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Homogenization of Elastic Thin Structures: A Measure-Fattening Approach

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We study the homogenization of vector problems on thin periodic structures in $\mathbb{R}^n$. The analysis is carried out within the same measure framework that we previously enforced in [4] for scalar problems, namely each periodic, low-dimensional structure is identified with the overlying positive Radon measure μ. Thus, we deal with a sequence of measures $\{\mu_\varepsilon\}$, whose periodicity cell has size ε converging to zero, and our aim is to identify the limit, in the variational sense of Γ-convergence, of the elastic energies associated to μ_ε. We show that the explicit formula for such homogenized functional can be obtained combining the application of a two-scale method with respect to measures, and a fattening approach; actually, it turns out to be crucial approximating μ by a sequence of measures $\{\mu_\delta\}$, where δ is an auxiliary, infinitesimal parameter, associated to the thickness of the structure. In particular, our main result is proved under the assumption that the structure is asymptotically not too thin (*i.e.* $\delta \gg \varepsilon$), and, for all $\delta > 0$, μ_δ satisfy suitable *fatness* conditions, which generalize the *connectedness* hypotheses needed in the scalar case. We conclude by pointing out some related problems and conjectures.

Keywords: Thin structures, homogenization, two-scale convergence, periodic measures

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1. Introduction

In this paper we deal with the homogenization of elasticity problems on thin structures. Following the approach proposed in [4], we identify periodic structures to periodic positive measures on $\mathbb{R}^n$; for instance, a periodic network of bars may be represented through the overlying one-dimensional Hausdorff measure. Thus, for a given periodic Radon measure μ on $\mathbb{R}^n$, we are led to study the asymptotic behaviour, as the positive parameter ε tends to zero, of a sequence of functionals of the type

$$J_\varepsilon(u) := \int_\Omega j\left(e(u)\right)\, d\mu_\varepsilon\ , \qquad u \in \mathcal{C}^1_0(\Omega; \mathbb{R}^n)\ , \tag{1}$$

where Ω is a bounded open subset of $\mathbb{R}^n$, $e(u)$ is the symmetric part of the gradient of u, and μ_ε are the rescaled measures $\mu_\varepsilon(B) := \varepsilon^n \mu\left(\frac{B}{\varepsilon}\right)$; the integrand $j(z)$ is assumed to be convex, to depend only on the symmetric part $z^* := \frac{z+z^T}{2}$ of the matrix $z \in \mathbb{R}^{n^2}$ and to

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satisfy standard p-growth conditions, i.e., for all $z \in \mathbb{R}^{n^2}$ and suitable positive constants c, C,

$$c\,|z^*|^p \;\leq\; j(z^*) \;=\; j(z) \;\leq\; C(1 + |z^*|^p)\,. \tag{2}$$

For instance, in the case of an isotropic linearly elastic body, j will take the form $j(z) = \beta|z^*|^2 + \frac{\alpha}{2}|\operatorname{tr}(z^*)|^2$, being α, β the so-called *Lamé constants*.

In [4] we treated the scalar case: when u belongs to $\mathcal{C}^1_0(\Omega)$ and $e(u)$ is replaced by ∇u in (1), we proved a quite general homogenization theorem for the limit of the energies $\{J_\varepsilon\}$ in the variational sense of Γ-convergence [8]. The method we used relies on a two-scale structure result for sequences of gradients, holding under suitable connectedness assumptions on the measure μ. Moreover, we investigated the case of reinforced structures. Such kind of periodic structures are made by bars (or layers) of a very small cross-section (or thickness) δ, so that the associated measures μ_δ are absolutely continuous with respect to the Lebesgue measure, and they weakly converge, as δ tends to zero, to a measure μ supported on a thin structure (the zero thick skeleton). More precisely, given a periodic measure μ concentrated on a low-dimensional set in $\mathbb{R}^n$, we think of μ_δ as the convolution of μ with a sequence of mollifiers $\{\rho_\delta\}$, or else as the rescaled Lebesgue measure over a δ-tubular neighbourhood of the support of μ. In this situation, we showed that the connectedness of μ guarantees the commutativity of the limit process with respect to ε (the *periodicity* parameter) and δ (the *fattening* parameter).

In occasion of the French-German-Italian Conference on Optimization held in Montpellier on September 2000, Gérard Michaille asked us a question about the possibility of extending our results to the vector case. As far as we knew, it would have been reasonable to expect a positive answer; this was partially motivated by the fact that a relaxation theorem for the stored energy functional of elasticity had already been proved in [2].

In this paper, we show that actually the same two-scale technique, considered with respect to a general measure μ, may be fruitfully employed to handle the problem also in the vector case; nevertheless, attention must be paid because some remarkable different features come into light.

Actually, one can adapt the approach of [2] to the periodic setting, thus recovering the space of all finite energy displacements; moreover a two-scale structure theorem for all the possible two-scale limits of symmetric parts of gradients can be proved. In spite, the assumptions that are needed on μ for the validity of such result are so stringent that in practise they are never satisfied by thin structures.

This fact leads to reconsider the δ-fattening approach, which in the scalar case was unessential in view of the above mentioned commutativity result [4, Theorem 6.1]. At this regard, the behaviour of the elasticity case can be summarized as follows:

- the δ-fattening approach is in some sense necessary to handle the problem, as otherwise the Γ-limit of the sequence of energies given by (1) may degenerate to zero, even for quite simple measures μ;
- the passage to the limit with respect to the periodicity and the thickness parameters fails in general to be commutative;
- for every fixed δ, a homogenization result holds in suitable generality replacing the measure μ in (1) by the approximating measures μ_δ defined as above; so the most appropriated procedure seems to be first apply such homogenization result to each

μ_δ, and secondly pass to the limit as δ tends to zero; let us mention that such a procedure has been extensively studied using quite different techniques (based on the existence of extension operators) for which we refer to [7] and references therein;

- if we let δ depend on ε, and we let ε (and $\delta(\varepsilon)$ at the same time) go to zero, it would be interesting to characterize the rate of convergence to zero of the function $\delta = \delta(\varepsilon)$ which ensures a non-vanishing limit energy; in this case, we would like such energy to coincide with the Γ-limit obtained by the method described in the previous item, that is making tend to zero ε and δ in the order.

Each of these aspects of the problem is discussed therein. The outline is the following.

In Section 2, we introduce some notation and preliminaries; in particular, we give a short survey of our earlier results on scalar homogenization.

Section 3 is devoted to the relaxation of elastic energies, via suitable notions of admissible displacements and tangential strain operator.

In Section 4, we deal with the homogenization of vector problems: in Section 4.1 we exploit the two-scale technique to find the Γ-limit of the functionals J_ε; this can be done under suitable assumptions on μ, which are shown to fail for thin structures in Section 4.2.

In Section 5, we focus on the fattening approach, which allows to use fruitfully the result found in Section 4.1 provided ε first tend to zero (see Theorem 5.2) or $\delta(\varepsilon) \gg \varepsilon$ (see Theorem 5.4).

In Section 6 we discuss some related problems and conjectures. Throughout the following sections, some of the proofs are omitted since they follow by minor changes from the results of our previous paper [4], to which the reader is referred.

2. Notation and preliminaries

We begin by introducing some notation which are used throughout the paper.

Let Ω be an open bounded subset of $\mathbb{R}^n$ with boundary $\partial\Omega$ of class $\mathcal{C}^1$, let $\mathbb{R}^{n^2}_{\rm sym}$ be the space of $n \times n$ symmetric real matrices, and let p, $p' \in (1, +\infty)$ be fixed conjugated exponents. If not otherwise specified, j will denote a real convex integrand on $\mathbb{R}^{n^2}_{\rm sym}$ satisfying (2).

In the following, μ will always be a positive, Y-periodic Radon measure on $\mathbb{R}^n$, where $Y := (0,1)^n$; we assume for simplicity the not restrictive conditions $\mu(Y) = 1$ and $\mu(\partial Y) = 0$. For every $\varepsilon > 0$, we let μ_ε be the (εY)-periodic measure defined by

$$\int_\Omega \varphi(x)\, d\mu_\varepsilon(x) := \varepsilon^n \int_{\Omega/\varepsilon} \varphi(\varepsilon x)\, d\mu(x) \qquad \forall \varphi \in \mathcal{C}_0(\Omega)\ , \tag{3}$$

being $\mathcal{C}_0(\Omega)$ the space of continuous and compactly supported functions on Ω.

We also set $\mathbf{T}$ the n-torus in $\mathbb{R}^n$. Whenever a μ-measurable function is Y-periodic (or kY-periodic for a positive integer k), its domain will be indicated by $\mathbf{T}$ (or $k\mathbf{T}$).

We recall that, for a given $\sigma \in L^1_{\mu,\rm loc}(\Omega, \mathbb{R}^{n^2}_{\rm sym})$, the vector-valued distribution ${\rm div}(\sigma\mu)$ is defined on Ω by

$$\langle \mathrm{div}(\sigma\mu), \psi \rangle := -\int_\Omega \sigma \cdot \nabla\psi \, d\mu \qquad \forall \psi \in \mathcal{C}_0^\infty(\Omega; \mathbb{R}^n) \ .$$

Here and in the following, the notation $A \cdot B$ stands for the scalar product between two matrices $A, B \in \mathbb{R}^{n^2}_{\mathrm{sym}}$, or sometimes between two vectors $A, B \in \mathbb{R}^n$.

When $\mathrm{div}(\sigma\mu)$ is a measure absolutely continuous with respect to μ, with a density belonging to $L^{p'}_\mu(\Omega; \mathbb{R}^n)$, we write for brevity $\mathrm{div}(\sigma\mu) \in L^{p'}_\mu(\Omega; \mathbb{R}^n)$, and we denote by $\mathrm{div}_\mu \sigma$ the derivative $\frac{d}{d\mu} \mathrm{div}(\sigma\mu)$. The analogous definitions are adopted if σ takes values in $\mathbb{R}^n$ (in this case, $\mathrm{div}(\sigma\mu)$ is a scalar distribution).

We set $T_\mu(x)$ the *tangent space to μ at x*, namely

$$T_\mu(x) := \mu - \mathrm{ess} \bigcup \left\{ \Phi(x) \ : \ \Phi \in X^{p'}_\mu(\Omega; \mathbb{R}^n) \right\} \qquad \text{for } \mu\text{-a.e. } x \in \Omega \ , \tag{4}$$

where the class $X^{p'}_\mu(\Omega; \mathbb{R}^n)$ of tangent fields to μ is given by

$$X^{p'}_\mu(\Omega; \mathbb{R}^n) := \left\{ \Phi \in L^{p'}_\mu(\Omega; \mathbb{R}^n) \ : \ \mathrm{div}(\Phi\mu) \in L^{p'}_\mu(\Omega) \right\} \ .$$

For more details on the properties of the above notion of T_μ, see [4] and references therein (in particular, see [5] for the meaning of the μ-essential union). We just remark that for usual measures the right hand side in (4) is independent of the exponent p'; further, T_μ coincides with the usual tangent space to S when μ is the Hausdorff measure $\mathcal{H}^k$ over a k-Lipschitz manifold S in $\mathbb{R}^n$. Let us now briefly recall the main Γ-convergence theorem proved in [4] for the sequence $\{F_\varepsilon\}$ which corresponds to (1) in the scalar case, that is

$$F_\varepsilon(u) := \int_\Omega f(\nabla u) \, d\mu_\varepsilon \ , \qquad u \in \mathcal{C}^1_0(\Omega) \ . \tag{5}$$

Here the integrand f is assumed to be convex on $\mathbb{R}^n$ and to satisfy the analogous of (2), *i.e.* $c|z|^p \le f(z) \le C(1 + |z|^p)$ for all $z \in \mathbb{R}^n$.

The Γ-limit of $\{F_\varepsilon\}$ will be computed with respect to the convergence $u_\varepsilon\mu_\varepsilon \rightharpoonup u\mathcal{L}^n$, intended in the sense of the weak $*$ convergence of measures on $\mathbb{R}^n$ (here and in the following, if u is a function on Ω vanishing on $\partial\Omega$, we still denote by u its extension to zero out of Ω).

