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Sensitivity of Dynamic Structures, Case of a Smart Beam

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Continuity of the solution with respect to data is a classical theme; for evolution equation this is usually restricted to continuity with respect to initial data and right hand side. Here we are interested in continuity with respect to coefficients appearing in bilinear forms (or in the coefficient of the operators involved in partial differential equations) or in the shape of the domain in which the system is posed. From a mathematical point of view, as noted in previous work of the author ["Quelques resultats en optimisation de domaine", These d'Etat, Univ. de Nice Sophia Antipolis (1982)], the classical implicit function theorem is not applicable for equations of Petrowsky type. We consider the sensitivity of a classical abstract optimal control problem to present the approach. As we consider beams of variable thickness, the partial differential equation involves variable coefficients; in the situation of exact control, the controlability issue seems still to be open. In this paper, we assume that controlability is satisfied. In the situation of classical optimal control, the existence of an optimal control may be obtained classically by lower semi continuity.

1. Introduction

As we said in [16], "the idea of well posed problems of mathematical physics was introduced and sudied by Hadamard in the 1920s [14]. The term 'well posed' for initial and boundary-value problems in the modern literature has come to be defined to mean that (i) the problem has a solution, (ii) the solution is unique, (iii) the solution depends continuously on data defining the problem. The first two parts of this definition have a precise meaning but the third is indefinite as to what constitutes 'data' ".

Continuity of the solution with respect to data is a classical theme; for evolution equation this is usually restricted to continuity with respect to initial data and right hand side. Here we are interested in continuty with respect to coefficients appearing in the bilinear forms (or in the coefficient of the operators involved in the partial differential equations) or in the shape of the domain Ω in which the system is posed. Many papers have adressed the sensitivity of static structures: see for examples [15], [13]; some examples of static structures with geometric non linearities and instabilities has been considered in [2], [1]; a case of a thermal problem in [6], [7], [8]. From a mathematical point of view, as noted previously in [25], the classical implicit function theorem is not applicable for equations of Petrowsky type; this is due to the following fact: with a right hand side in $L^2(0,T,L^2(\Omega))$, the solution is not in gneral in $L^2(0,T,H^{2m}(\Omega))$, see for example in paragraph 5.1.3 of [20]. In these conditions, the partial differential operator is not an isomorphism which prevents the use of a classical implicit theorem.

We present here some abstract lipschitz continuity and differentiablity results which would

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lead to the same kind of properties for shape sensitivity (already considered in [25], [24]) or thickness sensitivity (see [26]) of dynamically loaded structures. A recent related result may be found in [5].

But the goal of this paper is also to start to adress the sensitivity of controlled systems. In the oral presentation at the conference FGI 2000, we tried to adress the sensitivity of an exactly controled continuous system. In a private discussion, F. Bonnans pointed out that the approach is quite general as documented in his recent book [17]. We try to show the applicability with some examples. Independently design sensitivity has been considered in ([23], [21], [22]). We consider the sensitivity of a classical abstract optimal control problem to present the approach.

Shape sensitivity of piezoelectric wafers mounted on a plate has been considered in [10] for the approximation with a finite number of modes. We consider *two* situations: classical optimal control and exact control in finite time. The term exact control means that the control actuator should bring the system to rest in a given finite time T such that $y(T,x) = 0,\ y_t(T,x) = 0$. This is somewhat different of stabilizing the system and has been a field of active research for systems modeled by partial differentiable equations since the work of J. L. Lions ([18]) and numerical approximations and implementations ([11, 12]); the use of eigenfunctons for exact control (see [3]) enables to compute exact control within a reasonable computing time and so the optimal design of such exactly controled systems becomes also reasonable. In this paper we restrict the approach to a model of smart beam used in [9] but the approach is very general and can be applied quite easily for shape sensitivity of controled systems; such extensions are in preparation.

As we consider beams of variable thickness, the partial differential equation involves variable coefficients; in the situation of exact control, the controlability issue seems still open. In this paper, we *assume* that controlability is satisfied. In the situation of classical optimal control, the existence of an optimal control may be obtained classically by lower semi continuity.

2. Sensitivity of a controled system

Let us consider the classical abstract optimal control problem of minimizing:

$$J(u,y) = \frac{1}{2}(u,u) + \frac{r}{2}(y,y) \tag{1}$$

with the constraint

$$Ay = Bu + f \tag{2}$$

For simplicity, we may assume that $u \in \mathbb{R}^m, y \in \mathbb{R}^n$ and $B \in \mathcal{L}(\mathbb{R}^m, \in \mathbb{R}^n)$ and A symetric positive definite on $\mathbb{R}^n$. But the process is very general and A may be a partial diffential operator. Naturally in this case, the proofs may be very difficult. Introducing the Lagrangian

$$\mathcal{L}(y,u,p) = J(u,y) + (p, Ay - Bu - f) \tag{3}$$

with the Lagrange multiplier $p \in \mathbb{R}^n$, we get easily the optimality system:

$$\frac{\partial \mathcal{L}}{\partial p} = 0 \text{ yields } Ay = Bu + f \quad \text{(state)} \tag{4}$$

$$\frac{\partial \mathcal{L}}{\partial u} = 0 \text{ yields } u = B^* p \quad \text{(optimality)} \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial y} = 0 \text{ yields } Ap = -ry \quad \text{(adjoint state)} \tag{6}$$

