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# On Duality for Minmax Generalized $B$-vex Programming Involving $n$-set Functions

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We consider a minmax programming problem involving several generalized $B$-vex $n$-set functions. Some optimality results and Wolfe type duality theorems are given. As a special case, a $n$-set generalized minmax fractional programming problem is considered.

## 1. Introduction

Many problems containing set functions arise in situations dealing with optimal constrained selection of measurable subsets. Some problems of this type have been encountered in statistics [17, 24], fluid flow [9], electrical insulator design [12], optimal plasma confinement [32] and regional design (districting, facility location, warehouse layout, urban planning) [14, 15].

General theory for optimizing $n$-set functions was first developed by Morris [23] who, for fractions of a single set, obtained results that are similar to the standard mathematical programming problem. Corley [16] developed an optimization theory for programming problems with $n$-set functions, established optimality conditions, and obtained Lagrangian duality. Zalmai [33] considered several practical applications for a class of nonlinear programming problems involving a single objective and differentiable $n$-set functions, and established several sufficient and duality results under generalized $\rho$-convexity conditions.

In [19, 22, 27] there are presented different approaches to define and to characterize the notion of convexity for set or $n$-set functions, and optimality and duality results based on these approaches are obtained.

Bector et al. [4] unified the concept of $B$-vex functions and invex functions, naming such functions as $B$-invex functions. Independently, Jeyakumar and Mond [18] introduced the idea of $V$-invex functions which are similar to $B$-invex functions. Both $B$-invex functions and $V$-invex functions unify the duality of vector valued fractional programs [5, 8, 11, 18]. A useful consequence of $B$-vexity is that pseudolinear multiobjective and minmax programming problems and certain nonlinear multiobjective fractional and minmax (generalized) fractional programming problems do not require a separate treatment for duality, and all results on optimality conditions and duality for them can be derived by using the general concept of $B$-vexity.

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In Preda and Stancu-Minasian [28], and Preda [29, 26] the notions of ($\rho$,b)-vexity and strict ($\rho$,b)-vexity for nondifferentiable and differentiable set functions are introduced. Further, some duality results for multiobjective programming problems and multiobjective fractional programming problems which involve set vectorial functions are given. Also, recently, by using the second order differentiability for set functions defined by Chou, Hsia and Lee [13], Preda [25] defines the notions of bonvexity and generalized bonvexity for $n$-set functions and higher order duality results are established for nonlinear fractional programming involving these functions. A nonlinear multiple objective programming problem is studied by Preda in [30] where optimality and duality conditions are given in terms of the right differentials of the functions. The duality results are stated by using the concepts of generalized semilocally convex functions.

Other important results concerning generalized convexity are obtained by Penot and Sach [31], and a general duality scheme for nonconvex minimization problems is given by Lemaire and Volle in [20].

In the present paper we consider a minmax programming problem involving several generalized $B$-vex $n$-set functions for which we give some optimality results and Wolfe type duality theorems. As a special case, a $n$-set generalized minmax fractional programming problem is considered.

## 2. Notation, Definitions, and Preliminaries

Let $(X, A, \mu)$ be a finite atomless measure space with $L_1(X, A, \mu)$ a separable space. We assume that $S$ is a subset of $A^n = A \times A \times \cdots \times A$, the $n$-fold product of the $\sigma$-algebra $A$ of subsets of a given set $X$. Let $\delta$ be the pseudometric on $A^n$ defined by

$$\delta(R, S) = \left[\sum_{i=1}^{n} \mu^2 (R_i \Delta S_i)\right]^{\frac{1}{2}}$$

with $R = (R_1, R_2, ..., R_n)$, $S = (S_1, S_2, ..., S_n)$, $R_i, S_i \in A$, $\forall i = 1, 2, ..., n$, where $R_i \Delta S_i$ denotes the symmetric difference for $R_i$ and $S_i$. Thus $(A^n, \delta)$ is a pseudometric space which will serve as the domain for most of the functions used in the present paper. Thus $h \in L_i(X, A, \mu)$ and $Z \in A$ with indicator (characteristic) function $I_Z \in L_\infty(X, A, \mu)$; the general integral $\int_Z h \mathrm{d}\mu$ will be denoted by $\langle h, I_Z \rangle$.

We now give some definitions.

**Definition 2.1.** A set function $H : A \to \mathbb{R}$ is differentiable at $S^* \in A$ if there exists $DH(S^*) \in L_1(X, A, \mu)$, called the derivative of $H$ at $S^*$, such that

$$H(S) = H(S^*) + \langle DH(S^*), I_S - I_{S^*} \rangle + V_H(S^*, S)$$

where $V_H(S^*, S)$ is $o[\delta(S^*, S)]$, i.e. $\lim\limits_{\delta(S^*,S)\to 0} \dfrac{V_H(S^*, S)}{\delta(S^*, S)} = 0.$

We now define the differentiation for an $n$-set function.

**Definition 2.2.** Let $F : A^n \to \mathbb{R}$ and $S^* = (S_1^*, S_2^*, ..., S_n^*) \in A^n$. We say that $F$ has a partial derivative at $(S_1^*, S_2^*, ..., S_n^*)$ with respect to its $i$-th argument $S_i$ if the set function

$$H(S_i) = F\left(S_1^*, S_2^*, ..., S_{i-1}^*, S_i, S_{i+1}^*, ..., S_n^*\right)$$

has derivative $DH(S_i^*)$ at $S_i^*$. In this case we define the $i$-th partial derivative of $F$ at $S^*$ to be $D_iF(S^*) = DH(S_i^*)$, $i = 1, 2, ..., n$.

**Definition 2.3.** Let $F : A^n \to \mathbb{R}$ and $S^* = (S_1^*, S_2^*, ..., S_n^*) \in A^n$. We say that $F$ is differentiable at $S^*$ if all the partial derivatives $D_iF(S^*)$, $i = 1, 2, ..., n$, exist and satisfy

$$F(S) - F(S^*) = \sum_{i=1}^{n} \left\langle D_iF(S^*), I_{S_i} - I_{S_i^*} \right\rangle + W_F[S^*, S]$$

where $W_F[S^*, S]$ is $o[\delta(S^*, S)]$ for all $S \in A^n$.