More precisely, we preliminarily redefine F_ε on the class $\mathcal{M}$ of Radon measures on $\mathbb{R}^n$ by setting

$$F_\varepsilon(\lambda) = \begin{cases} \int_\Omega f\big(\nabla u\big), d\mu_\varepsilon & \text{if } \lambda = u\mu_\varepsilon \ , \ u \in \mathcal{C}^1_0(\Omega) \ , \\ +\infty & \text{otherwise.} \end{cases} \tag{6}$$

Then we say that $F_\varepsilon \xrightarrow{\Gamma} F^{\mathrm{hom}}$ if, for every $\lambda \in \mathcal{M}$, both the Γ-liminf and Γ-limsup inequalities hold, which read respectively:

$$\inf\left\{ \liminf_{\varepsilon\to 0} F_\varepsilon(\lambda_\varepsilon) \ : \ \lambda_\varepsilon \rightharpoonup \lambda \right\} \ge F^{\mathrm{hom}}(\lambda) \ ; \tag{7}$$

$$\inf\left\{ \limsup_{\varepsilon\to 0} F_\varepsilon(\lambda_\varepsilon) \ : \ \lambda_\varepsilon \rightharpoonup \lambda \right\} \le F^{\mathrm{hom}}(\lambda) \ . \tag{8}$$

In order to find the explicit expression of F^{hom}, suitable assumptions on μ are required. Their formulation involves the notions of *μ-tangential gradient* and related *Sobolev spaces associated to μ*. The μ-tangential gradient ∇_μ can be defined, for any smooth function ψ, by $P_{T_\mu}[\nabla\psi]$, being P_{T_μ} the orthogonal projector from $\mathbb{R}^n$ onto T_μ. If one considers the operator $\nabla_\mu : L^p_\mu(\Omega) \to L^p_\mu(\Omega;\mathbb{R}^n)$ as defined either on the domain $D(\nabla_\mu) := \mathcal{C}^\infty_0(\Omega)$, or on the domain $D(\nabla_\mu) := \mathcal{C}^\infty(\mathbf{T})$, in both cases it turns out to be closable. The domain of its unique closed extension gives in the former case the Banach space $H^{1,p}_{0,\mu}(\Omega)$ of μ-Sobolev functions vanishing at the boundary of Ω, and in the latter case the Banach space $H^{1,p}_\mu(\mathbf{T})$ of periodic μ-Sobolev functions. We also set $H^{1,p}_{\mu,\mathrm{loc}} := \{u \in L^p_{\mu,\mathrm{loc}}(\mathbb{R}^n) \ : \ u\psi \in H^{1,p}_{0,\mu}(\mathbb{R}^n) \ \forall\, \psi \in \mathcal{C}^\infty_0(\Omega)\}$.

We can now summarize the connectedness assumptions on μ introduced in [4]. They are given in relation to the growth exponent p of the integrand f in (5). For remarks and examples concerning definitions below, we refer to [4, Section 4]; c and C are supposed real constants.

- μ is *weakly p-connected on* $\mathbf{T}$ if:

  $(C1)$ $$u \in H^{1,p}_\mu(\mathbf{T}), \quad \nabla_\mu u = 0 \ \mu\text{-a.e.} \quad \Rightarrow \quad \exists\, c \ : \ u = c \ \mu\text{-a.e.}\,;$$

- μ is *weakly p-connected on* $\mathbb{R}^n$ if:

  $(C2)$ $$u \in H^{1,p}_{\mu,loc}, \quad \nabla_\mu u = 0 \ \mu\text{-a.e.} \quad \Rightarrow \quad \exists\, c \ : \ u = c \ \mu\text{-a.e.}\,;$$

- μ is *strongly p-connected on* $\mathbf{T}$ if:

  $(C3)$ $$\exists\, C \ : \ \int_Y |u|^p\, d\mu \le C \int_Y |\nabla_\mu u|^p\, d\mu \ , \qquad \forall u \in H^{1,p}_\mu(\mathbf{T}) \text{ with } \int_Y u\, d\mu = 0\,;$$

- μ is *strongly p-connected on* $\mathbb{R}^n$ if:

  $(C4)$ $$\exists\, C : \int_{kY} |u|^p\, d\mu \le C\, k^p \int_{kY} |\nabla_\mu u|^p\, d\mu \ \ \forall k \in \mathbb{N}\,, \forall u \in H^{1,p}_{\mu,loc} \text{ with } \int_{kY} u\, d\mu = 0\,.$$

Finally, we are in a position to state the homogenization theorem holding for measures which are strongly p-connected on $\mathbb{R}^n$. For the proof, as well as for weaker versions of this result, which are valid when μ satisfies $(C3)$ but possibly not $(C4)$ and $(C2)$, we refer to [4].

Theorem 2.1. *Let μ satisfy $(C4)$. Then the sequence $\{F_\varepsilon\}$ defined in (5) Γ-converges on $\mathcal{M}$ as $\varepsilon \to 0$ to the homogenized functional F^{hom} given by*

$$F^{\mathrm{hom}}(\lambda) = \begin{cases} \int_\Omega f^{\mathrm{hom}}(\nabla u(x))\, dx & \textit{if } \lambda = u\mathcal{L}^n \ , \ u \in W^{1,p}_0(\Omega) \\ +\infty & \textit{otherwise,} \end{cases} \tag{9}$$

where for any $z \in \mathbb{R}^n$ the integrand $f^{\mathrm{hom}}(z)$ is defined via the unit-cell problem

$$\begin{aligned} f^{\mathrm{hom}}(z) :&= \inf\Big\{ \int_Y f(z + \nabla u(y))\, d\mu \ : \ u \in \mathcal{C}^\infty(\mathbf{T})\Big\} \\ &= \inf\Big\{ \int_Y f_\mu(y, P_{T_\mu(y)} z + \nabla_\mu u(y))\, d\mu \ : \ u \in H^{1,p}_\mu(\mathbf{T})\Big\}\ , \end{aligned} \tag{10}$$

being

$$f_\mu(y,z) := \inf\big\{ f(z+\xi) \ : \ \xi \in [T_\mu(y)]^\perp\big\}\ , \qquad \forall (y,z) \in Y \times \mathbb{R}^n\ . \tag{11}$$

3. Relaxation of elastic energies

In order to extend Theorem 2.1 to the case of elasticity, we need to restate the unit cell problem (10) in a suitable space of *admissible periodic displacements* associated with the measure μ. Such functional space can be obtained by a procedure similar to the one adopted in [2] within the non-periodic setting. Therefore, for the sake of clearness, let us briefly recall the line followed in [2]: this will also enable us to give some explicit examples of relaxation which will be useful in Section 4.2.

In [2, Section 3], the authors define, for any positive Radon measure μ on $\mathbb{R}^n$ and μ-a.e. $x \in \Omega$, a linear subspace $M_\mu(x)$ of $\mathbb{R}^{n^2}_{\rm sym}$ as

$$M_\mu(x) := \mu - \mathrm{ess} \bigcup \left\{ \Phi(x) \ : \ \Phi \in X^{p'}_\mu(\Omega, \mathbb{R}^{n^2}_{\rm sym}) \right\} ,$$

where

$$X^{p'}_\mu(\Omega, \mathbb{R}^{n^2}_{\rm sym}) := \left\{ \Phi \in L^{p'}_\mu(\Omega, \mathbb{R}^{n^2}_{\rm sym}) \ : \ \mathrm{div}(\Phi\mu) \in L^{p'}_\mu(\Omega, \mathbb{R}^n) \right\} .$$

The *tangential strain operator* $e_\mu : L^p_\mu(\Omega; \mathbb{R}^n) \to L^p_\mu(\Omega, \mathbb{R}^{n^2}_{\rm sym})$ is defined by

$$D(e_\mu) := \mathcal{C}^\infty_0(\Omega; \mathbb{R}^n) , \qquad e_\mu u := P_{M_\mu}[e(u)] , \tag{12}$$

where $D(e_\mu)$ is the domain of e_μ, and P_{M_μ} is the orthogonal projector onto M_μ. In particular, one can show that

(i) if μ is the Lebesgue measure over an open subset of $\mathbb{R}^n$, then $e_\mu u$ coincides with the usual strain tensor $e(u)$;

(ii) if μ is the Hausdorff measure $\mathcal{H}^1$ or $\mathcal{H}^2$ over a one or two-dimensional structure in $\mathbb{R}^3$, then $e_\mu u = P_{T_\mu}(e(u))P_{T_\mu}$, being P_{T_μ} the orthogonal projector on T_μ defined in (4).

It turns out that the operator e_μ defined by (12) is closable (see Lemma 3.2 of [2]). Therefore, one can consider the completion of $\mathcal{C}^\infty_0(\Omega; \mathbb{R}^n)$ endowed with the norm $\|u\|_{p,\mu,\Omega} + \|e_\mu u\|_{p,\mu,\Omega}$, finding by this way the Banach space $\mathcal{D}^{1,p}_{0,\mu}(\Omega; \mathbb{R}^n)$ of *admissible displacements*.

Such space can also be characterized by duality and by relaxation. More precisely, using the duality Lemma 3.1 of [4], one obtains:

$$u \in \mathcal{D}^{1,p}_{0,\mu}(\Omega; \mathbb{R}^n) \ \Leftrightarrow \ \exists\, C > 0 \ : \ |\langle u, \mathrm{div}(P_{M_\mu}\sigma\mu)\rangle| \le C\|\sigma\|_{p',\mu,\Omega} \quad \forall\, \sigma \in D(e^*_\mu) . \tag{13}$$

Here e^*_μ denotes the adjoint operator of e_μ, and one can easily check that an element σ of $L^{p'}_\mu(\Omega; \mathbb{R}^{n^2}_{\rm sym})$ belongs to $D(e^*_\mu)$ if and only if $\mathrm{div}(P_{M_\mu}\sigma\mu) \in L^{p'}_\mu(\Omega; \mathbb{R}^n)$.

On the other hand, $\mathcal{D}^{1,p}_{0,\mu}(\Omega; \mathbb{R}^n)$ coincides with the finiteness domain of the relaxed functional $\overline{J}$ in the L^p_μ-topology of

$$J(u) = \begin{cases} \int_\Omega j(e(u))\, d\mu & \text{if } u \in \mathcal{C}^\infty_0(\Omega; \mathbb{R}^n) \\ +\infty & \text{if } u \in L^p_\mu(\Omega; \mathbb{R}^n) \setminus \mathcal{C}^\infty_0(\Omega; \mathbb{R}^n). \end{cases}$$

Indeed, one has

$$\overline{J}(u) = \begin{cases} \int_\Omega j_\mu(x, e_\mu u)\, d\mu & \text{if } u \in \mathcal{D}^{1,p}_{0,\mu}(\Omega; \mathbb{R}^n) \\ +\infty & \text{if } u \in L^p_\mu(\Omega; \mathbb{R}^n) \setminus \mathcal{D}^{1,p}_{0,\mu}(\Omega; \mathbb{R}^n) , \end{cases} \tag{14}$$

where

$$j_\mu(x,z) := \inf\left\{ j(z+\xi) \;:\; \xi \in [M_\mu(x)]^\perp \right\}, \qquad \forall (x,z) \in \Omega \times \mathbb{R}^{n^2}. \tag{15}$$

For instance, in the case of an isotropic linearly elastic body, when $j(e(u)) = \beta |e(u)|^2 + \frac{\alpha}{2} |\operatorname{tr}(e(u))|^2$, the explicit expression of j_μ becomes $j_\mu(y, e_\mu u) = \frac{\alpha\beta}{\alpha+2\beta} |\operatorname{tr}(e_\mu u)|^2 + \beta |e_\mu u|^2$ in the case of a membrane, and $j_\mu(y, e_\mu u) = \frac{\beta(3\alpha+2\beta)}{2(\alpha+\beta)} |e_\mu u|^2$ in the case of a string.

Of course, one is free to choose a periodic measure μ. In Examples 3.1 and 3.2 below, we take as a periodic measure μ the Hausdorff measure $\mathcal{H}^1$ over a truss structure composed of straight linear elastic beams with constant stiffness, which are linked one to each other at their ends (called nodes). It turns out that the relaxed energy sees only the longitudinal component of the displacement on each bar. In fact, the total displacement vector field u has an energetic meaning only at that nodes of the frame where the directions of outcoming bars span all $\mathbb{R}^n$. In that sense, when dealing with a frame made of trusses, it is natural to consider loads concentrated on the nodes and to write the total energy in terms of finitely many variables corresponding to the displacement of each node (see for instance [6], [11]). Unfortunately, such a representation is available only for one-dimensional structures.

