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Formulas for Subdifferentials of Sums of Convex Functions

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We discuss various formulas for the subdifferential of the sum of lower semicontinuous convex functions given in terms of certain topological closure operations on the sum of the subdifferentials of each function. We show how the accuracy of the formulas expressed by the closure operations can be improved when additional assumptions on the family of functions are available.

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1. Introduction

This paper is motivated by the recent contributions of Hiriart-Urruty and Phelps [10], Attouch, Baillon and Théra [1], Thibaut [19, 20, 21] and Penot [16] to the subdifferential calculus for convex functions. Its main objective is to point out where convexity intervenes in the elaboration of formulas for the subdifferential of the sum of lower semicontinuous convex functions, and how additional conditions on the family of functions (so-called "qualification conditions") act upon the accuracy of the formulas. Our approach enables us to recover, and even extend, most of the sum formulas established in the above papers, and provides a different insight on the classical exact sum rule under the Robinson-Rockafellar qualification condition, namely $0 \in \operatorname{cor}(\operatorname{dom} f_1 - \operatorname{dom} f_2)$.

To be more precise, this paper is devoted to the study of the formula

$$\partial\left(\sum_{i=1}^{k} f_i\right)(x) = \tau^*\!-\limsup_{x_i \xrightarrow{\alpha} x} \sum_{i=1}^{k} \partial f_i(x_i), \tag{1}$$

where ∂ is the subdifferential of convex analysis, the f_i's are lower semicontinuous convex functions from a Banach space X into $\mathbb{R} \cup \{\infty\}$ such that $\bigcap_{i=1}^{k} \operatorname{dom} f_i$ is not empty, x is any point in X, τ^* is either the strong topology $\|\,.\,\|_*$ or the weak* topology w^* on the dual space X^*, and $\tau^*\!-\limsup_{x_i \xrightarrow{\alpha} x} \sum_{i=1}^{k} \partial f_i(x_i)$ stands for the set of τ^*-limits of nets

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of the form $(x_{1,\nu}^* + \ldots + x_{k,\nu}^*)_\nu$ for which there exist nets $(x_{i,\nu})_\nu \subset X$, $i = 1, \ldots, k$, such that $x_{i,\nu}^* \in \partial f_i(x_{i,\nu})$, $x_{i,\nu} \to x$ and the couples $(x_{i,\nu}, x_{i,\nu}^*)$ satisfy various additional requirements.

Of course, formula (1) amounts to the following two inclusions

$$\partial\left(\sum_{i=1}^k f_i\right)(x) \supset \tau^*\!-\!\limsup_{x_i \xrightarrow{\alpha} x} \sum_{i=1}^k \partial f_i(x_i), \tag{2}$$

$$\partial\left(\sum_{i=1}^k f_i\right)(x) \subset \tau^*\!-\!\limsup_{x_i \xrightarrow{\alpha} x} \sum_{i=1}^k \partial f_i(x_i). \tag{3}$$

It turns out that inclusion (2) relies only on basic properties of convexity while inclusion (3) can be derived as a special case of the more general theory of subdifferentials for lower semicontinuous functions where convexity plays no role.

It is also clear that the exact sum rule for two lower semicontinuous convex functions,

$$\partial(f_1 + f_2)(x) = \partial f_1(x) + \partial f_2(x), \tag{4}$$

is equivalent to inclusion (3), with $k = 2$ and $\tau^* = \|\,.\,\|_*$, combined with the equality

$$\partial f_1(x) + \partial f_2(x) = \|\,.\,\|_*\!-\!\limsup_{x_i \xrightarrow{\alpha} x} (\partial f_1(x_1) + \partial f_2(x_2)). \tag{5}$$

This last formula is shown to be valid whenever the domains of the functions overlap sufficiently, that is $0 \in \operatorname{cor}(\operatorname{dom} f_1 - \operatorname{dom} f_2)$, by using the uniform boundedness theorem and the definition of the sudbifferential of convex analysis.

Inclusions (2) and (5) are studied in Section 2 whereas inclusion (3) is studied in Section 3. The last section sums up the various sum formulas obtained as consequences of the preceding results.

Notation. Throughout, X denotes a Banach space (except when otherwise specified) with norm $\|\,.\,\|$ and X^* its topological dual with $\langle .,. \rangle$ being the duality pairing on $X^* \times X$. The strong (norm) topology of X^* is denoted by $\|\,.\,\|_*$ and its weak* topology by w^*. We write $B_\lambda(x) = \{\, y \in X \mid \|y - x\| \le \lambda \,\}$ for the closed λ-ball centered at point x.