**Definition 2.4.** Let $F : A^n \to \mathbb{R}$ be a differentiable $n$-set function. We say that $F$ is convex (strictly convex) at $S$ if for any $R \in A^n$

$$F(R) - F(S) \geq (>) \sum_{i=1}^{n} \langle D_iF(S), I_{R_i} - I_{S_i} \rangle$$

**Definition 2.5.** We say that the differentiable $n$-set functions $F_j : A^n \to \mathbb{R}$, $j = 1, 2, ..., p$, are additively convex (additively strictly convex) at $S \in A^n$ if for any $R \in A^n$

$$\sum_{j=1}^{p} [F_j(R) - F_j(S)] \geq (>) \sum_{j=1}^{p} \sum_{i=1}^{n} \langle D_iF_j(S), I_{R_i} - I_{S_i} \rangle$$

Let $B_0, B_1, ..., B_p, \alpha_1, ..., \alpha_n : A^n \times A^n \to \mathbb{R}_+ \setminus \{0\}$, $\rho_0, \rho_1, ..., \rho_p : A^n \to \mathbb{R}$, $d : A^n \times A^n \to \mathbb{R}_+$, and let us denote $B = (B_1, ..., B_p)$, $\alpha = (\alpha_1, ..., \alpha_n)$ and $\rho = (\rho_1, ..., \rho_p)$. The function $d$ can (but must not) be the pseudometric $\delta$.

**Definition 2.6.** We say that a differentiable $n$-set function $F : A^n \to \mathbb{R}$ is $B_0$-vex (strictly $B_0$-vex) at $S \in A^n$ if for any $R \in A^n$

$$B_0(R, S)[F(R) - F(S)] \geq (>) \sum_{i=1}^{n} \langle D_iF(S), I_{R_i} - I_{S_i} \rangle$$

**Definition 2.7.** We say that the differentiable $n$-set functions $F_j : A^n \to \mathbb{R}$, $j = 1, 2, ..., p$, are additively $B$-vex (additively strictly $B$-vex) at $S \in A^n$ if for any $R \in A^n$

$$\sum_{j=1}^{p} B_j(R, S)[F_j(R) - F_j(S)] \geq (>) \sum_{j=1}^{p} \sum_{i=1}^{n} \langle D_iF_j(S), I_{R_i} - I_{S_i} \rangle$$

We introduce now more general "vex"-notions for $n$-set functions.

**Definition 2.8.** We say that a differentiable $n$-set function $F : A^n \to \mathbb{R}$ is $(\alpha, B_0, \rho_0; d)$-vex (strictly $(\alpha, B_0, \rho_0; d)$-vex) at $S \in A^n$ if for any $R \in A^n$ we have

$$B_0(R, S)[F(R) - F(S)] \geq (>) \sum_{i=1}^{n} \alpha_i(R, S) \langle D_iF(S), I_{R_i} - I_{S_i} \rangle + \rho_0(S)\, d(R, S)$$

**Definition 2.9.** We say that the differentiable $n$-set functions $F_j : A^n \to \mathbb{R}$, $j = 1, 2, ..., p$, are additively $(\alpha, B, \rho; d)$-vex (additively strictly $(\alpha, B, \rho; d)$-vex) at $S \in A^n$ if for any $R \in A^n$ we have

$$\sum_{j=1}^{p} B_j (R, S) [F_j (R) - F_j (S)] \geq (>)$$
$$\geq (>) \sum_{j=1}^{p} \left( \sum_{i=1}^{n} \alpha_i (R, S) \langle D_i F_j (S), I_{R_i} - I_{S_i} \rangle + \rho_j (S) d (R, S) \right)$$

We say that the properties of the functions given in the Definitions 2.4 - 2.9 are on $A^n$ if those properties hold at any $S \in A^n$.

We say that

1. the differentiable $n$-set function $F$ is concave, $B_0$-cave, or $(\alpha, B_0, \rho_0; d)$-cave on $A^n$ if $(-F)$ is convex, $B_0$-vex or $(\alpha, B_0, \rho_0; d)$-vex on $A^n$, respectively.
2. the differentiable $n$-set functions $F_1, ..., F_p$, are additively concave, additively $B$-cave or additively $(\alpha, B, \rho; d)$-cave on $A^n$ if $(-F_1), ..., (-F_p)$ are additively convex, additively $B$-vex or additively $(\alpha, B, \rho; d)$-vex on $A^n$, respectively.

We note that in the above definitions, if we refer to the "strict" notions, we must take $R \neq S$.

**Remark 2.10.**

1. If we put $B_0 \equiv 1$, Definition 2.6 reduces to Definition 2.4, the definition of a convex (strict convex) function;
2. If we put $B_j \equiv 1$ for $j = 1, 2, ..., p$, Definition 2.7 reduces to Definition 2.5, the definition of additive convex (additive strict convex) functions;
3. For $\rho_0 \equiv 0$ and $\alpha_i \equiv 1$, $i = 1, 2, ..., n$, the Definition 2.8 reduces to Definition 2.6 and Definition 2.9 to Definition 2.7.

In the sequel we shall use the following problem:

$$\begin{array}{rl} \text{Minimize} & F (S_1, S_2, ..., S_n) \\ \text{subject to} & H_k (S_1, S_2, ..., S_n) \leq 0, \quad k = 1, 2, ..., m \\ & (S_1, S_2, ..., S_n) \in A^n \end{array} \tag{NP}$$

**Definition 2.11.** We say that $S^* = (S_1^*, S_2^*, ..., S_n^*) \in A^n$ is a regular feasible solution for (NP) if there exists $\hat{S} = \left(\hat{S}_1, \hat{S}_2, ..., \hat{S}_n\right) \in A^n$ such that

$$H_k (S^*) + \sum_{i=1}^{n} \left\langle D_i H_k (S^*), I_{\hat{S}_i} - I_{S_i^*} \right\rangle < 0 \ , \quad k = 1, 2, ..., m.$$