Example 3.1. Let us detail what the above described tools give when μ is the periodic measure in $\mathbb{R}^3$ whose restriction to the unit cell is the Hausdorff measure $\mathcal{H}^1$ over a system of threads parallel to the three coordinate axes (see Figure 3.1 below); for simplicity, we normalize μ in order that $\mu(Y) = 1$.

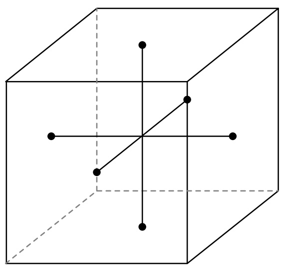

Figure 3.1

We denote by μ^j the measure associated to the fibers disposed along e^j, so that $\mu = \mu^1 + \mu^2 + \mu^3$. We claim that the space of admissible displacements can be characterized by:

$$u \in \mathcal{D}^{1,p}_{0,\mu}(\Omega; \mathbb{R}^3) \;\Leftrightarrow\; u^j \in H^{1,p}_{0,\mu^j}(\Omega) \quad \text{for every } j = 1,2,3\,. \tag{16}$$

Before proving (16), let us add few comments on it.

Note that (16) provides a well-defined notion of displacement at each node x_0 in the network determined by $\operatorname{spt}(\mu)$. Actually, for a given $u \in \mathcal{D}^{1,p}_{0,\mu}(\Omega, \mathbb{R}^3)$, the component of u along e^j at x_0 can be determined as the trace of u^j at x_0.

We also observe that, taking as a density energy $j(z) = \beta |z^*|^2 + \frac{\alpha}{2} |\operatorname{tr}(z^*)|^2$, the relaxation formula (14) reads

$$\overline{J}(u) = \begin{cases} \dfrac{\beta(3\alpha+2\beta)}{2(\alpha+\beta)} \left[\displaystyle\int_\Omega \left| \frac{\partial u^1}{\partial x_1} \right|^2 d\mu^1 + \int_\Omega \left| \frac{\partial u^2}{\partial x_2} \right|^2 d\mu^2 + \int_\Omega \left| \frac{\partial u^3}{\partial x_3} \right|^2 d\mu^3 \right] & \text{if } u \in \mathcal{D}^{1,2}_{0,\mu}(\Omega; \mathbb{R}^3), \\ +\infty & \text{otherwise,} \end{cases}$$

where, for every $u \in \mathcal{D}^{1,2}_{0,\mu}(\Omega;\mathbb{R}^3)$ and every $j = 1,2,3$, $\dfrac{\partial u^j}{\partial x_j}$ is well defined (as a $L^2_{\mu^j}(\Omega)$ function), in view of (16).

Proof of (16). One can adopt two different arguments, based respectively on:

(i) exploiting the characterization by duality given by (13);
(ii) using directly the definitions of $\mathcal{D}^{1,p}_{0,\mu}(\Omega;\mathbb{R}^3)$ and $H^{1,p}_{0,\mu^j}(\Omega)$.

For the reader's convenience, let us explain both methods. (i) Let $\sigma \in L^{p'}_\mu(\Omega, \mathbb{R}^{n^2}_{\rm sym})$. We have:

$$\operatorname{div}(P_{M_\mu}\sigma\mu) = \sum_{j=1}^3 \operatorname{div}\left(\sigma(e^j\otimes e^j)\mu^j\right) = \sum_{j=1}^3 \frac{\partial \sigma_{jj}(x)}{\partial x_j} e^j \mu^j \ .$$

Therefore, the requirement $\operatorname{div}(P_{M_\mu}\sigma\mu) \in L^{p'}_\mu(\Omega;\mathbb{R}^3)$ can be reformulated as $\operatorname{div}(\sigma(e^j \otimes e^j)\mu^j) \in L^{p'}_{\mu_j}(\Omega,\mathbb{R}^3)$ for $j=1,2,3$, or equivalently $(P_{T_{\mu^j}}\sigma^j\mu^j) \in X^{p'}_{\mu_j}(\Omega)$, where σ^j denotes the j^{th} column of σ. Inserting such condition into (13), we infer that (16) holds. (ii) In view of the definitions of $\mathcal{D}^{1,p}_{0,\mu}(\Omega;\mathbb{R}^3)$ and $H^{1,p}_{0,\mu^j}(\Omega)$, since the measures μ^j are mutually singular, it is immediate that the implication $\Rightarrow$ in (16) holds. To show the converse, assume that, for every $j=1,2,3$, the components u^j of u belong to $H^{1,p}_{0,\mu^j}(\Omega)$. Then we claim that u can be approximated in the L^p_μ-norm by a sequence $\{v_h\} \subset \mathcal{C}^\infty_0(\Omega;\mathbb{R}^3)$ such that $\sup_h \int_\Omega |e_\mu v_h|^p\,d\mu < +\infty$. Indeed, since $u^j \in H^{1,p}_{0,\mu^j}(\Omega)$, there exists $\{w^j_h\}_h \subset \mathcal{C}^\infty_0(\Omega)$, converging to u^j in the $L^p_{\mu^j}$-norm, such that $\sup_h \int_\Omega |\nabla_{\mu^j} w^j_h|^p\,d\mu^j < +\infty$. Choose a sequence of positive numbers $\{\rho_h\}$, converging to zero as $h \to +\infty$, such that $\lim\limits_{h\to+\infty}\left(\rho_h\,\|w^j_h\|^p_{\infty,\Omega}\right) = 0$ for $j=1,2,3$. For every $h \in \mathbb{N}$, let $\{B_{h,k}\}_k$ be family of the balls of radius ρ_h centered at the points of $\Omega\cap\mathbb{Z}^3$, with $k=1,\dots,K(\Omega)$. Consider the sequence of vector functions $\{v_h\}$ whose component v^j_h is obtained as the restriction of w^j_h to the set $F := \left[\operatorname{spt}(\mu)\setminus\bigcup_k B_{h,k}\right] \cup \left[\bigcup_k B_{h,k}\cap\operatorname{spt}(\mu^j)\right]$, for $j=1,2,3$. One can extend each v^j_h to a smooth function on Ω (still denoted by v^j_h), such that $\|v^j_h\|_{\infty,\Omega} \le \|w^j_h\|_{\infty,\Omega}$. By construction, the sequence of vector functions $\{v_h\}$ thus obtained lies in $\mathcal{C}^\infty_0(\Omega;\mathbb{R}^3)$, and satisfies $\sup_h \int_\Omega |e_\mu v_h|^p\,d\mu < +\infty$. Moreover, $\lim\limits_{h\to+\infty}\int_\Omega |v^j_h - u^j|^p d\mu = 0$, since, by the definition of v^j_h and the choice of the sequence $\{\rho_h\}$, it holds

$$\begin{aligned}\int_\Omega |v^j_h - w^j_h|^p d\mu &= \int_{\Omega\setminus F} |v^j_h - w^j_h|^p d\mu \\ &\le 2^{p-1}\left(\|v^j_h\|^p_{\infty,\Omega} + \|w^j_h\|^p_{\infty,\Omega}\right)\mu(\Omega\setminus F) \le 2^{p+2}K(\Omega)\rho_h\|w^j_h\|^p_{\infty,\Omega} \ .\end{aligned}$$

Example 3.2. Let us consider the case when μ is the periodic measure in $\mathbb{R}^2$ whose restriction to the unit cell is the Hausdorff measure $\mathcal{H}^1$ over a network containing also oblique bars (see Figure 3.2 below).

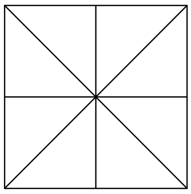

Figure 3.2

For $j = 1, 2, 3, 4$, we denote by μ^j the measure associated to the fibers along the direction ν^j, being $\nu^1 := e^1$, $\nu^2 := e^2$, $\nu^3 = e^1 + e^2$, $\nu^4 = e^1 - e^2$. In this case, if one takes four scalar functions u^j, each one belonging to $H^{1,p}_{0,\mu^j}(\Omega)$, in general it does not exist an admissible displacement $u \in \mathcal{D}^{1,p}_{0,\mu}(\Omega; \mathbb{R}^2)$ whose component $(u \cdot \nu^j)$ along ν^j equals u^j. The existence of such a displacement is subject to the additional compatibility conditions

$$\begin{cases} \operatorname{tr} u^3(x_0) = \operatorname{tr} u^1(x_0) + \operatorname{tr} u^2(x_0), \\ \operatorname{tr} u^4(x_0) = \operatorname{tr} u^1(x_0) - \operatorname{tr} u^2(x_0), \end{cases}$$

which must be satisfied at each node x_0 of $\operatorname{spt}(\mu)$. This can be obtained argueing in a similar way as in Example 3.1, for instance by duality. Moreover, still for $j(z) = \beta|z^*|^2 + \frac{\alpha}{2}|\operatorname{tr}(z^*)|^2$, the relaxed energy takes the form

$$\overline{J}(u) = \begin{cases} \dfrac{\beta(3\alpha + 2\beta)}{2(\alpha+\beta)} \displaystyle\sum_{j=1}^{4} \int_\Omega \Big| \frac{\partial(u \cdot \nu^j)}{\partial \nu^j} \Big|^2 d\mu^j & \text{if } u \in \mathcal{D}^{1,2}_{0,\mu}(\Omega; \mathbb{R}^2), \\ +\infty & \text{otherwise,} \end{cases}$$

where, for every $u \in \mathcal{D}^{1,2}_{0,\mu}(\Omega; \mathbb{R}^2)$ and every $j = 1, 2, 3, 4$, the expression $\dfrac{\partial(u \cdot \nu^j)}{\partial \nu^j}$ makes sense (as a $L^2_{\mu^j}(\Omega)$ function), as $u \cdot \nu^j$ belongs to $H^{1,2}_{0,\mu^j}(\Omega)$.

Periodic displacements. Let us adapt the previous framework to Y-periodic functions. Set

$$X^{p'}_\mu(\mathbf{T}, \mathbb{R}^{n^2}_{\mathrm{sym}}) := \Big\{ \Phi \in L^{p'}_\mu(\mathbf{T}, \mathbb{R}^{n^2}_{\mathrm{sym}}) \ : \ \operatorname{div}(\Phi\mu) \in L^{p'}_{\mu,\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n) \Big\} \ .$$

Notice that, since $\mu(\partial Y) = 0$, the periodicity condition satisfied by functions in $X^{p'}_\mu(\mathbf{T}, \mathbb{R}^{n^2}_{\mathrm{sym}})$ does not affect their μ-essential union, so that

$$\mu - \operatorname{ess} \bigcup \big\{ \Phi(x) \ : \ \Phi \in X^{p'}_\mu(\mathbf{T}, \mathbb{R}^{n^2}_{\mathrm{sym}}) \big\} = M_\mu(x) \qquad \text{for } \mu\text{-a.e. } x \in \Omega \ .$$

In particular, we can restrict the tangential strain operator to the class of smooth periodic functions, namely we can consider the operator $e_\mu : L^p_\mu(\mathbf{T}; \mathbb{R}^n) \to L^p_\mu(\mathbf{T}, \mathbb{R}^{n^2}_{\mathrm{sym}})$ defined by

$$D(e_\mu) := \mathcal{C}^\infty(\mathbf{T}; \mathbb{R}^n) \ , \qquad e_\mu u := P_{M_\mu}[e(u)] \ . \tag{17}$$

Argueing as in the proof of [4, Lemma 3.4], one can show that the integration by parts formula

$$\int_Y e_\mu u \cdot \Phi \, d\mu = - \int_Y u \ \operatorname{div}_\mu \Phi \, d\mu \tag{18}$$

holds for every $u \in \mathcal{C}^\infty(\mathbf{T}; \mathbb{R}^n)$ and every $\Phi \in X^{p'}_\mu(\mathbf{T}; \mathbb{R}^{n^2}_{\rm sym})$. This allows to adapt the proof of [2, Lemma 3.2] to the periodic setting, thus proving that the operator e_μ defined by (17) is still closable. Therefore, we can consider the completion of $\mathcal{C}^\infty(\mathbf{T}; \mathbb{R}^n)$ endowed with the norm $\|u\|_{p,\mu,Y} + \|e_\mu u\|_{p,\mu,Y}$. By this way, we find the Banach space of *admissible periodic displacements*, that we call $\mathcal{D}^{1,p}_\mu(\mathbf{T}; \mathbb{R}^n)$. It turns out to be a closed subspace of

$$\mathcal{D}^{1,p}_{\mu,\rm loc} := \left\{ u \in L^p_{\mu,\rm loc}(\mathbb{R}^n; \mathbb{R}^n) \ : \ u\psi \in \mathcal{D}^{1,p}_{0,\mu}(\mathbb{R}^n; \mathbb{R}^n) \ \forall \psi \in \mathcal{C}^\infty_0(\mathbb{R}^n) \right\} .$$

Extending the notion of tangential strain and the integration by parts formula (18) to every $u \in \mathcal{D}^{1,p}_\mu(\mathbf{T}; \mathbb{R}^n)$, one can easily generalize the relaxation formula (14) to the periodic case. More precisely, one finds that the relaxed functional of

$$J(u) = \begin{cases} \int_Y j(e(u))\, d\mu & \text{if } u \in \mathcal{C}^\infty(\mathbf{T}; \mathbb{R}^n) \\ +\infty & \text{if } u \in L^p_\mu(\mathbf{T}; \mathbb{R}^n) \setminus \mathcal{C}^\infty(\mathbf{T}; \mathbb{R}^n), \end{cases}$$

is given by

$$\overline{J}(u) = \begin{cases} \int_Y j_\mu(y, e_\mu u)\, d\mu & \text{if } u \in \mathcal{D}^{1,p}_\mu(\mathbf{T}; \mathbb{R}^n) \\ +\infty & \text{if } u \in L^p_\mu(\mathbf{T}) \setminus \mathcal{D}^{1,p}_\mu(\mathbf{T}; \mathbb{R}^n)\ , \end{cases} \tag{19}$$

where j_μ is still given by formula (15).

