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Upper Hőlder Continuity of Minimal Points

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In the present paper we derive criteria for upper Lipschitz/Hőlder continuity of the set of minimal points of a given subset $A \subset Y$ of a normed space Y when A is subjected to perturbations. To this aim we introduce the rate of containment of A, a real-valued function of one real variable, which measures the depart from minimality as a function of the distance from the minimal point set. The main requirement we impose is that for small arguments the rate of containment is a sufficiently fast growing function. The obtained results are applied to parametric vector optimization problems to derive conditions for upper Hőlder continuity of the performance multifunction.

Keywords: Minimal points, Hőlder multivalued mappings, parametric vector optimization

1. Introduction

Rate of convergence and Lipschitz type properties of solutions to perturbed optimization problems are fundamental topics of stability analysis in scalar optimization. The list of contributors to the subject is long and contains for instance [24], [14], [16], [26], [17], [27], [19], [20], [28], and many others. In vector optimization the results on Lipschitz continuity of solutions are scarce and concern mainly some special classes of problems, for linear case see eg.[10], [11], [12], for convex case see eg. [9], [13].

In the present paper we investigate upper Hőlder continuity of the set of minimal points $Min(A|\mathcal{K})$ with respect to cone $\mathcal{K} \subset Y$ of a given subset $A \subset Y$ of a normed space Y when A is subjected to perturbations. We express perturbations by a certain multivalued mapping Γ, defined on a space of perturbations U, with $\Gamma(u_0) = A$, and consider the family of problems (P_u) of finding $Min(\Gamma(u)|\mathcal{K})$. Upper Hőlder property at u_0 ensures that the distance of a solution of perturbed problem (P_u) to the set of solutions of unperturbed problem (P_{u_0}) can be estimated via the distance of perturbation $\|u - u_0\|$ raised to some power q. Hence, upper Hőlder property is of interest when it is impossible or too dificult to deal with the original problem and one wants to know the magnitude of the error made by accepting a solution of perturbed problem as a solution of the original problem. For instance, numerical representations of problems lead to perturbations due to finite precision. The upper Lipschitz property (upper Hőlder property with $q = 1$) has already appeared in investigation of stability of different problems, see eg [21], [22].

In Section 3 we introduce the rate of containment of a set A with respect to $\mathcal{K}$, which is a function of one variable and measures the depart from minimality as a function of the distance from the minimal point set. This is a nondecreasing function, in general nonconvex, even for convex sets. In Section 4 our main result is Theorem 4.2 which gives conditions for upper Hőlder continuity of minimal point multifunction $M(u) = Min(\Gamma(u)|\mathcal{K})$ at a

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given point u_0. The essential requirement of Theorem 4.2 is that the rate of containment of the set $\Gamma(u_0)$ is a sufficiently fast growing function for arguments close to zero. In Section 5 we apply the obtained results to vector optimization problems. In Theorem 5.5 we give conditions assuring upper Hőlder continuity of the performance mulifunction of parametric vector optimization problems. These conditions are expressed with the help of the notion of strong minimal solution which can be viewed as a generalization of the notion of $\phi-$local minimizer as defined by Attouch and Wets ([2],[3]).

Throughout the paper we assume that $(Y, \|\cdot\|)$ is a normed space and $\mathcal{K}$ is a closed convex pointed cone in Y. Let $A \subset Y$ be a subset of Y. We say that $y \in A$ is a minimal point of A with respect to $\mathcal{K}$ if $(y-\mathcal{K}) \cap A = \{y\}$. By $Min(A|\mathcal{K})$ we denote the set of all minimal points of A with respect to $\mathcal{K}$. We say that the domination property, (DP), holds for A if $A \subset Min(A|\mathcal{K}) + \mathcal{K}$.

2. Containment Property

By $B(a,r)$ we denote the open ball of radius r and centre a, $B(0,1) = B$. For any subset A of Y and any $y \in Y$ we have $d(y, A) = \inf_{a \in A} \|y-a\|$ and $B(A, \varepsilon) = \{y \in Y \mid d(y, A) < \varepsilon\}$.

Definition 2.1 ([6])(Containment property). We say that the **containment property** (CP) holds for a subset $A \subset Y$ of Y if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$[A \setminus B(Min(A|\mathcal{K}), \varepsilon)] + B(0, \delta) \subset Min(A|\mathcal{K}) + \mathcal{K}. \tag{1}$$

If $Min(A|\mathcal{K})$ is closed, (CP) implies (DP). Properties of (CP) are investigated in [6], [7].

The following proposition gives a purely topological proof of the equivalent form of condition (1) for cones with nonempty interior.

