 (Serrin). *Let f satisfy, in addition to* (2)*, one of the following conditions:*

(a) $f(x,s,\xi) \to +\infty$ *when* $|\xi| \to +\infty$, *for all* $(x,s) \in \Omega \times \mathbb{R}$;
(b) $f(x,s,\xi)$ *is strictly convex in* $\xi \in \mathbb{R}^n$ *for all* $(x,s) \in \Omega \times \mathbb{R}$;
(c) *the derivatives* $f_x(x,s,\xi)$, $f_\xi(x,s,\xi)$ *and* $f_{\xi x}(x,s,\xi)$ *exist and are continuous.*

Then $F(u,\Omega)$ *is lower semicontinuous in* $W^{1,1}_{loc}(\Omega)$ *with respect to the strong convergence in* L^1_{loc}.

Many attempts have been made to weaken the assumptions on the integrand f. Serrin himself gave in 1961 the following further result (see Theorem 11 in [16]).

Theorem 1.2 (Serrin). *Let us assume that f satisfies* (2) *and the following (uniform) continuity condition*

$$|f(x_1,s_1,\xi) - f(x_2,s_2,\xi)| \leq \lambda\left(|x_1 - x_2| + |s_1 - s_2|\right) \cdot \{1 + f(x_1,s_1,\xi)\}, \tag{3}$$

for every $(x_1,s_1),(x_2,s_2) \in \Omega \times \mathbb{R}$ *and for all* $\xi \in \mathbb{R}^n$, *where* λ *is a modulus of continuity. Then*

$$\liminf_{h \to +\infty} F(u_h,\Omega) \geq F(u,\Omega),$$

for every $u_h, u \in W^{1,1}_{loc}(\Omega)$ *such that* $u_h \to u$ *in* $L^1_{loc}(\Omega)$, *assuming in addition that* $u \in C^0(\Omega)$.

The aims of some further studies in the direction of Theorem 1.2 tempt to remove the assumption of continuity of u and to weaken the uniform continuity condition (3) on f. Dal Maso [3] in 1980 gave the following lower semicontinuity result, without continuity assumptions on the limit function u (and in fact Dal Maso was able to extend his analysis to $u \in BV(\Omega)$, the functional space of functions of class $L^1(\Omega)$ with *bounded variation*, also considering more generally sequences of integral functionals which Γ-*converge*). However Dal Maso had to introduce some *coercivity* and *growth conditions*, as follows.

Theorem 1.3 (Dal Maso). *Let us assume that f satisfies* (2), (3) *and that there exist functions* $m, q \in C^0(\Omega)$, *with* $m(x) > 0$ *for every* $x \in \Omega$, *and a positive constant M such that*

$$m(x)\,|\xi| \leq f(x,s,\xi) \leq M\,|\xi| + q(x),$$

for every $(x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$. *Then* $F(u,\Omega)$ *is lower semicontinuous in* $W^{1,1}_{loc}(\Omega)$ *with respect to the strong convergence in* L^1_{loc}.

Let us also mention that Dal Maso himself, revisiting the already quoted example by Aronszajn [15], emphasized that the continuity of f with respect to (x,u) alone is not sufficient for lower semicontinuity of $F(u,\Omega)$ in L^1. See Section 4 of this paper for further details.

A recent extension of Theorem 1.2 is due to Fonseca and Leoni (see Theorem 1.1 in [10], where the case $u \in BV(\Omega)$ is considered too; see also [11]).

Theorem 1.4 (Fonseca - Leoni). *Let $f(x,s,\xi)$ be a Borel function, convex with respect to $\xi \in \mathbb{R}^n$. Let us also assume that, for every $(x_0,s_0) \in \Omega \times \mathbb{R}$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$f(x_0,s_0,\xi) - f(x,s,\xi) \le \varepsilon \{1 + f(x,s,\xi)\}, \tag{4}$$

for every $(x,s) \in \Omega \times \mathbb{R}$ such that $|x-x_0|+|s-s_0| \le \delta$ and for all $\xi \in \mathbb{R}^n$. Then $F(u,\Omega)$ is lower semicontinuous in $W^{1,1}_{loc}(\Omega)$ with respect to the strong convergence in L^1_{loc}.

Notice that assumption (4) is a kind of lower semicontinuity of f with respect to $(x,s) \in \Omega \times \mathbb{R}$, *uniform* with respect to $\xi \in \mathbb{R}^n$. *Lower semicontinuity* of integrands of the type $f(x,\xi)$ *with respect to* $x \in \Omega$ has been pointed out by Fusco [12] in 1979, as a necessary condition for lower semicontinuity of the respective (one-dimensional) integrals with respect to the strong convergence in $L^1(\Omega)$, on discussing the case of *linear growth*, with $f(x,\xi) = a(x)\,|\xi|$.

As already said, Theorems 1.3 and 1.4 has been obtained, respectively by Dal Maso and by Fonseca and Leoni, in the more general setting of $BV(\Omega)$, the subspace of $L^1(\Omega)$ whose functions have *bounded variation*. We quoted above the particular case when $u_h, u \in W^{1,1}_{loc}(\Omega)$, for a better comparison with the other results presented in this paper.

Some researches, as in Theorem 1.4, had the aim to relax the assumptions on $f(x,s,\xi)$ related to the dependence on s, starting from a result in [7] by De Giorgi, Buttazzo and Dal Maso in 1983.

Theorem 1.5 (De Giorgi - Buttazzo - Dal Maso). *Let $f = f(s,\xi)$ be a nonnegative Borel function, convex with respect to $\xi \in \mathbb{R}^n$, only measurable with respect to $s \in \mathbb{R}$, although lower semicontinuous with respect to $s \in \mathbb{R}$ at $\xi = 0$. If*

$$\limsup_{|\xi|\to 0} \frac{(f(s,0) - f(s,\xi))^+}{|\xi|} \in L^1_{loc}(\mathbb{R}),$$

then $F(u,\Omega)$ is lower semicontinuous in $W^{1,1}_{loc}(\Omega)$ with respect to the strong convergence in L^1_{loc}.

Theorem 1.5 has been generalized in 1987 by Ambrosio [2], and in 1990-91 by De Cicco [4], [5] to the $BV(\Omega)$ setting. In [10] Fonseca and Leoni obtained the same conclusion of Theorem 1.5 for integrands $f(x,s,\xi)$, depending explicitly on the x variable too, under the assumption of *uniform continuity* of f with respect to $x \in \Omega$, similarly to (4).

From the above exposition it should be clear that the dependence of f on (x,s) must be treated carefully in studying lower semicontinuity of the integral $F(u,\Omega)$ with respect to the strong convergence in $L^1(\Omega)$. Of course (x,s) dependence gives some difficulties in the proofs, which are not only technical difficulties, since the existence of a counterexample to lower semicontinuity under explicit (x,s) dependence of the integrand f. In particular *measurability* of $f(x,s,\xi)$ with respect to (x,s), or only with respect to x, is not appropriate for strong lower semicontinuity. As already mentioned, Fusco [12] gave a one-dimensional example, related to the integrand $f(x,\xi) = a(x)\,|\xi|$, where lower semicontinuity with respect to x was a necessary condition; in this context we refer also to the example in [14], related to an integrand of the type $f(x,\xi) = a(x)\,|\xi|^2$, which may have a *relaxed functional* in the strong L^1 norm topology identical equal to zero, although the coefficient $a(x)$ is a nonnegative function, not identical equal to zero (thus the corresponding integral $F(u,\Omega)$ is not lower semicontinuous). In Section 5 of this paper we will give further details, as well as we will propose a coercivity condition also sufficient for lower semicontinuity in L^1_{loc}.

In this paper we consider specifically the dependence of $f(x,s,\xi)$ with respect to $x \in \Omega$. In the previous results some qualified assumptions of uniform continuity, or of uniform lower semicontinuity, of $f(x,s,\xi)$ with respect to x have been made (in the sense made more precise in the statements). On the contrary, in this paper we propose a new simple condition, in addition to (2), sufficient for lower semicontinuity. In fact we assume that $f(x,s,\xi)$ is Lipschitz continuous with respect to x, *locally* respect to (s,ξ) and not necessarily *globally*. That is, we do not assume that the Lipschitz constant is uniform for $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$. This main difference allows us to obtain, as a corollary, an improvement of Serrin's Theorem 1.1(c); in fact we get the lower semicontinuity of $F(u,\Omega)$ under the only assumption that the derivative $f_x(x,s,\xi)$ exists and is continuous, condition which clearly implies Lipschitz continuity of f on *compact subsets* of $\Omega \times \mathbb{R} \times \mathbb{R}^n$, but not necessarily Lipschitz continuity of f on the *full set* $\Omega \times \mathbb{R} \times \mathbb{R}^n$.

One of the main results of this paper (an other is Example 4.2) is the following Theorem 1.6. It has as a direct consequence Corollary 1.7, which of course generalizes Serrin's Theorem 1.1(c).

Theorem 1.6. *Assume that $f(x,s,\xi)$ satisfies* (2) *and that, for every compact set $K \subset \Omega \times \mathbb{R} \times \mathbb{R}^n$, there exists a constant $L = L(K)$ such that*

$$|f(x_1,s,\xi) - f(x_2,s,\xi)| \leq L\,|x_1 - x_2|\,, \tag{5}$$

for every $(x_1,s,\xi), (x_2,s,\xi) \in K$. Then the integral $F(u,\Omega)$ in (1) *is lower semicontinuous in $W^{1,1}_{loc}(\Omega)$ with respect to the strong convergence in $L^1_{loc}(\Omega)$.*

The proof of Theorem 1.6 is divided in several steps. In one of these steps we use an approximation procedure, due to De Giorgi [6], of the integrand f by a sequence f_j, each f_j being the maximum of a finite number of functions $g(x,s,\xi) = a_0(x,s) + \sum_{i=1}^n a_i(x,s)\xi_i$, linear with respect to ξ, with coefficients $a_0(x,s)$ and $a_i(x,s)$ $(i = 1,2,\ldots,n)$ as in (18), which can be reduced to have compact support in $\Omega \times \mathbb{R}$. By this reason it is enough to assume *local* Lipschitz continuity of f with respect to x as in (5), and not necessarily global Lipschitz continuity.

Direct consequence of the Theorem 1.6 is the following result.

Corollary 1.7. *Assume that $f(x,s,\xi)$ satisfies* (2) *and that the derivative $f_x(x,s,\xi)$ exists and is continuous (or only locally bounded). Then $F(u,\Omega)$ is lower semicontinuous in $W^{1,1}_{loc}(\Omega)$ with respect to the strong convergence in $L^1_{loc}(\Omega)$.*

In Sections 2 and 3 we will give the proof of Theorem 1.6. In Section 4 we will exhibit some examples showing that assumptions made in Theorem 1.6 (and in Corollary 1.7) are relevant for strong lower semicontinuity, in the sense that the only property of continuity of $f(x,s,\xi)$ with respect to $x \in \Omega$ (of course, together with the other conditions in (2)) is not enough for strong lower semicontinuity of $F(u,\Omega)$ in L^1_{loc}, although sufficient for weak lower semicontinuity in $W^{1,1}_{loc}(\Omega)$.

More precisely, in Example 4.2 we will compare the assumptions of local *Lipschitz continuity* of f with respect to x with the assumption of local *Hölder continuity* of f (with respect to x) with exponent $\alpha < 1$. In fact we will show that, for every exponent $\alpha \in (0,1)$, there exists an n-dimensional integral $F(u,\Omega)$ (the dimension n depends on α, precisely $n > 4\alpha/(1-\alpha)$) which is not lower semicontinuous in L^1_{loc} and whose integrand $f(x,s,\xi)$ is Hölder continuous with respect to x (and of course nonnegative, continuous for $(x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ and convex with respect to $\xi \in \mathbb{R}^n$).

In Section 5 we will show that lower semicontinuity results, corresponding to those of Theorem 1.6 and Corollary 1.7, do not hold in the vector-valued case, for applications $u \in W^{1,1}_{loc}(\Omega,\mathbb{R}^m)$, i.e., $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ with $m > 1$, not just for *quasiconvex* integrands, but under *convexity* conditions too. Finally, having in mind some *relaxation formulas* due to Marcellini [14] and Fusco [12], we give in Proposition 5.6 a *coercivity condition* sufficient for lower semicontinuity in L^1_{loc} of the integral $F(u,\Omega)$ in (1).