4. Homogenization of vector problems by the two-scale method

4.1. Results for p-fat structures.

This section is devoted to transpose into the vector setting the main homogenization result of [4] (*cf.* Theorem 4.2 below). This aim will be pursued by means of the very same two-scale technique used for scalar problems. We recall that a sequence $\{v_\varepsilon\} \in L^p_{\mu_\varepsilon}(\Omega; \mathbb{R}^d)$ two-scale converge to $v_0 \in L^p_{\mathcal{L}^n \otimes \mu}(\Omega \times \mathbf{T}; \mathbb{R}^d)$ if, for every test function $\varphi \in \mathcal{C}^\infty_0(\Omega; \mathcal{C}^\infty(\mathbf{T}; \mathbb{R}^d))$, it holds (componentwise)

$$\lim_{\varepsilon \to 0} \int_\Omega v_\varepsilon(x) \varphi\big(x, \frac{x}{\varepsilon}\big)\, d\mu_\varepsilon(x) = \int_{\Omega \times Y} v_0(x,y) \varphi(x,y)\, d\mathcal{L}^n(x) \otimes d\mu(y)\ ; \tag{20}$$

for more details and the properties of such convergence, we refer to [4, Section 2].

The key argument which is needed for applying the two-scale approach to the sequence $\{J_\varepsilon\}$ in (1), is an explicit characterization of all the possible two-scale limits of symmetric parts of gradients $\{e(u_\varepsilon)\}$, when $\{u_\varepsilon\} \subset \mathcal{C}^1_0(\Omega; \mathbb{R}^n)$ satisfy the uniform boundedness estimate $\sup_\varepsilon \int_\Omega |u_\varepsilon|^p + |e(u_\varepsilon)|^p\, d\mu_\varepsilon < +\infty$. Such characterization requires suitable assumptions on the periodic measure μ. We are thus led to introduce the following notions of *p-fatness* for periodic measures; they are the analogue in the vector case of the properties of p-connectedness introduced in [4] within the scalar setting. The reason of the terminology "p-fatness" comes from the behaviour of thin structures with respect to rigid orthogonal displacements, and it will be clarified in Section 4.2. Observe that the relations between the different notions of fatness below look similar to those holding for the connectedness hypotheses of Section 2, that is $(H4) \Rightarrow (H3) \Rightarrow (H1)$ and $(H4) \Rightarrow (H2) \Rightarrow (H1)$.

- μ is *weakly p-fat on* $\mathbf{T}$ if:

$(H1)$ $\qquad u \in \mathcal{D}^{1,p}_\mu(\mathbf{T}; \mathbb{R}^n), \quad e_\mu u = 0\ \mu\text{-a.e.} \ \Rightarrow \ \exists\, c \in \mathbb{R}^n \ : \ u = c\ \mu\text{-a.e.}\,;$

- μ is *weakly p-fat on* $\mathbb{R}^n$ if:

$$(H2)\; u \in \mathcal{D}^{1,p}_{\mu,\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n),\; e_\mu u = 0\; \mu\text{-a.e.} \;\Rightarrow\; \exists\, c \in \mathbb{R}^n,\; R \in \mathbb{R}^{n^2}_{\mathrm{skew}} \;:\; u(x) = Rx + c\; \mu\text{-a.e.}\,;$$

- μ is *strongly p-fat on* $\mathbf{T}$ if:

$$(H3) \quad \exists\, C > 0 : \int_Y |u|^p\, d\mu \le C \int_Y |e_\mu u|^p\, d\mu\;, \qquad \forall u \in \mathcal{D}^{1,p}_\mu(\mathbf{T};\mathbb{R}^n) \text{ with } \int_Y u\, d\mu = 0\,;$$

- μ is *strongly p-fat on* $\mathbb{R}^n$ if:

$$(H4)\; \exists C > 0 : \int_{kY} |u|^p d\mu \le C k^p \int_{kY} |e_\mu u|^p\, d\mu \;\forall k \in \mathbb{N}\,, \forall u \in \mathcal{D}^{1,p}_\mu(k\mathbf{T};\mathbb{R}^n) \text{ with } \int_{kY} u\, d\mu = 0\,.$$

A major role in the proof of Theorem 4.1 below, is played by the following orthogonality conditions.

– *Let*

$$V := \left\{ \mathrm{div}_\mu \Phi \;:\; \Phi \in X^{p'}_\mu(\mathbf{T};\mathbb{R}^{n^2}_{\mathrm{sym}}) \right\} \subset L^{p'}_\mu(\mathbf{T};\mathbb{R}^n)\;;$$

under (H1), the orthogonal space of V *in* $L^p_\mu(\mathbf{T};\mathbb{R}^n)$ *is given by*

$$V^\perp := \left\{ u \in \mathcal{D}^{1,p}_\mu(\mathbf{T};\mathbb{R}^n) \;:\; \exists\, c \text{ such that } u = c\; \mu\text{-a.e.} \right\}\;. \tag{21}$$

– *Let*

$$W := \left\{ \sigma \in L^{p'}_\mu(\mathbf{T};\mathbb{R}^{n^2}_{\mathrm{sym}}) \;:\; \mathrm{div}(P_{M_\mu}\sigma\mu) = 0 \right\} \subset L^{p'}_\mu(\mathbf{T};\mathbb{R}^{n^2}_{\mathrm{sym}})\;; \tag{22}$$

under (H3), the orthogonal space of W *in* $L^p_\mu(\mathbf{T};\mathbb{R}^{n^2}_{\mathrm{sym}})$ *is given by*

$$W^\perp = \left\{ e_\mu u \;:\; u \in \mathcal{D}^{1,p}_\mu(\mathbf{T};\mathbb{R}^n) \right\}\;. \tag{23}$$

In order to show (21) and (23), one has to transpose into the vector framework the proofs of Lemma 4.3 and Lemma 4.6 of [4]. Let us notice that condition (H3) ensures that the subspace $\left\{ e_\mu u \;:\; u \in \mathcal{D}^{1,p}_\mu(\mathbf{T};\mathbb{R}^n) \right\}$ is closed. If it is not the case, we simply have

$$W^\perp = \overline{\left\{ e_\mu u \;:\; u \in \mathcal{D}^{1,p}_\mu(\mathbf{T};\mathbb{R}^n) \right\}}\;, \tag{24}$$

where the closure is intended in the $L^p_\mu(\mathbf{T},\mathbb{R}^{n^2}_{\mathrm{sym}})$-norm.

The orthogonality relations (21) and (23) can be used as crucial steps to obtain the following two-scale structure result for symmetric parts of gradients.

Theorem 4.1. *Let* $\{u_\varepsilon\} \subset \mathcal{C}^1_0(\Omega;\mathbb{R}^n)$ *satisfy* $\int_\Omega \left(|u_\varepsilon|^p + |e(u_\varepsilon)|^p\right) d\mu_\varepsilon \le M$*; possibly passing to a subsequence, assume that* $u_\varepsilon \rightharpoonup\rightharpoonup u_0 \in L^p_{\mathcal{L}^n\otimes\mu}(\Omega \times Y;\mathbb{R}^n)$ *and* $e(u_\varepsilon) \rightharpoonup\rightharpoonup \chi \in L^p_{\mathcal{L}^n\otimes\mu}(\Omega \times Y;\mathbb{R}^{n^2})$*. Then:*

(i) *if* μ *satisfies* $(H1)$, $u_0(x,y) = u(x)$ *(i.e.* u_0 *is independent of* y*), where the function* u *belongs to* $W^{1,p}_0(\Omega;\mathbb{R}^n)$ *provided* μ *satisfies also* $(H2)$ *and* $(H3)$*;*

(ii) *under assumptions* $(H2)$ *and* $(H3)$ *on* μ*, there exists* $u_1 \in L^p\big(\Omega, \mathcal{D}^{1,p}_\mu(\mathbf{T};\mathbb{R}^n)\big)$ *such that* $\chi(x,y) = e(u)(x) + e_{\mu,y}u_1(x,y) + \xi(x,y)$*, with* u *as in (i), and* $\xi(x,y) \in L^p(\Omega;[M_\mu(y)]^\perp)$*. In addition,* $e_{\mu_\varepsilon}u_\varepsilon \rightharpoonup\rightharpoonup e(u)(x) + e_{\mu,y}u_1(x,y)$*.*

Proof (sketch). The result can be deduced following the same line of the proof as in [4, Theorem 4.2], taking care of adapting it to the vector framework. Mainly, one has to replace the operators ∇u and $\nabla_\mu u$ by $e(u)$ and $e_\mu u$, the space T_μ and the projector onto T_μ by the space M_μ and the projector onto M_μ, $X^{p'}_\mu(\mathbf{T})$ by $X^{p'}_\mu(\mathbf{T};\mathbb{R}^{n^2}_{\rm sym})$, and $H^{1,p}_\mu(\mathbf{T})$ by $\mathcal{D}^{1,p}_\mu(\mathbf{T};\mathbb{R}^n)$. We notice also that the assertion $u\in W^{1,p}_0(\Omega;\mathbb{R}^n)$ in the statement can be proved as follows: first show, by the generalization procedure indicated above, that $e(u)\in L^p(\Omega;\mathbb{R}^{n^2}_{\rm sym})$, then apply the Korn inequality. $\square$

We are now in a position to give the homogenization result for p-fat measures. We preliminarily extend J_ε to the class $\mathcal{M}^n$ of vector Radon measures on $\mathbb{R}^n$ by setting

$$J_\varepsilon(\lambda)=\begin{cases}\int_\Omega j\big(e(u)\big)\,d\mu_\varepsilon & \text{if } \lambda=u\mu_\varepsilon\ ,\ u\in\mathcal{C}^1_0(\Omega;\mathbb{R}^n)\ ,\\ +\infty & \text{otherwise;}\end{cases}\tag{25}$$

then we say that $\{J_\varepsilon\}$ Γ-converge to $J^{\rm hom}$ if, for every $\lambda\in\mathcal{M}^n$, the Γ- liminf and Γ- limsup inequalities (7) and (8) hold, with J_ε and $J^{\rm hom}$ in place of F_ε and $F^{\rm hom}$.