Proposition 2.2. *Let $\mathcal{K}$ be a closed convex pointed cone in Y, $\operatorname{int}\mathcal{K} \neq \emptyset$, and $A \subset Y$ a subset of Y. The following are equivalent:*

(i) *(CP) holds for A*

(ii) *for each $\varepsilon > 0$ there exists $\delta > 0$ such that each $y \in A \setminus B(Min(A|\mathcal{K}), \varepsilon)$ can be represented as*

$$y = \eta + k, \quad \textit{where} \quad \eta \in Min(A|\mathcal{K}), k + B(0, \delta) \subset \mathcal{K}. \tag{2}$$

Proof. $(i) \to (ii)$. For any $0-$neighbourhood O, define

$$\mathcal{K}_O = \{k \in \mathcal{K} \mid k + O \subset \mathcal{K}\}.$$

Clearly, we have $\operatorname{int}\mathcal{K} = \bigcup_{O \in \mathcal{N}} \mathcal{K}_O$. We show that for any $0-$neighbourhood Q there exists a $0-$neighbourhood O such that

$$(Min(A|\mathcal{K}) + \mathcal{K})_Q \subset Min(A|\mathcal{K}) + \mathcal{K}_O, \tag{3}$$

where $(Min(A|\mathcal{K})+\mathcal{K})_Q = \{y \in Y \mid y+Q \subset Min(A|\mathcal{K})+\mathcal{K}\}$. Indeed, let $a \in (Min(A|\mathcal{K})+\mathcal{K})_Q$, i.e., $a + Q \subset Min(A|\mathcal{K}) + \mathcal{K}$. Since $0 \in \operatorname{cl}(-\mathcal{K})$, for any $0-$neighbourhood Q there exists a $0-$neighbourhood O such that $Q \cap (-\mathcal{K}_O) \neq \emptyset$. Thus, there exists $q \in Q \cap (-\mathcal{K}_O)$ such that $a + q \in Min(A|\mathcal{K}) + \mathcal{K}$, and consequently $a \in Min(A|\mathcal{K}) + \mathcal{K}_O$.

Suppose now that (CP) holds for A, ie for each $0-$neighbourhood W there exists a $0-$neighbourhood Q such that for any $y \in A \setminus (Min(A|\mathcal{K}) + W)$

$$y \in (Min(A|\mathcal{K}) + \mathcal{K})_Q \,,$$

and by (3), for some $0-$neighbourhood O, $y \in Min(A|\mathcal{K}) + \mathcal{K}_O$.

$(ii) \to (i)$. Obvious. □

3. Rate of Containment

Denote

$$A(\varepsilon) = A \setminus B(Min(A|\mathcal{K}), \varepsilon) \,.$$

Definition 3.1 (Rate of containment). The function $\mu : Y \to R$ defined as

$$\mu(y) = \sup_{\substack{\delta \geq 0 : \\ y = \eta + k, \\ \eta \in Min(A|\mathcal{K}), \\ k + \delta B \subset \mathcal{K}}} \delta \,. \tag{4}$$

is the **rate of containment of y with respect to A and $\mathcal{K}$**.

The **rate of containment of a set A with respect to $\mathcal{K}$** is the function $\delta : R_+ \to R$ defined as

$$\delta(\varepsilon) = \inf_{y \in A(\varepsilon)} \mu(y) \,.$$

We have $\{y \in Y \mid \mu(y) > -\infty\} = Min(A|\mathcal{K}) + \mathcal{K}$. For $y \in Min(A|\mathcal{K})$, it is $\mu(y) = 0$. For $y \notin Min(A|\mathcal{K}$, the value $\mu(y)$ gives the maximal radius r such that $k + rB \subset \mathcal{K}$, where $k \in y - [Min(A|\mathcal{K}) \cap (y - \mathcal{K})] \subset \mathcal{K}$. In this sense, $\mu(y)$ can be viewed as a measure of depart from minimality of y. Consequently, $\delta(\varepsilon)$ is the minimal depart from minimality over all $y \in A$ whose distance from $Min(A|\mathcal{K})$ is not smaller than ε. If $\operatorname{int} \mathcal{K} = \emptyset$, then $\mu(y) = 0$ for any $y \in Min(A|\mathcal{K}) + \mathcal{K}$.

If $Min(A|\mathcal{K})$ is closed, then (DP) holds for A if and only if $\delta(\varepsilon) \geq 0$ for $\varepsilon > 0$.

Now we define an auxiliary function for $\mathcal{K}$. The function $cont : \mathcal{K} \to R_+$, defined as

$$cont(k) = \sup\{r \mid k + rB \subset \mathcal{K}\}$$

is called the **cone containment function**. The function $cont$ is positively homogeneous, i.e., $cont(\lambda k) = \lambda\, cont(k)$ for $\lambda \geq 0$, suplinear, i.e., $cont(k_1 + k_2) \geq cont(k_1) + cont(k_2)$, for $k_1, k_2 \in \mathcal{K}$, and hence $cont$ is concave on $\mathcal{K}$. Thus, we can rewrite the rate of containment δ as follows

$$\delta(\varepsilon) = \inf_{y \in A(\varepsilon)} \sup_{\eta \in Min(A|\mathcal{K}) \cap (y - \mathcal{K})} cont(y - \eta) \,.$$

In Proposition below we give conditions for the supremum in the definition of μ to be attained. These conditions allow us to express containment property (CP) through the rate of containment (see Proposition 3.4).

Proposition 3.2. *Let $Y = (Y, \|\cdot\|)$ be a normed space. Let $\mathcal{K}$ be a closed convex pointed cone in Y, $\operatorname{int}\mathcal{K} \neq \emptyset$, and let $A \subset Y$ be a subset of Y.*

Under one of the following conditions:

(i) *$Min(A|\mathcal{K})$ is weakly compact,*
(ii) *$Min(A|\mathcal{K})$ is bounded weakly closed, and $\mathcal{K}$ has a weakly compact base,*

for any $y \in Min(A|\mathcal{K})+\mathcal{K}$, there exists a representation $y = \eta_y+k_y$, with $\eta_y \in Min(A|\mathcal{K})$, and $k_y + \mu(y)B \subset \mathcal{K}$.