**Theorem 2.12.** *[16, 23] Let $S^* = (S_1^*, S_2^*, ..., S_n^*)$ be a regular optimal solution of (NP).*

*Then there exists* $u^* = (u_1^*, u_2^*, ..., u_m^*) \in \mathbb{R}_+^m$ *(nonnegative orthant of* $\mathbb{R}^m$*) such that*

$$\left\langle D_i F(S^*) + \sum_{k=1}^{m} u_k^* D_i H_k(S^*), I_{S_i} - I_{S_i^*} \right\rangle \geq 0, \quad \forall S_i \in A,\ i = 1, 2, ..., n$$

$$u_k^* H_k(S^*) = 0, \quad (k = 1, 2, ..., m)$$

$$H_k(S^*) \leq 0, \quad (k = 1, 2, ..., m)$$

## 3. Main Problem and Optimality Conditions

As in [7], we consider the following generalized minmax programming problem (P) involving differentiable $n$-set functions:

$$q^* = \min_{S \in A^n} \max_{1 \leq j \leq p} [Q_j(S)] \tag{P}$$

subject to

$$Q_{jk}(S) \leq 0, \quad j = 1, 2, ..., p \text{ and } k = 1, 2, ..., m \tag{1}$$

$$S = (S_1, S_2, ..., S_n) \in A^n \tag{2}$$

For the problem (P) we suppose the following:

(A1) $A^n$ is the $n$-fold product of a $\sigma$-algebra $A$ of subsets of a given set $X$ and $Q_j$, $Q_{jk}$, $j = 1, 2, ..., p$, $k = 1, 2, ..., m$ are real valued differentiable $n$-set functions defined on $A^n$;

(A2) Each $Q_j$ is $(\alpha, B_j, \rho_j, d)$-vex on $A^n$, and for any $k = 1, 2, ..., m$, each $Q_{jk}$ is $(\alpha, B_j, \rho_j'$, $d)$-vex on $A^n$, $j = 1, 2, ..., p$.

We now consider relative to problem (P), as in [7], the following programming problem (EP) which is equivalent to (P) in the sense of Lemmas 3.1 and 3.2 given below,

$$\text{Minimize } q \tag{EP}$$

subject to

$$\begin{aligned} Q_j(S) &\leq q, \quad j = 1, 2, ..., p, & (3) \\ Q_{jk}(S) &\leq 0, \quad j = 1, 2, ..., p,\ k = 1, 2, ..., m, & (4) \\ S &= (S_1, S_2, ..., S_n) \in A^n. & (5) \end{aligned}$$

**Lemma 3.1.** *[7] Let* $S \in A^n$ *be (P)-feasible. Then there exists* $q$ *such that* $(S, q) \in A^n \times \mathbb{R}$ *is (EP)-feasible, and if* $(S, q) \in A^n \times \mathbb{R}$ *is (EP)-feasible then* $S \in A^n$ *is (P)-feasible.*

**Lemma 3.2.** *[7] Let* $S^* \in A^*$ *be (P)-optimal. Then there exists* $q$ *such that* $(S^*, q) \in A^n \times \mathbb{R}$ *is (EP)-optimal, and if* $(S^*, q) \in A^n \times \mathbb{R}$ *is (EP)-optimal then* $(S^*) \in A^n$ *is (P)-optimal.*

**Remark 3.3.**

1. If (EP) is a convex programming problem, then we can easily derive optimality conditions and Wolfe type duality (see [16, 33]).

2. If we assume that $Q_j$ is as in (A2), then $Q_j - q$, in (3), is not a $(\alpha, B_j, \rho_j, d)$-vex function on $A^n \times \mathbb{R}$. Hence, the following unified frame of minmax fractional programming with $n$-set functions is very useful.

Relative to Remark 3.3(2) we can easily prove the following lemma.

**Lemma 3.4.** *Let $Q_j$ be $(\alpha, B_j, \rho_j, d)$-vex at $S \in A^n$, and let each $Q_{jk}$, $k = 1, 2, ..., m$, be $\left(\alpha, B_j, \rho'_j, d\right)$-vex at $S \in A^n$. We have:*

*(i1) If $\lambda_j \geq 0$, and $y_{jk} \geq 0$, $k = 1, 2, ..., m$, then the function $\lambda_j Q_j + \sum_{k=1}^{m} y_{jk} Q_{jk}$ is an $\left(\alpha, B_j, \lambda_j \left(\rho_j + \rho'_j\right), d\right)$-vex function at $S$;*

*(i2) If $Q_j$, for which $\lambda_j > 0$, is strictly $(\alpha, B_j, \rho_j, d)$-vex at $S$, and/or at least one of $Q_{jk}$, $k = 1, 2, ..., m$, for which the corresponding $y_{jk} > 0$, is strictly $\left(\alpha, B_j, \rho'_j, d\right)$-vex at $S$, then $\lambda_j Q_j + \sum_{k=1}^{m} y_{jk} Q_{jk}$ is an $\left(\alpha, B_j, \lambda_j \left(\rho_j + \rho'_j\right), d\right)$-vex function at $S$.*

In the following theorem we consider a necessary optimality condition for problem (P) stated by Bector and Singh [7].

**Theorem 3.5 (Necessary optimality condition).** *Let $S^* = (S_1^*, ..., S_n^*)$ be a regular (P)-optimal solution. Then there exist $q^* \in \mathbb{R}$, $\lambda^* = \left(\lambda_1^*, ..., \lambda_p^*\right)$, and $y^{*k} = \left(y_{1k}^*, y_{2k}^*, ..., y_{pk}^*\right)$, $k = 1, 2, ..., m$, such that*

$$\left\langle \sum_{j=1}^{p} \lambda_j^* D_i Q_j \left(S^*\right) + \sum_{j=1}^{p} \sum_{k=1}^{m} y_{jk}^* D_i Q_{jk} \left(S^*\right), I_{S_i} - I_{S_i^*} \right\rangle \geq 0 \qquad \forall S_i \in A,\ i = 1, 2, ..., n \tag{6}$$

$$\begin{aligned}
\lambda_j^* \left[Q_j \left(S^*\right) - q^*\right] &= 0, \quad j = 1, 2, ..., p, &(7)\\
y_{jk}^* Q_{jk} \left(S^*\right) &= 0, \quad j = 1, 2, ..., p;\ k = 1, 2, ..., m, &(8)\\
Q_j \left(S^*\right) &\leq q^*, \quad j = 1, 2, ..., p, &(9)\\
Q_{jk} \left(S^*\right) &\leq 0, \quad j = 1, 2, ..., p;\ k = 1, 2, ..., m, &(10)\\
\sum_{j=1}^{p} \lambda_j^* &= 1, &(11)\\
\lambda^* &\in \mathbb{R}_+^p, \quad y^{*k} \in \mathbb{R}_+^p,\ k = 1, 2, ..., m. &(12)
\end{aligned}$$

We state the following hypothesis:

(A3) There exist $\lambda^* \in \mathbb{R}_+^p$, $y^{*k} \in \mathbb{R}_+^p$ $(k = 1, 2, ..., m)$ and $(S^*, q^*) \in A^n \times \mathbb{R}$ such that the relations (6) - (12) are satisfied.