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2. A preliminary lemma

In this section we give some preliminary results that will be used in the proof of Theorem 1.6. The first lemma is a modification of an argument introduced by Serrin (see the proof of Theorem 12 in [16]).

Lemma 2.1. *Let us assume that f satisfies* (2) *and that the derivative $f_\xi(x,s,\xi)$ exists and is a continuous function in $\Omega \times \mathbb{R} \times \mathbb{R}^n$. Let us also assume that, for every compact set $K \subset \Omega \times \mathbb{R} \times \mathbb{R}^n$, there exists a constant $L = L(K)$ such that*

$$|f_\xi(x_1,s,\xi) - f_\xi(x_2,s,\xi)| \leq L\,|x_1 - x_2|\,, \quad \forall\ (x_1,s,\xi)\,,(x_2,s,\xi) \in K, \tag{6}$$

and, for every compact set $K_1 \subset \Omega \times \mathbb{R}$, there exists a constant $L_1 = L_1(K_1)$ such that

$$\begin{cases} |f_\xi(x,s,\xi)| \leq L_1, \ \ \forall\ (x,s) \in K_1, \ \ \forall\, \xi \in \mathbb{R}^n, \\ \\ |f_\xi(x,s,\xi_1) - f_\xi(x,s,\xi_2)| \leq L_1\,|\xi_1 - \xi_2|\,, \ \ \forall\ (x,s) \in K_1, \ \ \forall\, \xi_1,\xi_2 \in \mathbb{R}^n. \end{cases} \tag{7}$$

Then $F(u,\Omega)$ is lower semicontinuous in $W^{1,1}_{loc}(\Omega)$ with respect to the strong convergence in L^1_{loc}.

Proof. Let us consider a sequence $f_i(x,s,\xi) = \alpha_i(x,s)f(x,s,\xi)$, $i = 1,2,\ldots$, where $\{\alpha_i\}_{i\in\mathbb{N}}$ is an increasing sequence of smooth functions with compact support in $\Omega \times \mathbb{R}$, converging pointwise to 1 in $\Omega \times \mathbb{R}$. It is clear that, for all $i \in \mathbb{N}$, f_i still satisfies the hypothesis of Lemma 2.1 and also vanishes if (x,s) vary outside a compact set of $\Omega \times \mathbb{R}$. Moreover f_i is an increasing sequence of functions which pointwise converge to f. Thus, by the monotone convergence theorem we can go to the limit as $i \to +\infty$ and, since the supremum of a family of lower semicontinuous functionals is lower semicontinuous, it is sufficient to prove the stated lower semicontinuity for the integral functional associated to a generic integrand $\alpha_i f$. In other words, in the proof of Lemma 2.1, without loss of generality, we can assume that the integrand $f(x,s,\xi)$ vanishes if (x,s) vary outside a compact set of $\Omega\times\mathbb{R}$. For the same reason in the exposition below we will always assume that (x,s) vary on a compact set of $\Omega\times\mathbb{R}$ and that f is equal to zero in the complement of this compact set.

Let $u_h, u \in W^{1,1}_{loc}(\Omega)$ such that $u_h \to u$ in $L^1_{loc}(\Omega)$. We will prove that

$$\liminf_{h\to+\infty} F(u_h,\Omega) \geq F(u,\Omega). \tag{8}$$

Without loss of generality, we can assume that u_h converges to u almost everywhere in Ω and that

$$\liminf_{h\to+\infty} F(u_h,\Omega) = \lim_{h\to+\infty} F(u_h,\Omega).$$

Let Ω' be an open set whose closure is contained in Ω, such that $f(x,s,\xi)=0$ when $x\in\Omega-\Omega'$; thus in particular $F(u_h,\Omega)=F(u_h,\Omega')$ and $F(u,\Omega)=F(u,\Omega')$. Let us denote by α_ρ a mollifier and by $u_\rho = u*\alpha_\rho$ the mollified function of u with step ρ. Since $u\in W^{1,1}_{loc}(\Omega)$, for every $\varepsilon>0$ there exists $\rho>0$ such that

$$\int_{\Omega'} |Du - Du_\rho|\,dx \leq \varepsilon. \tag{9}$$

By Fatou lemma we can choose ρ small enough such that u_ρ satisfies also the condition

$$\int_{\Omega'} f(x,u,Du_\rho)\,dx \geq \int_{\Omega'} f(x,u,Du)\,dx - \varepsilon. \tag{10}$$

We estimate the difference of the integrands in (8)

$$\begin{aligned} f(x,u_h,Du_h) - f(x,u,Du) &= f(x,u_h,Du_h) - f(x,u_h,Du_\rho) \\ &+ f(x,u_h,Du_\rho) - f(x,u,Du_\rho) + f(x,u,Du_\rho) - f(x,u,Du). \end{aligned} \tag{11}$$

From (11), by the convexity of $f(x,s,\xi)$ with respect to ξ, we have

$$\begin{gathered} f(x,u_h,Du_h) - f(x,u,Du) \geq (f_\xi(x,u_h,Du_\rho), Du_h - Du_\rho) \\ + \{f(x,u_h,Du_\rho) - f(x,u,Du_\rho)\} + \{f(x,u,Du_\rho) - f(x,u,Du)\} \\ = (f_\xi(x,u_h,Du_\rho), Du_h) - (f_\xi(x,u,Du_\rho), Du) \\ + (f_\xi(x,u,Du_\rho), Du - Du_\rho) + (f_\xi(x,u,Du_\rho) - f_\xi(x,u_h,Du_\rho), Du_\rho) \\ + \{f(x,u_h,Du_\rho) - f(x,u,Du_\rho)\} + \{f(x,u,Du_\rho) - f(x,u,Du)\}. \end{gathered}$$

We observe that $(x,s)\to f_\xi(x,s,Du_\rho(x))$ is a continuous function with compact support in $\Omega\times\mathbb{R}$ and that, by (7), $|f_\xi(x,s,Du_\rho(x))| \leq L_1$ for every $(x,s)\in\Omega\times\mathbb{R}$ and for every ρ. We obtain

$$f(x,u_h,Du_h) - f(x,u,Du) \geq (f_\xi(x,u_h,Du_\rho), Du_h) - (f_\xi(x,u,Du_\rho), Du)$$

$$-L_1 |Du - Du_\rho| + (f_\xi(x,u,Du_\rho) - f_\xi(x,u_h,Du_\rho), Du_\rho)$$
$$+ \{f(x,u_h,Du_\rho) - f(x,u,Du_\rho)\} + \{f(x,u,Du_\rho) - f(x,u,Du)\} \,.$$

We integrate both sides over Ω'. By (9) and (10) we have

$$\begin{aligned} &\int_{\Omega'} \{f(x,u_h,Du_h) - f(x,u,Du)\}\, dx \\ \geq\ &\int_{\Omega'} \{(f_\xi(x,u_h,Du_\rho), Du_h) - (f_\xi(x,u,Du_\rho), Du)\}\, dx \\ &+ \int_{\Omega'} (f_\xi(x,u,Du_\rho) - f_\xi(x,u_h,Du_\rho), Du_\rho)\, dx \\ &+ \int_{\Omega'} \{f(x,u_h,Du_\rho) - f(x,u,Du_\rho)\}\, dx \; - (1+L_1)\,\varepsilon\,. \end{aligned} \tag{12}$$

We can go to the limit as $h \to +\infty$. First we observe that, since $(x,s) \to f(x,s,Du_\rho(x))$ and $(x,s) \to f_\xi(x,s,Du_\rho(x))$ are bounded functions (in fact are continuous function with compact support), by the Lebesgue's dominated convergence theorem we obtain

$$\begin{cases} \lim_{h\to+\infty} \int_{\Omega'} (f_\xi(x,u,Du_\rho) - f_\xi(x,u_h,Du_\rho), Du_\rho)\, dx \, = 0\,, \\ \\ \lim_{h\to+\infty} \int_{\Omega'} \{f(x,u_h,Du_\rho) - f(x,u,Du_\rho)\}\, dx \, = 0\,. \end{cases}$$

Since ε in (12) can be arbitrarily small, to obtain the conclusion (8) it remains to prove that

$$\lim_{h\to+\infty} \int_{\Omega'} \{(f_\xi(x,u_h,Du_\rho), Du_h) - (f_\xi(x,u,Du_\rho), Du)\}\, dx \, = 0\,. \tag{13}$$

With the aim to prove (13), similarly to Tonelli [17], we denote by $g(x,s)$ the continuous vector-valued function with compact support ($g : \Omega \times \mathbb{R} \to \mathbb{R}^n$) defined by $g(x,s) = f_\xi(x,s,Du_\rho(x))$, i.e., more precisely,

$$g(x,s) = \left(g^{(i)}(x,s)\right)_{i=1}^n = \left(f_{\xi_i}(x,s,Du_\rho(x))\right)_{i=1}^n \,.$$

Let us first prove that g is Lipschitz continuous with respect to $x \in \Omega'$. Recall that (x,s) vary on a compact set of $\Omega \times \mathbb{R}$ (and out of this compact set f is equal to zero; in particular $f(x,s,\xi) = 0$ when $x \in \Omega - \Omega'$); also recall that $\xi = Du_\rho$ is bounded. For all $x_1, x_2 \in \Omega'$ and $s \in \mathbb{R}$, by using the assumptions (6) and (7), we have

$$\begin{aligned} |g(x_1,s) - g(x_2,s)| &= |f_\xi(x_1,s,Du_\rho(x_1)) - f_\xi(x_2,s,Du_\rho(x_2))| \\ &\leq |f_\xi(x_1,s,Du_\rho(x_1)) - f_\xi(x_2,s,Du_\rho(x_1))| \\ &\quad + |f_\xi(x_2,s,Du_\rho(x_1)) - f_\xi(x_2,s,Du_\rho(x_2))| \\ &\leq L\,|x_1 - x_2| + L_1\,|Du_\rho(x_1) - Du_\rho(x_2)| \leq L_2\,|x_1 - x_2| \,, \end{aligned}$$

where $L_2 = \max\left\{L; L_1 \|Du_\rho\|_{W^{1,\infty}(\Omega')}\right\}$. As before, let us denote by α_σ a mollifier with parameter $\sigma \to 0^+$ ($\alpha_\sigma \in C_c^\infty(\mathbb{R}^n)$, with $\alpha_\sigma \geq 0$ and $\int_{\mathbb{R}^n} \alpha_\sigma(\eta)\, d\eta = 1$); then we pose

$$g_\sigma(x,s) = \int_{\mathbb{R}^n} \alpha_\sigma(y)\, g(x-y,s)\, dy = \left(\int_{\mathbb{R}^n} \alpha_\sigma(y)\, g^{(i)}(x-y,s)\, dy\right)_{i=1}^n \,.$$

Let us observe that, if σ is sufficiently small, then $g_\sigma \in C_c^0(\Omega' \times \mathbb{R})$, because $g \in C_c^0(\Omega' \times \mathbb{R})$ too. Moreover, for every $x_1, x_2 \in \Omega'$ and for every $s \in \mathbb{R}$, we obtain

$$|g_\sigma(x_1,s) - g_\sigma(x_2,s)| = \left| \int_{\mathbb{R}^n} \alpha_\sigma(y)\ [g(x_1 - y, s) - g(x_2 - y, s)]\ dy \right|$$

$$\leq \int_{\mathbb{R}^n} \alpha_\sigma(y)\ |g(x_1 - y, s) - g(x_2 - y, s)|\ dy \leq L_2\, |x_1 - x_2|\ .$$

Therefore we also have

$$\left| \frac{\partial g_\sigma^{(i)}}{\partial x_j}(x,s) \right| \leq L_2\,, \qquad \forall\, i,j = 1,2,\dots n, \quad \forall\,(x,s) \in \Omega' \times \mathbb{R}. \tag{14}$$