Proof. Let $y \in Min(A|\mathcal{K})+\mathcal{K}$. For any $\alpha > 0$, one can find a representation $y = \eta_\alpha+k_\alpha$, $\eta_\alpha \in Min(A|\mathcal{K})$, $k_\alpha \in \mathcal{K}$, $k_\alpha + cont(k_\alpha)B \subset \mathcal{K}$, satisfying

$$cont(k_\alpha) \leq \mu(y) \quad \text{and} \quad cont(k_\alpha) > \mu(y) - \alpha\,.$$

We start by proving that under any one of the conditions (i), (ii) an element y can be represented in the form

$$y = \eta_0 + k_0\,, \tag{5}$$

where $\eta_0 \in Min(A|\mathcal{K})$, $k_0 \in \mathcal{K}$, $\eta_0 = \lim_\alpha \eta_\alpha$, $k_0 = \lim_\alpha k_\alpha$.

In the case (i), since $Min(A|\mathcal{K})$ is weakly compact, there exists a weakly convergent subnet of the net $\{\eta_\alpha\}$. Without loss of generality we can assume that the net $\{\eta_\alpha\}$ weakly converges to some $\eta_0 \in Min(A|\mathcal{K})$. Since $\mathcal{K}$ is closed and convex, the net $\{k_\alpha\}$, $k_\alpha = y - \eta_\alpha$, converges weakly to $k_0 \in \mathcal{K}$, and $y = \eta_0 + k_0$.

To prove (5) in the case (ii) suppose that Θ is a weakly compact base of $\mathcal{K}$, $k_\alpha = \lambda_\alpha\theta_\alpha$, $\lambda_\alpha \geq 0$, and $\{\theta_\alpha\} \subset \Theta$ contains a weakly convergent subnet. Without loss of generality we can assume that $\{\theta_\alpha\}$ converges weakly to some $\theta_0 \in \Theta$. Since $Min(A|\mathcal{K})$ is bounded and $\|\theta\| \geq M_0$ for all $\theta \in \Theta$ we get

$$M_1 \geq \|y - \eta_\alpha\| = \lambda_\alpha\|\theta_\alpha\| \geq M_0\lambda_\alpha\,,$$

for some positive constants M_0, M_1. This implies that $\{\lambda_\alpha\}$ is bounded, and thus the net $\{k_\alpha\}$ contains a convergent subnet, i.e., we can assume that $\{k_\alpha\}$ weakly converges to some $k_0 = \lambda_0\theta_0 \in \mathcal{K}$. In consequence, by the weak closedness of $Min(A|\mathcal{K})$, $\eta_\alpha = y - k_\alpha$ converges weakly to some $\eta_0 \in Min(A|\mathcal{K})$ and we get a representation $y = \eta_0 + k_0$.

To complete the proof we show that $k_0 + \mu(y)B \subset \mathcal{K}$. On the contrary, if it were $k_0 + \mu(y)b \notin \mathcal{K}$, for some $b_0 \in B$, by separation arguments it would be

$$f(k_0 + \mu(y)b_0) < 0 < f(k) \quad \text{for } k \in \mathcal{K}\,,$$

for some $f \in \mathcal{K}^*$, $\mathcal{K}^* = \{f \in Y^* \mid f(k) \geq 0\}$. By the weak convergence of $\{k_\alpha\}$ to k_0, and $\{(cont(k_\alpha) - \mu(y))b_0\}$ to zero we would have

$$f(k_\alpha + cont(k_\alpha)b_0) = f(k_0 + \mu(y)b_0) + f(k_\alpha - k_0) + f([cont(k_\alpha) - \mu(y)]b_0) < 0\,,$$

which would contradict the fact that $k_\alpha + cont(k_\alpha)B \subset \mathcal{K}$. □

In the example below we calculate $\mu(y)$ for y from the closed unit ball.

Example 3.3. Let $Y = R^2$, and $A = \text{cl}B$, and $\mathcal{K} = \{(y_1, y_2) \in R^2 \mid y_1 \geq 0 \quad y_2 \geq 0\}$. Clearly, (DP) and (CP) holds for A, and

$$Min(A|\mathcal{K}) = \{(\eta_1, \eta_2) \in A \mid \eta_2 = -\sqrt{1-\eta_1^2}, \quad -1 \leq \eta_1 \leq 0\}.$$

For any $y \in A$, put $Min(A|\mathcal{K})_y = Min(A|\mathcal{K}) \cap (y - \mathcal{K})$. For any representation of 0 in the form $0 = \eta + k_\eta$, where $\eta \in Min(A|\mathcal{K})$, $k_\eta \in \mathcal{K}$, we have $\eta = (\eta_1, \eta_2) \in Min(A|\mathcal{K})_0 = Min(A|\mathcal{K})$,

$$cont(k_\eta) = \min\{-\eta_1, \sqrt{1-\eta_1^2}\} = \begin{cases} \sqrt{1-\eta_1^2} & \text{for} \quad -1 \leq \eta_1 \leq -1/\sqrt{2} \\ -\eta_1 & \text{for} \quad -1/\sqrt{2} \leq \eta_1 \leq 0 \end{cases}.$$

and $\mu(0) = \sup_{\{-1 \leq \eta_1 \leq 0\}} cont(k_\eta) = 1/\sqrt{2}$. For $y = (y_1, y_2) \in A$, with $y_2 \geq 0$,

$$Min(A|\mathcal{K})_y = \{(\eta_1, \eta_2) \in Min(A|\mathcal{K}) \mid \eta_2 = -\sqrt{1-\eta_1^2}, \quad -1 \leq \eta_1 \leq \min\{0, y_1\}\},$$

and

$$\mu(y) = \max_{\{-1 \leq \eta_1 \leq \min\{0, y_1\}\}} \min\{y_1 - \eta_1, y_2 + \sqrt{1-\eta_1}\}.$$