(A4) $\sum_{j=1}^{p} \lambda_j^* \left(\rho_j \left(S^*\right) + \rho'_j \left(S^*\right)\right) \geq 0.$

Now we give a sufficient optimality condition for problem (P).

**Theorem 3.6 (Sufficient Optimality Condition).** *Assume that (A1), (A2), (A3) and (A4) are satisfied. Then $S^*$ is (P)-optimal.*

**Proof.** Using assumption (A2), Lemma 3.4 and Definition 2.8, for all (EP)-feasible $(S, q)$ we have

$$\begin{aligned} B_j(S,S^*)\left[\left(\lambda_j^* Q_j(S) + \sum_{k=1}^m y_{jk}^* Q_{jk}(S)\right) - \left(\lambda_j^* Q_j(S^*) + \sum_{k=1}^m y_{jk}^* Q_{jk}(S^*)\right)\right] \geq \\ \geq \sum_{i=1}^n \alpha_i(S,S^*)\left\langle D_i\left(\lambda_j^* Q_j(S^*) + \sum_{k=1}^m y_{jk}^* Q_{jk}(S^*)\right), I_{S_i} - I_{S_i^*}\right\rangle + \\ + \lambda_j^*\left(\rho_j(S^*) + \rho_j'(S^*)\right) d(S,S^*) \quad \text{for } j = 1,2,...,p \end{aligned} \tag{13}$$

Now, summing both sides of (13) over $j = 1, 2, ..., p$, we get

$$\begin{aligned} \sum_{j=1}^p B_j(S,S^*)\left[\left(\lambda_j^* Q_j(S) + \sum_{k=1}^m y_{jk}^* Q_{jk}(S)\right) - \left(\lambda_j^* Q_j(S^*) + \sum_{k=1}^m y_{jk}^* Q_{jk}(S^*)\right)\right] \geq \\ \geq \sum_{j=1}^p \sum_{i=1}^n \alpha_i(S,S^*)\left\langle D_i\left(\lambda_j^* Q_j(S^*) + \sum_{k=1}^m y_{jk}^* Q_{jk}(S^*)\right), I_{S_i} - I_{S_i^*}\right\rangle + \\ + \sum_{j=1}^p \lambda_j^*\left(\rho_j(S^*) + \rho_j'(S^*)\right) d(S,S^*) \end{aligned} \tag{14}$$

Since $\alpha_i(S, S^*) > 0$ for any $i = 1, 2, ..., n$, and $S, S^* \in A^n$, by (6) we obtain

$$\alpha_i(S,S^*)\left\langle \sum_{j=1}^p \left(\lambda_j^* D_i Q_j(S^*) + \sum_{k=1}^m y_{jk}^* D_i Q_{jk}(S^*)\right), I_{S_i} - I_{S_i^*}\right\rangle \geq 0$$

Summing both sides of this inequality over $i = 1, 2, ..., n$, we get

$$\sum_{i=1}^n \alpha_i(S,S^*)\left\langle \sum_{j=1}^p \left(\lambda_j^* D_i Q_j(S^*) + \sum_{k=1}^m y_{jk}^* D_i Q_{jk}(S^*)\right), I_{S_i} - I_{S_i^*}\right\rangle \geq 0 \tag{15}$$

Using assumption (A4) and inequality (15), the inequality (14) yields

$$\sum_{j=1}^p B_j(S,S^*)\left[\left(\lambda_j^* Q_j(S) + \sum_{k=1}^m y_{jk}^* Q_{jk}(S)\right) - \left(\lambda_j^* Q_j(S^*) + \sum_{k=1}^m y_{jk}^* Q_{jk}(S^*)\right)\right] \geq 0 \tag{16}$$

Now we proceed as in Bector and Singh [7, Theorem 3.2] and obtain that $S^*$ is an optimal solution for problem (P).

**Remark 3.7.**

1. From (13) we note that the Theorem 3.6 can still be proved if assumptions (A2) and (A4) are replaced by:

(A2)' Each $\lambda_j Q_j(S) + \sum_{k=1}^{m} y_{jk} Q_{jk}(S)$ is $(\alpha, B_j, \rho_j, d)$-vex on $A^n$, for $\lambda_j \geq 0$, and $y_{jk} \geq 0$, $j = 1, 2, ..., p$, $k = 1, 2, ..., m$.

(A4)' $\sum_{j=1}^{p} \rho_j(S^*) \geq 0$.

2. From (14) we observe that the Theorem 3.6 also can be proved if the assumptions (A2) and (A4) are replaced by:

(A2)" The functions $\lambda_j Q_j(S) + \sum_{k=1}^{m} y_{jk} Q_{jk}(S)$, for $\lambda_j \geq 0$, and $y_{jk} \geq 0$, $j = 1, 2, ..., p$, $k = 1, 2, ..., m$, are additively $(\alpha, B, \bar{\rho}_0, d)$-vex on $A^n$, where $\bar{\rho}_0 = (\rho_0, ..., \rho_0)$.

(A4)" $\rho_0(S^*) \geq 0$.

3. We observe also that the Theorem 3.6 remains still true if the assumptions (A2) and (A4) are replaced by:

(A2)"' The functions $\lambda_1 Q_1, ..., \lambda_p Q_p$ are additively $(\alpha, B, \rho, d)$-vex on $A^n$ and for any $k = 1, ..., m$, the functions $y_{1k} Q_{1k}, ..., y_{pk} Q_{pk}$ are additively $(\alpha, B, \rho', d)$-vex on $A^n$.

(A4)"' $\sum_{j=1}^{p} \left(\rho_j(S^*) + \rho'_j(S^*)\right) \geq 0$.