Let us consider the case where A and B are dependent on a design parameter $e \in \mathbb{R}^k$; in practice it may be a design variable taking into account the amount of material involved; it is then natural to look for the minimization of J with a given quantity of material:

$$\sum_{i=1}^{k} e_i = c \quad \text{a given constant} \tag{7}$$

To write optimality conditions or for optimization algorithms the sensitivity is needed:

$$j(e) = J(u(e), y(e)) \tag{8}$$

$$\frac{\partial j}{\partial e} de = \frac{\partial J}{\partial y} dy + \frac{\partial J}{\partial u} du \quad \text{or here} \tag{9}$$

$$\frac{\partial j}{\partial e} de = r(y, \frac{\partial y}{\partial e} de) + (u, \frac{\partial u}{\partial e} de) \tag{10}$$

where $\frac{\partial y}{\partial e}, \frac{\partial u}{\partial e}$ are derivatives of implicit functions and an explicit expression of this derivative is needed. These derivatives are solution of the optimality system differentiated with respect to the design parameter e:

$$Ady - Bdu = dAy - dBu \tag{11a}$$

$$du - B^* dp = -dB^* p \tag{11b}$$

$$rdy + Adp = -dAp \tag{11c}$$

This system has the same operators as the ones involved in the optimality system (4, 5, 6); if the optimal control system is well posed, they define an isomorphism and the implicit function theorem gives the differentiability of y, u, p with respect to e.

We note that $\frac{\partial y}{\partial e}$ and $\frac{\partial u}{\partial e}$ may be obtained by solving this system but it is more convenient to use directly the Lagrangian

$$j(e) = \mathcal{L}(y, u, p) = \frac{1}{2}(u, u) + \frac{r}{2}(y, y) + (p, Ay - Bu - f) \tag{12}$$

where $y(e), u(e), p(e)$ are implicit functions of of e defined by the optimality system. By differentiation we get:

$$dj = (u, du) + r(y, dy) + \tag{13}$$

$$(p, Ady - Bdu + dAy - dBu) + \tag{14}$$

$$(dp, Ay - Bu - f) \tag{15}$$

or manipulating:

$$dj = (u - B^* p, du) + (ry + Ap, dy) + \tag{16}$$

$$(dp, Ay - Bu - f) + (p, dAy - dBu) \tag{17}$$

and we notice that

$$dj = \frac{\partial \mathcal{L}}{\partial u} du + \frac{\partial \mathcal{L}}{\partial y} dy + \frac{\partial \mathcal{L}}{\partial p} dp + \frac{\partial \mathcal{L}}{\partial e} de \tag{18}$$

so that using the optimality system (4):

$$dj = \frac{\partial \mathcal{L}}{\partial e} de = (p, dAy - dBu) \tag{19}$$

We note that the approach is very general.

We can easily derive necessary optimality conditions by introducing a Lagrange multiplier for the constraint (7):

$$(p, \frac{\partial A}{\partial e_i} y - \frac{\partial B}{\partial e_i} u) + \lambda = 0 \quad i = 1, ...k \tag{20}$$

3. Second order evolution equation

We denote Ω, a bounded open subset of $\mathbb{R}^n$; it will be usually with a $\mathcal{C}^\infty$ boundary Γ and situated locally on one side of Γ. We consider a Hilbert space $V(\Omega)$ with $H_0^m(\Omega) \subseteq V(\Omega) \subseteq H^m(\Omega)$; m is also the order of derivatives appearing in the bilinear form $a(y, p)$; it is assumed to be symmetric, continuous, coercive on V. It depends on a design variable h which may be either a parameter appearing in the equation such as thickness of a structure of variable thickness or the shape of the boundary Γ. For equations arising in solid mechanics, **a** denotes virtual work of internal forces, **c**, the virtual work of inertial forces and (f, p), the virtual work of external forces with p the virtual displacement.

The following hypothesis will be assumed throughout the paper.

Hypothesis 3.1. **a** is a symmetric bilinear form coercive on V

Hypothesis 3.2. **c** is a symmetric bilinear form coercive on $L^2(\Omega)$

Hypothesis 3.3. for simplicity, we assume that the bilinear forms are not time dependent.

We consider the abstract equation:

$$\frac{d}{dt}\mathbf{c}(\frac{\partial y}{\partial t}, p) + \mathbf{a}(y, p) = (f, p) \tag{21}$$

We recall the following theorems from [19]; in fact we are mainly recalling a priori inequalities which are in the proof of the quoted theorems.

Theorem 3.4. *Theorem 3.8.1 of [19]*

$$\|y\|_V^2 + \|\frac{\partial y}{\partial t}\|_{L^2(\Omega)}^2 \leq c \left(\|y_0\|_V^2 + \|y_1\|_{L^2(\Omega)}^2 + \int_0^T \|f(t)\|_{L^2(\Omega)}^2 \right) \tag{22}$$

Theorem 3.5. *(from proof of Theorem 3.9.3 of [19])*

$$\|y\|_{L^2(\Omega)}^2 + \|\frac{\partial y}{\partial t}\|_{V'}^2 \leq c \left(\|y_0\|_{L^2(\Omega)}^2 + \|y_1\|_{V'}^2 + \int_0^T \|f(t)\|_{V'}^2 \right) \tag{23}$$

