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Convexity, Differential Equations and Games

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Theoretical and experimental studies of noncooperative games increasingly recognize Nash equilibrium as a limiting outcome of players' repeated interaction. This note, while sharing that view, illustrates and advocates combined use of convex optimization and differential equations, the purpose being to render equilibrium both plausible and stable.

Keywords: Noncooperative games, Nash equilibrium, repeated play, differential equations, stability

1. Introduction

While economics has grown game-theoretic, the demanding nature of the central solution concept has increasingly been recognized. That concept, the Nash equilibrium, captures, in one shot, rationality, optimality and foresight. But, precisely by achieving so much, it cries out for justification in dynamic terms. Indeed, legitimacy for making Nash equilibria key items of inquiry can only be produced by dynamics which eventually converge to such focal points. It is unsatisfactory to study stability *after* equilibrium reigns without exploring *first* what process brought that distinguished state into being. Common and unifying features of such processes are that players

- have imperfect foresight, knowledge, or understanding of possibilities, intentions, and consequences, yet
- steadily seek to improve own payoff.

Thus real, repeated play is likely to unfold with manifold imperfections in the short run. To analyze possible long-run convergence this paper advocates use of convex analysis, differential equations, and (stochastic) approximation. After defining the infinitely repeated stage game in Section 2, I synthesize some generic instances and indicate extensions. For the sake of illustration the classical Cournot oligopoly will come on stage time and again. Since the main concerns of this paper are with modelling, some technicalities get limited attention.

2. The Stage Game

There is a finite, fixed set I of economic agents who play the same game repeatedly. At every stage individual $i \in I$ seeks to maximize - or merely improve - his payoff $\pi_i(x_i, x_{-i}) \in \mathbb{R} \cup \{-\infty\}$ with respect own strategy $x_i \in \mathbb{E}_i$. Here $\mathbb{E}_i$ is a Euclidean space, endowed with inner product $\langle \cdot, \cdot \rangle_i$, and x_{-i} stands for the strategy profile $(x_j)_{j \neq i}$ implemented by i's

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rivals. The value $-\infty$ accounts for constraints if any. A point $x = (x_i)$ is then declared a *Nash equilibrium* iff each $x_i \in \arg\max \pi_i(\cdot, x_{-i})$; that is, iff for all i

$$0 \in m_i(x) := \frac{\partial}{\partial x_i} \pi_i(x). \tag{1}$$

Here $\frac{\partial}{\partial x_i}$ denotes the partial superdifferential operator of convex analysis, namely:

$$g_i \in \frac{\partial}{\partial x_i} \pi_i(x) \Leftrightarrow \pi_i(\hat{x}_i, x_{-i}) \leq \pi_i(x_i, x_{-i}) + \langle g_i, \hat{x}_i - x_i \rangle_i, \forall \hat{x}_i \in \mathbb{E}_i.$$

I henceforth take existence of at least one Nash equilibrium for granted and posit that each payoff $\pi_i(x_i, x_{-i})$ be concave in own variable x_i.

Ever since von Neumann's first study [30] there has been some predilection with finite-strategy games.[1] That restriction seems not fully fortunate though. First, it often entails approximations. Second, apart from facilitating learning schemes [5], [32], it can hardly generate smooth dynamics. Third, it seems a paradox that although players are commonly supposed to respond optimally, they make virtually no use of optimization theory or methods. By contrast, the classical Cournot oligopoly [9], featuring a continuum of strategies and no approximation, begs for calculus, optimality conditions, dynamics, and convex analysis. So, to illustrate and motivate use of such analysis, I shall often activate, here below, that workhorse model of applied game theory.