For every $\sigma > 0$ we denote by $G_{\sigma,h}(x) = \left(G_{\sigma,h}^{(i)}(x) \right)_{i=1}^n$ the sequence of vector-valued functions defined by

$$G_{\sigma,h}(x) = \int_{u(x)}^{u_h(x)} g_\sigma(x,s)\, ds\,.$$

By the chain rule we can compute the trace of the $n \times n$ matrix DG_h ; we have

$$\operatorname{trace} DG_{\sigma,h}(x) = \sum_{i=1}^n \frac{\partial G_{\sigma,h}^{(i)}}{\partial x_i} = \left(g_\sigma\left(x, u_h(x)\right), Du_h(x) \right) \tag{15}$$

$$- \left(g_\sigma\left(x, u(x)\right), Du(x) \right) + \sum_{i=1}^n \int_{u(x)}^{u_h(x)} \frac{\partial g_\sigma^{(i)}}{\partial x_i}(x,s)\, ds\,.$$

Since also $G_{\sigma,h}(x)$ vanishes outside Ω', it results

$$\int_{\Omega'} DG_{\sigma,h}(x)\, dx\ = 0\,, \qquad \forall\, h \in \mathbb{N}\,. \tag{16}$$

From (15), (16) and (14) we deduce that

$$\left| \int_{\Omega'} \left\{ \left(g_\sigma\left(x, u_h(x)\right), Du_h(x) \right) - \left(g_\sigma\left(x, u(x)\right), Du(x) \right) \right\}\, dx \right|$$

$$= \left| - \int_{\Omega'} \left\{ \sum_{i=1}^n \int_{u(x)}^{u_h(x)} \frac{\partial g_\sigma^{(i)}}{\partial x_i}(x,s)\, ds \right\}\, dx \right| \leq nL_2 \int_{\Omega'} |u_h(x) - u(x)|\ dx\ .$$

We go first to the limit as $\sigma \to 0^+$ (and h fixed). We obtain the same inequality when in the left hand side g_σ is replaced by g. Thus we also have

$$\left| \int_{\Omega'} \left\{ (f_\xi(x,u_h,Du_\rho), Du_h) - (f_\xi(x,u,Du_\rho), Du) \right\}\, dx \right|$$

$$= \left| \int_{\Omega'} \left\{ \left(g\left(x, u_h(x)\right), Du_h(x) \right) - \left(g\left(x, u(x)\right), Du(x) \right) \right\}\, dx \right|$$

$$\leq nL_2 \int_{\Omega'} |u_h(x) - u(x)| \, dx \,,$$

which goes to zero as $h \to +\infty$, since, by assumption, $u_h \to u$ in $L^1_{loc}(\Omega)$. Therefore (13) holds and the proof of Lemma 2.1 is complete. $\square$

The following approximation result has been given by De Giorgi (see [6]).

Lemma 2.2. *If $f = f(x, s, \xi)$ satisfies (2) and vanishes outside a compact set of $\Omega \times \mathbb{R}$, there exists an increasing sequence of functions $\{f_j(x, s, \xi)\}_{j\in\mathbb{N}}$ that converges to $f(x, s, \xi)$ uniformly on the compact sets of $\Omega \times \mathbb{R} \times \mathbb{R}^n$ and such that, for all $j \in \mathbb{N}$, f_j satisfies (2) and*

$$|f_j(x, s, \xi_1) - f_j(x, s, \xi_2)| \leq L_j \, |\xi_1 - \xi_2| \,, \tag{17}$$

for some constant L_j and for every $(x, s, \xi_1), (x, s, \xi_2) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$.

Proof. We do not give all the details and we refer to the original proof by De Giorgi [6]. By using the support tangent hyperplanes to the graph of $f(x, s, \xi)$, up to a regularization procedure, De Giorgi shows that, for every $j \in \mathbb{N}$, f_j can be defined as the maximum between the zero function and a finite number of (affine with respect to $\xi \in \mathbb{R}^n$) functions of the type

$$g(x, s, \xi) = a_0(x, s) + \sum_{i=1}^{n} a_i(x, s)\xi_i \,.$$

For the use that we will make in the next section, we recall that the coefficients a_i $(i = 0, 1, 2, \ldots, n)$ are given by

$$\begin{cases} a_i(x, s) = -\int_{\mathbb{R}^n} f(x, s, \eta) D_i\alpha(\eta) \, d\eta \,, \quad \forall \, i = 1, 2, \ldots, n, \\[2ex] a_0(x, s) = \int_{\mathbb{R}^n} f(x, s, \eta) \left\{(n+1)\alpha(\eta) + \sum_{i=1}^{n} \eta_i D_i\alpha(\eta)\right\} \, d\eta \,, \end{cases} \tag{18}$$

for some mollifier $\alpha \in C_c^\infty(\mathbb{R}^n)$, with $\alpha \geq 0$ and $\int_{\mathbb{R}^n} \alpha(\eta) \, d\eta = 1$. $\square$

3. Proof of Theorem 1.6

In this section we will prove Theorem 1.6. With the same argument used at the beginning of the proof of Lemma 2.1, without loss of generality, we can assume that $f(x, s, \xi)$ vanishes outside a compact set of $\Omega \times \mathbb{R}$. Therefore we can also assume that Ω is a set with *finite measure* (we will use this remark in the definition (20) of the integral $F_j(u, \Omega)$).

Let $\{f_j(x, s, \xi)\}_{j\in\mathbb{N}}$ be the increasing sequence that pointwise converges to $f(x, s, \xi)$, as in Lemma 2.2. Let us denote by φ_ρ a mollifier ($\varphi_\rho \in C_c^\infty(\mathbb{R}^n)$, $\varphi_\rho \geq 0$, $\varphi_\rho(\eta) = 0$ if $|\eta| \geq \rho$, $\int_{\mathbb{R}^n} \varphi_\rho(\eta) \, d\eta = 1$) and, for all $j \in \mathbb{N}$, by $f_{j,\rho} = f_j * \varphi_\rho$ the mollified function of f_j, with respect to the variable $\xi \in \mathbb{R}^n$, with step ρ. That is, for every $j \in \mathbb{N}$ and $\rho > 0$, the function $f_{j,\rho} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is defined by

$$f_{j,\rho}(x, s, \xi) = \int_{\mathbb{R}^n} f_j(x, s, \xi - \eta)\varphi_\rho(\eta) \, d\eta \,.$$

By the Lipschitz continuity (17) of f_j with respect to $\xi \in \mathbb{R}^n$, we have

$$|f_{j,\rho}(x,s,\xi) - f_j(x,s,\xi)| \le \int_{\mathbb{R}^n} |f_j(x,s,\xi-\eta) - f_j(x,s,\xi)|\, \varphi_\rho(\eta)\, d\eta$$

$$\le \int_{\operatorname{supp}\varphi_\rho} L_j\, |\eta|\, \varphi_\rho(\eta)\, d\eta \le L_j \rho\,.$$

Thus we can choose $\rho = \rho_j =: 1/\left(jL_j\right) \to 0$ so that

$$f_j(x,s,\xi) - \frac{2}{j} \le f_{j,\rho_j}(x,s,\xi) - \frac{1}{j} \le f_j(x,s,\xi) \le f(x,s,\xi)\,, \tag{19}$$

for every $(x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$. By the monotone convergence theorem we have

$$\lim_{j\to+\infty} \int_\Omega f_j\left(x,u(x),Du(x)\right)\, dx \;=\; \int_\Omega f\left(x,u(x),Du(x)\right)\, dx\,.$$

Thus, if we consider the sequence of integrals

$$F_j(u,\Omega) = \int_\Omega \left\{ f_{j,\rho_j}\left(x,u(x),Du(x)\right) - \frac{1}{j} \right\} dx\,, \tag{20}$$

by (19) we obtain that $F_j(u,\Omega)$ converges, as $j \to +\infty$, to the main integral $F(u,\Omega) = \int_\Omega f\left(x,u(x),Du(x)\right) dx$, and at the same time $F_j(u,\Omega) \le F(u,\Omega)$ for every $j \in \mathbb{N}$. Therefore $F(u,\Omega)$, being the supremum of the family of functionals $\left\{F_j(u,\Omega)\right\}_{j\in\mathbb{N}}$, will be lower semicontinuous if every of such $F_j(u,\Omega)$ is lower semicontinuous.

Thus we must prove that, for every fixed $j \in \mathbb{N}$, the integral functional F_j in (20) is lower semicontinuous in $W^{1,1}_{loc}(\Omega)$ with respect to the strong convergence in L^1_{loc}. To this aim we apply Lemma 2.1. Of course $f_{j,\rho_j}(x,s,\xi)$ satisfies (2) and (17); thus it satisfies also the bound for the derivative in (7). It remains to verify that $f_{j,\rho_j}(x,s,\xi)$ also satisfies the second assumption in (7) and (6). We first compute the n partial derivatives of f_{j,ρ_j} with respect to the gradient variable ξ, i.e., we compute the vector field $\partial f_{j,\rho_j}/\partial\xi$:

$$\frac{\partial f_{j,\rho_j}}{\partial \xi}(x,s,\xi) = \int_{\mathbb{R}^n} f_j(x,s,\xi-\eta) \frac{\partial \varphi_{\rho_j}}{\partial \xi}(\eta)\, d\eta\,.$$

Then from (17) we deduce that

$$\left| \frac{\partial f_{j,\rho_j}}{\partial \xi}(x,s,\xi_1) - \frac{\partial f_{j,\rho_j}}{\partial \xi}(x,s,\xi_2) \right| \le M_j L_j \left|\xi_1 - \xi_2\right|\,,$$

for every $(x,s,\xi_1),(x,s,\xi_2) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, where

$$M_j = \int_{\mathbb{R}^n} \left| \frac{\partial \varphi_{\rho_j}}{\partial \xi}(\eta) \right| d\eta\,. \tag{21}$$

Therefore the second assumption in (7) is satisfied. To prove (6), we recall that $f_j(x,s,\xi)$ is the maximum between the zero function and a finite number of affine functions, with respect to $\xi \in \mathbb{R}^n$, of the type

$$g(x,s,\xi) = a_0(x,s) + \sum_{i=1}^n a_i(x,s)\xi_i\,, \tag{22}$$

where the coefficients $a_i(x,s)$ $(i=0,1,2,\dots,n)$ are given in (18). From assumption (5), of local Lipschitz continuity of f with respect to x, for every compact set $K\subset\Omega\times\mathbb{R}\times\mathbb{R}^n$ and for every $(x_1,s,\xi),(x_2,s,\xi)\in K$ we have

$$|f(x_1,s,\xi)-f(x_2,s,\xi)|\le L\,|x_1-x_2|\,,$$

for some constant $L=L(K)$. Then the coefficients a_i $(i=0,1,2,\dots,n)$ in (18) are locally Lipschitz continuous with respect to x too; in fact, for example, for every $i=1,2,\dots,n$,

$$\begin{aligned}|a_i(x_1,s)-a_i(x_2,s)| &= \left|\int_{\mathbb{R}^n}\{f(x_1,s,\eta)-f(x_2,s,\eta)\}\,D_i\alpha(\eta)\,d\eta\right|\\ &\le m_iL\,(K)\,|x_1-x_2|\ ,\end{aligned}$$

for every $(x_1,s),(x_2,s)$ which vary on a compact set K_0 of $\Omega\times\mathbb{R}$ (in fact the two points $(x_1,s,\eta),(x_2,s,\eta)$ vary in the compact set $K=K_0\times\operatorname{supp}\alpha$) and m_i is given by

$$m_i=\int_{\mathbb{R}^n}\left|\frac{\partial\alpha}{\partial\xi_i}(\eta)\right|\,d\eta\,.$$