Theorem 3.6. *(from proofs of Theorem 5.2.1 and Lemma 5.2.1 of [20])*

$$\|\frac{\partial y}{\partial t}\|^2_{H^m(\Omega)} + \|\frac{\partial^2 y}{\partial t^2}\|^2_{L^2(\Omega)} \leq c\left[\|y_0\|^2_{H^{2m}(\Omega)} + \|y_1\|^2_{H^m(\Omega)} + \int_0^T \|\frac{\partial f}{\partial t}\|^2_{L^2(\Omega)}\right] \tag{24}$$

$$\|y\|^2_{H^{2m}(\Omega)} \leq c\left[\|y_0\|^2_{H^{2m}(\Omega)} + \|y_1\|^2_{H^m(\Omega)} + \int_0^T \|f(t)\|^2_{L^2(\Omega)} + \int_0^T \|\frac{\partial f}{\partial t}\|^2_{L^2(\Omega)}\right] \tag{25}$$

Corollary 3.7. *With zero initial data:*

$$\|\frac{\partial y}{\partial t}\|^2_{H^{2m}(\Omega)} \leq c\left[\int_0^T \|f(t)\|^2_{H^m(\Omega)} + \int_0^T \|\frac{\partial^2 f}{\partial t^2}\|^2_{L^2(\Omega)}\right] \tag{26}$$

The proof is based on Theorem 5.7.1 of [20] and the use of Theorem of intermediate derivatives (Theorem 1.2.3 of [19]).

4. Lipschitz continuity of the solution with respect to design variables

This section is devoted to Lipschitz continuity of the solution with respect to design variables while the following is considering differentiability with respect to design variables.

Two situations will be considered: one with additional regularity on the data, one with basic regularity on the data.

Notations: We denote h, the **design variable**; $\mathbf{a}_h(z,p)$, the bilinear form for the value h of the design variable; $\delta\mathbf{a} = \mathbf{a}_{h+\delta h} - \mathbf{a}_h$ and similarly $\delta y = y_{h+\delta h} - y_h$, $\delta f = f_{h+\delta h} - f_h$.

4.1. With basic regularity

Hypothesis 4.1 (Lipschitz 1).

$$f \in L^2(0,T,L^2(\Omega)), \quad y_0 \in V,\ y_1 \in L^2(\Omega) \tag{27}$$

$$\forall z \in V\ \forall p \in V \quad |\delta\mathbf{a}(z,p)| \leq c\|\delta h\|_X \|z\|_V \|p\|_V \tag{28}$$

$$\|\delta C z\|_{V'} \leq c\|\delta h\|_X \|z\|_{V'} \quad \forall p \in V \quad |(\delta f, p)| \leq c\|\delta h\|_X \|p\|_V \tag{29}$$

Proposition 4.2. *With Hypothesis 4.1 we have:*

$$\|\delta y\|^2_{L^2(\Omega)} + \|\delta\dot{y}\|^2_{V'} \leq c\|\delta h\|^2_X \int_0^T \|y\|^2_V \tag{30}$$

or $h \longmapsto y$ is lipschitzian from X to $L^\infty(0,T,L^2(\Omega))$ and $h \longmapsto \dot{y}$ is lipschitzian from X to $L^\infty(0,T,V')$.

4.2. With additional regularity

Let us denote A, the operator from $H^{2m}(\Omega)$ to $L^2(\Omega)$ associated to the bilinear form a and C, the oerator in $L^2(\Omega)$) associated to the bilinear form c.

Hypothesis 4.3 (Lipschitz 2).

$$f \in L^2(0,T,L^2(\Omega)), \quad \frac{\partial f}{\partial t} \in L^2(0,T,L^2(\Omega)), \quad y_0 \in H^{2m}(\Omega) \cap D(A), \quad y_1 \in H^m(\Omega) \tag{31}$$

$$\forall z \in H^{2m}(\Omega) \|\delta Az\|_{L^2(\Omega)} \leq c\|\delta h\|_X \|z\|_{H^{2m}(\Omega)} \tag{32}$$

$$|(\delta f, p)| \leq c\|\delta h\|_X \|p\|_{L^2(\Omega)} \tag{33}$$

$$\|\delta Cz\|_{L^2(\Omega)} \leq c\|\delta h\|_X \|z\|_{L^2(\Omega)} \tag{34}$$

Proposition 4.4. *With Hypothesis 4.3 we have:*

$$\|\delta y\|^2_{H^m(\Omega)} + \|\delta \dot{y}\|^2_{L^2(\Omega)} \leq c\|\delta h\|^2_X \int_0^T \|y\|^2_{H^{2m}(\Omega)} \tag{35}$$

or $e \longmapsto y$ *is lipschitzian from* X *to* $L^\infty(0,T,V)$ *and* $e \longmapsto \dot{y}$ *is lipschitzian from* X *to* $L^\infty(0,T,L^2(\Omega))$

With still some more regularity, we get the following proposition; it was proved in the context of shape sensitivity in [25], chapter 4 and presented in [24],[26]. We state it here in an abstract setting.