For any $y = (y_1, y_2) \in A$, $y_2 < 0$,

$$Min(A|\mathcal{K})_y = \{(\eta_1, \eta_2) \in Min(A|\mathcal{K}) \mid \eta_2 = -\sqrt{1-\eta_1^2}, \quad -\sqrt{1-y_2^2} \leq \eta_1 \leq \min\{0, y_1\}\},$$

and

$$\mu(y) = \max_{\{-\sqrt{1-y_2^2} \leq \eta_1 \leq \min\{0, y_1\}\}} \min\{y_1 - \eta_1, y_2 + \sqrt{1-\eta_1^2}\}.$$

Let

$$\text{dom}\,\delta = \{\varepsilon \in R \mid \delta(\varepsilon) < +\infty\}.$$

The following properties of the rate of containment are direct consequences of the definition.

1. The rate of containment $\delta : R_+ \to R$ is nondecreasing. Indeed, let $\varepsilon_1, \varepsilon_2 \in \text{dom}\delta$, $\varepsilon_1 > \varepsilon_2 > 0$. Then $A(\varepsilon_1) \subset A(\varepsilon_2)$, and consequently $\delta(\varepsilon_1) = \inf_{y \in A(\varepsilon_1)} \mu(y) \geq \delta(\varepsilon_2)$.

2. Assume that there exists at least one $\eta \in Min(A|\mathcal{K})$ which is not an isolated point of A. Suppose that one of the conditions hold:
(i) $Min(A|\mathcal{K})$ is weakly compact,
(ii) $Min(A|\mathcal{K})$ is bounded and weakly closed, and $\mathcal{K}$ has a weakly compact base.
Then $\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0$. Indeed, suppose that

$$+\infty > \delta(\varepsilon_n) = \inf_{y \in A(\varepsilon_n)} \mu(y) > c$$

for some $\varepsilon_n \to 0$ and $c > 0$. Hence, for each n and $y \in A(\varepsilon_n)$ it is $\mu(y) > c$. By assumption, one can choose $y_n \in A(\varepsilon_n)$, $y_n \to \eta \in Min(A|\mathcal{K})$. Since $\mu(y_n) > c$, there exists a representation $y_n = \eta_n + k_n$, $\eta_n \in Min(A|\mathcal{K})$, $k_n + cB \in \mathcal{K}$. By (i) or (ii), $\eta = \eta_0 + k_0$, where $\lim_n \eta_n = \eta_0 \in Min(A|\mathcal{K})$, $\lim_n k_n = k_0 \in \mathcal{K}$ (for details see the proof of Proposition 3.2 above). Consequently, $k_0 = 0$, but on the other hand, $k_0 + c/2B \subset \mathcal{K}$, which is a contradiction. This proves the assertion.

3. Let $A \subset Y$ be a subset of Y. Let $\mathcal{K} \subset Y$ be a closed convex cone in Y, $\operatorname{int}\mathcal{K} \neq \emptyset$. Then, (CP) holds for A if and only if $\delta(\varepsilon) > 0$ for $\varepsilon > 0$. The "only if" part follows directly from Proposition 2.2. To prove the "if" part take any $\varepsilon > 0$. We have $\delta(\varepsilon) = c > 0$, and consequently, $\mu(y) \geq c$, for any $y \in A(\varepsilon)$, which means that there exists a representation $y = \eta_y + k_y$, $\eta_y \in Min(A|\mathcal{K})$, $k_y + cB \subset \mathcal{K}$. Thus, (CP) holds.

4. Let $A \subset Y$ be a convex subset of Y. Under one of the conditions:

(i) $Min(A|\mathcal{K})$ is weakly compact,

(ii) $Min(A|\mathcal{K})$ is weakly bounded, and $\mathcal{K}$ has a weakly compact base

we have

$$\delta(\varepsilon) = \inf_{y \in A_{eq}(\varepsilon)} \mu(y) = \inf_{y \in A_{\beta}(\varepsilon)} \mu(y)\,,$$

where $A_{eq}(\varepsilon) = \{y \in A \mid d(y, Min(A|\mathcal{K})) = \varepsilon\}$, and $A_\beta = \{y \in A \mid \beta > d(y, Min(A|\mathcal{K})) \geq \varepsilon\}$.