The Cournot oligopoly goes as follows: Firm $i \in I$ produces quantity $x_i \in \mathbb{R}$ of one and the same perfectly divisible, homogeneous good to obtain a profit

$$\pi_i(x) = P(a)x_i - c_i(x_i)$$

which incorporates a convex cost function $x_i \mapsto c_i(x_i) \in \mathbb{R} \cup \{+\infty\}$ and a smooth price curve $a \mapsto P(a)$. Specifically, $P(a)$ is the price at which consumers will demand the aggregate quantity $a := \sum_{i \in I} x_i$. Assuming concavity of individual objectives - and suitable differentiability as well - a Cournot-Nash equilibrium obtains iff for all i

$$0 \in m_i(x) := P(a) + P'(a)x_i - \partial c_i(x_i), \tag{2}$$

∂ denoting here the customary subdifferential of convex analysis.[2]

3. Parametric Interaction

Motivated by (2) suppose optimality condition (1) assumes the form

$$0 \in m_i(x_i, a), \tag{3}$$

featuring an endogenously determined parameter a that belongs to a nonempty compact convex set $\mathbb{A} \subset \mathbb{R}^n$. We posit that a results from a continuous *aggregation* mechanism $x \mapsto a = Ax \in \mathbb{A}$, maybe nonlinear and/or unknown. Also suppose that each inclusion (3)

[1] Then payoff $\pi_i(x)$ becomes multilinear and equals $-\infty$ whenever x_i falls outside the probability simplex of mixed strategies.

[2] Note that (2) fits the parametrized variational inclusions studied in [8].

has a parameter-dependent, continuous solution $a \mapsto x_i(a)$. Taken together these solutions admit common observation of a new aggregate outcome, called $f(a)$, via the following string:

$$a \in \mathbb{A} \mapsto [x_i(a)] =: x(a) \mapsto Ax(a) =: f(a) \in \mathbb{A}. \tag{4}$$

Consequently, a steady state prevails iff $a = f(a)$. That is, any fixed point of f, in confirming expectations, supports a Nash equilibrium.

In many instances, aggregation like (4) may simplify play, reduce the complexity of strategic interaction, and lower the perceived dimensionality. To wit, instead of players having to learn about each other, they need only form predictions about the parameter a. While still out of equilibrium, such predictions are likely to be wrong - and refuted by observations. Whenever so, they had better be improved. One way of intentional improvement is modelled next. It involves a sequence of step sizes $s_k \in (0, 1]$ that tend to zero, but so slowly that $\sum s_k = +\infty$. Agents will

start at an initial point $a^0 \in \mathbb{A}$ determined by guesswork, accident, or historical factors not elaborated here;
update the current prediction a^k iteratively at stages $k = 0, 1, ...$ by the mean-value rule

$$a^{k+1} := (1 - s_k)a^k + s_k f(a^k); \tag{5}$$

continue until convergence (if ever).

Evidently, s_k strikes a balance between the state a^k, which prevails at stage k, and the fresh observation $f(a^k)$. In other words, compromise (5) reflects on-going learning. The requirement that $\sum s_k = +\infty$ ensures that learning never comes to a halt. Property $s_k \to 0^+$ accounts for increasing experience or maturation as $k \to +\infty$. [3]

Theorem 3.1 (Global convergence to equilibrium). *Suppose f maps a compact convex set $\mathbb{A} \subset \mathbb{R}^n$ continuously into itself. Also suppose the flow $\dot{a} = f(a) - a$ has unique integral curves, and that each minimal invariant set must be an isolated point. Then, for arbitrary initial $a^0 \in \mathbb{A}$, process* (5) *converges to a fixed point of f.*

Proof. Let $\mathcal{L}$ denote the nonempty set of accumulation points of the sequence $\{a^k\}$. By the Limit Set Theorem in [4] $\mathcal{L}$ is compact, connected, invariant under $\dot{a} = f(a) - a$, and does not contain a proper attractor. But then, by assumption, $\mathcal{L}$ reduces to a singleton whence there is convergence. □

Pemantle [31], in exploring stochastically perturbed versions of (5), provides conditions under which linearly unstable equilibria almost surely cannot be limit points; see also [5]. David and Jonathan Borwein show that given $n = 1$, that is, if $\mathbb{A} \subset \mathbb{R}$, then a constant step size could be applicable:

Proposition 3.2 (One-dimensional convergence with constant step sizes [6]). *Suppose f is Lipschitz with modulus L, i.e., $|f(a) - f(a')| \leq L|a - a'|$ for all $a, a' \in \mathbb{A} \subset \mathbb{R}$. Also suppose that $s_k \equiv s \in (0, \frac{2}{L+1})$. Then the sequence $\{a^k\}$ generated by* (5) *converges to a fixed point of f.* □

[3]Earlier studies of (5) include [21], [24], [25]. Multi-valued mappings $a \rightsquigarrow f(a)$ may also be accommodated; see e.g. [13]. Format (5) is the one which dominates in studies of so-called fictitious play [7], [22], [32].