Hypothesis 4.5 (Lipschitz 3).

$$f \in L^2(0,T,L^2(\Omega)), \quad \frac{\partial f}{\partial t} \in L^2(0,T,L^2(\Omega)), \quad \frac{\partial^2 f}{\partial t^2} \in L^2(0,T,L^2(\Omega)) \tag{36}$$

$$y_0 \in H^{2m}(\Omega) \cap D(A), \quad y_1 \in H^m(\Omega) \tag{37}$$

$$\forall z \in H^{2m}(\Omega) \quad \|\delta Az\|_{L^2(\Omega)} \leq c\|\delta h\|_X \|z\|_{H^{2m}(\Omega)} \tag{38}$$

$$|(\delta f, p)| \leq c\|\delta h\|_X \|p\|_{L^2(\Omega)} \tag{39}$$

$$|(\delta \frac{\partial f}{\partial t}, p)| \leq c\|\delta h\|_X \|p\|_{L^2(\Omega)} \tag{40}$$

$$\|\delta Cz\|_{L^2(\Omega)} \leq c\|\delta h\|_X \|z\|_{L^2(\Omega)} \tag{41}$$

Proposition 4.6. *With Hypothesis 4.5 we have:*

$$\|\delta y\|^2_{H^{2m}(\Omega)} + \|\delta \dot{y}\|_{H^m(\Omega)^2} + \|\delta \ddot{y}\|^2_{L^2(\Omega)} \leq c\|\delta h\|^2_X \int_0^T \left(\|y\|^2_{H^{2m}(\Omega)} + \|\dot{y}\|^2_{H^{2m}(\Omega)} \right) \tag{42}$$

or $e \longmapsto y$ *is lipschitzian from* X *to* $L^\infty(0,T,H^{2m}(\Omega))$ *and* $e \longmapsto \dot{y}$ *is lipschitzian from* X *to* $L^\infty(0,T,V)$

Proof of previous propositions. We denote with $\dot{}$ the time derivative. The proof is based on the simple remark that δy is solution of:

$$\forall p \in V \quad \mathbf{c}(\delta \ddot{y}, p) + \mathbf{a}(\delta y, p) = -\delta \mathbf{c}(\ddot{y}_{h+\delta h}, p) - \delta \mathbf{a}(y_{h+\delta h}, p) + (\delta f, p) \tag{43}$$

For Proposition 4.2, we first use Theorem 3.4; we obtain that y is in $L^2(0,T,V)$ so that the right hand side of (43) is in $L^2(0,T,V')$. Now we apply Estimation (23) to Equation 43; the initial conditions of Equation 43 are zeros so that:

$$\|\delta y\|^2_{L^2(\Omega)} + \|\delta \dot{y}\|^2_{V'} \leq c \int_0^T \|\delta Ay\|^2_{V'} + \|\delta f\|_{V'} \tag{44}$$

with Hypothesis 4.1 (Lipschitz 1) we get (30).

For Proposition 4.4, we first use Theorem 3.6; we obtain that y is in $L^2(0,T,H^{2m}(\Omega))$ so that the right hand side of (43) is in $L^2(0,T,L^2(\Omega))$; Now we apply Estimation (22) to Equation 43; the initial conditions of Equation 43 are zeros so that:

$$\|\delta y\|^2_{H^m(\Omega)} + \|\delta \dot{y}\|^2_{L^2(\Omega)} \leq c \int_0^T \|\delta A y\|^2_{L^2(\Omega)} + \|\delta f\|_{L^2(\Omega)} \tag{45}$$

For Proposition 4.6 , we first use Theorem 3.6 ; we obtain that y is in $L^2(0,T,H^{2m}(\Omega))$; with Corollary 3.7, we get also that $\dot{y}$ is in $L^2(0,T,H^{2m}(\Omega))$; now we get that the right hand side of (43) is in $L^2(0,T,L^2(\Omega))$ and also its time derivative; now we use again Theorem 3.6 to obtain:

$$\|\delta y\|^2_{H^{2m}(\Omega)} + \|\delta \dot{y}\|^2_{H^m(\Omega)} + \|\delta \ddot{y}\|^2_{L^2(\Omega)} \leq \tag{46}$$

$$c\|\delta h\|^2_X \int_0^T \{\|\delta A y\|^2_{L^2(\Omega)} + \|\delta C \ddot{y}\|^2_{L^2(\Omega)} + \|\delta f\|^2_{L^2(\Omega)} + \tag{47}$$

$$\|\delta A \dot{y}\|^2_{L^2(\Omega)} + \|\delta C \dddot{y}\|^2_{L^2(\Omega)} + \|\delta \dot{f}\|^2_{L^2(\Omega)}\} \tag{48}$$

and we conclude with the estimations assumed in Hypothesis 4.5 of the proposition.

5. Differentiability with respect to design variables

5.1. Differentiablity of the solution

Let us first consider Equation (21) formally differentiated with respect to an abstract design variable $h \in X$. We denote $d\mathbf{a}(z,p) = \frac{\partial \mathbf{a}(z,p)}{\partial h} dh$ where the derivative is considered with z and p being fixed; similarly $dy = \frac{\partial y}{\partial h} dh$ denotes a differential considered with x being fixed; it is the derivative of $h \longmapsto y_h(x)$.

$$\forall p \in V \quad \mathbf{c}(d\ddot{y}, p) + \mathbf{a}(dy, p) = -d\mathbf{c}(\ddot{y}, p) - d\mathbf{a}(y, p) + (df, p) \tag{49}$$

Let us first consider this equation with Hypothesis 4.1; with this assumption, Theorem 3.4 states that $y \in V$ and $\ddot{y} \in V'$; the right hand side of (49) satisfies hypothesis suitable for Theorem 3.5 and we get that this equation has a solution $dy \in L^2(\Omega)$ and $d\dot{y} \in V'$. Unfortunately, the proof, with the Hypothesis 4.1 that dy is indeed the differential of $h \longmapsto y_h(x)$ would rely on estimates which are not classical. It will be possible to prove it under Hypothesis 5.1.