## 4. A Dual Problem and some Duality Results

In this section we consider a dual problem of (EP) used by Bector and Singh [7] and under general assumptions on $Q_j$ and $Q_{jk}$, $j = 1, 2, ..., p$; $k = 1, 2, ..., m$, some duality results are given.

We shall use, as in [7], the following notational convenience:

$$\lambda = (\lambda_1, \lambda_2, ..., \lambda_p) \in \mathbb{R}^p_+, \quad T = (T_1, T_2, ..., T_n) \in A^n,$$

$$S = (S_1, S_2, ..., S_n) \in A^n$$

$$Y = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1m} \\ y_{21} & y_{22} & \cdots & y_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ y_{p1} & y_{p2} & \cdots & y_{pm} \end{bmatrix} \in \mathbb{R}^{p \times m}$$

is the matrix of Lagrange multipliers for the constraints of (EP),

$$\begin{aligned} y^j &= (y_{j1}, y_{j2}, ..., y_{jm}), \quad j = 1, 2, ..., p \\ Q_j(S) &= Q_j(S_1, S_2, ..., S_n), \quad j = 1, 2, ..., p \\ Q_{jk}(S) &= Q_{jk}(S_1, S_2, ..., S_n), \quad j = 1, 2, ..., p;\ k = 1, 2, ..., m \\ L_j\left(T, \lambda_j, y^j\right) &= \lambda_j Q_j(T_1, T_2, ..., T_n) + \sum_{k=1}^{m} y_{jk} Q_{jk}(T_1, T_2, ..., T_n) \end{aligned}$$

In view of the above notation, (EP) becomes

$$\text{Minimize } q \tag{EP}$$

subject to

$$\begin{aligned} Q_j(S) &\leq q, \quad j = 1,2,...,p && (17) \\ Q_{jk}(S) &\leq 0, \quad j = 1,2,...,p; \ k = 1,2,...,m && (18) \\ S &\in A^n \end{aligned}$$

The dual problem (ED) for the minimization problem (EP) is the following maximization problem

$$\text{Maximize } v \tag{ED}$$

subject to

$$\left\langle \sum_{j=1}^{p} D_i \left( L_j \left( T, \lambda_j, y^j \right) \right), I_{S_i} - I_{T_i} \right\rangle \geq 0, \quad \text{for all } S \in A^n, \quad i = 1,2,...,n \tag{19}$$

$$L_j \left( T, \lambda_j, y^j \right) \geq \lambda_j v, \quad (j = 1,2,...,p) \tag{20}$$

$$\sum_{j=1}^{p} \lambda_j = 1 \tag{21}$$

$$\lambda \in \mathbb{R}_+^p, \ Y \in \mathbb{R}_+^{p \times m}, \ v \in \mathbb{R}, \ y^j \in \mathbb{R}^m, \ T \in A^n \tag{22}$$

We state the following conditions:

(H1) The functions $Q_j$, $j = 1,2,...,p$, are $(\alpha, B_j, \rho_j, d)$-vex and $Q_{jk}$ $(k = 1,2,...,m)$ are $\left(\alpha, B_j, \rho'_{jk}, d\right)$-vex on all feasible solutions of (P) and (ED).

(H2) $\displaystyle\sum_{j=1}^{p} \left( \lambda_j \rho_j(T) + \sum_{k=1}^{m} y_{jk} \rho'_{jk}(T) \right) \geq 0.$

**Theorem 4.1 (Weak Duality).** *Let $S \in A^n$ be (P)-feasible and $(\lambda, v, T, Y)$ be (ED)-feasible. We suppose that (A1), (H1) and (H2) are satisfied. Then $v \leq q$.*

**Proof.** Using (H1), for $j = 1,2,...,p$ we have

$$B_j(S,T)\left(Q_j(S) - Q_j(T)\right) \geq \sum_{i=1}^{n} \alpha_i(S,T) \left\langle D_i Q_j(T), I_{S_i} - I_{T_i} \right\rangle + \rho_j(T) d(S,T) \tag{23}$$

and further, for $k = 1,2,...,m$ we have

$$B_j(S,T)\left(Q_{jk}(S) - Q_{jk}(T)\right) \geq \sum_{i=1}^{n} \alpha_i(S,T) \left\langle D_i Q_{jk}(T), I_{S_i} - I_{T_i} \right\rangle + \rho'_{jk}(T) d(S,T) \tag{24}$$

Using (22) and the relations (23) and (24) we get

$$\begin{aligned} &\sum_{j=1}^{p} B_j(S,T)\left[L_j\left(S, \lambda_j, y^j\right) - L_j\left(T, \lambda_j, y^j\right)\right] \geq \\ &\geq \sum_{i=1}^{n} \alpha_i(S,T) \sum_{j=1}^{p} \left\langle D_i\left(L_j\left(T, \lambda_j, y^j\right)\right), I_{S_i} - I_{T_i} \right\rangle + \\ &\quad + \sum_{j=1}^{p} \left( \lambda_j \rho_j(T) + \sum_{k=1}^{m} y_{ij} \rho'_{jk}(T) \right) d(S,T) \end{aligned} \tag{25}$$

According to (19), (25), (H2) and $\alpha > 0$, we obtain

$$\sum_{j=1}^{p} B_j(S,T)\left[L_j\left(S,\lambda_j,y^j\right) - L_j\left(T,\lambda_j,y^j\right)\right] \geq 0 \tag{26}$$

Now, using (26), we proceed as in Bector and Singh [7, Theorem 4.1] and the theorem is proved.

**Remark 4.2.** The Theorem 4.1 holds if we replace (H1) and (H2) by any of the following couple of hypotheses:

(H1)' For any $j = 1, 2, ..., p$, $L_j(T, \lambda_j, y^j)$ with $\lambda_j \geq 0$, $y^j \in \mathbb{R}^m_+$, is an $(\alpha, B_j, \rho_j, d)$-vex function on all feasible solutions of (P) and (ED).

(H2)' $\sum_{j=1}^{p} \rho_j(T) \geq 0$

(H1)" For $\lambda_j \geq 0$, $y^j \in \mathbb{R}^m_+$, $j = 1, 2, ..., p$, the functions $L_1(T, \lambda_1, y^1), ..., L_p(T, \lambda_p, y^p)$ are additively $(\alpha, B, \bar{\rho}_0, d)$-vex functions, where $\bar{\rho}_0 = (\rho_0, ..., \rho_0)$.