Example 3.3 (One-dimensional interaction, repeated Cournot play [13]). Suppose each oligopolist $i \in I$ knows the price curve $P(\cdot)$ albeit nothing about his rivals. But presumably he is able to solve (2) for the unknown $x_i = x_i(a)$, depending continuously on the predicted aggregate supply a. That supply belongs to a nonempty compact interval $\mathbb{A} \subset \mathbb{R}$. If $\mathbb{A}$ is invariant under $a \mapsto \sum_i x_i(a) =: f(a)$, and f has isolated fixed points, then (5) converges to a rational-expectation, market-clearing, aggregate demand $a = f(a)$ which complies with Cournot-Nash equilibrium.

Continuous dependence can here be derived via an auxiliary problem [28], namely: Since $P'(a) < 0$, maximization of the strictly concave, coercive objective

$$\sum_{i \in I} \left\{ p x_i + \frac{p'}{2} x_i^2 - c_i(x_i) \right\}, \tag{6}$$

featuring short notation $p = P(a)$, $p' = P'(a)$, will produce a unique, continuously dependent, optimal solution $x(p, p')$. Since $a \mapsto (p, p') = [P(a), P'(a)]$ is already presumed continuous, the desired overall continuity follows by composition. □

Still with $\mathbb{A} \subset \mathbb{R}$ equation (5) also fits well to models concerned with price predictions; see [1], [12]. We next let $\mathbb{A} \subset \mathbb{R}^2$.

Proposition 3.4 (Convergence with two-dimensional interaction). *Suppose* $\mathbb{A} \subset \mathbb{R}^2$ *is nonempty compact convex and that* $f : \mathbb{A} \to \mathbb{A}$ *is* C^1 *with isolated fixed points and* $div f := \frac{\partial f_1}{\partial a_1} + \frac{\partial f_2}{\partial a_2} \neq 2$. *Then, for arbitrary initial* $a^0 \in \mathbb{A}$, *process* (5) *converges to a fixed point of* f.

Proof. By the Bendixon-Poincaré theorem $\dot{a} = f(a) - a$ accumulates to a fixed point of f, or to a periodic solution (possibly a limit cycle) [29]. The latter possibility is excluded, however, by Green's theorem. This shows that minimal invariant sets are isolated singletons, and then Theorem 3.1 applies. □

Example 3.5 (Two-dimensional interaction; repeated Cournot play [17]). We continue with the Cournot oligopoly. But more realistically than in Example 3.3, suppose now that each producer knows neither the price curve $P(\cdot)$ nor his rivals. Everybody then forms a belief $a := (p, p') \in \mathbb{R}^2$ about the upcoming price $p = P(\sum x_i) > 0$ and the associated slope $p' = P'(\sum x_i) < 0$. Consequently, $x(a) = x(p, p')$ is the unique solution of (6). Under appropriate hypotheses the implicit function theorem certifies that $\sum x_i(a)$ becomes C^1 whence so is $f(a) := [P, P'] (\sum x_i(a))$. One may argue, or reasonably assume, that a higher predicted price $a_1 := p$, inspires increased supply $\sum x_i(a)$ and thereby lower realized price $f_1(a)$, i.e. $\frac{\partial f_1}{\partial a_1} \leq 0$. Similarly, assuming $P'(\cdot)$ concave, a flatter price curve (that is, a more moderate slope $a_2 := p'$) incites greater supply and thereby smaller realized p', i.e. $\frac{\partial f_2}{\partial a_2} \leq 0$. Taken together the last two inequalities largely suffice for $div f < 2$. □

Proposition 3.4 seemingly applies to fictitious play of 2×2 games; see [5], [7], [14], [22], [32]. But smoothness is then absent. That much studied instance attests to the need for a qualitative theory of planar vector fields with discontinuous right hand side.