All functions are assumed to take their values in $\mathbb{R} \cup \{\infty\}$. For $f, g : X \to \mathbb{R} \cup \{\infty\}$, $x \in X$ and $A \subset X$, we write

$$\operatorname{dom} f = \{\, y \in X \mid f(y) < +\infty \,\},$$

$$(f \triangledown g)(x) = \inf\{\, f(y) + g(x - y) \mid y \in X \,\},$$

$$f_A(x) = \begin{cases} f(x) & \text{if} \quad x \in A \\ +\infty & \text{if} \quad x \notin A, \end{cases}$$

$$f_A^-(x) = f_A(-x),$$

$$\delta_A(x) = \begin{cases} 0 & \text{if} \quad x \in A \\ +\infty & \text{if} \quad x \notin A, \end{cases}$$

$$\operatorname{diam}(A) = \sup\{ \|y - z\| \mid z, y \in A \},$$

$$\tau\text{–cl}\, A = \text{the closure of } A \text{ w.r.t. the topology } \tau,$$

$$\operatorname{int} A = \text{the topological interior of } A \text{ in } X,$$

$$\operatorname{cor} A = \text{the algebraic interior of } A;$$

hence, $0 \in \operatorname{cor} A$ means that $X = \mathbb{R}_+ A$.

We recall that for a proper function $f : X \to \mathbb{R} \cup \{\infty\}$, i.e. with $\operatorname{dom} f$ nonempty, the subdifferential of f of convex analysis at $x \in X$ is defined to be the set

$$\partial f(x) = \{ x^* \in X^* \mid \langle x^*, y - x\rangle + f(x) \leq f(y),\ \forall y \in X \};$$

we let $\partial f = \{ (x, x^*) \in X \times X^* \mid x^* \in \partial f(x) \}$ denote the graph of the subdifferential of f.

If τ is a topology on X, we say that f is *τ-inf-compact on the ball $B_\lambda(x)$* if for every $s \in \mathbb{R}$ the set

$$\{ x \in X \mid f(x) \leq s \} \cap B_\lambda(x)$$

is τ-compact.

2. Closures of sums of convex subdifferentials

2.1. Closures of the convex subdifferential

The graph of the subdifferential of a lower semicontinuous convex function is clearly closed in $X \times X^*$ supplied with the topology $\|\,.\,\| \times \|\,.\,\|_*$. Said differently, for any lower semicontinuous convex function $f : X \to \mathbb{R} \cup \{\infty\}$ one has

$${}_{\|\,.\,\|\times\|\,.\,\|_*}\text{–cl}\, \partial f \subset \partial f,$$

or, equivalently,

$${}_{\|\,.\,\|_*}\text{–}\limsup_{\bar{x}\to x} \partial f(\bar{x}) \subset \partial f(x), \qquad \forall\, x \in X, \tag{6}$$

where $\tau^*\text{–}\limsup_{\bar{x}\to x} \partial f(\bar{x}) = \{ x^* \in X^* \mid (x, x^*) \in {}_{\|\,.\,\|} \times \tau^*\text{–cl}\,\partial f \}$. This pleasant closure property does not remain true when $X \times X^*$ is given with the topology $\|\,.\,\| \times w^*$. To see this, it suffices to consider the following example constructed by S. Fitzpatrick: take $X = \ell^2([0,1])$, identify X with X^* and define a proper lower semicontinuous convex function $f : X \to \mathbb{R}$ by letting

$$f(x) := \max(\langle e_0, x\rangle + 1,\ \sup\{r^{-1}\langle e_r, x\rangle : 0 < r \leq 1\}),$$

where $e_r \in X$ is given by $e_r(s) = 0$ if $s \neq r$ and $e_r(r) = 1$ for $0 \leq r, s \leq 1$; it can be verified that $(0,0)$ is not in ∂f but is in the $\|\,.\,\| \times w^*$ closure of the set $\{ (re_r, r^{-1}e_r) \mid 0 < r \leq 1 \}$ which is contained in ∂f. More specifically, Borwein-Fitzpatrick-Girgensohn [4] recently proved that the graph of the subdifferential of every lower semicontinuous convex function is $\|\,.\,\| \times w^*$ closed in $X \times X^*$ if and only if X is finite dimensional. Thus, given an arbitrary convex lower semicontinuous function f on an infinite dimensional Banach space X and an arbitrary point $x \in X$, the set

$$w^*\text{–}\limsup_{\bar{x}\to x} \partial f(\bar{x})$$

need not be included in $\partial f(x)$.