(H2)" $\rho_0(T) \geq 0$.

(H1)"' The functions $\lambda_1 Q_1(T), ..., \lambda_p Q_p(T)$ are additively $(\alpha, B, \rho, d)$-vex and for any $k = 1, ..., m$, the functons $\lambda_{1k} Q_{1k}(T), ..., \lambda_{pk} Q_{pk}(T)$ are additively $(\alpha, B, \rho'_k, d)$-vex, where $\rho'_k = \left(\rho'_{1k}, ..., \rho'_{pk}\right)$.

(H2)"' $\sum_{j=1}^{p}\left(\rho_j(T) + \sum_{k=1}^{m} \rho'_{jk}(T)\right) \geq 0.$

Using the Theorem 4.1 we get the following result.

**Corollary 4.3.** *We suppose that the conditions in the weak duality Theorem 4.1 hold. Let $(S^*, q^*)$ be (EP)-feasible and $(\lambda^*, v^*, T^*, Y^*)$ be (ED)-feasible with $q^* = v^*$. Then $S^*$ is (P)-optimal, and $(\lambda^*, v^*, T^*, Y^*)$ is (ED)-optimal.*

Using the above corollary and following the lines of Bector and Singh [7, Theorem 4.2], we obtain the following strong duality result.

**Theorem 4.4 (Strong Duality).** *We suppose that between (P) and (ED) hold a week duality result, and that $(S^*, q^*) \in A^n \times \mathbb{R}$ is (EP)-optimal. Then there exist $\lambda^* \in \mathbb{R}^p$, $Y^* \in \mathbb{R}^{p \times m}$, $\lambda^* \geq 0$, $Y^* \geq 0$ such that $(\lambda^*, q^*, S^*, Y^*)$ is (ED)-optimal, and the objective value of (EP) at $(S^*, q^*)$ is equal to the objective value of (ED) at $(\lambda^*, q^*, S^*, Y^*)$.*

To obtain a strict converse duality result of Mangasarian type [21], we consider two other hypotheses:

(H3) The functions $Q_j$, $j = 1, 2, ..., p$, are $(\alpha, B_j, \rho_j, d)$-vex and $Q_{jk}$, $k = 1, 2, ..., m$, are $\left(\alpha, B_j, \rho'_{jk}, d\right)$-vex.

(H4) $\sum_{j=1}^{p}\left(\lambda_j \rho_j(T) + \sum_{k=1}^{m} y_{jk} \rho'_{jk}(T)\right) > 0.$

**Theorem 4.5 (Strict Converse Duality).** *Let $(S^*, q^*) \in A^n \times \mathbb{R}$ be an optimal solution of (EP) and $(\lambda^*, v^*, T^*, Y^*)$ be (ED)-optimal. We assume that for all feasible solutions of (P) and (ED) the conditions (H3) and (H4) hold. Then $(T^*, v^*) = (S^*, q^*)$, i.e., $(T^*, v^*)$ is (EP)-optimal.*

**Proof.** We proceed by contradiction, i.e. we suppose that $(T^*, v^*) \neq (S^*, q^*)$. Since $(\lambda^*, v^*, T^*, Y^*)$ is (ED)-optimal and $(S^*, q^*)$ is (EP)-optimal, we have: $q^* = v^*$.

Now, from $(T^*, v^*) \neq (S^*, q^*)$ we have $T^* \neq S^*$. Also, since $(S^*, q^*)$ is optimal for (EP), there exist $\hat{\lambda} \in \mathbb{R}^p$, $\hat{Y} \in \mathbb{R}^{p \times m}$ such that $\left(\hat{\lambda}, v^*, T^*, \hat{Y}\right)$ is (ED)-optimal, and the relations (6) - (12) hold at $\left(\hat{\lambda}, v^*, T^*, \hat{Y}\right)$.

Using the Lemma 3.4 and the assumptions (H3) and (H4), we obtain

$$\begin{aligned}
&\sum_{j=1}^{p} B_j\left(S^*, T^*\right)\left[L_j\left(S^*, \lambda_j^*, y^{*j}\right) - L_j\left(T^*, \hat{\lambda}_j, \hat{y}^j\right)\right] \geq \\
&\geq \sum_{i=1}^{n} \alpha_i\left(S^*, T^*\right) \sum_{j=1}^{p} \left\langle D_i L_j\left(T^*, \hat{\lambda}_j, \hat{y}^j\right), I_{S_i^*} - I_{T_i^*}\right\rangle + \\
&\quad + \sum_{j=1}^{p}\left(\lambda_j \rho_j\left(T^*\right) + \sum_{k=1}^{m} y_{jk} \rho_{jk}'\left(T^*\right)\right) > 0
\end{aligned}$$

Using this relation we now proceed as in [7, Theorem 4.3] and the theorem is proved.

## 5. Application to the Case of the generalized fractional programming

As an application of the results stated in the previous sections, we consider in this section the case of a minmax generalized fractional programming problem involving $n$-set functions, defined by

$$q^* = \min_{S \in A^n} \max_{1 \leq j \leq p} \left(\frac{F_j(S)}{G_j(S)}\right) \tag{GFP}$$

subject to

$$\begin{aligned}
H_k(S) &\leq 0, \quad k = 1, 2, ..., m \\
S &\in A^n
\end{aligned}$$

Relative to (GFP) we consider, according to Bector [1, 6], Bector et al. [2], Chandra et al. [10], and Bector and Singh [7], the following transformed generalized fractional programming problem

$$q^* = \min_{S \in A^n} \max_{1 \leq j \leq p} \left(\frac{F_j(S)}{G_j(S)}\right) \tag{TGFP}$$

subject to

$$\begin{aligned}
Q_{jk}(S) &\leq 0, \quad j = 1, 2, ..., p, \; k = 1, 2, ..., m \\
S &\in A^n
\end{aligned}$$

where $Q_{jk}(S) = \dfrac{H_k(S)}{G_j(S)}$.

The following lemma relates (GFP) and (TGFP).

**Lemma 5.1.** *[7] (i1) $S \in A^n$ is (GFP)-feasible if and only if it is (TGFP)-feasible.*
*(i2) $S \in A^n$ is (GFP)-optimal if and only if it is (TGFP)-optimal.*

Now we give the following lemmas that are useful in the (GFP)-duality. The proofs of these lemmas are not too difficult, and hence we omit them.