When $\mathbb{A} \subset \mathbb{R}^n$ with $n \geq 3$, matters become more difficult. But sometimes $a \mapsto f(a)$ has monotonicity properties caused by substitution or complementarity. The following is a well known result in that vain:

Proposition 3.6 (Convergence under monotone interaction). *Suppose $f : \mathbb{A} \to \mathbb{A} \subset \mathbb{R}^n$ is strictly monotone in the sense that*

$$\langle f(a) - f(\bar{a}), a - \bar{a} \rangle \leq \|a - \bar{a}\|^2 - \mu(\|a - \bar{a}\|) \tag{7}$$

where $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, increasing, vanishes only at 0, and $\bar{a}$ is any fixed point of f. Then f has a unique fixed point to which (5) converges.

Proof. Existence of two distinct fixed points $\bar{a}, a \in \mathbb{A}$ would contradict (7). So let $\bar{a}$ be the unique fixed point, $\dot{a} = f(a) - a$, and $L(t) := \|a(t) - \bar{a}\|^2 /2$. Then inequality $\dot{L} = \langle a - \bar{a}, f(a) - a \rangle \leq -\mu(\|a - \bar{a}\|)$ implies $a(t) \to \bar{a}$. Invoke Theorem 3.1 to conclude. □

Example 3.7 (Cournot play with many commodities). In (1) suppose $x_i \in \mathbb{R}^n$. This means that each firm can produce n homogeneous goods - to be sold at common markets. Suppose now that the "price slope" $P'(\sum x_i)$ is a constant, known, symmetric, negative definite $n \times n$ matrix, denoted $-S$. Then, regarding the upcoming price vector $p \in \mathbb{R}^n$ as the aggregate parameter a, solutions $x = (x_i)$ to (1) coincide with those of $p \in \partial \left[c_i(x_i) + \langle x_i, Sx_i \rangle_i /2\right], \forall i \in I$. Thus, letting C_i^* denote the Fenchel conjugate of $x_i \mapsto c_i(x_i) + \langle x_i, Sx_i \rangle_i /2$, it holds that $x_i = x_i(p) \in \partial C_i^*(p)$ for all i. This implies, quite naturally, that aggregate supply increases with more favorable price predictions, i.e. $\langle \sum_i x_i(p) - \sum_i x_i(\hat{p}), p - \hat{p} \rangle \geq 0$. It seems reasonable therefore, in this context, to assume by "the law of demand" that $f(p) := P(\sum x_i(p))$ is decreasing, that is, inequality $\langle f(p) - f(\hat{p}), p - \hat{p} \rangle \leq 0$ (or a good approximation) should be satisfied. The upshot is that (7) holds with $a = p$ and $\mu(r) = r^2$. □

4. Non-parametric Interaction

Nash equilibrium leaves the impression that each player foresees perfectly and responds optimally. Must human-like, rational agents really acquire both these faculties? This section argues that in some instances neither is ever needed. To show this repeated play is modelled here in various ways as processes driven by noncoordinated pursuit of better payoff.

For simplicity take $\pi_i(x_i, x_{-i})$ to be differentiable in x_i and sufficiently smooth. I begin by considering a first-order gradient process:

Proposition 4.1 (Convergence of a gradient method). *Suppose there is ball $\mathcal{B}$ around a point $\bar{x}$ such that the standard inner product $\langle x - \bar{x}, m(x) \rangle$ is negative and upper semicontinuous on $\mathcal{B} \diagdown \bar{x}$. Then, solution trajectories of*

$$\dot{x}_i = m_i(x), \forall i \in I,$$

emanating from any initial point $x(0) \in \mathcal{B}$, will converge to $\bar{x}$, this point being a Nash equilibrium.