Hypothesis 5.1. We assume Hypothesis 4.5 and moreover

$$\forall z \in H^{2m}(\Omega) \quad \|dAz\|_{L^2(\Omega)} \leq c\|dh\|_X \|z\|_{H^{2m}(\Omega)} \tag{50}$$

$$\|dCz\|_{L^2(\Omega)} \leq c\|dh\|_X \|z\|_{L^2(\Omega)} \tag{51}$$

$$\|\delta^2 Az\|_{L^2(\Omega)} \leq c\|dh\|^2_X \|z\|_{H^{2m}(\Omega)} \tag{52}$$

$$\|\delta^2 Cz\|_{L^2(\Omega)} \leq c\|dh\|^2_X \|z\|_{L^2(\Omega)} \tag{53}$$

$$|(\delta^2 f, p)| \leq c\|dh\|^2_X \|p\|_{L^2(\Omega)} \tag{54}$$

Theorem 5.2. *Under Hypothesis 5.1, the solution of* (49) *is the differential of*

$$h \longmapsto y \qquad\qquad h \longmapsto \dot{y} \tag{55}$$
$$X \longrightarrow L^\infty(0,T,V) \quad X \longrightarrow L^\infty(0,T,L^2(\Omega)) \tag{56}$$

Proof starts by expanding all quantities up to second order: for example, the bilinear form: $a_{h+dh} = a_h + da + \delta^2 a$; we consider Equation (21) for the value $h + dh$ of the design variable and expand it up to second order; we simplify the expansion with Equation (21) for the value h of the design variable and with the differentiated Equation (49); we obtain after some manipulations:

$$\forall p \in V \quad \mathbf{c}_h(\delta^2 \ddot{y}, p) + \mathbf{a}_h(\delta^2 y, p) = \tag{57}$$
$$-d\mathbf{c}(\ddot{\delta y}, p) - d\mathbf{a}(\delta y, p) + (\delta^2 f, p) - \delta^2 \mathbf{c}(\ddot{y}_{h+dh}, p) - \delta^2 \mathbf{a}(y_{h+dh}, p) \tag{58}$$

If we look at the right hand side, we notice that Hypothesis 4.1 provides $\delta y \in L^2(\Omega)$ so that $dA\delta y \in dAL^2(\Omega)$; the latter space is larger than V'; however it seems that the solution of this equation would be well defined by using transposition theory (see [20]); however estimates of the solution does not seem standard so that we restrict the situation to Hypothesis 5.1; with this hypothesis, we know with Proposition 4.6 that $y,\ \delta y \in H^{2m}(\Omega)$ so that the right hand side is in $L^2(\Omega)$ and using Theorem 3.4:

$$\|\delta^2 y\|_V^2 + \|\delta^2 \dot{y}\|_{L^2(\Omega)}^2 \leq \tag{59}$$
$$\leq \quad c\int_0^T \left(\|dC\delta\ddot{y}\|^2_{L^2(\Omega)} + \|dA\delta y\|^2_{L^2(\Omega)} + (\delta^2 f, p)\|\delta^2 C\ddot{y}_{h+dh}\|^2_{L^2(\Omega)} + \|\delta^2 A y_{h+dh}\|^2_{L^2(\Omega)} \right)$$

and we conclude with the estimates of the Hypothesis 5.1

6. A smart beam model:

6.1. Euler Bernoulli beam model

Without transverse shear, the strain work is:

$$\mathbf{a}_e(y, \hat{y}) = \int_0^l EI(e) y_{xx} \hat{y}_{xx} \tag{60}$$

where y denotes the normal deflection and $\hat{y}$ the virtual normal deplacement. Work of quantity of accelerations (neglecting rotational inertia):

$$\mathbf{c}_\mathbf{e}(\frac{\partial^2 y}{\partial t^2}, \hat{y}) = \int_0^l \rho e \frac{\partial^2 y}{\partial t^2} \hat{y} \tag{61}$$

Work of external control forces: u_1, u_2, u_3:

$$L(\hat{y}) = Au_1 \hat{y}(t,a) + Bu_2 \frac{\partial \hat{y}}{\partial t}(t,a) + \int_a^b Cu_3 \hat{y} \tag{62}$$

This is a possible mathematical model of the action of a piezoelectric wafer on a beam as proposed in [9]. Let $H^2_0(\Omega) \subset V \subset H^2(\Omega)$. Principle of virtual work

$$\forall \hat{y} \in V \quad \mathbf{c}_\mathbf{e}(\frac{\partial^2 y}{\partial t^2}, \hat{y}) + \mathbf{a}_e(y, \hat{y}) = L(\hat{y}) \tag{63}$$

with initial conditions:

$$y(0,x) = y_0(x), \;\; \frac{\partial y}{\partial t}(0,x) = y_1(x) \tag{64}$$

6.2. Control

If $u_1, u_2 \in L^2(0,T)$ $u_3 \in L^2(0,T,L^2(\Omega))$, $y_0 \in L^2(\Omega)$, $y_1 \in V'$ as a consequence of Theorem 3.5, the solution of (63) is in $\mathcal{V}^0$ where

$$\mathcal{V}^0 = \{\phi \in L^2(0,T,L^2(\Omega)),\; \dot{\phi} \in L^2(0,T,V')\} \tag{65}$$