Proposition 3.4. *Let $\mathcal{K} \subset Y$ be a closed convex cone in Y, $\operatorname{int}\mathcal{K} \neq \emptyset$. Let A be a nonempty subset of Y and let (CP) holds for A. Under one of the following conditions:*

- *$Min(A|\mathcal{K})$ is weakly compact,*
- *$Min(A|\mathcal{K})$ is bounded and weakly closed, and $\mathcal{K}$ has a weakly compact base,*

for any $\varepsilon > 0$ we have

(i) *$A(\varepsilon) + \delta(\varepsilon)B \subset Min(A|\mathcal{K}) + \mathcal{K}$,*

(ii) *each $y \in A(\varepsilon)$ can be represented in the form $y = \eta_y + k_y$, where $\eta_y \in Min(A|\mathcal{K})$, $k_y + \delta(\varepsilon) \cdot B \subset \mathcal{K}$.*

Proof. (ii). Follows directly from Proposition 3.2.

(i). Follows from (ii). □

4. Upper Hőlder Continuity of Minimal Points

Let $U = (U, \|\cdot\|)$ be a normed space and let $\Gamma : U \to Y$ be a multivalued mapping.

In this section we investigate the multivalued mapping $M : U \to Y$ defined as

$$M(u) = Min(\Gamma(u)|\mathcal{K})\,,$$

where $\Gamma : U \to Y$ is a given multivalued mapping. M is called the **minimal point multivalued mapping**. As defined in Section 5 the performance multivalued mapping of a given parametric vector optimization problem is a minimal point multivalued mapping M for some Γ.

A multivalued mapping $F : U \to Y$ is said to be upper Lipschitz at u_0 with constant L (see eg [21, 22, 23]) if there exists a neighbourhood U_0 of u_0 such that $F(u) \subset F(u_0) + L\|u - u_0\|B$ for $u \in U_0$. This property has been used in [21, 23] to investigate behaviour of parametric generalized equations.

Definition 4.1. Let $F : U \to Y$ be a multivalued mapping. We say that F is:

- **upper Hőlder at u_0 with order q and constant L** if there exists a neighbourhood U_0 of u_0 such that $F(u) \subset F(u_0) + L\|u - u_0\|^q B$ for $u \in U_0$,

- **lower Hőlder at u_0 with order q and constant** L if there exists a neighbourhood U_0 of u_0 such that $F(u_0) \subset F(u) + L\|u - u_0\|^q B$ for $u \in U_0$,
- **lower Lipschitz at u_0 with constant** L if F is lower Hőlder at u_0 with order $q = 1$ and constant L,

Following [4], we say that F is Lipschitz around u_0 with constant L if there exists a neighbourhood U_0 of u_0 such that $F(u_1) \subset F(u_2) + L\|u_1 - u_2\|B$ for $u_1, u_2 \in U_0$. We say that F is Hőlder around u_0 with order q and constant L if there exists a neighbourhood U_0 of u_0 such that $F(u_1) \subset F(u_2) + L\|u_1 - u_2\|^q B$ for $u_1, u_2 \in U_0$.

Lipschitzian/Hőlderian continuity of F around u_0 implies that F is lower and upper Lipschitz/Hőlder at u_0, but not conversely.

In the theorem below we give sufficient conditions for upper Hőlder continuity of minimal point multifunction M.

Theorem 4.2. *Let $Y = (Y, \|\cdot\|)$ and $U = (U, \|\cdot\|)$ be normed spaces. Let $\mathcal{K} \subset Y$ be a closed convex pointed cone in Y, $\operatorname{int}\mathcal{K} \neq \emptyset$. Let $\Gamma : U \to Y$ be a multivalued mapping which is upper Hőlder with order ℓ_1 and constant L_1 and lower Hőlder with order ℓ_2 and constant L_2 at u_0.*

Suppose that one of the following conditions hold:

(i) *$Min(\Gamma(u_0)|\mathcal{K})$ is weakly compact,*

(ii) *$Min(\Gamma(u_0)|\mathcal{K})$ is bounded and weakly closed, and $\mathcal{K}$ has a weakly compact base.*

If the rate of containment δ of $\Gamma(u_0)$ satisfies the condition $\delta(\varepsilon) \geq c \cdot \varepsilon^p$, with $c > 0$, then

$$M(u) \subset M(u_0) + (L_1 + \left(\frac{L_1 + L_2}{c}\right)^{\frac{1}{p}})\|u - u_0\|^{\min\{\ell_1, \frac{\min\{\ell_1,\ell_2\}}{p}\}} \cdot B.$$

for all u in some neighbourhood of u_0.

Proof. By the upper Hőlder continuity of Γ,

$$\begin{aligned}\Gamma(u) &\subset \Gamma(u_0) + L_1\|u - u_0\|^{\ell_1} \cdot B \\ &\subset [M(u_0) + L_1 \cdot \|u - u_0\|^{\ell_1} \cdot B + \left(\tfrac{L_1+L_2}{c}\right)^{\frac{1}{p}} \cdot \|u - u_0\|^{\frac{\min\{\ell_1,\ell_2\}}{p}} \cdot B] \cup \\ &\cup [(\Gamma(u_0) \setminus (M(u_0) + \left(\tfrac{L_1+L_2}{c}\right)^{\frac{1}{p}} \|u - u_0\|^{\frac{\min\{\ell_1,\ell_2\}}{p}} \cdot B)) + L_1 \cdot \|u - u_0\|^{\ell_1} \cdot B],\end{aligned}$$

for u in a neighbourhood U_0 of u_0. By the lower Hőlder continuity of Γ, there exists a neighbourhood U_1 of u_0 such that $\Gamma(u_0) \subset \Gamma(u) + L_2\|u - u_0\|^{\ell_2} B$ for $u \in U_1$.