Theorem 4.2. *Let μ satisfy $(H4)$. Then the sequence $\{J_\varepsilon\}$ defined in (25) Γ-converges on $\mathcal{M}^n$ as $\varepsilon\to 0$ to the homogenized functional $J^{\rm hom}$ defined by*

$$J^{\rm hom}(\lambda)=\begin{cases}\int_\Omega j^{\rm hom}(e(u)(x))\,dx & \text{if } \lambda=u\mathcal{L}^n\ ,\ u\in W^{1,p}_0(\Omega;\mathbb{R}^n)\\ +\infty & \text{otherwise,}\end{cases}\tag{26}$$

where for any $z\in\mathbb{R}^{n^2}_{\rm sym}$ the integrand $j^{\rm hom}(z)$ is defined via the unit-cell problem

$$\begin{aligned} j^{\rm hom}(z) :&= \inf\Big\{\int_Y j(z+e(u)(y))\,d\mu\ :\ u\in\mathcal{C}^\infty(\mathbf{T};\mathbb{R}^n)\Big\}\\ &=\inf\Big\{\int_Y j_\mu(y,P_{M_\mu(y)}z+e_\mu u(y))\,d\mu\ :\ u\in\mathcal{D}^{1,p}_\mu(\mathbf{T};\mathbb{R}^n)\Big\}\ .\end{aligned}\tag{27}$$

Proof. It can be derived with minor changes from the proof of [4, Theorem 5.2]. In particular, the Γ- liminf inequality follows from the assumption (H4), Theorem 4.1, and the relaxation formula (19); the Γ- limsup inequality can be obtained considering the sequence $u_\varepsilon(x):=u(x)+\varepsilon\varphi(x,\frac{x}{\varepsilon})$ for φ varying in $\mathcal{C}^\infty_0(\Omega;\mathcal{C}^\infty(\mathbf{T};\mathbb{R}^n))$. $\square$

Remark 4.3. A weaker version of Theorem 4.2 holds for measures μ which do not enjoy $(H4)$ or $(H2)$. In case of lackness of $(H4)$, one has to restrict the statement to sequences $\{u_\varepsilon\}$ such that $\sup_\varepsilon\int_\Omega|u_\varepsilon|^p\,d\mu_\varepsilon<+\infty$. In case of lackness of $(H2)$, the homogenized integrand is still given by formula (27), but it can degenerate along some directions. In particular, the space $W^{1,p}_0(\Omega;\mathbb{R}^n)$ in the domain of $J^{\rm hom}$ has to be replaced by

$$W^{1,p}_{0,M}(\Omega;\mathbb{R}^n):=\{u\in L^p(\Omega;\mathbb{R}^n):\forall\, z\in M,\ z\cdot e(u)\in L^p(\Omega)\text{ and } u\cdot(z\,\nu_\Omega)=0\ \text{ on } \partial\Omega\}\,,$$

being ν_Ω the unit normal to $\partial\Omega$, and M the linear space generated in $\mathbb{R}^{n^2}_{\rm sym}$ by the relative interior of the convex set $K:=\Big\{\int_Y\Phi\,d\mu:\Phi\in X^{p'}_\mu(\mathbf{T};\mathbb{R}^{n^2}_{\rm sym}),\ {\rm div}_\mu\Phi=0,\ \|\Phi\|_{p',\mu,Y}\le 1\Big\}$. The space M corresponds to the set of non-degeneracy directions of $j^{\rm hom}$; for knowing how it can be obtained, we refer to the proof of Theorem 4.2 in [4].

Let us notice that, when μ is the measure chosen in Example 3.1, the space M is given by the class of all diagonal matrices, so that the corresponding j^{hom} is not coercive on $\mathbb{R}^9_{\text{sym}}$ (see (41)). In spite, for the measure of Example 3.2, one can check that M equals $\mathbb{R}^2$, so that the corresponding j^{hom} is coercive on $\mathbb{R}^4_{\text{sym}}$.

4.2. Thin structures are not p-fat.

We wish now to investigate the behaviour of low-dimensional structures in regard to the p-fatness properties introduced in the previous section. Let us recall that, if u is a smooth vector field on Ω, and e_{ij}, ω_{ij} denote respectively the symmetric and skew-symmetric parts of the Jacobian matrix of u, for every x', x'' in Ω the Cesaro formula holds (which can be proved by elementary integration by parts):

$$u_i(x'') = u_i(x') + \omega_{ij}(x')(x''_j - x'_j) + \int_{x'}^{x''} e_{ir}(x) + (x''_j - x_j)\left(\frac{\partial e_{ir}(x)}{\partial x_j} - \frac{\partial e_{jr}(x)}{\partial x_i}\right) dx_r\ . \quad (28)$$

Here the notation $\int_{x'}^{x''} \psi(x)\, dx_r$ stands for the one-dimensional integral $\int_0^t \psi(x' + s\nu)\nu_r\, ds$, being $x'' = x' + t\nu$.

In particular, it follows from (28) that, if u satisfies the equation $e(u) = 0$ on Ω, then it can be written as $u(x) = Rx + c$, being R a skew-symmetric $n \times n$ matrix, and c a constant vector. Such kind of vector fields u are usually referred in the elasticity literature as *rigid displacements*.

Let us now focus attention on the two model examples when μ is given on the unit cell respectively by the measure μ_1 equal to $\mathcal{H}^1 \llcorner F$, where F is the vertical fiber $Y \cap \{y_1 = y_2 = 0\}$, or by the measure μ_2 equal to $\mathcal{H}^2 \llcorner S$, where S is the horizontal plane $Y \cap \{y_3 = 0\}$ (see Figures 4.1 and 4.2 below). In these cases, the condition $e_\mu u = 0$ appearing in the assumption $(H1)$ yields respectively:

$$\frac{\partial u^3}{\partial y_3} = 0 \quad \mu_1\text{-a.e.} \qquad \text{and} \qquad \frac{\partial u^1}{\partial y_1} = \frac{\partial u^2}{\partial y_2} = \frac{\partial u^1}{\partial y_2} + \frac{\partial u^2}{\partial y_1} = 0 \quad \mu_2\text{-a.e.}\ , \qquad (29)$$

where u^j stands for the component of u along the direction e^j.

In case of measure μ_1, we infer from (29) that the third component u^3 of u must be constant along F (Figure 4.1 represents the case when such constant equals zero). Similarly, in case of measure μ_2, the first components (u^1, u^2) of u turn out to be given by a rigid displacement on the plane S, that is $(u^1, u^2)(x) = Rx + c$ for μ_2-a.e. x, where R is a constant skew-symmetric element of $\mathbb{R}^4$, and c is a constant vector in $\mathbb{R}^2$; recalling that u must also satisfy a periodicity condition (*cf.* (H1)), this yields that u^1 and u^2 are necessarily constant on S (Figure 4.2 represents the case when such constants are both zero).

The above examples show why low-dimensional structures are not p-fat, even in the weakest sense of assumption (H1); indeed, more generally, each of the p-connected thin structures that we considered in [4] for the scalar case, are not p-fat.

Thus, since there is no hope to apply Theorem 4.1 to the associated measures (and also by stronger reasons discussed in Example 4.4 below), we shall need to hang on the fattening method, as it is illustrated in the next section. We stress once more that this is in full contrast with the behaviour of the scalar case, in which we showed that an advantageous approach for reinforced structures is letting the thickness parameter δ tend to zero *before* making the homogenization procedure.

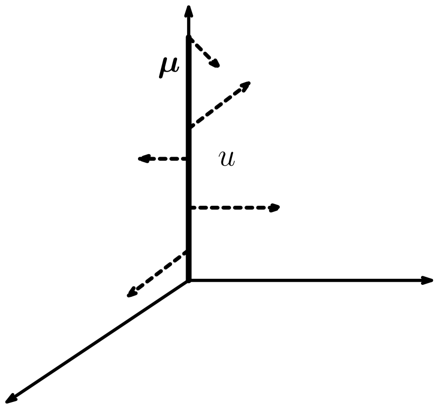

Figure 4.1

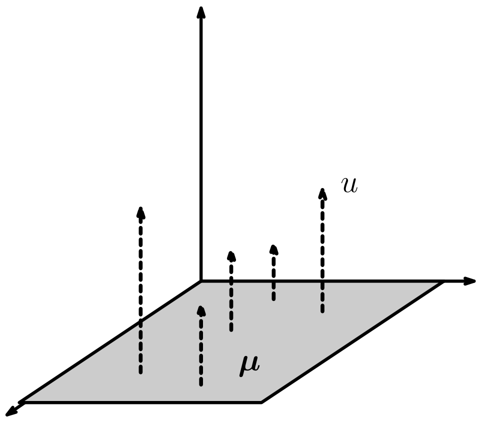

Figure 4.2

Example 4.4. Further differences from the scalar case appear looking at the Γ-limit of the functionals (1), which can be found by explicit computations in case of simple measures μ, having the property to be p-connected but not p-fat. For instance, let μ be the periodic 1-dimensional measure in $\mathbb{R}^3$ considered in Example 3.1, and let us take as a density energy $j(z) = \beta|z^*|^2 + \frac{\alpha}{2}|\operatorname{tr}(z^*)|^2$. We claim that

$$\Gamma - \lim_{\varepsilon \to 0} J_\varepsilon(u) = 0 \qquad \forall\, u \in W_0^{1,2}(\Omega)\ . \tag{30}$$

The validity of (30), that we are going to prove, puts in evidence how stringent is the need of the fattening approach when dealing with the homogenization of elasticity problems on thin structures.

Proof of (30). We begin by noticing that, since $j(z)$ is nonnegative for every $z \in \mathbb{R}^9$, the Γ- liminf inequality is trivially satisfied. Therefore, we focus on the Γ- limsup inequality. In view of the density of $\mathcal{C}_0^\infty(\Omega;\mathbb{R}^3)$ into $W_0^{1,2}(\Omega;\mathbb{R}^3)$, by a standard diagonalization argument it is not restrictive to assume that $u \in \mathcal{C}_0^\infty(\Omega;\mathbb{R}^3)$; moreover, by a well-known property of Γ-convergence, it is enough to prove that $\Gamma - \lim_{\varepsilon\to 0} \overline{J}_\varepsilon = 0$, being $\overline{J}_\varepsilon(u) = \int_\Omega j_{\mu_\varepsilon}(x, \nabla_{\mu_\varepsilon} u)\, d\mu_\varepsilon$ the relaxed functional of J_ε in the $L^2_{\mu_\varepsilon}$-norm. We stress that the explicit expression of $\overline{J}_\varepsilon$ is analogous to the one found in Example 3.1, replacing μ and μ^j respectively by their ε-scalings μ_ε and μ^j_ε.

Let $u \in \mathcal{C}_0^\infty(\Omega;\mathbb{R}^3)$, and let us show that we can find $\{u_\varepsilon\} \subset \mathcal{D}^{1,2}_{0,\mu_\varepsilon}(\Omega;\mathbb{R}^3)$ such that

$$u_\varepsilon \mu_\varepsilon \rightharpoonup u\,\mathcal{L}^3\ , \qquad \lim_{\varepsilon\to 0} \int_\Omega |e_{\mu_\varepsilon} u_\varepsilon|^2\, d\mu_\varepsilon = 0\ . \tag{31}$$

We let $\{u_\varepsilon\}$ be the sequence whose components u^j_ε are defined μ_ε-a.e. by

$$u^j_\varepsilon = \begin{cases} 0 & \text{on } \operatorname{spt}(\mu^j_\varepsilon)\ , \\ \frac{3}{2} u^j & \text{on } \operatorname{spt}(\mu_\varepsilon) \setminus \ \operatorname{spt}(\mu^j_\varepsilon)\ . \end{cases} \tag{32}$$

It is easy to check that u_ε satisfy both conditions in (31). It remains to show that, for fixed ε, u_ε belong to $\mathcal{D}^{1,2}_{0,\mu_\varepsilon}(\Omega;\mathbb{R}^3)$. In fact, it is immediate that $u^j_\varepsilon \in H^{1,2}_{0,\mu^j_\varepsilon}(\Omega)$ for every $j =$

$1, 2, 3$; therefore, since the equivalence (16) proved in Example 3.1 still holds replacing μ by μ_ε, we obtain that $\{u_\varepsilon\}$ lies in $\mathcal{D}^{1,2}_{0,\mu_\varepsilon}(\Omega; \mathbb{R}^3)$.

5. The fattening approach

The aim of this section is to show that, even when the underlying periodic measure μ does not satisfy the p-fatness properties, the limit energy defined in (26) can still be used for the homogenization of elastic thin structures. This can be justified by two different arguments, both involving the fattening approach.