Therefore the affine functions $g(x,s,\xi)$ in (22), since ξ vary on a bounded set, are local Lipschitz continuous with respect to x, for $(x,s,\xi)\in K$. Finally $f_j(x,s,\xi)$, being the maximum between the zero function and a finite number of affine functions of the type of $g(x,s,\xi)$ in (22), is local Lipschitz continuous with respect to x, i.e., for every compact set $K\subset\Omega\times\mathbb{R}\times\mathbb{R}^n$, there exists a constant $L_j'=L_j'(K)$ such that

$$|f_j(x_1,s,\xi)-f_j(x_2,s,\xi)|\le L_j'(K)\,|x_1-x_2|\ ,$$

for every $(x_1,s,\xi),(x_2,s,\xi)\in K$. For the same values of (x_i,s,ξ), $i=1,2$, we deduce that

$$\begin{aligned}&\left|\frac{\partial f_{j,\rho_j}}{\partial\xi}(x_1,s,\xi)-\frac{\partial f_{j,\rho_j}}{\partial\xi}(x_2,s,\xi)\right|\\ &\le\int_{\mathbb{R}^n}|f_j(x_1,s,\xi-\eta)-f_j(x_2,s,\xi-\eta)|\cdot\left|D\varphi_{\rho_j}(\eta)\right|\,d\eta\\ &\le M_jL_j'(K')\,|x_1-x_2|\ ,\end{aligned}$$

where M_j is the constant in (21) and K' is a suitable compact subset of $\Omega\times\mathbb{R}\times\mathbb{R}^n$ containing K. Therefore also the assumption (6) is satisfied and the proof of Theorem 1.6 is complete. □

4. Aronszajn-Dal Maso's example revisited

In this section, and in the next one, we collect some examples, partially known, partially new or revisited, modified and adapted to a more general context, with the aim to introduce some parameters which will allow us to test more carefully the assumptions. More precisely, we will show that the assumption of *continuity* alone of $f(x,s,\xi)$ with respect to $x\in\Omega$ (together with (2)) is not sufficient for the lower semicontinuity of the integral functional $F(u,\Omega)$ with respect to the strong convergence in $L^1_{loc}(\Omega)$ (see Example 4.1). Then, with a more precise analysis of some parameters and with the study of the

n-dimensional context, we compare in Example 4.2 assumptions of *Lipschitz continuity* of f with respect to x with the weaker assumption of *Hölder continuity.*

The first example that we propose in this section has been inspired by an old example by Aronszajn in 1941 (see Pauc [15], starting from page 54), more recently exploited by Dal Maso (see Section 4 in [3]). In the new version proposed here we consider, in particular, a simplified sequence u_h; as already said, this simplification will allow us to compare *Lipschitz continuity* versus *Hölder continuity* of f with respect to x. Notice also that the original example by Aronszajn is related to a *multiple integral*, i.e., with $n = 2$, although Aronszajn's integrand $f(x, \xi)$ does not explicitly depends on s. A *one-dimensional* version of Aronszajn's example was known to Dal Maso, who gave us some handwritten notes on the subject.

In this and in the following section we propose several examples; for completeness we also mention the case considered by Acerbi, Buttazzo and Fusco [1], with the main difference that their example is posed in the vector-valued setting of *polyconvex* integrands, but similar to the next two examples, at least for two aspects: the fact that the integrand $f(x, s, \xi) = |a(x, s)\,\xi - 1|$ has not minimum at $\xi = 0$ and the L^∞ convergence of the sequence u_h. Other similarities seem to exist, and we hope to come back to the vectorial setting in the future.

Example 4.1. Let Ω be the open interval $(0, 2\pi)$. Let u_h be the sequence (converging to $u \equiv 0$ in $L^\infty(\Omega)$, but not in the weak topology of $W^{1,1}(\Omega)$) defined by

$$u_h(x) = \frac{1}{2^h}\left(1 - \frac{1}{4}\cos\left(4^h x\right)\right).$$

Then there exists a function $a(x, s)$, bounded and uniformly continuous for $(x, s) \in \Omega \times \mathbb{R}$, such that, if we define

$$f(x, s, \xi) = |a(x, s)\,\xi - 1|, \qquad x \in \Omega \subset \mathbb{R},\ s \in \mathbb{R},\ \xi \in \mathbb{R},$$

then

$$\lim_{h \to +\infty} F(u_h, \Omega) = \lim_{h \to +\infty} \int_\Omega f\left(x, u_h, u_h'\right)\,dx = 0,$$

while of course

$$F(u, \Omega) = \int_\Omega f(x, 0, 0)\,dx = 2\pi.$$

Thus in this case the integral F is not lower semicontinuous with respect to the strong convergence in $L^\infty(\Omega)$, although $f(x, s, \xi)$ is a continuous nonnegative function, convex with respect to ξ, i.e., f satisfies (2).

Proof. Let us observe that, for every $h \in \mathbb{N}$,

$$\min\{u_h(x) :\ x \in [0, 2\pi]\} = \frac{1}{2^h}\left(1 - \frac{1}{4}\right)$$

$$> \frac{1}{2^{h+1}}\left(1 + \frac{1}{4}\right) = \max\{u_{h+1}(x) :\ x \in [0, 2\pi]\};$$

thus the graph $G_h = \{(x,s) \in \Omega \times \mathbb{R} : \; s = u_h(x)\}$ of u_h is disjoint from the graph of u_{h+1}, more precisely, they have positive distance each other. We will first define the function $a(x,s)$ on a subset of the union $\bigcup_{h\in\mathbb{N}} G_h$.

With the aim to define this subset, we notice that, for periodicity reasons, for every positive λ the measure of the set $\{x \in [0,2\pi] : \; |\sin(mx)| < \lambda\}$ does not depend on the integer m. Moreover, since for $x > 0$ sufficiently small $\sin x > x/2$, then for the same x values (i.e., for $\lambda > 0$ sufficiently small) the following inclusion holds $\{x : \sin x < \lambda\} \subset \{x : \; x/2 < \lambda\}$. This implies that the measure of the set $\{x > 0$ close to zero such that $\sin x < \lambda\}$ is less than 2λ. Taking into account the three zeroes $x = 0, \pi, 2\pi$ of the sinus function in the interval $[0, 2\pi]$, finally we have

$$\operatorname{meas}\{x \in [0,2\pi] : \; |\sin(mx)| < \lambda\} < 8\lambda, \quad \forall\, m \in \mathbb{N}. \tag{23}$$

Let us denote by E_h the open subset of $[0,2\pi]$ given by

$$E_h = \left\{x \in [0,2\pi] : \; \left|\sin\left(4^h x\right)\right| < h2^{-h}\right\}.$$

By (23) we have $\operatorname{meas}(E_h) < h2^{-h+3}$. For every $h \in \mathbb{N}$ we compute the derivative $u_h'(x) = 2^{h-2}\sin\left(4^h x\right)$. We define the function $a(x,s)$ on the following subset of the union $\{G_h\}_{h\in\mathbb{N}}$: if $x \in [0,2\pi] - E_h$ and $s = u_h(x)$, then

$$a(x,s) = \frac{1}{u_h'(x)} = \frac{1}{2^{h-2}\sin\left(4^h x\right)}.$$

Since h by h the graphs G_h are disjoint sets of $\overline{\Omega}\times\mathbb{R}$, then the above definition is consistent. Here we use the relevant fact that the derivative u_h' of u_h, as $h \to +\infty$, diverges (in absolute value) for $x \in [0,2\pi] - E_h$ (otherwise we should expect lower semicontinuity of the integral F); in fact $2^{h-2}\left|\sin\left(4^h x\right)\right| \geq h/4$ for every $x \in [0,2\pi] - E_h$. Therefore $1/u_h'(x)$ converges to zero as $h \to +\infty$ and we can also define by continuity

$$a(x,s) = 0, \quad \text{if } \; x \in [0,2\pi] \; \text{ and } s = 0.$$

At this stage the function $a(x,s)$ has been defined as a continuous function on a compact subset of $\overline{\Omega}\times\mathbb{R}$. Then it can be extended to the full $\overline{\Omega}\times\mathbb{R}$ remaining uniformly continuous (and bounded) on $\overline{\Omega}\times\mathbb{R}$.

By definition, for every $h \in \mathbb{N}$ and for $x \in [0,2\pi] - E_h$, we have

$$f\left(x, u_h(x), u_h'(x)\right) = \left|a(x,u_h(x))\, u_h'(x) - 1\right| = 0.$$

Thus, since $|u_h'(x)| = 2^{h-2}\left|\sin\left(4^h x\right)\right| \leq 2^{h-2}h2^{-h} = h/4$ for $x \in E_h$, and $\operatorname{meas}(E_h) < h2^{-h+3}$, if we denote by $M > 0$ a bound for $a(x,s)$ in $\overline{\Omega}\times\mathbb{R}$, we obtain

$$\begin{aligned}
\int_\Omega f\left(x, u_h, u_h'\right) dx &= \int_{E_h} f\left(x, u_h, u_h'\right) dx \\
&= \int_{E_h} \left|a(x,u_h(x))\, u_h'(x) - 1\right| dx \\
&\leq \int_{E_h} \left\{M\left|u_h'(x)\right| + 1\right\} dx \leq \left(\frac{Mh}{4} + 1\right) h2^{-h+3},
\end{aligned}$$

which converges to zero as $h \to +\infty$. $\square$

Let us go back to the main Theorem 1.6, in particular to the assumption that $f(x,s,\xi)$ is *Lipschitz continuous* with respect to x, locally respect to (x,s,ξ), i.e., for every compact set $K \subset \Omega \times \mathbb{R} \times \mathbb{R}^n$, there exists a constant $L = L(K)$ such that

$$|f(x_1,s,\xi) - f(x_2,s,\xi)| \le L\,|x_1 - x_2|\,,$$

for every $(x_1,s,\xi)\,,(x_2,s,\xi) \in K$. We may ask if we can assume a less restrictive local continuity assumption of f with respect to x. For example, we may ask if Theorem 1.6 holds under the only assumption that $f(x,s,\xi)$ satisfies (2) and is *Hölder continuous* with respect to x, locally respect to (x,s,ξ), i.e., there exists a real number $\alpha \in (0,1)$ with the property that, for every compact set $K \subset \Omega \times \mathbb{R} \times \mathbb{R}^n$, there exists a constant $L = L(K)$ such that

$$|f(x_1,s,\xi) - f(x_2,s,\xi)| \le L\,|x_1 - x_2|^{\alpha}\,, \tag{24}$$

for every $(x_1,s,\xi)\,,(x_2,s,\xi) \in K$.

By the next example we give an answer to this question; in fact we will prove that, for every exponent $\alpha \in (0,1)$, it is possible to find a nonnegative continuous integrand $f = f_\alpha(x,s,\xi)$ (depending on α too), convex with respect to $\xi \in \mathbb{R}^n$, satisfying the Hölder continuity property (24), but whose corresponding integral is not lower semicontinuous, even in $C^\infty(\overline{\Omega})$, with respect to the strong $L^\infty(\Omega)$ convergence.

We emphasize that, in the next example, we do not consider an arbitrary independent dimension $n \ge 1$, but we impose the constraint

$$n > \frac{4\alpha}{1-\alpha} \tag{25}$$

on the dimension, or equivalently the constraint

$$\alpha < \frac{n}{n+4} \tag{26}$$

on the Hölder exponent α. The less restrictive constraint $\alpha < 1/3$ is assumed when $n = 1$. Thus it remains open the interesting question to know if, for every $n \in \mathbb{N}$, there exists a critical exponent $\alpha(n)$ such that the integral (1) is lower semicontinuous with respect to the strong convergence in L^1_{loc} under the usual condition (2) and the Hölder continuity property (24) for some exponent α such that $\alpha(n) \le \alpha < 1$. In particular we may ask if, for example, the integral (1) is lower semicontinuous in the one-dimensional case $n = 1$, when the integrand f is Hölder continuous with exponent $\alpha \ge 1/3$.