Take any

$$y \in \Gamma(u) \cap [(\Gamma(u_0) \setminus (M(u_0) + \left(\frac{L_1 + L_2}{c}\right)^{\frac{1}{p}} \|u - u_0\|^{\frac{\min\{\ell_1,\ell_2\}}{p}} \cdot B)) + L_1\|u - u_0\|^{\ell_1} \cdot B],$$

where $u \in U_0 \cap U_1$. We have $y = \gamma + b_1$, where $\gamma \in \Gamma(u_0) \setminus (M(u_0) + \left(\frac{L_1+L_2}{c}\right)^{\frac{1}{p}} \|u - u_0\|^{\frac{\min\{\ell_1,\ell_2\}}{p}} \cdot B)$, $b_1 \in L_1\|u - u_0\|^{\ell_1} \cdot B$.

In view of (i), and (ii), by Proposition 3.4, any $z \in \Gamma(u_0) \setminus [M(u_0) + \varepsilon \cdot B]$, $\varepsilon > 0$, can be represented in the form $z = \eta_z + k_z$, $\eta_z \in Min(\Gamma(u_0)|\mathcal{K})$, $k_z + \delta(\varepsilon) \cdot B \subset \mathcal{K}$. Hence,

$$\gamma = \eta_\gamma + k_\gamma\,,\ \eta_\gamma \in M(u_0)\,,\ k_\gamma + \delta(\left(\frac{L_1 + L_2}{c}\right)^{\frac{1}{p}} \|u - u_0\|^{\frac{\min\{\ell_1,\ell_2\}}{p}}) \cdot B \subset \mathcal{K}\,.$$

By the lower Hőlder continuity of Γ,

$$\eta_\gamma = \gamma_1 + b_2\,,\ \gamma_1 \in \Gamma(u)\,,\ b_2 \in L_2\|u - u_0\|^{\ell_2} \cdot B\,,$$

and consequently, since $\delta(\varepsilon) \geq c \cdot \varepsilon^p$,

$$\begin{aligned} y - \gamma_1 &= \gamma + b_1 - \eta_\gamma + b_2 = \eta_\gamma + k_\gamma + b_1 - \eta_\gamma + b_2 \\ &\subset k_\gamma + (L_1 + L_2)\|u - u_0\|^{\min\{\ell_1,\ell_2\}} \cdot B \\ &\subset k_\gamma + \delta((\tfrac{L_1+L_2}{c})^{\frac{1}{p}} \|u - u_0\|^{\frac{\min\{\ell_1,\ell_2\}}{p}}) \cdot B \subset \mathcal{K}\,. \end{aligned} \tag{6}$$

This proves that for $u \in U_0 \cap U_1$we have

$$M(u) \cap [(\Gamma(u_0) \setminus (M(u_0) + \left(\frac{L_1 + L_2}{c}\right)^{\frac{1}{p}} L\|u - u_0\|^{\frac{\min\{\ell_1,\ell_2\}}{p}} \cdot B)) + L_1\|u - u_0\|^{\ell_1} \cdot B] = \emptyset\,.$$

Hence,

$$\begin{aligned} M(u) &\subset M(u_0) + L_1 \cdot \|u - u_0\|^{\ell_1} \cdot B + (\tfrac{L_1+L_2}{c})^{\frac{1}{p}} \cdot \|u - u_0\|^{\frac{\min\{\ell_1,\ell_2\}}{p}} \cdot B \\ &\subset M(u_0) + (L_1 + (\tfrac{L_1+L_2}{c})^{\frac{1}{p}})\|u - u_0\|^{\min\{\ell_1, \frac{\min\{\ell_1,\ell_2\}}{p}\}} \cdot B\,, \end{aligned}$$

for $u \in U_0 \cap U_1$, which completes the proof. □

Corollary 4.3. *Let $Y = (Y, \|\cdot\|)$ and $U = (U, \|\cdot\|)$ be normed spaces. Let $\mathcal{K} \subset Y$ be a closed convex pointed cone in Y, $\operatorname{int}\mathcal{K} \neq \emptyset$. Let $\Gamma : U \to Y$ be a Lipschitz multivalued mapping around u_0 with constant L.*

Suppose that one of the following conditions hold:

(i) *$Min(\Gamma(u_0)|\mathcal{K})$ is weakly compact,*
(ii) *$Min(\Gamma(u_0)|\mathcal{K})$ is bounded and weakly closed, and $\mathcal{K}$ has a weakly compact base.*

If the rate of containment δ of $\Gamma(u_0)$, satisfies the condition $\delta(\varepsilon) \geq c \cdot \varepsilon$, with $c > 0$, then the minimal point multivalued mapping M is upper Lipschitz at u_0 with constant $\frac{(2+c)L}{c}$.