**Lemma 5.2.** *Let $F, G : A^n \to \mathbb{R}$ be differentiable functions and let $Q = \dfrac{F}{G}$. We suppose*
*(i1) $F$ is $(\alpha, B, \rho', d)$-vex at $S$ and nonnegative;*
*(i2) $G$ is $(\alpha, B, \rho'', d)$-cave at $S$ and strict positive.*
*Then $Q$ is a $(\alpha, B', \rho'', d)$-vex at $S$ where*

$$B'(R,S) = \frac{G(R)}{G(S)},\ \rho''(S) = \frac{\rho(S)\,G(S) + \rho'(S)\,F(S)}{G^2(S)}$$

*Further, if the function in the numerator is strictly $(\alpha, B, \rho', d)$-vex and/or the function in the denominator is strictly $(\alpha, B, \rho'', d)$-cave at $S$, then $Q$ is a strictly $(\alpha, B', \rho'', d)$-vex at $S$.*

**Lemma 5.3.** *Let us consider, as in (GFP), that for $j = 1, 2, ..., p$ we have: $F_j \geq 0$ and $\left(\alpha, B, \rho_j^1, d\right)$-vex; $G_j > 0$ and $\left(\alpha, B, \rho_j^2, d\right)$-cave; and for $k = 1, 2, ..., m$, $H_k$ is $(\alpha, B, \rho_k^3, d)$-vex. Then $\lambda_j F_j(S) + \sum_{k=1}^{m} y_{jk} H_k(S)$ is an $(\alpha, B, \rho_j, d)$-vex function, where*

$$\rho_j(S) = \lambda_j \rho_j^1(S) + \sum_{k=1}^{m} y_{jk} \rho_j^2(S). \tag{27}$$

**Lemma 5.4.** *Let $F_j$, $G_j$ and $H_k$, $j = 1, 2, ..., p$, $k = 1, 2, ..., m$, be as in (GFP). Then $U_j(S) = \dfrac{\lambda_j F_j(S) + \sum_{k=1}^{m} y_{jk} H_k(S)}{G_j(S)}$ is an $\left(\alpha, B_j, \rho_j', d\right)$-vex function with $B_j(R,S) = \dfrac{B(R,S)\,G_j(R)}{G_j(S)}$ and*

$$\rho_j'(S) = G_j(S)\,\rho_j(S) + \frac{\left(\lambda_j F_j(S) + \sum_{k=1}^{m} y_{jk} H_k(S)\right)\rho_j^2(S)}{G_j^2(S)},$$

*where $\rho_j(S)$ is given by (27).*
*For any $j = 1, 2, ..., p$, if at least one function in the numerator is of strict type, and/or at least one function in the denominator is also of strict type on $A^n$, then the functions $U_j(S)$, $j = 1, 2, ..., p$ are of additive strict type on $A^n$.*

Taking for $j = 1, 2, ..., p$,

$$L_j\left(T, \lambda_j, y^j\right) = \frac{\lambda_j F_j(T) + \sum_{k=1}^{m} y_{jk} H_k(T)}{G_j(T)} ,$$

we can easily consider some assumptions as in Lemmas 5.1 - 5.4, and the results of Sections 2, 3 and 4 become applicable to (GFP).

Also, using the (ED)-problem, we can easily remark that the problems (GFD1), (GFD2), (GFD3) and (GFD) are dual problems to (GFP)-problem:

$$\text{Maximize } v \tag{GFD1}$$

subject to

$$\begin{aligned} \left\langle \sum_{j=1}^{p} D_i\left(L_j\left(T, \lambda_j, y^j\right)\right), I_{S_i} - I_{T_i} \right\rangle &\geq 0 \quad \text{for all } S \in A^n, \quad i = 1, 2, ..., n \\ \left(\lambda_j F_j(T) + \sum_{k=1}^{m} y_{jk} H_k(T)\right) \Big/ G_j(T) &\geq \lambda_j v , \quad j = 1, 2, ..., p \\ \sum_{j=1}^{p} \lambda_j &= 1 \end{aligned}$$

$$\lambda \in \mathbb{R}_+^p,\ Y \in \mathbb{R}_+^{p\times m},\ v \in \mathbb{R},\ y^j \in \mathbb{R}^m,\ T \in A^n.$$

$$\max \sum_{j=1}^{p} L_j\left(T, \lambda_j, y^j\right) = \sum_{j=1}^{p} \left[\left(\lambda_j F_j(T) + \sum_{k=1}^{m} y_{jk} H_k(T)\right) \Big/ G_j(T)\right] \tag{GFD2}$$

subject to

$$\begin{aligned} \left\langle \sum_{j=1}^{p} D_i\left(L_j\left(T, \lambda_j, y^j\right)\right), I_{S_i} - I_{T_i} \right\rangle &\geq 0 \quad \text{for all } S \in A^n, \quad i = 1, 2, ..., n \\ \sum_{j=1}^{p} \lambda_j &= 1 \end{aligned}$$

$$\lambda \in \mathbb{R}_+^p,\ Y \in \mathbb{R}_+^{p\times m},\ v \in \mathbb{R},\ y^j \in \mathbb{R}^m,\ T \in A^n.$$

$$\max \left\{ \sum_{j=1}^{p} \left(\lambda_j F_j(T) + \sum_{k=1}^{m} y_{jk} H_k(T)\right) \Big/ \left(\sum_{j=1}^{p} \lambda_j G_j(T)\right) \right\} \tag{GFD3}$$

subject to

$$\begin{aligned} \left\langle \sum_{j=1}^{p} D_i\left(L_j\left(T, \lambda_j, y^j\right)\right), I_{S_i} - I_{T_i} \right\rangle &\geq 0 \quad \text{for all } S \in A^n, \quad i = 1, 2, ..., n \\ \sum_{j=1}^{p} \lambda_j &= 1 \end{aligned}$$

$$\lambda \in \mathbb{R}_+^p,\ Y \in \mathbb{R}_+^{p\times m},\ v \in \mathbb{R},\ y^j \in \mathbb{R}^m,\ T \in A^n.$$

$$\min_{T\in A^n} \max_{1\le j\le p} \left(\lambda_j F_j(T) + \sum_{k=1}^{m} y_{jk} H_k(T)\right) \Bigg/ \left(\sum_{j=1}^{p} \lambda_j G_j(T)\right) \tag{GFD}$$

subject to

$$\begin{aligned} \left\langle \sum_{j=1}^{p} D_i\left(L_j\left(T,\lambda_j,y^j\right)\right), I_{S_i} - I_{T_i} \right\rangle &\geq 0 \quad \text{for all } S \in A^n, \quad i = 1,2,...,n \\ \sum_{j=1}^{p} \lambda_j &= 1 \end{aligned}$$

$$\lambda \in \mathbb{R}_+^p,\ Y \in \mathbb{R}_+^{p\times m},\ v \in \mathbb{R},\ y^j \in \mathbb{R}^m,\ T \in A^n.$$

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## References

[1] C. R. Bector: Duality in nonlinear fractional programming, Z. Oper. Res. 17 (1973) 183–193.