Example 4.2. We use notations similar to the previous Example 4.1, with some parameters. Thus let Ω be the open hyper-rectangle $(0,2\pi)^n \subset \mathbb{R}^n$. For $h \in \mathbb{N}$ let $u_h(x) = u_h(x_1,x_2,\ldots,x_n)$ be defined by

$$u_h(x) = \sum_{i=1}^{n} \frac{1}{a_h}\left(1 - \frac{1}{4}\cos\left(b_h x_i\right)\right), \tag{27}$$

with $\{a_h\}$, $\{b_h\}$ sequences of positive real numbers diverging to $+\infty$ as $h \to +\infty$. Thus u_h converges to $u \equiv 0$ in $L^\infty(\Omega)$. Then, for every $\alpha \in (0,1)$, if

$$n > \frac{4\alpha}{1-\alpha}\,, \tag{28}$$

there exist $\{a_h\}$, $\{b_h\}$ and a vector-valued function $a_\alpha : \overline{\Omega} \times \mathbb{R} \subset \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ with the properties:

(i) $a_\alpha(x,s)$ is bounded and uniformly continuous for $(x,s) \in \overline{\Omega} \times \mathbb{R}$;

(ii) for every $s \in \mathbb{R}$, $a_\alpha(x,s)$ is Hölder continuous (of exponent α) with respect to $x \in \overline{\Omega}$; more precisely, there exists a constant L such that

$$|a_\alpha(x_1,s) - a_\alpha(x_2,s)| \le L\,|x_1 - x_2|^\alpha\,, \quad \forall\, x_1, x_2 \in \overline{\Omega} \subset \mathbb{R}^n,\ \forall\, s \in \mathbb{R};$$

(iii) if we denote by $(\cdot,\cdot)$ the scalar product in $\mathbb{R}^n$ and we define

$$f_\alpha(x,s,\xi) = |(a_\alpha(x,s),\xi) - 1|\,, \qquad x \in \overline{\Omega},\ s \in \mathbb{R},\ \xi \in \mathbb{R}^n,$$

then $f_\alpha(x,s,\xi)$ satisfies (2), the Hölder continuity property (24) and

$$\liminf_{h \to +\infty} \int_\Omega f_\alpha\,(x, u_h, Du_h)\; dx = 0\,; \qquad \int_\Omega f_\alpha\,(x,0,0)\; dx = (2\pi)^n\ . \tag{29}$$

The same construction for the one-dimensional case $n = 1$ has all the stated properties, under the less restrictive assumption (instead of (28)) that $\alpha < 1/3$.

Proof.

Step 1 (definition of a_α): passing to a subsequence if necessary (this is the reason to have in (29) the limit inferior, instead of the limit, that, however, can be easily reduced to became a limit), we can assume that the graphs of the functions u_h are disjoint; more precisely that

$$\min\{u_h(x):\ x \in [0,2\pi]^n\} = \frac{3n}{4a_h} > \frac{5n}{4a_{h+1}} = \max\{u_{h+1}(x):\ x \in [0,2\pi]^n\}$$

for every $h \in \mathbb{N}$. With similar notations as in the previous example we define

$$E_h^i = \{x_i \in [0,2\pi]:\ |\sin(b_h x_i)| < \lambda_h\}\,, \quad i = 1,2,\ldots,n,$$

$$E_h = \left\{x = (x_1,x_2,\ldots,x_n) \in [0,2\pi]^n:\ \sum_{i=1}^n \sin^2(b_h x_i) < \lambda_h^2\right\},$$

where $\{\lambda_h\}$ is a sequence of positive real numbers converging to 0 as $h \to +\infty$. We have

$$\operatorname{meas}\left(E_h^i\right) < 8\lambda_h\,, \quad \forall\, h \in \mathbb{N},\ \forall\, i = 1,2,\ldots,n, \tag{30}$$

and, since $E_h \subset E_h^1 \times E_h^2 \times \ldots \times E_h^n$,

$$\operatorname{meas}(E_h) < 8^n\,(\lambda_h)^n\,, \quad \forall\, h \in \mathbb{N}. \tag{31}$$

The partial derivatives of $u_h(x)$ are equal to

$$\frac{\partial u_h}{\partial x_i} = \frac{b_h}{4a_h}\sin(b_h x_i)\,, \quad i = 1,2,\ldots,n,$$

and, for $x \in [0,2\pi]^n - E_h$, we have

$$|Du_h(x)| = \frac{b_h}{4a_h}\left\{\sum_{i=1}^n \sin^2(b_h x_i)\right\}^{1/2} \ge \frac{b_h}{4a_h}\lambda_h\,. \tag{32}$$

Similarly to the previous example, we define the vector-valued function $a_\alpha(x,s)$ on a part of the graphs of the functions u_h. Precisely, for every $h \in \mathbb{N}$ and for $x \in [0,2\pi]^n - E_h$ and $s = u_h(x)$, we define

$$a_\alpha(x,s) = \frac{Du_h}{|Du_h|^2}\,, \tag{33}$$

i.e., with the notation $a_\alpha(x,s) = (a_\alpha^i(x,s))_{i=1}^n$, we define

$$a_\alpha^i(x,s) = \frac{\partial u_h}{\partial x_i} \cdot \frac{1}{|Du_h|^2} = \frac{4a_h \sin(b_h x_i)}{b_h \sum_{j=1}^n \sin^2(b_h x_j)}\,.$$

We have

$$|a_\alpha(x,s)| = \frac{4a_h}{b_h \left\{\sum_{j=1}^n \sin^2(b_h x_j)\right\}^{1/2}}\,. \tag{34}$$

We also define $a_\alpha(x,0)$ by continuity

$$a_\alpha(x,s) = 0, \quad \text{if } x \in [0,2\pi]^n \text{ and } s = 0;$$

to this aim, since $u_h(x) \to 0$ as $h \to +\infty$, we impose the condition

$$\max\{|a_\alpha(x,s)| : x \in [0,2\pi]^n - E_h\,, s = u_h(x)\} = \frac{4a_h}{b_h\lambda_h} \to 0\,, \quad \text{as } h \to +\infty. \tag{35}$$

At this stage the vector-valued function $a_\alpha(x,s)$ has been defined as a continuous function on a closed subset of $\overline{\Omega} \times \mathbb{R}$. In Step 2 we will extend it to the full $\overline{\Omega} \times \mathbb{R}$.

Step 2 (extension of a_α to $\overline{\Omega} \times \mathbb{R}$): fixed $h \in \mathbb{N}$, the vector-valued function $a_\alpha(x,s)$ has been defined in Step 1 at the points $(x,s) \in \overline{\Omega} \times \mathbb{R}$ related by the conditions $x \in [0,2\pi]^n - E_h$ and $s = u_h(x)$ (the function $a_\alpha(x,s)$ has been also defined for $s = 0$ with the zero value). In fact, for every fixed $h \in \mathbb{N}$, $a_\alpha(x,s)$ has been defined at the points (x,s) of the set $R_h \subset \overline{\Omega} \times \mathbb{R}$ (a subset of the graph of u_h) given by

$$R_h = \{(x,s) : \ x \in [0,2\pi]^n - E_h, \quad s = u_h(x)\}\,.$$

Recalling the analytic expression of u_h in (27), the set R_h is contained in the hyper-rectangle

$$\left\{(x,s) : \ x \in [0,2\pi]^n\,, \quad \frac{3}{4} \cdot \frac{n}{a_h} < s < \frac{5}{4} \cdot \frac{n}{a_h}\right\}.$$

We will extend $a_\alpha(x,s)$ to the larger hyper-rectangle $R'_h \subset \overline{\Omega} \times \mathbb{R}$, given by

$$R'_h = \left\{(x,s) : \ x \in [0,2\pi]^n\,, \quad \frac{1}{2} \cdot \frac{n}{a_h} \le s \le \frac{3}{2} \cdot \frac{n}{a_h}\right\}.$$

First we define $a_\alpha(x,s) = 0$ when $s = \frac{n}{2a_h}$ and $s = \frac{3n}{2a_h}$, so that a_α will be continuously defined passing from an hyper-rectangle to an other; in fact we extend a_α also equal to zero out of the union $\cup_h R'_h$. Note that, passing possibly to a subsequence, we can assume that $R'_h \cap R'_k = \emptyset$ if $h \neq k$.

In order to estimate the oscillation $|a_\alpha(x_1,s_1)-a_\alpha(x_2,s_2)|$ when (x_1,s_1), (x_2,s_2) vary in R'_h, we first consider (x_1,s_1), $(x_2,s_2)\in R_h$ and we prove the following Lipschitz estimate (with constant depending on h)

$$|a_\alpha(x_1,s_1)-a_\alpha(x_2,s_2)|\leq \frac{16n\cdot a_h}{\lambda_h^4}\cdot|x_1-x_2|\ . \tag{36}$$

In fact, under the conditions $x_1,x_2\in[0,2\pi]^n-E_h$, $s_1=u_h(x_1)$, $s_2=u_h(x_2)$ and with the notations $a_\alpha(x,s)=\left(a_\alpha^i(x,s)\right)_{i=1}^n$, $x_1=\left(x_1^i\right)_{i=1}^n$, $x_2=\left(x_2^i\right)_{i=1}^n$, for every $i=1,2,\ldots,n$ we have

$$\begin{aligned}\left|a_\alpha^i(x_1,s_1)-a_\alpha^i(x_2,s_2)\right| &= \frac{4a_h}{b_h}\left|\frac{\sin\left(b_hx_1^i\right)}{\sum_{j=1}^n\sin^2\left(b_hx_1^j\right)}-\frac{\sin\left(b_hx_2^i\right)}{\sum_{j=1}^n\sin^2\left(b_hx_2^j\right)}\right|\\ &\leq \frac{4a_h}{b_h\cdot\sum_{j=1}^n\sin^2\left(b_hx_1^j\right)}\left|\sin\left(b_hx_1^i\right)-\sin\left(b_hx_2^i\right)\right|\\ &+\frac{4a_h\cdot\left|\sin\left(b_hx_2^i\right)\right|}{b_h\sum_{j=1}^n\sin^2\left(b_hx_1^j\right)\cdot\sum_{j=1}^n\sin^2\left(b_hx_2^j\right)}\left|\sum_{j=1}^n\left\{\sin^2\left(b_hx_1^j\right)-\sin^2\left(b_hx_2^j\right)\right\}\right|.\end{aligned}$$

Since $x_1,x_2\notin E_h$, we also have $\sum_{j=1}^n\sin^2(b_hx_1^j)\geq\lambda_h^2$ and $\sum_{j=1}^n\sin^2(b_hx_2^j)\geq\lambda_h^2$; therefore, by the Lipschitz continuity of the *sinus* function, we obtain

$$\left|a_\alpha^i(x_1,s_1)-a_\alpha^i(x_2,s_2)\right|\leq\frac{4a_h}{b_h\lambda_h^2}\cdot b_h\left|x_1^i-x_2^i\right|+\frac{4a_h}{b_h\lambda_h^4}\cdot 2b_h\sum_{j=1}^n\left|x_1^j-x_2^j\right|.$$

Since this estimate holds for every $i=1,2,\ldots,n$, for the modulus of the vector field $a_\alpha(x_1,s)-a_\alpha(x_2,s)$ we obtain

$$|a_\alpha(x_1,s_1)-a_\alpha(x_2,s_2)|\leq\frac{4a_h}{\lambda_h^2}\left(1+\frac{2n}{\lambda_h^2}\right)|x_1-x_2|\,.$$

The sequence λ_h converges to zero as $h\to+\infty$; therefore, as h is sufficiently large, we get the proof of (36). Of course (36) also gives

$$|a_\alpha(x_1,s_1)-a_\alpha(x_2,s_2)|\leq\frac{16\,n\cdot a_h}{\lambda_h^4}\cdot(|x_1-x_2|+|s_1-s_2|)\ , \tag{37}$$

for every $h\in\mathbb{N}$ and for every (x_1,s_1), $(x_2,s_2)\in R_h$ with $s_1=u_h(x_1)$, $s_2=u_h(x_2)$.

Recalling (34), we have the bound

$$\max\left\{|a_\alpha(x,s)|:\ (x,s)\in R_h\right\}=\frac{4a_h}{b_h\lambda_h}\,;$$

therefore, if $(x_1,s_1)\in R_h$ and $(x_2,s_2)\in R'_h$ with either $s=\frac{n}{2a_h}$ or $s=\frac{3n}{2a_h}$, we deduce that

$$|a_\alpha(x_1,s_1)-a_\alpha(x_2,s_2)|=|a_\alpha(x_1,s_1)|\leq\frac{4a_h}{b_h\lambda_h}$$

and, since $|s_1 - s_2| \geq \frac{1}{4} \cdot \frac{n}{a_h}$,

$$|a_\alpha(x_1, s_1) - a_\alpha(x_2, s_2)| \leq \frac{16\, a_h^2}{n b_h \lambda_h} \cdot |s_1 - s_2| \leq \frac{16\, a_h^2}{n b_h \lambda_h} \cdot (|x_1 - x_2| + |s_1 - s_2|) \ .$$

By (35) we have $\frac{a_h}{b_h \lambda_h} \to 0$ as $h \to +\infty$; therefore $\frac{a_h}{b_h} \lambda_h^3 \to 0$ too, which implies

$$\frac{a_h^2}{n b_h \lambda_h} \leq \frac{n \cdot a_h}{\lambda_h^4}$$

for every h sufficiently large. This proves that the Lipschitz estimate (37) (with constant depending on h) holds at every (x_1, s_1), $(x_2, s_2) \in R'_h$ where $a_\alpha(x, s)$ has been already defined.