Corollary 4.4. *Let $Y = (Y, \|\cdot\|)$ and $U = (U, \|\cdot\|)$ be normed spaces. Let $\mathcal{K} \subset Y$ be a closed convex pointed cone in Y, $\operatorname{int}\mathcal{K} \neq \emptyset$. Let $\Gamma : U \to Y$ be a Hőlder multivalued mapping with order q and constant L around u_0.*

Suppose that one of the following conditions hold:

(i) *$Min(\Gamma(u_0)|\mathcal{K})$ is weakly compact,*
(ii) *$Min(\Gamma(u_0)|\mathcal{K})$ is bounded and weakly closed, and $\mathcal{K}$ has a weakly compact base.*

If the rate of containment δ of $\Gamma(u_0)$, satisfies the condition $\delta(\varepsilon) \geq c \cdot \varepsilon^p$, with $p > 1$ and $c > 0$, then the minimal point multivalued mapping M is upper Hőlder at u_0 with constant $\left(L + (\frac{2L}{c})^{1/p}\right)$ and order $\frac{\ell}{p}$.

In the Introduction we indicated some situations where upper Hőlder continuity has a natural significance. One more example of such a situation comes from parametric vector optimization. Theorem 6.4 of [6] and Theorem 6.2 of [7], reveal the importance of upper type continuities of the performance multivalued mapping (see Section 5) in deriving criteria for continuity of solutions to parametric vector optimization problems. We should mention here that Hőlder continuity of the minimal point multivalued mapping M around a given point u_0, which is also of interest, constitutes the subject of a separate study.

5. Vector Optimization Problems

In a series of publications Attouch and Wets [1],[2], [3] developed an approach to investigation of quantitative stability of variational systems as defined by Rockafellar and Wets [25]. These authors prove Lipschitz and Hölder continuity of solutions to scalar minimization problems under perturbations for $\phi-$local minimizers. Given a function $f : X \to R$ an element $x_f \in X$ is called a $\phi-$local minimizer of f if $f(y) \geq f(x_f) + \phi(\|y - x_f\|)$ for all y in some ball around x_f, with ϕ being an **admissible** function, i.e. $\phi : R_+ \to R_+$, $\phi(t_n) \to 0$ implies $t_n \to 0$.

In this section we use similar approach to investigate stability of vector optimization problems.

Let X be a normed space. Let $f : X \to Y$, and $A_0 \subset X$. The vector optimization problem

$$\begin{array}{c} \mathcal{K} - minf(x) \\ \text{subject to } x \in A_0 \end{array} \tag{7}$$

consists in finding all $x \in S(f, A_0, \mathcal{K}) = \{x \in A_0 \mid f(x) \in Min(f(A_0)|\mathcal{K})\}$, $Min(f(A_0)|\mathcal{K}) = \{y \in f(A_0) \mid (y - \mathcal{K}) \cap f(A_0) = \{y\}\}$, (see Jahn [15], Luc [18]).

Definition 5.1. The solution set $S(f, A_0, \mathcal{K})$ is called $\phi-$**strong** or $\phi-$**dominated** if for each $x \in A_0$, $d(x, S(f, A_0, \mathcal{K})) < \rho$, there exists $s_x \in S(f, A_0, \mathcal{K})$ such that

$$f(x) \geq f(s_x) + \phi(\|x - s_x\|) \cdot B\,, \text{ i.e., } f(x) - f(s_x) - \phi(\|x - s_x\|) \cdot B \in \mathcal{K}\,,$$

for some admissible function $\phi : R_+ \to R_+$ and $\rho > 0$.

Definition 5.2. The solution set $S(f, A_0, \mathcal{K})$ is strong of order p with constant $c > 0$ if for each $x \in A_0, d(x, S(f, A_0, \mathcal{K})) < \rho, \rho > 0$, there exists $s_x \in S(f, A_0, \mathcal{K}), \|x - s_x\| < \delta$, such that

$$f(x) \geq f(s_x) + c\|x - s_x\|^p \cdot B\,, \text{ i.e., } f(x) - f(s_x) - c\|x - s_x\|^p \cdot B \in \mathcal{K}\,.$$

Proposition 5.3. *Let $X = (X, \|\cdot\|)$ and $Y = (Y, \|\cdot\|)$ be normed spaces. Let $\mathcal{K} \subset Y$ be a closed convex pointed cone in Y, $\operatorname{int}\mathcal{K} \neq \emptyset$, and let $A_0 \subset X$ be a subset of X.*

Let $f : X \to Y$ be a Hőlder mapping with constant L and order q, and let $f(A_0)$ be convex. Assume that there exists $\varepsilon_0 > 0$ such that for any $x \in A_0$

$$\varepsilon_0 > d(f(x), Min(f(A_0)|\mathcal{K})) \text{ implies } d(x, S(f, A_0, \mathcal{K})) < \delta\,, \tag{$*$}$$

and one of the conditions hold

(i) *$Min(A|\mathcal{K})$ is weakly compact,*

(ii) *$Min(A|\mathcal{K})$ is weakly bounded, and $\mathcal{K}$ has a weakly compact base.*

If the solution set $S(f, A_0, \mathcal{K})$ is strong of order $p > q$ with constant c, the rate of containment of the set $f(A_0)$ satisfies

$$\delta(\varepsilon) \geq \frac{c}{L^{p/q}} \varepsilon^{\frac{p}{q}},$$

for $\varepsilon < \varepsilon_0$.