In a similar way as in [4], we consider a sequence of Y-periodic measures $\{\mu_\delta\}$, each one associated with a structure of thickness δ, in such way that $\{\mu_\delta\}$ converge weakly * to μ as δ tends to zero. When the size of the periodicity cell of μ_δ is scaled to ε, the elastic energy of the corresponding δ thick, ε periodic structure can be modeled by the functional

$$J_{\delta,\varepsilon}(u) = \int_\Omega j\big(e(u)\big)\, d\mu_{\delta,\varepsilon} \ , \qquad u \in \mathcal{C}^1_0(\Omega; \mathbb{R}^n) \ , \tag{33}$$

where $\mu_{\delta,\varepsilon}$ is the ε-periodization of μ_δ according to (3).

The simplest procedure in order to study the asymptotic behaviour of the sequence defined in (33), consists first in homogenizing with respect to each μ_δ, and then let δ tend to zero.

We are going to prove that, under the assumption that for each fixed δ the measure μ_δ satisfies (H4), the effective energy obtained by this method coincides with the homogenized functional defined by (26).

In what follows, the Γ-limit of $J_{\delta,\varepsilon}$ as $\varepsilon \to 0$ is computed with respect to the analogous convergence as in Theorem 4.2, namely $u_\varepsilon \mu_{\delta,\varepsilon} \rightharpoonup u\mathcal{L}^n$. Moreover, in order to simplify the presentation, we use as measures μ_δ the approximations of μ by convolution. More precisely, for every $\delta > 0$, we let ρ_δ be a convolution kernel $\rho_\delta(x) := \frac{1}{\delta^n}\rho\left(\frac{x}{\delta}\right)$, where ρ is assumed to be a smooth, positive, even function, with support compactly contained into Y, and such that $\int_{\mathbb{R}^n} \rho\, dx = 1$. We set $\mu_\delta := (\rho_\delta \star \mu)\mathcal{L}^n$, being $\rho_\delta \star \mu$ the smooth function $\rho_\delta \star \mu(x) := \int_{\mathbb{R}^n} \rho_\delta(x-y)\, d\mu(y)$; in particular, we observe that the sequence of measures $\{\mu_\delta\}$ converge weakly * to μ as $\delta \to 0^+$. Further, due to the periodicity of μ, it holds

$$\rho_\delta \star \mu(x) = \int_Y \rho^\sharp_\delta(x-y)\, d\mu(y) \ , \tag{34}$$

being $\rho^\sharp_\delta$ the Y-periodic function obtained by ρ_δ through

$$\rho^\sharp_\delta(y) := \sum_{i \in \mathbb{Z}^n} \rho_\delta(y-i) \ , \qquad y \in Y \ .$$

For $v \in L^1_{\rm loc}(\mathbb{R}^n)$, $\rho_\delta \star v$ denotes the usual convolution, that is $\rho_\delta \star v(x) = \int_{\mathbb{R}^n} \rho_\delta(x-y)v(y)\, dy$. Similarly as above, when v is Y-periodic, $\rho_\delta \star v$ can be written as an integral over Y, *i.e.*

$$\rho_\delta \star v(x) = \int_Y \rho^\sharp_\delta(x-y) v(y)\, dy \ , \qquad \forall v \in L^1(\mathbf{T}) \ . \tag{35}$$

We keep the notation $j^{\rm hom}$ for the function on $\mathbb{R}^{n^2}_{\rm sym}$ associated to μ according to (27); when in such formula μ is replaced by μ_δ, we call $j^{\rm hom}_\delta$ the integrand thus obtained.

Lemma 5.1. *For every $z \in \mathbb{R}^{n^2}_{\rm sym}$, it holds*

$$\lim_{\delta \to 0^+} j_\delta^{\rm hom}(z) = \inf_{\delta > 0} j_\delta^{\rm hom}(z) = j^{\rm hom}(z) \ . \tag{36}$$

Proof. Let $z \in \mathbb{R}^{n^2}_{\rm sym}$ be fixed, and let us show that $\limsup_{\delta \to 0^+} j_\delta^{\rm hom}(z) \leq j^{\rm hom}(z)$. It is easy to check that

$$\limsup_{\delta \to 0^+} \inf_{u \in \mathcal{C}^\infty(\mathbf{T};\mathbb{R}^n)} \int_Y j(z + e(u)(y))\, d\mu_\delta(y) \leq \inf_{u \in \mathcal{C}^\infty(\mathbf{T};\mathbb{R}^n)} \limsup_{\delta \to 0^+} \int_Y j(z + e(u)(y))\, d\mu_\delta(y) \ . \tag{37}$$

For every $u \in \mathcal{C}^\infty(\mathbf{T};\mathbb{R}^n)$, due to the weak * convergence of μ_δ to μ and to the continuity of the mapping $y \mapsto j(z + e(u)(y))$, we have

$$\lim_{\delta \to 0^+} \int_Y j(z + e(u)(y))\, d\mu_\delta(y) = \int_Y j(z + e(u)(y))\, d\mu(y) \ . \tag{38}$$

Combining (37) and (38), we deduce that $\limsup_{\delta \to 0^+} j_\delta^{\rm hom}(z) \leq j^{\rm hom}(z)$.

On the other hand, the inequality $j^{\rm hom}(z) \leq j_\delta^{\rm hom}(z)$ holds for every fixed $\delta > 0$. To prove such claim, it is enough to show that, for every $u \in \mathcal{C}^\infty(\mathbf{T};\mathbb{R}^n)$, there exists $u_\delta \in \mathcal{C}^\infty(\mathbf{T};\mathbb{R}^n)$ such that

$$\int_Y j(z + e(u_\delta)(y))\, d\mu(y) \leq \int_Y j(z + e(u)(y))\, d\mu_\delta(y) \ .$$

Set $u_\delta := \rho_\delta \star u$. Using in the order the commutation between convolution and symmetric gradient, the Jensen's inequality, (35), Fubini's theorem, and (34), we obtain

$$\begin{aligned}
&\int_Y j(z + e(u_\delta)(y))\, d\mu(y) = \int_Y j\big[\rho_\delta \star \big(z + e(u)(y)\big)\big]\, d\mu(y) \\
&\leq \int_Y \rho_\delta \star j\big[z + e(u)(y)\big]\, d\mu(y) = \int_Y \Big\{ \int_Y \rho_\delta^\sharp(x-y) j[z + e(u)(y)]\, dy \Big\}\, d\mu(x) \\
&= \int_Y j[z + e(u)(y)] \Big\{ \int_Y \rho_\delta^\sharp(x-y)\, d\mu(x) \Big\}\, dy = \int_Y j(z + e(u)(y))\, d\mu_\delta(y) \ ,
\end{aligned}$$

which concludes the proof. □

As a consequence of Lemma 5.1, we obtain the following result, which gives a first rigorous justification for using (26) as an effective energy, whenever μ belongs to a large class of (possibly non-fat) measures.

Theorem 5.2. *Let μ be a Y-periodic measure such that, for every $\delta > 0$, $\mu_\delta := (\rho_\delta \star \mu)\mathcal{L}^n$ satisfies $(H4)$. Moreover, assume that the integrand $j^{\rm hom}$ defined by (27) is coercive on $\mathbb{R}^{n^2}_{\rm sym}$. Then*

$$\Gamma\!-\!\lim_{\delta \to 0} \Big(\Gamma\!-\!\lim_{\varepsilon \to 0} J_{\delta,\varepsilon} \Big) = J^{\rm hom} \ , \tag{39}$$

with $J^{\rm hom}$ defined by (26).

Proof. By Theorem 4.2, for every $\delta > 0$ there holds $\Gamma - \lim_{\varepsilon\to 0} J_{\delta,\varepsilon} = J_\delta^{\hom}$, where

$$J_\delta^{\hom}(\lambda) = \begin{cases} \int_\Omega j_\delta^{\hom}(e(u)(x))\, dx & \text{if } \lambda = u\mathcal{L}^n\ ,\ u \in W_0^{1,p}(\Omega;\mathbb{R}^n) \\ +\infty & \text{otherwise.} \end{cases}$$

Hence we are reduced to prove that $J_\delta^{\hom}$ Γ-converges to $J^{\hom}$ as $\delta \to 0$. Since by Lemma 5.1 we have $j_\delta^{\hom} \geq j^{\hom}$, by the coerciveness assumption on $j^{\hom}$ and the classical Korn inequality, the functionals $J_\delta^{\hom}$, $J^{\hom}$ are lower semicontinuous and equi-coercive on $W_0^{1,p}(\Omega;\mathbb{R}^n)$; thus we may substitute the weak star topology on measures λ with the strong topology on $L^p(\Omega;\mathbb{R}^n)$. We thus have $\Gamma-\liminf_{\delta\to 0} J_\delta^{\hom} \geq J^{\hom}$. On the other hand, for every $u \in W_0^{1,p}(\Omega;\mathbb{R}^n)$, using $u_\delta := u$ as an approximating sequence, we deduce from (36) and the monotone convergence theorem that

$$\Gamma-\limsup_{\delta\to 0} J_\delta^{\hom}(u) \leq \limsup_{\delta\to 0} J_\delta^{\hom}(u) \leq J^{\hom}(u)\ .$$

□

Remark 5.3. (i) The assumption (H4) on μ_δ can be recovered in practice by proving that Korn inequality holds on a connected periodic open subset of $\mathbb{R}^n$ whose thickness is uniformly minorized (by δ). In particular, considering one unit cell, it implies that there exists a constant C_δ such that

$$\int_Y |u|^p\, d\mu_\delta \leq C_\delta \int_Y |e(u)|^p\, d\mu_\delta\ , \qquad \forall u \in \mathcal{D}^{1,p}_{\mu_\delta}(\mathbf{T};\mathbb{R}^n) \text{ with } \int_Y u\, d\mu_\delta = 0\,; \tag{40}$$

(ii) Equality (39) remains true for all functions $u \in \mathcal{C}_0^\infty(\Omega;\mathbb{R}^n)$ also when $j^{\hom}$ fails to be coercive. In this case, if one would find the explicit expression of the left hand side of (39) on its whole finiteness domain, one should extend by relaxation the functional obtained as the restriction of $J^{\hom}$ to $\mathcal{C}_0^\infty(\Omega;\mathbb{R}^n)$. For instance, (5.7) remains true on $\mathcal{C}_0^\infty(\Omega;\mathbb{R}^n)$ when μ is the measure of Example 3.1. In this case one can easily check that

$$j^{\hom}(z) = j_\mu(P_{M_\mu} z^*) = \frac{\beta(3\alpha+2\beta)}{2(\alpha+\beta)}(z_{11}^2 + z_{22}^2 + z_{33}^2)\ . \tag{41}$$

In particular, taking into account (30) and (39), equation (41) enlights the non-commutativity of the limit process in ε and δ.

(iii) By a standard diagonalization argument, it turns out from Theorem 5.2 that, for a suitable sequence $\delta(\varepsilon) \to 0$, we have $\Gamma-\lim_{\varepsilon\to 0} J_{\delta(\varepsilon),\varepsilon} = J^{\hom}$. A more precise statement concerning the results which can be obtained when δ depends on ε will be given in Theorem 5.4 below.

Now we focus attention on the limiting behaviour of the sequence $\{J_{\delta,\varepsilon}\}$ when the parameters δ and ε tend to zero at the same time, and in particular on the Γ-limit as $\varepsilon \to 0$ of $\{J_{\delta(\varepsilon),\varepsilon}\}$, being $\delta(\varepsilon)$ an assigned sequence infinitesimal with ε. In light of the examples made in Section 4, when the limit measure μ does not satisfy assumption $(H4)$, we cannot expect the commutativity of the passage to the limit in (33) as ε and δ tend to zero.

Anyhow, we are going to show that, when the dependence of δ on ε is such that $\delta(\varepsilon) \gg \varepsilon$, the Γ-limit of $J_{\delta(\varepsilon),\varepsilon}$ as $\varepsilon \to 0$ still coincides with the functional $J^{\rm hom}$ in (26), provided a suitable additional assumption on the sequence $\{\mu_\delta\}$ holds. Such assumption concerns the constant C_δ in (40). Indeed, if μ does not satisfy any fatness condition, it is clear that C_δ will tend to infinity as $\delta \to 0$. But we want to drive our attention on the precise behaviour of C_δ: in fact one can observe that, in most situations where fattened structures of thickness δ are considered, (40) holds with $C_\delta = \dfrac{C}{\delta^p}$, being C a positive constant independent of δ. Otherwise said, in most cases the following assumption is satisfied by the measures μ_δ:

$$(H3)_\delta \quad \exists C > 0 : \int_Y |u|^p \, d\mu_\delta \le \frac{C}{\delta^p} \int_Y |e(u)|^p \, d\mu_\delta \,, \qquad \forall u \in \mathcal{D}^{1,p}_{\mu_\delta}(\mathbf{T}; \mathbb{R}^n) \text{ with } \int_Y u \, d\mu_\delta = 0 \,.$$

The main result of this section is the following. It yields a second reason to retain formula (26) still valid for several non-fat measures.