By using Kirszbraun theorem (see Theorem 2.10.43 in Federer [9]) for the vector-valued function a_α, or in a simpler way by applying Mac Shane lemma to each component of $a_\alpha(x, s) = (a_\alpha^i(x, s))_{i=1}^n$, we can extend it to the hyper-rectangle R'_h with the same Lipschitz constant as in (37), or, in case of extension of every component separately, with the same Lipschitz constant times $\sqrt{n}$. That is we have

$$|a_\alpha(x_1, s_1) - a_\alpha(x_2, s_2)| \leq \frac{16\, n \sqrt{n}\, a_h}{\lambda_h^4} \cdot (|x_1 - x_2| + |s_1 - s_2|) \ , \tag{38}$$

for every $h \in \mathbb{N}$ and for every (x_1, s_1), $(x_2, s_2) \in R'_h$. Moreover, by truncating each component a_α^i, we can assume that the following bound holds

$$\max \{|a_\alpha(x, s)| : \ (x, s) \in R'_h\} = \frac{4\sqrt{n}\, a_h}{b_h \lambda_h} \ , \tag{39}$$

for every $h \in \mathbb{N}$ and for every $(x, s) \in R'_h$.

Step 3 (Hölder continuity of a_α): now the parameter α enters. To test Hölder continuity of $a_\alpha(x, s)$ with respect to x we fix $h \in \mathbb{N}$ and $\frac{n}{2a_h} \leq s \leq \frac{3n}{2a_h}$ and we estimate

$$\sup \left\{ \frac{|a_\alpha(x_1, s) - a_\alpha(x_2, s)|}{|x_1 - x_2|^\alpha} : \ (x_1, s), (x_2, s) \in R'_h \right\} . \tag{40}$$

Let $t > 0$ be a new real parameter that we will choose later. We estimate the supremum in (40) separately for $|x_1 - x_2| \geq t$ and for $|x_1 - x_2| \leq t$.

Under the further condition $|x_1 - x_2| \geq t$, the supremum in (40) can be estimate by computing separately the maximum value of the numerator and the minimum value of the denominator. By (39) we have

$$\begin{aligned} &\max \{|a_\alpha(x_1, s) - a_\alpha(x_2, s)| : \ (x_1, s), (x_2, s) \in R'_h\} \\ &\leq 2 \max \{|a_\alpha(x, s)| : \ (x, s) \in R'_h\} = \frac{8\sqrt{n}\, a_h}{b_h \lambda_h} \ . \end{aligned}$$

For the same s-values, since $|x_1 - x_2| \geq t$, we obtain

$$\begin{aligned} &\sup \left\{ \frac{|a_\alpha(x_1, s) - a_\alpha(x_2, s)|}{|x_1 - x_2|^\alpha} : \ (x_1, s), (x_2, s) \in R'_h \ |x_1 - x_2| \geq t \right\} \\ &\qquad \leq \frac{8\sqrt{n}\, a_h}{b_h \lambda_h\, t^\alpha} \ . \end{aligned} \tag{41}$$

While, if $|x_1 - x_2| \le t$, we use the Lipschitz estimate (38) (with constant depending on h) with $s_1 = s_2 \equiv s$

$$|a_\alpha(x_1, s) - a_\alpha(x_2, s)| \le \frac{16\, n\sqrt{n}\, a_h}{\lambda_h^4} \cdot |x_1 - x_2|$$

and we obtain

$$\sup\left\{ \frac{|a_\alpha(x_1, s) - a_\alpha(x_2, s)|}{|x_1 - x_2|^\alpha} : \ (x_1, s), (x_2, s) \in R'_h \ \ |x_1 - x_2| \le t \right\}$$
$$\le \frac{16\, n\sqrt{n}\, a_h}{\lambda_h^4} \cdot t^{1-\alpha} . \tag{42}$$

From (41) and (42) we deduce that

$$\sup\left\{ \frac{|a_\alpha(x_1, s) - a_\alpha(x_2, s)|}{|x_1 - x_2|^\alpha} : \ \ (x_1, s), (x_2, s) \in R'_h \right\}$$
$$\le \frac{16\, n\sqrt{n}\, a_h}{\lambda_h} \cdot \max\left\{ \frac{1}{b_h\, t^\alpha} ; \frac{t^{1-\alpha}}{\lambda_h^3} \right\} . \tag{43}$$

The above inequality is valid for every $t > 0$. We consider the minimum of the right hand side with respect to $t > 0$, which is assumed when $\frac{1}{b_h\, t^\alpha} = \frac{t^{1-\alpha}}{\lambda_h^3}$, i.e., when $t = \frac{\lambda_h^3}{b_h}$.We obtain that the Hölder quotient in the left hand side of (43) is less than or equal to

$$\frac{16\, n\sqrt{n}\, a_h}{\lambda_h b_h\, t^\alpha} = \frac{16\, n\sqrt{n}\, a_h}{(b_h)^{1-\alpha}\, (\lambda_h)^{1+3\alpha}} . \tag{44}$$

Previously we estimated the Hölder continuity with respect to x of $a_\alpha(x, s)$ in R'_h, for every fixed $h \in \mathbb{N}$. Thus, to obtain the Hölder continuity of $a_\alpha(x, s)$ with respect to x, with $(x, s) \in \overline{\Omega} \times \mathbb{R}$, we impose the further condition that the sequence in (44) remains bounded, i.e., that there exists $L > 0$ such that

$$\frac{a_h}{(b_h)^{1-\alpha}\, (\lambda_h)^{1+3\alpha}} \le L , \quad \forall\, h \in \mathbb{N}. \tag{45}$$

Step 4 (lower semicontinuity test): let us prove that

$$\lim_{h\to+\infty} \int_\Omega f\left(x, u_h, Du_h\right)\, dx = 0 . \tag{46}$$

By the definition $f_\alpha(x, s, \xi) = |(a_\alpha(x, s), \xi) - 1|$ and by (33), for every $h \in \mathbb{N}$ we obtain

$$f\left(x, u_h, Du_h\right) = |(a_\alpha(x, u_h), Du_h) - 1| = 0 , \quad \forall\, x \in [0, 2\pi]^n - E_h .$$

Thus, since $|Du_h| < \frac{b_h \lambda_h}{4 a_h}$ for $x \in E_h$ (see (32)) and $\text{meas}\,(E_h) < 8^n\, (\lambda_h)^n$ (see (31)), we obtain (46); in fact, if we denote by $M > 0$ a bound for $a_\alpha(x, s)$ in $\overline{\Omega} \times \mathbb{R}$, we have

$$\int_\Omega f\left(x, u_h, Du_h\right)\, dx = \int_{E_h} f\left(x, u_h, Du_h\right)\, dx$$

$$\leq \int_{E_h} \{M |Du_h| + 1\}\, dx \leq \left\{ \frac{M}{4} \frac{b_h \lambda_h}{a_h} + 1 \right\} 8^n (\lambda_h)^n , \tag{47}$$

which converges to zero as $h \to +\infty$, if we assume that $\frac{b_h}{a_h} (\lambda_h)^{n+1} \to 0$ as $h \to +\infty$.

Step 5 (compatibility conditions; i.e., necessary conditions): looking above, we required the following limit relations (see in particular (35), (45) and (47))

$$\begin{cases} a_h \to +\infty\,, \quad b_h \to +\infty\,, \quad \lambda_h \to 0\,, \\ \frac{a_h}{b_h \lambda_h} \to 0\,, \quad \frac{b_h}{a_h} (\lambda_h)^{n+1} \to 0\,, \\ \frac{a_h}{(b_h)^{1-\alpha} (\lambda_h)^{1+3\alpha}} \leq L\,, \quad \forall\, h \in \mathbb{N}\,, \end{cases}$$

which, since

$$\frac{a_h}{(b_h)^{1-\alpha} (\lambda_h)^{1+3\alpha}} = \frac{a_h}{b_h \lambda_h} \cdot (b_h)^{\alpha} \cdot (\lambda_h)^{-3\alpha} ,$$

can be reduced to

$$\begin{cases} a_h \to +\infty\,, \quad b_h \to +\infty\,, \quad \lambda_h \to 0\,, \\ \frac{b_h}{a_h} (\lambda_h)^{n+1} \to 0\,, \\ \frac{a_h}{(b_h)^{1-\alpha} (\lambda_h)^{1+3\alpha}} \leq L\,, \quad \forall\, h \in \mathbb{N}\,. \end{cases} \tag{48}$$

Then, from the identity

$$\frac{a_h}{(b_h)^{1-\alpha} (\lambda_h)^{1+3\alpha}} = \left\{ \frac{b_h}{a_h} (\lambda_h)^{n+1} \right\}^{\alpha-1} \cdot (a_h)^{\alpha} (\lambda_h)^{n(1-\alpha)-4\alpha} ,$$

we find out the following compatibility condition

$$\begin{cases} \left\{ \frac{b_h}{a_h} (\lambda_h)^{n+1} \right\}^{\alpha-1} \to +\infty \\ (a_h)^{\alpha} \to +\infty \\ \frac{a_h}{(b_h)^{1-\alpha} (\lambda_h)^{1+3\alpha}} \leq L \end{cases} \quad \Longrightarrow \quad (\lambda_h)^{n(1-\alpha)-4\alpha} \to 0\,,$$

and, since $\lambda_h \to 0$, we must have that $n(1-\alpha) - 4\alpha > 0$, i.e., (28) is a necessary condition to let this construction work.

Step 6 (sufficient conditions): it remains to exhibit sequences of real parameters which satisfy the limit relations (48). Fixed $\alpha \in (0,1)$ we consider $n > \frac{4\alpha}{1-\alpha}$ so that $\alpha < \frac{n}{n+4}$. Then there exists $\beta \in \mathbb{N}$ ($\beta > 1$) large enough so that

$$\alpha < \frac{n(\beta - 1)}{(n+4)\,\beta - 3}\,. \tag{49}$$

By a simple computation we can see that inequality (49) is equivalent to

$$\frac{\beta - 1}{n+1} < \frac{\beta\,(1-\alpha) - 1}{1 + 3\alpha}\,. \tag{50}$$

We are ready to choose

$$a_h = 2^h, \quad b_h = 2^{\beta h}, \quad \lambda_h = 2^{-\gamma h},$$

with $\frac{\beta-1}{n+1} < \gamma \leq \frac{\beta(1-\alpha)-1}{1+3\alpha}$, a possibility that we can take since (50) holds. Then we can verify that all the conditions in (48) are satisfied. In fact $a_h, b_h \to +\infty$, $\lambda_h \to 0$, and

$$\frac{b_h}{a_h} (\lambda_h)^{n+1} = 2^{h[\beta-1-\gamma(n+1)]} \to 0 ,$$

since $\gamma > \frac{\beta-1}{n+1}$. Finally

$$\frac{a_h}{(b_h)^{1-\alpha} (\lambda_h)^{1+3\alpha}} = 2^{h[1-\beta(1-\alpha)+\gamma(1+3\alpha)]} \leq 1, \quad \forall\, h \in \mathbb{N} ,$$

since $\gamma \leq \frac{\beta(1-\alpha)-1}{1+3\alpha}$.