Proof. Put $B = f(A_0)$. Take any $0 < \varepsilon < \varepsilon_0$, and $y \in B_{\varepsilon_0}(\varepsilon)$. Hence, $y = f(x), x \in A_0$, $\varepsilon_0 > d(f(x), Min(f(A_0)|\mathcal{K})) \geq \varepsilon$. By $(*)$, $d(x, S(f, A_0, \mathcal{K})) < \delta$. Since the solution set $S(f, A_0, \mathcal{K})$ is strong of order p, $p > q$, there exists $s_x \in S(f, A, \mathcal{K}), d(x, S(f, A_0, \mathcal{K})) < \rho, \rho > 0$, such that

$$f(x) - f(s_x) + c\|x - s_x\|^p \cdot B \subset \mathcal{K}.$$

By Hőlder continuity of f,

$$\frac{c}{L^{p/q}} \|f(x) - f(s_x)\|^{\frac{p}{q}} \leq c\|x - s_x\|^p,$$

and we obtain

$$f(x) - f(s_x) + \frac{c}{L^{p/q}} \|f(x) - f(s_x)\|^{\frac{p}{q}} \cdot B \subset f(x) - f(s_x) + c\|x - s_x\|^p \cdot B \subset \mathcal{K}.$$

Finally,

$$f(x) - f(s_x) + \frac{c}{L^{p/q}} \varepsilon^{\frac{p}{q}} \cdot B \subset f(x) - f(s_x) + \frac{c}{L^{p/q}} \|f(x) - f(s_x)\|^{\frac{p}{q}} \cdot B \subset \mathcal{K}.$$

By the convexity of B, $\delta(\varepsilon) \geq \frac{c}{L^{p/q}} \varepsilon^{\frac{p}{q}}$. □

Remark 5.4. We say that a norm $\|\cdot\|$ in Y is nondecreasing if for any $y \in Y$ and $k \in \mathcal{K}$ we have $\|y + k\| \geq \|y\|$. If norm $\|\cdot\|$ in Y is nondecreasing, and the solution set $S(f, A_0, \mathcal{K})$ is strong of order p with constant c, then $\|f(x) - f(s_x)\| \geq c\|x - s_x\|^p$, for any $x \in A_0$, $d(x, S(f, A_0, \mathcal{K})) < \rho$, and some $s_x \in S(f, A_0, \mathcal{K})$.

Let $f : X \to Y$, $A_0 \subset X$. Let $A : U \to Y$ be a set-valued mapping defined on a normed space U such that $A(u_0) = A_0$. We consider parametric vector optimization problem (P_u) of the form

$$\begin{array}{c} \mathcal{K} - min\ f(x) \\ \text{subject to } x \in A(u) \end{array} \tag{8}$$

(P_{u_0}) coincides with problem (7). The **performance multivalued mapping** $P : U \to Y$ is of the form

$$P(u) = \{y \in f(A(u)) \mid (y - f(A(u))) \cap f(A(u)) = \{y\}\},$$

and $P(u_0) = Min(f(A_0)|\mathcal{K})$. Note that the performance multivalued mapping P is a minimal point multivalued mapping M with $\Gamma(u) = f(A(u))$. The solution multivalued mapping $S : U \to X$ takes the form

$$S(u) = \{x \in X \mid f(x) \in P(u)\},$$

and $S(u_0) = S(f, A_0, \mathcal{K})$.

Theorem 5.5. *Let* $X = (X, \|\cdot\|)$, $Y = (Y, \|\cdot\|)$, $U = (U, \|\cdot\|)$ *be normed spaces. Let* $\mathcal{K} \subset Y$ *be a closed convex pointed cone in* Y, $\operatorname{int}\mathcal{K} \neq \emptyset$, $A_0 \subset X$ *a subset of* X, *and let* $f : X \to Y$ *be a Lipschitz mapping defined on* X, *with constant* L. *Let* $f(A_0)$ *be convex and condition* $(*)$ *of Proposition 5.3 hold.*

Assume that the solution set $S(f, A_0, \mathcal{K})$ *of (7) is strong of order* p, $p > 1$, *with constant* $c > 0$, *and one of the following conditions holds:*

- *Min*$(f(A_0)|\mathcal{K})$ *is weakly compact,*
- *Min*$(f(A_0)|\mathcal{K})$ *is bounded and weakly closed, and* $\mathcal{K}$ *has a weakly compact base.*

For any parametric problem of the form (8) such that A *is Hőlder around* u_0 *with order* ℓ *and constant* L_1, *the performance multivalued mapping* P *is upper Hőlder at* u_0 *with order* $\frac{\ell}{p}$ *and constant* $\left(LL_1 + \left(\frac{2LL_1}{c}\right)^{\frac{1}{p}}\right)$.

Proof. For the proof it is enough to observe that $f(A) : U \to Y$, being the image of the Hőlder multivalued mapping $A : U \to X$ under the Lipschitz mapping $f : X \to Y$, is a Hőlder multivalued mapping.

Since A is Hőlder around u_0, there exists a neighbourhood U_0 of u_0 such that $A(u_1) \subset A(u_2) + L_1\|u_1 - u_2\|^\ell B$, for $u_1, u_2 \in U_0$, i.e., for each $a_1 \in A(u_1)$ there exists $a_2 \in A(u_2)$ such that $\|a_1 - a_2\| < L_1\|u_1 - u_2\|^\ell$. Since f is Lipschitzian with constant L we obtain

$$\|f(a_1) - f(a_2)\| \leq L\|a_1 - a_2\| < LL_1\|u_1 - u_2\|^\ell,$$

for $u_1, u_2 \in U_0$. The conclusion follows from Corollary 4.4 and Proposition 5.3. □

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