Proof. The function $L(t) := \frac{1}{2} \|x(t) - \bar{x}\|^2$ becomes Lyapunov provided $x(0) \in \mathcal{B}$. Indeed, omitting explicit mention of time, $\dot{L} = \langle x - \bar{x}, \dot{x} \rangle = \langle x - \bar{x}, m(x) \rangle \leq 0$ with strict inequality when $x \neq \bar{x}$. Thus $\|x(t) - \bar{x}\|$ tends monotonically downwards to a limit $r \geq 0$. Therefore, $r \leq \|x(t) - \bar{x}\| \leq \|x(0) - \bar{x}\|$ for all $t \geq 0$. Let $\mu := \max \{\langle x - \bar{x}, m(x) \rangle : r \leq \|x - \bar{x}\|$

$\leq \|x(0) - \bar{x}\|\}$. If r were positive, the upper semicontinuity of $\langle x - \bar{x}, m(x)\rangle$ would entail $\dot{L} \leq \mu < 0$, whence the absurdity $L(t) \searrow -\infty$. Thus $r = 0$, and the last assertion follows immediately. □

Example 4.2 (Gradient play of Cournot oligopoly [10]). Suppose $P(a) = P(0) - Sa$ for positive constants $P(0)$ and S. Then, with differentiable convex cost c_i, monotonicity obtains because

$$\langle m(x) - m(\hat{x}), x - \hat{x}\rangle = -S\left\{(a-\hat{a})^2 + \|x - \hat{x}\|^2\right\} - \sum_i \left[\partial c_i(x_i) - \partial c_i(\hat{x}_i)\right][x_i - \hat{x}_i]. \quad \square$$

Gradient dynamic enjoys many appealing properties: It is decentralized and proceeds in parallel; it is easy to discretize and implement; it can incorporate constraints and nonsmooth data [3], [10], [11], [15], [16]. At times, however, such dynamics do not quite satisfy natural expectations: First, the monotonicity assumption $\langle x - \bar{x}, m(x)\rangle \leq 0$ may fail, and second, convergence often comes slowly, if at all. These features lead me to consider briefly a method that uses not only m but derivatives of m as well. Specifically, let

$$\dot{x}_i = m_i(x) + \lambda_i \dot{m}_i(x) \text{ for all } i \in I. \tag{8}$$

(8) incorporates some extrapolation via the term $\dot{m}_i(x) = m_i'(x)\dot{x}$. Such behavior mirrors that player i moves in the direction $m_i(x)$ of steepest payoff ascent, modified somewhat by how rapidly that direction changes. Note how little information or expertise the concerned parties need to keep process (8) going. It suffices that every individual i continuously observes and appropriately reacts to "his" current data x_i, m_i, and $\dot{m}_i$. The numbers λ_i are positive and typically rather large. Intuitively, a large λ_i serves to mitigate the slowdown of gradient dynamics near stationary points.

Proposition 4.3 (Convergence of an extrapolative system). *Suppose the trajectory $x(t), t \geq 0$, solves (8) in a domain where $m'(x)$ is non-singular. Then, if all λ_i are sufficiently large, any accumulation point $\bar{x}$ of $x(t)$ must be a Nash equilibrium. In particular, if $\bar{x}$ is an isolated solution to (1), then $x(t) \to \bar{x}$.*

Proof. For completeness we reproduce the argument in [35]. Denote by λ the diagonal matrix having λ_i along the diagonal in block i. Then, with short notation $m = m(x), m' = m'(x)$, system (8) can be rewritten as

$$\dot{x} = m + \lambda m'\dot{x}, \quad \text{that is,} \quad \dot{x} = \left[\mathbb{I} - \lambda m'\right]^{-1} m.$$

Let $L(t) := \|m(x(t))\|^2/2$ and observe that

$$\dot{L} = \langle m, m'\dot{x}\rangle = \left\langle m, m'\left[\mathbb{I} - \lambda m'\right]^{-1} m\right\rangle = \left\langle m, \lambda^{-1}\lambda m'\left[\mathbb{I} - \lambda m'\right]^{-1} m\right\rangle.$$

At this point use the matrix identity $\left[\mathbb{I} - \lambda m'\right]^{-1} - \lambda m'\left[\mathbb{I} - \lambda m'\right]^{-1} = \mathbb{I}$ to get $\dot{L} = \left\langle m, \lambda^{-1}\left\{-\mathbb{I} + \left[\mathbb{I} - \lambda m'\right]^{-1}\right\} m\right\rangle$. Thus, for λ sufficiently large $m(x) \neq 0 \Rightarrow \dot{L} < 0$. □