Moreover if $A = 0$ $B = 0$ $y_0 \in V$, $y_1 \in L^2(\Omega)$, as a consequence of Theorem 3.4 the solution of (63) is in $\mathcal{V}^1$ where

$$\mathcal{V}^1 = \{\phi \in, L^2(0,T,V),\; \dot{\phi} \in L^2(0,T,L^2(\Omega))\} \tag{66}$$

For control purposes, it is convenient to write the equations in weak form with a test function that depends on time and space; state equation with initial conditions:

$$y(0,x) = y_0(x) \;\; \frac{\partial y}{\partial t}(0,x) = y_1(x) \tag{67}$$

$$\begin{aligned} &\textstyle\int_0^T \int_0^l \rho e \frac{\partial^2 y}{\partial t^2}\phi + \int_0^T \int_0^l EI(e) y_{xx}\phi_{xx} = \\ &\textstyle\int_0^T Au_1\phi(t,a) + Bu_2\phi_x(t,a) + \int_0^T \int_a^b Cu_3\phi \end{aligned} \tag{68}$$

We consider *two* situations

6.2.1. Optimal control

In this case we can consider the functional to minimize:

$$J(y,u) = \frac{1}{2}\int_a^b \|y(x,T)\|^2 + \frac{1}{2}\int_a^b \|\frac{\partial}{\partial t}y(x,T)t\|^2_{V'} + \int_0^T (u_1^2 + u_2^2) + \frac{1}{2}\int_0^T \int_a^b u_3^2 \tag{69}$$

The existence of an optimal control may be obtained classically by using the lower semi-continuity of the functional (see for example [4]. In order to derive the optimality conditions, we use the Lagrangian:

$$\begin{aligned} &\mathcal{L}(y,u;p,p_0,p_1) = J(y,u) + \qquad (70)\\ + \quad &\int_0^T \int_0^l \rho e \frac{\partial^2 y}{\partial t^2} p + \int_0^T \int_0^l EI(e) y_{xx} p_{xx} + \\ + \quad &\int_0^T \{Au_1 p(t,a) + Bu_2 p_x(t,a)\} + \int_0^T \int_a^b Cu_3 p + \\ + \quad &\int_0^l \left\{ p_1[y(0,x) - y_0(x)] \quad - p_0[\frac{\partial y}{\partial t}(0,x) - y_1(x)] \right\} \qquad (71) \end{aligned}$$

The optimality conditions may be written:

$$\frac{\partial \mathcal{L}}{\partial u} = 0 \quad \text{which gives:} \tag{72}$$

$$u_1(t) + Ap(t,a) = 0; \quad u_2(t) + Bp_x(t,a) = 0; \tag{73}$$

$$u_3(t,x) + Cp(t,x)\chi_{[a,b]} = 0 \tag{74}$$

where the adjoint state p is solution of:

$$\frac{\partial \mathcal{L}}{\partial y} z = 0 \quad \text{which gives by integration by parts:} \tag{75}$$

$$-\int_0^T \int_0^l \rho e \frac{\partial^2 p}{\partial t^2} z - \int_0^T \int_0^l EI(e) z_{xx} p_{xx} + \tag{76}$$

$$+\int_0^l < z_t(T,x), y_t(T,x) - \rho\, e\, p(T,x) >_{V'} + \int_0^l < z(T,x), y(T,x) + \rho\, e\, p_{,t}(T,x) + \tag{77}$$

$$+\int_0^l < p_1 - \rho\, e\, p_{,t}(0,x), z(0,x) > + \int_0^l < -p_0 + \rho\, e\, p(0,x), z_{,t}(0,x) >= 0 \tag{78}$$

from which we get easily the final conditions for p and relations for p_0, p_1.

6.2.2. Exact control in final time

Functional to minimize:

$$J(v) = \frac{1}{2} \int_0^T (u_1^2 + u_2^2) + \frac{1}{2} \int_0^T \int_a^b u_3^2 \tag{79}$$

We minimize J with the final conditions.

$$y(T,x) = 0; \quad \frac{\partial y}{\partial t}(T,x) = 0 \tag{80}$$

Controlability of plates and beam have been adressed by Ph. Destuynder ([9]); here we assume that there exists u such that (80) is satisfied. A way to prove it, is to penalise the final conditions as proposed in ([9]) for constant thickness beams. Here beams are of variable thikness ; in this case, the operator involves variable coefficients and exact control seems to be an open problem; in this paper, we *assume* that exact controlability is satisfied.

The **derivation** of the expression of necessary optimality conditions including the adjoint state is provided in an appendix; it uses the same Lagrangian as for optimal control but with a different functional space: it involves $y(T,x) = 0$ and $\dot{y}(T,x) = 0$.