[2] C. R. Bector, S. Chandra, M. K. Bector: Generalized fractional programming duality: a parametric approach, J. Optim. Theory Appl. 60 (1989) 243–260.

[3] C. R. Bector, C. Singh: $B$-vex functions, J. Optim. Theory Appl. 71 (1991) 237–253.

[4] C. R. Bector, S. K. Suneja, C. S. Lalitha: Generalized $B$-vex functions and generalized $B$-vex programming, J. Optim. Theory Appl. 76 (1993) 561–576.

[5] C. R. Bector, M. Singh: Nonlinear multiobjective duality involving $B$-vex $n$-set functions, J. Math. Anal. Appl. 202 (1996) 701–726.

[6] C. R. Bector: Duality in nondifferentiable generalized fractional programming involving $n$-set functions, Utilitas Math. 45 (1994) 91–96.

[7] C. R. Bector, M. Singh: Duality for minmax $B$-vex programming involving $n$-set functions, J. Math. Anal. Appl. 215 (1997) 112–131.

[8] C. R. Bector, D. Bhatia, S. Pandey: Duality for multiobjective fractional programming involving $n$-set functions, J. Math. Anal. Appl. 186 (1994) 747–768.

[9] D. Begis, R. Glowinski: Application de la méthode des elements fini a l'approximation d'un problème de domaine optimal. Méthodes de résolution de problèmes approchés, Applied Math. Optim. 2 (1975) 130–169.

[10] S. Chandra, B. D. Craven, B. Mond: Generalized fractional programming duality: a ratio game approach, J. Austral. Math. Soc., Ser. B 28 (1986) 170–180.

[11] S. Chandra, B. D. Craven, B. Mond: Vector-valued Lagrangian and multiobjective fractional programming duality, Numer. Funct. Anal. Optim. 11 (1990) 239–254.

[12] J. Cea, A. Gioan, J. Michel: Quelque resultats sur l'identification de domaines, Calcolo 10 (1973) 133–145.

[13] J. H. Chou, W. S. Hsia, T. Y. Lee: On multiple objective programming problems with set functions, J. Math. Anal. Appl. 15 (1985) 333–394.

[14] H. W. Corley, S. D. Roberts: Duality relationships for a partitioning problem, SIAM J. Appl. Math. 23 (1972) 490–494.

[15] H. W. Corley, S. D. Roberts: A partitioning problem with applications in regional design, Operations Res. 20 (1982) 1010–1019.

[16] H. W. Corley: Optimization theory for $n$-set functions, J. Math. Anal. Appl. 127 (1987) 193–205.

[17] G. Dantzig: A. Wald, On the fundamental lemma of Neyman and Pearson, Ann. Math. Stat. 22 (1951) 87–93.

[18] V. Jeyakumar, B. Mond: On generalized convex mathematical programming, J. Austral. Math. Soc., Ser. B 34 (1992) 43–53.

[19] H. C. Lai, S. S. Yang, G. R. Hwang: Duality in mathematical programming of set functions: on Fenchel duality theorem, J. Math. Anal. Appl. 95(1) (1983) 223–234.

[20] B. Lemaire, M. Volle: A general duality scheme for nonconvex minimization problems with a strict inequality constraint, J. Global Optimization 13 (1998) 317–327.

[21] O. L. Mangasarian: Nonlinear Programming, McGraw-Hill, New York (1969).

[22] P. Mazzoleni: On constrained optimization for convex set functions, in: Survey of Mathematical Programming, Vol. 1, A. Prekopa (ed.), North-Holland, Amsterdam (1979) 273–290.

[23] R. J. T. Morris: Optimal constrained selection of a measurable subset, J. Math. Anal. Appl. 70 (1979) 546–562.

[24] J. Neyman, E. S. Pearson: On the problem of the most efficient tests of statistical hypotheses, Philos. Trans. Roy. Soc. London, Ser. A 231 (1933) 289–337.

[25] V. Preda: Duality for multiobjective fractional programming problems involving $n$-set functions, in: Analysis and Topology, C. Andreian Cazacu, W. E. Lehto, T. M. Rassias (eds.) (1998) 569–583.

[26] V. Preda: Some optimality conditions for multiobjective programming problems with set functions, Rev. Roumaine Math. Pures Appl. 39(3) (1994) 233–247.

[27] V. Preda: On minmax programming problems containing $n$-set functions, Optimization 22(4) (1991) 527–537.

[28] V. Preda, I. M. Stancu-Minasian: On duality for multiobjective mathematical programming of $n$-set functions, Report no. 85, Pisa, Italy (1994).

[29] V. Preda: On duality of multiobjective fractional measurable subsets selection problems, J. Math. Anal. Appl. 196 (1995) 514–525.

[30] V. Preda: Optimality conditions and duality in multiple objective programming involving semilocally convex and related functions, Optimization 36(3) (1996) 219–230.

[31] P. H. Sach, J. P. Penot: Characterizations of generalized convexities via generalized directional derivatives, Numer. Funct. Anal. Optimization 19(5-6) (1998) 615–634.

[32] P. K. C. Wang: On a class of optimization problems involving domain variations, in: Lecture Notes in Control and Information Sciences 2, Springer Verlag, Berlin (1977) 49–60.

[33] G. J. Zalmai: Sufficiency criteria and duality for nonlinear programs involving $n$-set functions, J. Math. Anal. Appl. 149 (1990) 322–338.