Theorem 5.4. *Under the same assumptions of Theorem 5.2, assume in addition that condition $(H3)_\delta$ is satisfied by the measures μ_δ. Then, for every sequence $\delta(\varepsilon)$ such that $\delta(\varepsilon) \gg \varepsilon$, we have*

$$\Gamma - \lim_{\varepsilon \to 0} J_{\delta(\varepsilon),\varepsilon} = J^{\rm hom} \,, \tag{42}$$

with $J^{\rm hom}$ defined by (26).

In order to prove Theorem 5.4, we need to restate the structure Theorem 4.1 about two-scale convergence. The notion of convergence defined by (20) can be extended in a natural way to the framework of varying measures $\mu_{\delta(\varepsilon)}$ converging weakly * to μ. A sequence $\{v_\varepsilon\} \subset L^p_{\mu_{\delta(\varepsilon),\varepsilon}}(\Omega; \mathbb{R}^d)$ will be said two-scale converging to $v_0 \in L^p_{\mathcal{L}^n \otimes \mu}(\Omega \times \mathbf{T}; \mathbb{R}^d)$ if, for every test function $\varphi \in \mathcal{C}^\infty_0(\Omega; \mathcal{C}^\infty(\mathbf{T}; \mathbb{R}^d))$, it holds

$$\lim_{\varepsilon \to 0} \int_\Omega v_\varepsilon(x) \varphi\big(x, \frac{x}{\varepsilon}\big) \, d\mu_{\delta(\varepsilon),\varepsilon} = \int_{\Omega \times Y} v_0(x,y) \varphi(x,y) \, d\mathcal{L}^n(x) \otimes d\mu(y) \,. \tag{43}$$

We set N the subspace of the space W in (22) defined by

$$N := W \cap X^{p'}_\mu(\mathbf{T}; \mathbb{R}^{n^2}_{\rm sym}) = \Big\{ \sigma \in L^{p'}_\mu(\mathbf{T}; \mathbb{R}^{n^2}_{\rm sym}) \ : \ {\rm div}(\sigma\mu) = 0 \Big\} \,. \tag{44}$$

Denoting by $\mathcal{P}ot_\mu(\mathbf{T}; \mathbb{R}^{n^2}_{\rm sym})$ the space of periodic potential matrices

$$\mathcal{P}ot_\mu(\mathbf{T}; \mathbb{R}^{n^2}_{\rm sym}) := \overline{\big\{ e_\mu u \ : \ u \in \mathcal{D}^{1,p}_\mu(\mathbf{T}; \mathbb{R}^n) \big\}} \,, \tag{45}$$

and recalling (24), we find that the orthogonal space of N in $L^p_\mu(\mathbf{T}; \mathbb{R}^{n^2}_{\rm sym})$ is

$$N^\perp = \mathcal{P}ot_\mu(\mathbf{T}; \mathbb{R}^{n^2}_{\rm sym}) + L^p_\mu(\mathbf{T}; M_\mu(y)^\perp) \,. \tag{46}$$

Let K be the convex subset of $\mathbb{R}^{n^2}_{\rm sym}$ defined by

$$K := \Big\{ \int_Y \Psi d\mu \quad : \quad \Psi \in N \,, \ \|\Psi\|_{L^\infty(\mathbf{T}; \mathbb{R}^{n^2}_{\rm sym})} \le 1 \Big\} \,. \tag{47}$$

The next result can be obtained following the same line of proof as in [4, Lemma 4.5].

Lemma 5.5. *Let $j^{\rm hom}$ be the convex function on $\mathbb{R}^{n^2}_{\rm sym}$ defined by (27), and let $(j^{\rm hom})^*$ be its Fenchel conjugate. Then*

$$
\begin{aligned}
(j^{\rm hom})^*(z^*) &= \min\left\{\textstyle\int_Y j^*(\sigma)d\mu \ : \ \sigma \in N \ , \ \int_Y \sigma d\mu = z^*\right\} \ , \qquad \forall z^* \in \mathbb{R}^{n^2}_{\rm sym} \ , \quad (48)\\
j^{\rm hom}(z) &= \min\left\{\textstyle\int_Y j(z+\zeta(y))d\mu \ : \ \zeta \in N^\perp\right\} \ , \qquad \forall z \in \mathbb{R}^{n^2}_{\rm sym} \ . \quad (49)
\end{aligned}
$$

Moreover, if $j^{\rm hom}$ is coercive on $\mathbb{R}^{n^2}_{\rm sym}$, then the set K in (47) has a non-empty interior, i.e. there exists $r>0$ such that

$$\tau \in \mathbb{R}^{n^2}_{\rm sym} \quad , \quad |\tau| < r \quad \Rightarrow \quad \tau \in K \ . \tag{50}$$

We are now in a position to state the variant of Theorem 4.1 holding for the generalized notion (43) of two-scale convergence. For notational simplicity, in the following we omit to indicate the dependence of δ on ε.

Theorem 5.6. *Let $\{u_\varepsilon\} \subset \mathcal{C}^1_0(\Omega;\mathbb{R}^n)$ satisfy $\int_\Omega \left(|u_\varepsilon|^p + |e(u_\varepsilon)|^p\right) d\mu_{\delta,\varepsilon} \le M$; possibly passing to a subsequence, assume that $u_\varepsilon \rightharpoonup\rightharpoonup u_0 \in L^p_{\mathcal{L}^n\otimes\mu}(\Omega\times Y;\mathbb{R}^n)$ and $e(u_\varepsilon) \rightharpoonup\rightharpoonup \chi \in L^p_{\mathcal{L}^n\otimes\mu}(\Omega\times Y;\mathbb{R}^{n^2})$. Then, under the assumptions of Theorem 5.4, we have*

(i) *u_0 is independent of y and $u_0(x,y)=u(x)$ where the function u belongs to $W^{1,p}_0(\Omega;\mathbb{R}^n)$.*

(ii) *there exists $\eta \in L^p\big(\Omega;\mathcal{P}ot_\mu(\mathbf{T};\mathbb{R}^{n^2}_{\rm sym})\big)$ such that $\chi(x,y) = e(u)(x)+\eta(x,y)+\xi(x,y)$, with u as in (i), and $\xi(x,y) \in L^p\big(\Omega;\left[M_\mu(y)\right]^\perp\big)$.*

Proof of (i). Let v_ε be the unique function on $\mathbb{R}^n$ constant on each small cell $Y_{i,\varepsilon} = \varepsilon(i+Y)$, and such that $\int_{Y_{i,\varepsilon}} v_\varepsilon \, d\mu_{\delta,\varepsilon} = \int_{Y_{i,\varepsilon}} u_\varepsilon \, d\mu_{\delta,\varepsilon}$, for all $i \in \mathbb{Z}^n$. Then, by a change of variable and making use of $(H3)_\delta$, we infer

$$\int_{Y_{i,\varepsilon}} |u_\varepsilon - v_\varepsilon|^p \, d\mu_{\delta,\varepsilon} \le C\frac{\varepsilon^p}{\delta^p}\int_{Y_{i,\varepsilon}} |e(u_\varepsilon)|^p \, d\mu_{\delta,\varepsilon} \ ,$$

so that, summing with respect to i over $\mathbb{Z}^n$:

$$\int_\Omega |u_\varepsilon - v_\varepsilon|^p \, d\mu_{\delta,\varepsilon} \le C\frac{\varepsilon^p}{\delta^p}\int_\Omega |e(u_\varepsilon)|^p \, d\mu_{\delta,\varepsilon} \ .$$

As $\delta \gg \varepsilon$, it follows that the sequence $\{v_\varepsilon\}$ has the same two-scale limit as $\{u_\varepsilon\}$. Let us check the two-scale convergence (43) of $\{v_\varepsilon\}$ choosing a test function of the kind $\varphi(x,y) = \theta(x)\Psi(y)$, with $\theta \in \mathcal{D}(\Omega;\mathbb{R}^n)$, and $\Psi \in \mathcal{C}^\infty(\mathbf{T})$. Since v_ε is constant on each cell $Y_{i,\varepsilon}$, there exists an infinitesimal $o(\varepsilon)$ associated with the Lipschitz constant of θ such that, for all $i \in \mathbb{Z}^n$:

$$\int_{Y_{i,\varepsilon}} v_\varepsilon(x)\cdot\theta(x)\Psi(\frac{x}{\varepsilon})\mu_{\delta,\varepsilon} = \left(\int_{Y_{i,\varepsilon}} v_\varepsilon(x)\cdot\theta(x)\, d\mu_{\delta,\varepsilon}\right)\left(\int_Y \Psi d\mu_\delta\right)\big(1+o(\varepsilon)\big) \ .$$

Thus, summing the previous equality over $i \in \mathbb{Z}^n$ and passing to the limit as $\varepsilon \to 0$, we infer that

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon(x)\cdot\theta(x)\Psi(\frac{x}{\varepsilon})\,d\mu_{\delta,\varepsilon} &= \lim_{\varepsilon\to 0}\int_\Omega v_\varepsilon(x)\cdot\theta(x)\Psi(\frac{x}{\varepsilon})\,d\mu_{\delta,\varepsilon}\\ &= \left(\lim_{\varepsilon\to 0}\int_\Omega v_\varepsilon(x)\theta(x)\,d\mu_{\delta,\varepsilon}\right)\left(\lim_{\delta\to 0}\int_Y \Psi d\mu_\delta\right)\\ &= \left(\int_{\Omega\times Y} u_0(x,y)\cdot\theta(x)\,dx\otimes d\mu(y)\right)\left(\int_Y \Psi d\mu\right)\ .\end{aligned}$$

Thus, setting $u(x) := \int_Y u_0(x,y)\,d\mu(y)$, we deduce the first assertion of (i) from the following equality holding for every pair $(\theta,\Psi)\in\mathcal{D}(\Omega;\mathbb{R}^n)\times\mathcal{C}^\infty(\mathbf{T})$:

$$\int_{\Omega\times Y}[u_0(x,y)-u(x)]\cdot\theta(x)\,\Psi(y)\ dx\otimes d\mu(y)=0\ .$$

We prove now that u belongs to $W_0^{1,p}(\Omega;\mathbb{R}^n)$. For $\varphi\in\mathcal{C}^\infty(\overline{\Omega})$ and $\Psi\in N$, one has

$$\operatorname{div}\left(\varphi(x)\Psi(\frac{x}{\varepsilon})\mu_{\delta,\varepsilon}\right)=\Psi(\frac{x}{\varepsilon})\nabla\varphi(x)\mu_{\delta,\varepsilon}\ .$$

Thus, applying (43) to the sequence $\{e(u_\varepsilon)\}$ with $\varphi(x)\Psi(y)$ as test function, enforcing the integration by parts formula (18), and using the two-scale convergence of $\{u_\varepsilon\}$ to $u_0(x,y)=u(x)$, we obtain:

$$\begin{aligned}\int_{\Omega\times Y}\chi(x,y)\varphi(x)\Psi(y)\,dx\otimes d\mu(y) &= \lim_{\varepsilon\to 0}\textstyle\int_\Omega e(u_\varepsilon)\varphi(x)\Psi(\frac{x}{\varepsilon})d\mu_{\delta,\varepsilon}\\ &= -\lim_{\varepsilon\to 0}\textstyle\int_\Omega u_\varepsilon(x)\cdot\Psi(\frac{x}{\varepsilon})\nabla\varphi(x)\,d\mu_{\delta,\varepsilon} \qquad (51)\\ &= -\textstyle\int_{\Omega\times Y}u(x)\cdot\Psi(y)\nabla\varphi(x)dx\otimes d\mu(y)\\ &= -\left(\textstyle\int_Y\Psi d\mu\right)\cdot\left(\textstyle\int_\Omega u(x)\underset{s}{\otimes}\nabla\varphi(x)\,dx\right)\ ,\end{aligned}$$

where the symbol $\underset{s}{\otimes}$ denotes the symmetrized tensor product of two vectors in $\mathbb{R}^n$.