6.3. Sensitivity of a smart beam

We consider design problems in which the fonctional is the one considered for control. This is quite natural: the functional involved for control is usually a combination of the energy provided to the actuator and a quadratic mean of the displacement. We distinguish the case of optimal control and exact control in finite time. In both cases the result may be obtained with the Lagrangian we introduced in (70) for optimal control and in the appendix for exact control. The design variable is the thickness e which depends on x.

$$\begin{aligned}
&\mathcal{L}(y,u;p,p_0,p_1) = J(y,u) + && (81)\\
&-\int_0^T\int_0^l \rho e\frac{\partial^2 y}{\partial t^2}p - \int_0^T\int_0^l EI(e)y_{xx}p_{xx} + \\
+\quad &\int_0^T \{Au_1 p(t,a) + Bu_2 p_x(t,a)\} + \int_0^T\int_a^b Cu_3 p + \\
+\quad &\int_0^l \left\{p_1[y(0,x)-y_0(x)] \quad - p_0[\frac{\partial y}{\partial t}(0,x) - y_1(x)]\right\} && (82)
\end{aligned}$$

The *sensitivity* of $J(y,u)$ may be obtained by noticing that when y_e is solution of the state equation, the following equality holds:

$$J(y_e,u_e) = \mathcal{L}(y_e,u_e;p_e,p_{0e},p_{1e}) \text{ so that the derivative of} \tag{83}$$

$$j(e) = J(y_e,u_e) \text{ may be obtained with} \tag{84}$$

$$\frac{\partial j}{\partial e} = \frac{\partial \mathcal{L}}{\partial e} + \frac{\partial \mathcal{L}}{\partial y}\frac{\partial y}{\partial e} + \frac{\partial \mathcal{L}}{\partial u}\frac{\partial u}{\partial e} + \frac{\partial \mathcal{L}}{\partial p}\frac{\partial p}{\partial e} + \tag{85}$$

$$\frac{\partial \mathcal{L}}{\partial p_0}\frac{\partial p_0}{\partial e} + \frac{\partial \mathcal{L}}{\partial p_1}\frac{\partial p_1}{\partial e} \tag{86}$$

When $y_e, u_e, p_e, p_{0e}, p_{1e}$ are solutions of the optimality conditions, the only non zero term is the first one; so, the sensitivity of $J(y,u)$ or the differential of $j(e)$ is provided by:

$$\frac{\partial j}{\partial e} = \frac{\partial \mathcal{L}}{\partial e} \tag{87}$$

6.3.1. Optimal control

For optimal control, $J(y,u)$ is given by (69). The design variable is the thickness e which depends on x. As indicated in the sensitivity of an abstract optimal control problem, the sensitivity of $j(e) = J(y(e),u(e))$ may be computed simply with (19);

Proposition 6.1.

$$\begin{aligned}
&\frac{\partial J}{\partial e}\tilde{e} = \frac{\partial \mathcal{L}}{\partial e}\tilde{e} \quad \textit{or here} && (88)\\
&\frac{\partial J}{\partial e}\tilde{e} = -\int_0^T\int_0^l \tilde{e}\rho\frac{\partial^2 y}{\partial t^2}p - \int_0^T\int_0^l E\frac{\partial I(e)}{\partial e}\tilde{e}y_{xx}p_{xx}
\end{aligned}$$

where the adjoint state is solution of the adjoint equation (76).

6.3.2. Exact control

The design variable is still the thickness e which depends on x.

Proposition 6.2.

$$\begin{aligned}
&\frac{\partial J}{\partial e}\tilde{e} = \frac{\partial \mathcal{L}}{\partial e}\tilde{e} \quad \textit{or here} && (89)\\
&\frac{\partial J}{\partial e}\tilde{e} = -\int_0^T\int_0^l \tilde{e}\rho\frac{\partial^2 y}{\partial t^2}p - \int_0^T\int_0^l E\frac{\partial I(e)}{\partial e}\tilde{e}y_{xx}p_{xx}
\end{aligned}$$

where the adjoint state is solution of (92), (93), (94)

The proof uses the differentiated state equation in which we set $\phi = p$ and integration by parts. Justification relies on the case without control presented in section 5 ; shape and stuctural sensitivity of dynamic structures was presented first in ([25, 24, 26]). For design sensitivity, the equations to be considered are the state equation (68) in which we replace the control by its expression (101), the adjoint state equation (102), and equation (106).

$$\begin{aligned} & \int_0^T \int_0^l \rho e \frac{\partial^2 y}{\partial t^2} \phi - \int_0^T \int_0^l EI(e) y_{xx} \phi_{xx} + \\ + \quad & \int_0^T \{A^2 p(t,a)\phi(t,a) + B^2 p_x(t,a)\phi_x(t,a)\} + \int_0^T \int_a^b C p \phi = 0 \qquad (90) \\ & y(T,x) = 0 = \dot{y}(T,x) \qquad (91) \\ & \int_0^T \int_0^l \rho e \frac{\partial^2 p}{\partial t^2} z + \int_0^T \int_0^l EI(e) y_{xx} p_{xx} = 0 \qquad (92) \\ & \rho e p(0,x) - p_0(x) = 0, \;\; \rho e \frac{\partial p}{\partial t}(0,x) - p_1(x) = 0, \qquad (93) \\ & p(t,x) = 0, \quad \frac{\partial p}{\partial x} = 0 \text{ for } (t,x) \in [0,T] \times \{0,l\} \qquad (94) \\ & \Lambda \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} = \begin{pmatrix} y_1 \\ -y_0 \end{pmatrix} \qquad \text{which is equivalent to} \qquad (95) \\ & \quad y(0,x) = y_0(x) \;\; \dot{y}(0,x) = y_1(x) \qquad (96) \end{aligned}$$

We aknowledge that the sensitivity of the state equation is justified only in the case $A = 0 = B$ by using Theorem 5.2; then the righ hand side is in $L^2(0,T,L^2(\Omega))$. Nethertheless the computation is formally the same. The proof of the general case is in preparation.