Step 7 (the case $n = 1$): in the one-dimensional case we can go faster. In fact to obtain Hölder continuity of $a_\alpha(x, s)$ with respect to x we can estimate the Hölder quotient and we can compute separately the maximum value of the numerator and the minimum value of the denominator, as follows:

$$\begin{aligned} &\max \{|a_\alpha(x_1, s) - a_\alpha(x_2, s)| : x_1, x_2 \in [0, 2\pi] - E_h\,,\ s = u_h(x_1) = u_h(x_2)\} \\ &\qquad \leq 2 \max \{|a_\alpha(x, s)| : x \in [0, 2\pi] - E_h\,,\ s = u_h(x)\} = \frac{8a_h}{b_h\lambda_h} . \end{aligned}$$

The minimum value of $|x_1 - x_2|$ is equal to the measure of E_h^1 divided by the number of connected components of E_h^1, i.e., $|x_1 - x_2| = \text{meas}\,(E_h^1) / (2b_h)$. By (30) we obtain

$$\begin{aligned} &\min \{|x_1 - x_2| : x_1, x_2 \in [0, 2\pi] - E_h\,,\ s = u_h(x_1) = u_h(x_2)\} \\ &\qquad = \frac{\text{meas}\,(E_h^1)}{2b_h} < \frac{4\lambda_h}{b_h} . \end{aligned}$$

Therefore, when $n = 1$, we have

$$\begin{aligned} &\sup \left\{ \frac{|a_\alpha(x_1, s) - a_\alpha(x_2, s)|}{|x_1 - x_2|^\alpha} : x_1, x_2 \in [0, 2\pi] - E_h\,, s = u_h(x_1) = u_h(x_2) \right\} \\ &\qquad \leq \frac{8a_h}{b_h\lambda_h} \cdot \left(\frac{4\lambda_h}{b_h} \right)^{-\alpha} = \frac{2^{3-2\alpha} a_h}{(b_h)^{1-\alpha} (\lambda_h)^{1+\alpha}} . \end{aligned}$$

In order to obtain the Hölder continuity of $a_\alpha(x, s)$ with respect to x, we impose the further condition that there exists $L > 0$ such that

$$\frac{a_h}{(b_h)^{1-\alpha} (\lambda_h)^{1+\alpha}} \leq L\,, \quad \forall\, h \in \mathbb{N}.$$

Following this estimate, with similar computations as in the step 2 above, we can extend the function $a_\alpha(x, s)$ to $\overline{\Omega} \times \mathbb{R}$ (or, in a simpler way, we could extend it linearly). As in Step 5, we obtain the compatibility conditions

$$\begin{cases} a_h \to +\infty\,, \quad b_h \to +\infty\,, \quad \lambda_h \to 0\,, \\ \frac{a_h}{b_h\lambda_h} \to 0\,, \quad \frac{b_h}{a_h} (\lambda_h)^2 \to 0\,, \\ \frac{a_h}{(b_h)^{1-\alpha}(\lambda_h)^{1+\alpha}} \leq L\,, \quad \forall\, h \in \mathbb{N}\,, \end{cases}$$

(note the exponent in $(\lambda_h)^{1+\alpha}$ instead of $(\lambda_h)^{1+3\alpha}$), which gives, as in Steps 5 and 6, the constraint $\alpha < \frac{1}{3}$ for $n = 1$. □

5. Some other examples

In this section first we show in Example 5.1 that the *local Lipschitz continuity* of $f(x,s,\xi)$ with respect to $x \in \Omega$, although sufficient for the lower semicontinuity of $F(u,\Omega)$ in L^1_{loc} in the scalar case, as proved by our Theorem 1.6, it is not sufficient in the *vector-valued case*, i.e., for applications $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, when $m > 1$. With Example 5.2 we show that also Lemma 2.1 cannot be extended to the vectorial setting. By Example 5.5 we emphasize the role of *lower semicontinuity* of f with respect to $x \in \Omega$ in the case of *linear growth* of $f(x,s,\xi)$ as $|\xi| \to +\infty$, while in Example 5.3 we show that, however, neither continuity, nor even lower semicontinuity, of $f(x,s,\xi)$ with respect to $x \in \Omega$ are necessary in the case of *superlinear growth*, also if the usual *coercivity* condition $f(x,s,\xi) \geq \text{const}\,|\xi|^p$, for some $p > 1$, is not satisfied. We will start from this example to formulate (see Proposition 5.6) at the end of this section a sufficient condition for lower semicontinuity of $F(u,\Omega)$ in L^1_{loc}.

We can give all the examples below in the *one dimensional* case $n = 1$.

Eisen [8] showed with an example that Theorem 1.1(c) is false in the vectorial case. The same example shows that also Theorem 1.6 and Corollary 1.7 do not hold in the vectorial case. We recall this example, related to an integrand f independent of the variable x.

Example 5.1. Let Ω be the open interval $(0,1)$. Let us consider the function $f : \Omega \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,s_1,s_2,\xi_1,\xi_2) = (s_1\xi_2)^2$; thus, and for all $u = (u_1,u_2) \in W^{1,1}(\Omega,\mathbb{R}^2)$, the functional F is given by

$$F(u,\Omega) = \int_\Omega (u_1 \cdot u_2')^2 \, dx\,.$$

Then there exists a sequence $u_h = (u_{1,h}, u_{2,h}) : \Omega \to \mathbb{R}^2$ which converges to a function $u \in W^{1,1}(\Omega,\mathbb{R}^2)$ in the strong topology of $L^1(\Omega,\mathbb{R}^2)$, such that

$$F(u_h,\Omega) = 0, \quad \forall\, h \in \mathbb{N}; \qquad F(u,\Omega) = 1.$$

Proof. Let $u_h : \Omega \to \mathbb{R}^2$ be the sequence defined by

$$u_{1,h}(x) = \begin{cases} 0 \text{ if } x \in \left(\frac{m}{h}, \frac{m}{h} + 2^{-h}\right] \\ 2^{h+2}\left(x - \frac{m}{h} - 2^{-h}\right) \text{ if } x \in \left(\frac{m}{h} + 2^{-h}, \frac{m}{h} + 2^{-h} + 2^{-h-2}\right] \\ 1 \text{ if } x \in \left(\frac{m}{h} + 2^{-h} + 2^{-h-2}, \frac{m+1}{h} - 2^{-h-2}\right] \\ 1 - 2^{h+2}\left(x - \frac{m+1}{h} + 2^{-h-2}\right) \text{ if } x \in \left(\frac{m+1}{h} - 2^{-h-2}, \frac{m+1}{h}\right] \end{cases},$$

$$u_{2,h}(x) = \begin{cases} \frac{m}{h} + \frac{2^h}{h}\left(x - \frac{m}{h}\right) \text{ if } x \in \left(\frac{m}{h}, \frac{m}{h} + 2^{-h}\right] \\ \frac{m+1}{h} \text{ if } x \in \left(\frac{m}{h} + 2^{-h}, \frac{m+1}{h}\right] \end{cases},$$

where $m = 0, \ldots, (h-1)$. Then u_h is Lipschitz continuous in $(0,1)$ for all $h \in \mathbb{N}$. With a simple calculation we get

$$u_{2,h}'(x) = \begin{cases} \frac{2^h}{h} \text{ if } x \in \left(\frac{m}{h}, \frac{m}{h} + 2^{-h}\right) \\ 0 \text{ if } x \in \left(\frac{m}{h} + 2^{-h}, \frac{m+1}{h}\right) \end{cases},$$

thus $u_{1,h}(x) \cdot u_{2,h}'(x) = 0$ for almost every $x \in (0,1)$. If we denote by $u_1(x) = 1$, $u_2(x) = x$, we have $u_h \to u = (1,x)$ in $L^1((0,1),\mathbb{R}^2)$; in fact

$$\int_0^1 |u_{1,h}(x) - 1| \, dx = \frac{h-1}{2^{h-2}} \to 0\,, \qquad \sup_{x \in (0,1)} \{|x - u_{2,h}(x)|\} \leq \frac{1}{h} - \frac{1}{2^h} \to 0\,.$$

Finally the lower semicontinuity of F does not hold, since $F(u_h, \Omega) = 0$ for every $h \in \mathbb{N}$, while $F(u, \Omega) = 1$. $\square$

With a similar computation as in the previous example, following Eisen [8], we can show that also Lemma 2.1 cannot be extended to the vector-valued setting.

Example 5.2. Let Ω be the open interval $(0,1)$. Let us consider the function $f : \Omega \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, s_1, s_2, \xi_1, \xi_2) = a(x) \cdot b\left(\sqrt{s_1^2 + s_2^2}\right) s_1^2 \cdot c(\xi_2),$$

where $a(x)$ is a Lipschitz continuous function with compact support in $(0,1)$, not identically equal to zero and such that $0 \le a(x) \le 1$; $b : \mathbb{R} \to [0,1]$ is a Lipschitz continuous function with compact support and such that $b(t) = 1$ for every $t \in [1,2]$; finally $c : \mathbb{R} \to \mathbb{R}$ is defined by

$$c(t) = \begin{cases} \frac{1}{2}t^2 & \text{if } |t| \le 1 \\ |t| - \frac{1}{2} & \text{if } |t| > 1 \end{cases}.$$

Then, the function f satisfies the assumption of Lemma 2.1 while, on the same sequence $u_h : \Omega \to \mathbb{R}^2$ of the previous example, the integral $\int_\Omega f(x, u_1, u_2, u_1', u_2')\, dx$ is not lower semicontinuous.

If there exists $p > 1$ such that $f(x,s,\xi) \ge \text{const}\, |\xi|^p$, for some positive constant (i.e., a *coercivity condition* holds for f), then it is clear that the lower semicontinuity in $W^{1,p}_{loc}(\Omega)$ of the integral functional $F(u,\Omega) = \int_\Omega f(x,u,Du)\, dx$ with respect to the strong convergence in $L^1_{loc}(\Omega)$ is equivalent to the weak-$W^{1,p}_{loc}(\Omega)$ lower semicontinuity of $F(u,\Omega)$. Therefore, in this case the lower semicontinuity in $L^1_{loc}(\Omega)$ holds under the only assumption that $f(x,s,\xi)$ is a *Carathéodory function*, i.e., f is *measurable* with respect to $x \in \Omega$ and continuous in $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$, and of course f is also convex with respect to $\xi \in \mathbb{R}^n$. By the next examples 5.3 and 5.4, following [14], we will show that neither continuity nor even lower semicontinuity of $f(x,s,\xi)$ with respect to $x \in \Omega$ are necessary in the case of *superlinear growth* $p > 1$; this fact may happen also if the usual coercivity condition is not satisfied.

Example 5.3. Let Ω be the open interval $(0,1)$. Let $f(x,\xi) = a(x)\, |\xi|^p$ for some $p > 1$, where $a(x)$ is a bounded measurable function in $(0,1)$, with $a(x) \ge 0$ for almost every $x \in (0,1)$. Then the maximum lower semicontinuous (in the strong norm topology of $L^1_{loc}(\Omega)$) functional $\overline{F}_p(u,\Omega)$, less than or equal to $F(u,\Omega) = \int_\Omega a(x)\, |u'|^p\, dx$, is given by

$$\overline{F}_p(u,\Omega) = \int_\Omega b_p(x)\, |u'|^p\, dx, \tag{51}$$

for every $u \in W^{1,p}_{loc}(\Omega)$, where b_p is the bounded measurable function defined in $(0,1)$ by

$$b_p(x) = \liminf_{\varepsilon \to 0^+} \left\{ \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} a(t)^{-\frac{1}{p-1}}\, dt \right\}^{-(p-1)}. \tag{52}$$

Moreover, for every $p > 1$, b_p satisfies the estimates

$$0 \le \overline{a}(x) \le b_p(x) \le a(x), \quad \text{a.e. } x \in \Omega, \tag{53}$$

where $\overline{a}$ is the maximum lower semicontinuous function less than or equal to a in Ω. Thus in particular two sufficient conditions so that the integral

$$F(u,\Omega)=\int_{\Omega}a(x)\,|u'|^{p}\,dx$$

is lower semicontinuous in $W^{1,p}_{loc}(\Omega)$ with respect to the strong norm topology of $L^{1}_{loc}(\Omega)$, are: (i) the coefficient $a(x)$ is lower semicontinuous in Ω; (ii) $a^{-1/(p-1)}\in L^{1}_{loc}(\Omega)$.

Proof. The representation formulas (51), (52) have been established by Marcellini in [14], in the case $p=2$. The proof for general $p\in(1,+\infty)$ is similar.