Process (8) is not straightforward to discretize, and constraints are not quite easy to account for; see [18], [19]. More convenient in both regards is another procedure, inspired by an important, recent paper of Attouch et al. [2]. To convey the main idea assume first that each payoff function π_i be finite-valued. This means that there are no constraints. The approach is then motivated as follows: Whenever player i - and others similarly - sees $\dot{x}_i \neq m_i(x)$, he attempts to restore equality by way of suitable acceleration/retardation $\ddot{x}_i = m_i(x) - \dot{x}_i$. Broadly speaking, if $m_i(x)$ exceeds x_i in some coordinate, then that velocity component should increase. The resulting motion defines a differential system

$$\ddot{x} = m(x) - \dot{x} \tag{9}$$

which has each Nash equilibrium as a rest point. It also retains the merit of being decentralized and simple.

Given this motivation I step back now and reintroduce constraints of the following sort: For each i suppose $x_i \in X_i \subseteq \mathbb{E}_i$ where X_i is nonempty closed convex. Suppose that agent i, while using strategy $x_i \in X_i$, worries about feasibility as follows. Whatever be his contemplated rate of change - that is, his desired velocity - v_i, its normal component, if any, must be suppressed. Otherwise that component would lead outside X_i. Consequently, what should be retained of the proposed v_i is only its tangential part. Formally, let $P_{T_i x_i}$ denote the *orthogonal projection* onto the *tangent cone* $T_i x_i := cl\mathbb{R}_+(X_i - x_i)$, and posit

$$\dot{x}_i := P_{T_i x_i}[v_i] \quad \text{for all} \quad i. \tag{10}$$

This operation bends (projects) any tentative velocity v_i onto the local tangent cone $T_i x_i$ so as to avoid straying out of X_i.[4] In sum, projection takes care of feasibility but leaves the dynamics of v_i unspecified. For such specification I imitate (9) and posit that v_i evolves according to

$$\dot{v}_i = P_{T_i x_i}[m_i(x)] - P_{T_i x_i}[v_i] \quad \text{for all} \quad i. \tag{11}$$

Since $Tx = \Pi_{i\in I} T_i x_i$ is the tangent cone of the product set $X := \Pi_{i\in I} X_i$ at $x = (x_i)$, the differential equations (10), (11) can be assembled into system form

$$\left.\begin{array}{lcl} \dot{x} & = & P_{Tx}[v] \\ \dot{v} & = & P_{Tx}[m(x)] - P_{Tx}[v] \end{array}\right\} \tag{12}$$

By a solution to this system is understood an absolutely continuous profile $[x(t), v(t)]$, $t \geq 0$, that satisfies (12) almost everywhere. Since Tx is empty whenever $x \notin X$, it goes without saying that $x(\cdot)$ must be *viable* in the sense that $x(t) \in X$ for all $t \geq 0$. The *total energy*

$$E(t) := \|v(t)\|^2/2 - \int_0^t \left\langle P_{Tx(\tau)}[m(x(\tau))], \dot{x}(\tau)\right\rangle d\tau \tag{13}$$

is defined as the sum of *kinetic* and *potential energy.* The latter is a line integral

$$\int_{x(0)}^{x(t)} \langle P_{Tx} m(x), dx\rangle = \int_0^t \left\langle P_{Tx(\tau)}[m(x(\tau))], \dot{x}(\tau)\right\rangle d\tau., \tag{14}$$

calculated along the path of play.

The next result spells out the stability often inherent in (12). By incorporating constraints it extends Theorem 3.1 in Attouch et al. (2000). For simple notations and statements, when $1 \leq p \leq \infty$, let $L^p := L^p(\mathbb{R}_+, \mathbb{E})$ be the space of (equivalence classes of) measurable functions $0 \leq t \mapsto x(t) \in \mathbb{E} := \Pi_i \mathbb{E}_i$ such that $\int_0^\infty \|x(t)\|^p dt < +\infty$. In particular, $x \in L^\infty$ iff x is essentially bounded on $\mathbb{R}_+$.