Applying (51) for Ψ running over the elements of $N^\perp$ whose $L^{p'}_\mu$-norm is smaller than 1, by the Hölder inequality and by (50), we deduce that, for a suitable constant $C>0$, there holds

$$\left|\int_\Omega u(x)\underset{s}{\otimes}\nabla\varphi(x)\,dx\right|\le C\,\|\varphi\|_{L^{p'}(\Omega)}\ ,\qquad\forall\,\varphi\in\mathcal{C}^\infty(\overline{\Omega})\ . \tag{52}$$

This proves that the distributional strain $e(u)$ belongs to $L^p(\Omega;\mathbb{R}^{n^2}_{\rm sym})$, hence, by the classical Korn inequality, $u\in W^{1,p}(\Omega;\mathbb{R}^n)$. We may now integrate by parts to find

$$\int_\Omega u(x)\underset{s}{\otimes}\nabla\varphi(x)\,dx\ =\ -\int_\Omega e(u)(x)\varphi(x)\,dx\ +\ \int_{\partial\Omega}(u(x)\underset{s}{\otimes}n(x))\,\varphi(x)\ d\mathcal{H}^{n-1}\ , \tag{53}$$

where $n(x)$ denotes the unit exterior normal vector to $\partial\Omega$. Then, from (52) and (53), it easily follows that the trace of u vanishes on $\partial\Omega$.

Proof of (ii). By (51) and (53), the equality

$$\int_{\Omega\times Y} [\chi(x,y) - e(u)(x)]\, \varphi(x)\Psi(y)\, dx \otimes d\mu(y) = 0$$

holds for every $\varphi \in \mathcal{C}^\infty(\overline{\Omega})$ and $\Psi \in N$. Localizing with respect to x, we infer that, for almost all $x \in \Omega$, $\chi(x,\cdot)$ belongs to $N^\perp$. Then we conclude deducing (ii) from (46). □

Proof of Theorem 5.4 (sketch). The proof of the $\Gamma-$limsup inequality runs exactly as for Theorem 4.2 by considering an approximating sequence of the kind $u_\varepsilon(x) = u(x) + \varepsilon\varphi(x, \frac{x}{\varepsilon})$ for φ varying in $\mathcal{C}_0^\infty(\Omega; \mathcal{C}^\infty(\mathbf{T}; \mathbb{R}^n))$.

In order to prove the $\Gamma-$liminf inequality, let us consider a sequence $\{u_\varepsilon\}$ such that $u_\varepsilon \mu_{\delta,\varepsilon} \rightharpoonup u\mathcal{L}^n$ and $\sup_\varepsilon \big\{J_{\delta(\varepsilon),\varepsilon}(u_\varepsilon)\big\} < +\infty$. Then by the p-growth condition from below satisfied by j, the sequence $\int_\Omega |e(u_\varepsilon)|^p\, d\mu_{\delta,\varepsilon}$ remains bounded, so that we may apply Theorem 5.6 (which holds in fact under the sole assumption of an uniform control on $\int_\Omega |u_\varepsilon| d\mu_{\delta,\varepsilon}$ instead of $\int_\Omega |u_\varepsilon|^p d\mu_{\delta,\varepsilon}$) to find that u belongs to $W_0^{1,p}(\Omega)$ and that, possibly passing to a subsequence, we have:

$$e(u_\varepsilon) \rightharpoonup\rightharpoonup e(u)(x)+\eta(x,y)+\xi(x,y), \text{ with } \eta \in L^p\big(\Omega; \mathcal{P}ot_\mu(\mathbf{T}; \mathbb{R}^{n^2}_{\rm sym})\big),\ \xi \in L^p\big(\Omega; \big[M_\mu(y)\big]^\perp\big).$$

Then, using a straighforward convexity argument [see [4], Proposition 2.5], we deduce that

$$\begin{aligned}
\liminf_{\varepsilon\to 0} J_{\delta(\varepsilon),\varepsilon}(u_\varepsilon) &= \liminf_{\varepsilon\to 0} \int_\Omega j(e(u_\varepsilon))\, d\mu_{\delta,\varepsilon} \\
&\geq \int_{\Omega\times Y} j\big(e(u)(x) + \eta(x,y) + \xi(x,y)\big)\, dx \otimes d\mu(y) \\
&\geq \int_{\Omega\times Y} j_\mu\big(e(u)(x) + \eta(x,y)\big)\, dx \otimes d\mu(y) \\
&\geq \int_\Omega j^{\rm hom}(e(u))\, dx.
\end{aligned}$$

□

6. Related problems and conjectures

We wish to conclude the paper by pointing out some related problems, which arise in the framework of both scalar and vector homogenization. First of all, we think it could be interesting to study the asymptotic behaviour of the sequence $\{J_{\delta,\varepsilon}\}$ in (33), when δ depends on ε, but we don't have $\delta(\varepsilon) \gg \varepsilon$ for $\varepsilon \to 0$ (as in Theorem 5.4). The critical scale for $\delta(\varepsilon)$ below which the Γ-limit degenerates to zero seems to be $\delta(\varepsilon) \sim \varepsilon$. As far as we know, this problem is fully open in any general framework. Further, two more topics are in our opinion worth of investigation, as they would provide respectively possible applications in mechanics, and a full theoretical understanding of non-local homogenization phenomena.

The former matter is the investigation of optimal bounds for the effective matrices or tensors, which can be obtained homogenizing scalar or vector problems on low-dimensional structures. Actually, a wide part of the classical literature on homogenization, is devoted to the following problem: characterize the matrices $A^{\rm hom}$ or the fourth-order tensors $B^{\rm hom}$ which can be found starting respectively from a symmetric matrix $A(x)$ or fourth-order tensor $B(x)$, whose elements are periodic and bounded functions on a sub-domain of $\mathbb{R}^n$. To that aim, one has to prove accurate estimates satisfied by $A^{\rm hom}$ or $B^{\rm hom}$, and possibly show that they are attained, namely that the involved inequalities turn into equalities when A or B are suitably chosen. For instance, if one restricts the analysis to the case of isotropic matrices associated to two-phase media, whose periodicity cell contains two given materials in fixed proportions, the inequalities satisfied by the corresponding homogenized matrix are the famous *Hashin-Shtrikman bounds* [9], which are regarded as a central result in the theory of composite materials; it is also well-known that optimal media in this respect are stratified composites of rank one. We refer to Chapters 6 and 13 of the book [10], and references quoted therein, for a review on this subject.

The same kind of problem may be considered, within our measure approach, for low-dimensional structures. Indeed, one can wonder what kind of matrices or fourth-order tensors may be found, via the homogenization formulae (10) or (27), when the integrands f and j are prescribed quadratic forms, and μ belongs by assumption to a certain class of measures.

We address to a forthcoming paper for the answer to such question under dimension and mass constraints on μ [3].

The latter topic we would probe, is the scalar homogenization, by fattening approach, of measures μ which are not p-connected in the sense of $(C1)$. More precisely, let us consider the sequence of two-parameter integrals $\{F_{\delta,\varepsilon}\}$ defined as in (6), replacing the measure $\mu_\varepsilon(x) = \mu(\frac{x}{\varepsilon})$ by $\mu_{\delta,\varepsilon}(x) = \mu_\delta(\frac{x}{\varepsilon})$, where $\mu_\delta := \rho_\delta \star \mu$.

If μ is p-connected, we have shown in [4, Section 6] that the passage to the limit with respect to the two parameters ε and δ is commutative; thus, the homogenized energy $F_0 := \Gamma - \lim_{\varepsilon\to 0} F_{\delta(\varepsilon),\varepsilon}$ is independent of the asymptotic behaviour of $\delta(\varepsilon)$ as ε tends to zero, and it can be obtained simply applying Theorem 2.1 to the measure μ.

On the other hand, if μ is not p-connected, the commutativity of the limit process becomes false, and F_0 may be a non-local functional, depending on the rate of convergence to zero of $\delta = \delta(\varepsilon)$. The explicit computation of F_0 when μ is associated to a fibred structure in $\mathbb{R}^3$, has been performed by capacitary methods in [1], where it was firstly pointed out the appearance of non-local homogenized functionals.

The range of this phenomenon is in our opinion an extended one, as non-locality is strictly linked to the lack of connectedness (see [4, Section 6] for a reinterpretation of the main result of [1] in the framework of measures). Thus, we have tried to characterize the functional F_0 when μ is non-connected, and we have got convinced of the following conjecture, which so far we are not able to prove.

Suppose that we can write μ as $\mu = \mu_1 + \cdots + \mu_m$, where the measures μ_i are mutually singular, and each of them is p-connected. Take a sequence $\{u_\varepsilon\} \subset \mathcal{C}^1_0(\Omega)$ such that $\sup_\varepsilon\{F_{\delta(\varepsilon),\varepsilon}\} < +\infty$. The two-scale limit $u_0(x,y)$ of u_ε (intended in the sense of (43)), will be given, for μ_i-a.e. $y \in Y$, by a function $u_i(x)$ belonging to $W^{1,p}_0(\Omega)$. Therefore, we expect that F_0 will contain the sum of the diffusion energies related to each μ_i, that is $\sum_{i=1}^m \int_\Omega f_i^{\rm hom}(\nabla u_i)\, dx$, where $f_i^{\rm hom}$ are defined by formula (10) with $\mu = \mu_i$.

Nevertheless, one should also take into account that, if u_ε oscillate at scale ε, say $u_\varepsilon(x) = u_0(x, \frac{x}{\varepsilon}) + \varepsilon v(x, \frac{x}{\varepsilon})$, a term of order -1 in ε appears in the gradient of u_ε, due to the dependence of u_0 on the rapid variable $\frac{x}{\varepsilon}$. We think that such term is responsible for the interaction between the different "connected components" μ_i of μ, and we guess that it can be somehow decoupled from the other terms, in such way that it produces the non-local effect in the expression of F_0.

In order to describe in a proper mathematical way the energy created by the gaps between the different values $u_1(x), u_2(x), \ldots, u_m(x)$ of $u_0(x, \cdot)$, we are led to consider an auxiliary sequence of variational integrals on the unit cell. For every λ in the space $\mathcal{M}(\mathbf{T})$ of signed periodic measures, set

$$G_\varepsilon(\lambda) := \begin{cases} \displaystyle\int_Y f\Big(\frac{\nabla w}{\varepsilon}\Big)\, d\mu_{\delta(\varepsilon)} & \text{if } \lambda = w\mu_{\delta(\varepsilon)},\ w \in \mathcal{C}^\infty(\mathbf{T}) \ , \\ +\infty & \text{otherwise;} \end{cases}$$

and assume that the sequence $G_\varepsilon(\lambda)$ Γ-converges with respect to the weak * convergence on $\mathcal{M}(\mathbf{T})$ to a functional $G_0(\lambda)$. Then, by the growth condition from below satisfied by f, it is easy to check that $G_0(\lambda)$ is finite only if λ is absolutely continuous with respect to μ, with a density w satisfying $(\nabla_\mu w)\mu = 0$, hence constant on $\mathrm{spt}(\mu_i)$, for $i = 1, \ldots, m$. Therefore G_0 is completely described in terms of the real function g defined on $\mathbb{R}^m$ by setting

$$g(c_1, \ldots, c_m) := G_0\Big(\sum_{i=1}^m c_i\mu_i\Big) \ .$$

The statement of the conjecture is the following:

$$F_0(\lambda_1, \ldots, \lambda_m) = \begin{cases} \displaystyle\sum_{i=1}^m \int_\Omega f_i^{\mathrm{hom}}(\nabla u_i(x))\, dx + \int_\Omega g(u_1(x), \ldots, u_m(x))\, dx & \text{if } \lambda_i = u_i\mathcal{L}^n, \\ & u_i \in W_0^{1,p}(\Omega); \\ +\infty & \text{otherwise.} \end{cases}$$

Let us finally remark that, to our mind, the validity of this conjecture can be extended to the case in which μ_δ, instead of being $\rho_\delta \star \mu$, is any approximating sequence for μ which guarantees suitable properties of convergence for the related functional spaces $X^{p'}_{\mu_\delta}(\Omega)$. In particular, the above expression of F_0 can be used to recover the non-local homogenized functional found by Bellieud and Bouchitté in [1].

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