Let us prove (53). Since $\overline{F}_{p}(u,\Omega)\le F(u,\Omega)$ for every $u\in W^{1,p}_{loc}(\Omega)$, then $b_{p}(x)\le a(x)$ for almost every $x\in\Omega$. If $c(x)$ is a lower semicontinuous function less than or equal to a in Ω, then, for every $x\in\Omega$ and every $\mu>0$ there exists $\delta>0$ such that

$$c(x)\le c(t)+\mu\le a(t)+\mu\,,\qquad \text{a.e.}\, t\in\Omega\cap(x-\varepsilon,x+\varepsilon)\,,\ \ \forall\,\varepsilon\le\delta.$$

Again, for $0<\varepsilon\le\delta$, we deduce that

$$\int_{x-\varepsilon}^{x+\varepsilon}a(t)^{-\frac{1}{p-1}}\,dt\le 2\varepsilon\,(c(x)-\mu)^{-\frac{1}{p-1}}$$

and, from the definition (52) of b_{p},

$$b_{p}(x)=\liminf_{\varepsilon\to0^{+}}\left\{\frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon}a(t)^{-\frac{1}{p-1}}\,dt\right\}^{-(p-1)}\ge c(x)-\mu\,.$$

Thus $b_{p}(x)\ge c(x)$ and also $b_{p}(x)\ge\overline{a}(x)$ for almost every $x\in\Omega$.

From the estimates in (53) we obtain the conclusion (i), i.e., that the integral $F(u,\Omega)=\int_{\Omega}a(x)\,|u'|^{p}\,dx$, being equal to $\overline{F}_{p}(u,\Omega)$, is lower semicontinuous in L^{1}_{loc} if the coefficient a is lower semicontinuous. Finally, if $a^{-\frac{1}{p-1}}\in L^{1}_{loc}(\Omega)$, then, by using the Lebesgue points of this function, we have $b(x)=a(x)$ for almost every $x\in(0,1)$; thus again $F=\overline{F}_{p}$, which proves (ii). □

As in [14], we give below an explicit application of Example 5.3. In particular, given $p>1$ and the integral $F(u,\Omega)=\int_{\Omega}a_{p,s}(x)\,|u'|^{p}\,dx$, where the nonnegative measurable function $a_{p,s}$ is defined below in (54), we show that there exist some values of the real parameter s such that $F(u,\Omega)$ is not lower semicontinuous in the strong norm topology of $L^{1}_{loc}(\Omega)$.

Example 5.4. Let Ω be the open interval $(0,1)$. Let us denote by $\{x_i\}_{i\in\mathbb{N}}$ the set of rational numbers in $(0,1)$ ordered in a sequence and let s be a real parameter. Let $f(x,\xi)=a_{p,s}(x)\,|\xi|^{p}$ for some $p>1$, where $a_{p,s}(x)$ is the bounded measurable nonnegative function in $(0,1)$ defined by

$$a_{p,s}(x)=\frac{1}{\left(1+\sum_{i=1}^{\infty}2^{-i}\,|x-x_i|^{-s}\right)^{p-1}}\,,\qquad x\in(0,1)\tag{54}$$

if the denominator is finite, otherwise we pose $a_{p,s}(x)=0$. Then, for every $s\in\mathbb{R}$, the measurable function $a_{p,s}(x)$ is not identically equal to zero (more precisely, the set

$\{x \in \Omega : a_{p,s}(x) \neq 0\}$ has positive measure), while the integral $F(u,\Omega) = \int_\Omega a_{p,s}(x) |u'|^p \, dx$ is lower semicontinuous in $W^{1,p}_{loc}(\Omega)$ with respect to the strong norm topology of $L^1_{loc}(\Omega)$ if and only if $s < 1$.

Proof. With reference to the representation formulas (51), (52), we will prove that

$$b_{p,s}(x) = \liminf_{\varepsilon \to 0^+} \left\{ \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} a_{p,s}(t)^{-\frac{1}{p-1}} \, dt \right\}^{-(p-1)} = \begin{cases} a_{p,s}(x) & \text{if } s < 1 \\ 0 & \text{if } s \geq 1 \end{cases} .$$

In fact (for simplicity of notations we integrate over the whole interval $(0,1)$)

$$\int_0^1 a_{p,s}(x)^{-\frac{1}{p-1}} \, dx = \int_0^1 \left\{ 1 + \sum_{i=1}^{\infty} 2^{-i} |x - x_i|^{-s} \right\} dx$$

$$= 1 + \sum_{i=1}^{\infty} 2^{-i} \int_0^1 |x - x_i|^{-s} \, dx \leq 1 + \frac{1}{1-s} < +\infty$$

if $s < 1$ (and, in this case $b_{p,s}(x) = a_{p,s}(x)$ for almost every x in $(0,1)$), otherwise the integral is equal to $+\infty$, when computed on any subinterval $(x_0 - \varepsilon, x_0 + \varepsilon)$ too (and, in this case $b_{p,s}(x) = 0$ for almost every x in $(0,1)$).

It remains to show that the set $\{x \in \Omega : a_{p,s}(x) \neq 0\}$ has positive measure. To this aim we observe that, if $s < 1$, by the above condition

$$\int_0^1 a_{p,s}(x)^{-\frac{1}{p-1}} \, dx < +\infty$$

we deduce that $a_{p,s}(x)^{-\frac{1}{p-1}}$ is finite almost everywhere in Ω; therefore $a_{p,s}(x)$ is different from zero almost everywhere in Ω. Otherwise, if $s \geq 1$, we compute similarly

$$\int_0^1 a_{p,s}(x)^{-\frac{1}{2s(p-1)}} \, dx = \int_0^1 \left\{ 1 + \sum_{i=1}^{\infty} 2^{-i} |x - x_i|^{-s} \right\}^{\frac{1}{2s}} dx$$

$$\leq \int_0^1 \left\{ 1 + \sum_{i=1}^{\infty} \left(2^{-i} |x - x_i|^{-s} \right)^{\frac{1}{2s}} \right\} dx \leq 1 + \sum_{i=1}^{\infty} 2^{\frac{-i}{2s}} \int_0^1 |x - x_i|^{-\frac{1}{2}} \, dx < +\infty \, ;$$

thus again $a_{p,s}(x)^{-\frac{1}{2s(p-1)}}$ is finite almost everywhere and $a_{p,s}(x)$ is different from zero almost everywhere in Ω. □

In the next example we consider the limit case $p = 1$. Example 5.5 is due to Fusco [12]. We emphasize here that Example 5.5 can be considered as a passage to the limit from the case $p > 1$ in the formulas of Example 5.3. In fact, analogously to the well known limit relation $\lim_{r \to +\infty} \|v\|_{L^r(\Omega)} = \|v\|_{L^\infty(\Omega)}$, when we replace v by $1/v$, as well known we also have

$$\lim_{r \to +\infty} \left\| v^{-1} \right\|_{L^r(\Omega)}^{-1} = \inf \{ |v(x)| : x \in \Omega \} .$$

Therefore, from (52), we obtain the following representation formula for the maximum lower semicontinuous function $\overline{a}$ less than or equal to a in Ω

$$\begin{aligned} \overline{a}(x) &= \lim_{\varepsilon\to 0^+} \inf\{a(t) : \ t\in(x-\varepsilon,x+\varepsilon)\} \qquad (55)\\ &= \lim_{\varepsilon\to 0^+}\lim_{p\to 1^+}\left\{\int_{x-\varepsilon}^{x+\varepsilon} a(t)^{-\frac{1}{p-1}}\,dt\right\}^{-(p-1)}\\ &= \lim_{\varepsilon\to 0^+}\lim_{p\to 1^+}\left\{\frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} a(t)^{-\frac{1}{p-1}}\,dt\right\}^{-(p-1)}. \end{aligned}$$

As already said, the proof of the statement of the following Example 5.5 can be found in Fusco [12].

Example 5.5. Let Ω be the open interval $(0,1)$. Let $f(x,\xi)=a(x)\,|\xi|$, where $a(x)$ is a bounded measurable function in $(0,1)$, with $a(x)\geq 0$ for almost every $x\in(0,1)$. Then the maximum lower semicontinuous (in L^1_{loc}) functional $\overline{F}_1(u,\Omega)$, less than or equal to $F(u,\Omega)=\int_\Omega a(x)\,|u'|\,dx$, is given by

$$\overline{F}_1(u,\Omega)=\int_\Omega \overline{a}(x)\,|u'|\;dx\,, \qquad (56)$$

for every $u\in W^{1,1}_{loc}(\Omega)$, where $\overline{a}$ is the maximum lower semicontinuous function less than or equal to a in Ω. Thus in particular the integral $F(u,\Omega)=\int_\Omega a(x)\,|u'|\,dx$ is lower semicontinuous in $L^1_{loc}(\Omega)$ if and only if the coefficient a is lower semicontinuous in Ω (that is, if the measurable function a is almost everywhere equal to a lower semicontinuous function in Ω).

Having in mind Examples 5.3 and 5.4, we give the following sufficient condition for lower semicontinuity of $F(u,\Omega)$ in L^1_{loc}. Here we go back to the general n-dimensional case, under the assumption that $f(x,s,\xi)$ is a *Carathéodory* function, i.e., that f is *measurable* with respect to $x\in\Omega\subset\mathbb{R}^n$ and continuous with respect to $(s,\xi)\in\mathbb{R}\times\mathbb{R}^n$. We mention explicitly that the following result holds in the vector-valued setting too.

We thank Giovanni Leoni, who pointed out to us an improvement of a previous version of the following Proposition 5.6.

Proposition 5.6. *Assume that $f(x,s,\xi)$ is a Carathéodory function, convex with respect to ξ, which satisfies the coercivity condition*

$$f(x,s,\xi)\geq a(x)\,|\xi|^p\,,\qquad a.e.\ x\in\Omega,\ \ \forall\,(s,\xi)\in\mathbb{R}\times\mathbb{R}^n, \qquad (57)$$

for some $p>1$, where $a(x)$ is a measurable function in an open set $\Omega\subset\mathbb{R}^n$, $a(x)\geq 0$ for almost every $x\in\Omega$, and such that

$$a^{-\frac{1}{p-1}}\in L^1_{loc}(\Omega). \qquad (58)$$

Then the integral $F(u,\Omega)=\int_\Omega f(x,u,Du)\,dx$ is lower semicontinuous in $W^{1,1}_{loc}(\Omega)$ with respect to the strong convergence in $L^1_{loc}(\Omega)$.

Proof. By Hölder inequality and by the coercivity assumption (57), for every open set Ω' compactly contained in Ω, we have

$$\int_{\Omega'} |Du|\, dx = \int_{\Omega'} a(x)^{1/p}\, |Du| \cdot a(x)^{-1/p}\, dx$$

$$\leq \left\{ \int_{\Omega'} a(x)\, |Du|^p\, dx \right\}^{1/p} \cdot \left\{ \int_{\Omega'} a(x)^{-\frac{1}{p}\cdot\frac{p}{p-1}}\, dx \right\}^{(p-1)/p}$$

$$\leq \left\{ \int_{\Omega'} f\left(x, u, Du\right)\, dx \right\}^{1/p} \cdot \left\{ \int_{\Omega'} a(x)^{-\frac{1}{p-1}}\, dx \right\}^{(p-1)/p}.$$

Let $u_h, u \in W^{1,1}_{loc}(\Omega)$ such that $u_h \to u$ in $L^1_{loc}(\Omega)$. Let us also assume that

$$\liminf_{h\to+\infty} F\left(u_h, \Omega\right) = \lim_{h\to+\infty} F\left(u_h, \Omega\right) = C < +\infty\,.$$

Under such conditions Du_h is a sequence locally equi-integrable in Ω and so u_h weakly converges to u in $W^{1,1}_{loc}(\Omega)$; in fact a satisfies (58) and we have

$$\int_{\Omega'} |Du_h|\, dx \leq C^{1/p} \cdot \left\{ \int_{\Omega'} a(x)^{-\frac{1}{p-1}}\, dx \right\}^{(p-1)/p}.$$

Therefore we can apply the original lower semicontinuity theorem by De Giorgi [6]. $\square$

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