[4]Clearly, given continuous time, projection is required only when x_i resides at the boundary of X_i.

Proposition 4.4 (Asymptotic stability and convergence of constrained play). *Consider the second-order process* (12) *with* $m(\cdot)$ *Lipschitz continuous on bounded sets. Suppose the potential energy*

$$\int_0^t \left\langle P_{Tx(\tau)}\left[m(x(\tau))\right], \dot{x}(\tau)\right\rangle d\tau$$

is bounded above along any solution trajectory.[5] Then,

- from any admissible initial state $[x(0), v(0)] \in X \times \mathbb{E}$ there emanates an infinitely extendable, feasible solution $0 \leq t \mapsto [x(t), v(t)] \in X \times \mathbb{E}$ *of* (12);
- the total energy $E(t)$ *converges monotonically downwards to a limiting finite level* $E(\infty)$ *and* $v \in L^\infty$, $\dot{x} \in L^\infty \cap L^2$;
- it holds that $\dot{x}, \dot{v} \in L^\infty$ *and, provided* $\lim_{t\to+\infty} P_{Tx(t)}\left[m(x(t))\right]$ *exists, all points* $\dot{x}(t), \dot{v}(t), P_{Tx(t)}\left[m(x(t))\right]$ *tend to* 0 *as* $t \to +\infty$, *this saying that every cluster point of* $x(t), t \geq 0$, *is a Nash equilibrium.* □

Since no player acts continually, it is mandatory to recast (12) in discrete time. As discretization we propose

$$\begin{array}{rcl} x^{k+1} & := & P\left[x^k + s_k v^k\right] \\ v^{k+1} & := & v^k + P\left[x^k + s_k m(x^k)\right] - P\left[x^k + s_k v^k\right] \end{array} \tag{15}$$

Here P is short notation for the orthogonal projection onto X, and s_k, $k = 0, 1, \ldots$ are the step sizes mentioned earlier. Evidently, in our context, (15) amounts to a much decentralized system in which, iteratively at stages $k = 0, 1, ..$, each individual i updates his strategy and velocity by the rule

$$\begin{array}{rcl} x_i^{k+1} & := & P_i\left[x_i^k + s_k v_i^k\right] \\ v_i^{k+1} & := & v_i^k + P_i\left[x_i^k + s_k m_i(x^k)\right] - P_i\left[x_i^k + s_k v_i^k\right] \end{array}$$

Here P_i denotes orthogonal projection onto X_i. The initial points $(x_i^0, v_i^0), i \in I$, are determined by accident or historical factors better discussed in each particular setting.

Theorem 4.5 (Convergence of discrete-time, constrained, repeated play). *Suppose system* (12) has unique solution trajectories. Then, under the hypotheses of Proposition 4.4 and the assumption that $m(\cdot)$ *has isolated roots, any bounded discrete-time trajectory* (x^k, v^k) *generated by* (15) *must be such that* x^k *converges to a Nash equilibrium.* □

For proof of Proposition 4.4 and Theorem 4.5 see [20].

5. Concluding Remarks

Noncooperative game theory cannot - and quite reasonably, does not - claim that real, human-like players, when facing unfamiliar situations, will settle in Nash equilibrium right away. That theory rather invites two questions: First, save a unique solution, *which*

[5] In the unconstrained case, the potential energy becomes upper bounded when $m = P'$ for some differentiable, upper bounded potential $P : \mathbb{E} \to \mathbb{R}$; see [26], [27].

principle can select a specific equilibrium? Second, *what plausible sort of process could eventually bring the parties there?*

The literature already offers several models of learning to play Nash over time.[6] Common to these is the prime position - and somewhat overwhelming attention - given to finite-strategy games and best responses. By contrast, this paper used continuous strategy spaces and quite often dispensed with best responses. In applying differential equations (and related approximation theory) it subscribes to a tradition that goes back to Brown (1951) and Rosen (1965). That tradition is also eminently pursued in [23] and [37].

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