7. Appendix

In order to derive necessary optimality conditions for control problem (79), (80) , we use a Lagrangian; this is in contrast with Ph. Destuynder ([9]) and the introduction of exact controlabity by J. L. Lions ([18]) where penalty arguments are mainly used. We consider the state equation with final conditions and use Lagrange multipliers to enforce initial conditions; to simplify we assume that $V = H_0^2(\Omega)$: mechanically it means that the beam is clamped at both ends. The Lagrangian is written:

$$\begin{aligned} & \mathcal{L}(y,u;p,p_0,p_1) = \qquad (97) \\ = \quad & J(u) - \int_0^T \int_0^l \rho e \frac{\partial^2 y}{\partial t^2} p - \int_0^T \int_0^l EI(e) y_{xx} p_{xx} + \\ = \quad & \int_0^T \{A u_1 p(t,a) + B u_2 p_x(t,a)\} + \int_0^T \int_a^b C u_3 p + \\ = \quad & \int_0^l \left\{ p_1[y(0,x) - y_0(x)] \;\; - p_0[\frac{\partial y}{\partial t}(0,x) - y_1(x)] \right\} \qquad (98) \end{aligned}$$

with

$$y \in \mathcal{C}^0([0,T], L^2(]0,l[) \cap \mathcal{C}^1([0,T], V') \text{ and } \quad y(T,x) = 0 = \dot{y}(T,x) \tag{99}$$

$$u = (u_1, u_2, u_3) \in L^2(0,T) \times L^2(0,T) \times L^2(0,T, L^2(]0,l[), \tag{100}$$

then, $\frac{\partial \mathcal{L}}{\partial u_i} = 0$ yields:

$$u_1(t) = -Ap(t,a), \;\; u_2(t) = -B\frac{\partial p}{\partial x}(t,a), \;\text{ and for }\; x \in [b,c], \; u_3 = -Cp \tag{101}$$

$\frac{\partial \mathcal{L}}{\partial p} = 0$ gives back the state equation and after integration by parts: $\frac{\partial \mathcal{L}}{\partial y} z = 0$ yields the adjoint state equation:

$$\int_0^T \int_0^l \rho e \frac{\partial^2 p}{\partial t^2} z + \int_0^T \int_0^l EI(e) y_{xx} p_{xx} = 0 \tag{102}$$

with initial conditions:

$$\rho e p(0,x) - p_0(x) = 0 \;\; \rho e \frac{\partial p}{\partial t}(0,x) - p_1(x) = 0 \text{ and boundary conditions:} \tag{103}$$

$$p(t,x) = 0 \quad \frac{\partial p}{\partial x} = 0 \;\; x \in [0,T] \times \{0,l\} \tag{104}$$

as we have assumed for simplicity that the beam is clamped at both ends.

What are the differences with the situation of optimal control?
There are no final conditions for the adjoint state and in the relations (103) p_0 and p_1 have to be determined in order to fullfill $y(0,x) = y_0(x)$ and $\dot{y}(0,x) = y_1(x)$.

To study the necesary conditions, following J. L. Lions ([18]), we introduce:

$$\Lambda : \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} \longmapsto \begin{pmatrix} \frac{\partial y}{\partial t}(0,x) \\ -y(0,x) \end{pmatrix} \tag{105}$$

The choice of the order of the target insures symetry and coercivity of Λ; and this implies controlability (ie existence of a control which brings the state from given initial conditions to final conditions). The exact control is given by (101) with p the adjoint state satisfying (103) and with p_0, p_1 being solution of:

$$\Lambda \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} = \begin{pmatrix} y_1 \\ -y_0 \end{pmatrix} \tag{106}$$

Lemma 7.1.

$$\int_0^l \rho\, e(\tilde{p_0}, \tilde{p_1}) \Lambda \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} = \tag{107}$$

$$= \int_0^l -\tilde{p_1} y_0(x) \;\; + \tilde{p_0} \frac{\partial y}{\partial t}(0,x) =$$

$$= -\left(\int_0^T A\tilde{u_1} p(t,a) + B\tilde{u_2} p_x(t,a) + \int_0^T \int_a^b C u_3 p \right)$$

and

$$J(u_1,u_2,u_3) = \frac{1}{2}(p_0,p_1)\Lambda\begin{pmatrix}p_0\\p_1\end{pmatrix} = \tag{108}$$

$$= \quad \frac{1}{2}\int_0^T A^2p(a)^2 + B^2p'(a)^2 + \int_0^T\int_a^b C^2p^2$$

This identity is derived from the adjoint state equation multiplied by the state y and integrated by parts.

8. Conclusion

We have presented a general abstract approach for design sensitivity of systems governed by variational second order evolution equations; it encompasses some previous results of the author and is widely applicable. Its use for design sensitivity of a model of a smart beam has been presented for illustration. In formulas (88) and (89), it should be emphasized that the design sensitivity involves the adjoint state used for control but no adjoint state for design sensitivity; in this respect the situation is analogous to the one encountered when we consider the sensitivity of the compliance of a static structure. Other cases of application are in preparation.

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