It is therefore natural to ask what would be sufficient to add to the topology $\|\,.\,\| \times w^*$ on $X \times X^*$ to guarantee this inclusion. For example, consider the simple case where $\operatorname{dom} f = X$ and supply $X \times X^*$ with the topology $\beta(w^*)$ generated by the family of semi-distances:

$$d_p((x,x^*),(y,y^*)) := \|x-y\| + |f(x)-f(y)| + p(x^*-y^*) + |\langle x^*,x\rangle - \langle y^*,y\rangle|,$$

where p runs through the family of semi-norms defining the w^*-topology on X^*. A net $(x_\nu, x_\nu^*)_\nu$ converges towards (x,x^*) for this topology $\beta(w^*)$ if and only if one has:

(i) $x_\nu \longrightarrow x, \quad f(x_\nu) \longrightarrow f(x)$;

(ii) $x_\nu^* \xrightarrow{w^*} x^*$;

(iii) $\langle x_\nu^*, x_\nu - x\rangle \longrightarrow 0$.

It is easy to see that the strengthening of $\|\,.\,\| \times w^*$-convergence brought by property (iii) forces the closure of ∂f. Indeed, if a net $(x_\nu, x_\nu^*)_\nu$ in ∂f converges to (x,x^*) for the above topology $\beta(w^*)$, then for every y in X

$$\langle x_\nu^*, y-x\rangle + f(x_\nu) - \langle x_\nu^*, x_\nu - x\rangle = \langle x_\nu^*, y - x_\nu\rangle + f(x_\nu) \le f(y), \tag{7}$$

so, passing to the limit, we conclude that (x,x^*) belongs to ∂f.

To express the localization at some point x of the closure property of ∂f w.r.t. $\beta(w^*)$, we use the following notation by analogy with the classical lim sup operation:

$$\begin{aligned} w^*\!-\!\limsup_{\bar{x}\xrightarrow{+}x} \partial f(\bar{x}) &:= \{\, x^* \in X^* \mid (x,x^*) \in \beta(w^*)\!-\!\operatorname{cl}\partial f \,\} \\ &= \{\, x^* \in X^* \mid \exists\, (x_\nu, x_\nu^*)_\nu \subset \partial f \text{ verifying (i)–(iii)} \,\}. \end{aligned}$$

The closure of ∂f with respect to $\beta(w^*)$ may then be rewritten as

$$w^*\!-\!\limsup_{\bar{x}\xrightarrow{+}x} \partial f(\bar{x}) \subset \partial f(x), \qquad \forall\, x \in X.$$

Actually, the above proof shows that it is possible to slightly weaken the assertions (i)–(iii) defining the $\beta(w^*)$-convergence while conserving the closure property of ∂f. Consider the following variants:

(i′) $x_\nu \longrightarrow x$;

(iii-c) $\limsup_\nu \langle x_\nu^*, x_\nu - x\rangle \le 0$;

(iii-d) $f(x) \le \limsup_\nu \, (f(x_\nu) - \langle x_\nu^*, x_\nu - x\rangle)$;

and write:

$$w^*\!-\!\limsup_{\bar{x}\xrightarrow{c}x} \partial f(\bar{x}) := \{\, x^* \in X^* \mid \exists\, (x_\nu, x_\nu^*)_\nu \subset \partial f \text{ verifying (i′), (ii) and (iii-c)} \,\},$$

$$w^*\!-\!\limsup_{\bar{x}\xrightarrow{d}x} \partial f(\bar{x}) := \{\, x^* \in X^* \mid \exists\, (x_\nu, x_\nu^*)_\nu \subset \partial f \text{ verifying (i′), (ii) and (iii-d)} \,\}.$$

We then have the following sequence of inclusions:

$$w^*\!-\!\limsup_{\bar{x}\xrightarrow{+}x} \partial f(\bar{x}) \subset w^*\!-\!\limsup_{\bar{x}\xrightarrow{c}x} \partial f(\bar{x}) \subset w^*\!-\!\limsup_{\bar{x}\xrightarrow{d}x} \partial f(\bar{x}) \subset \partial f(x). \tag{8}$$

Indeed, (iii) clearly implies (iii-c). On the other hand, (iii-c) implies (iii-d) because, f being lower semicontinuous at x, we can write

$$\begin{aligned} f(x) &\leq \liminf_{\nu} f(x_\nu) \\ &\leq \limsup_{\nu} \left(f(x_\nu) - \langle x^*_\nu, x_\nu - x\rangle\right) + \limsup_{\nu} \langle x^*_\nu, x_\nu - x\rangle \\ &\leq \limsup_{\nu} \left(f(x_\nu) - \langle x^*_\nu, x_\nu - x\rangle\right). \end{aligned}$$

Finally, if (i′), (ii) and (iii-d) hold, taking the upper limit on both sides of inequality (7) yields that (x, x^*) belongs to ∂f.

Remark 2.1. If a net $(x_\nu, x^*_\nu)_\nu \subset \partial f$ converges to (x, x^*) w.r.t. $\|\,.\,\| \times w^*$ and if $(x^*_\nu)_\nu$ happens to be bounded from a certain index, then (i′), (ii) and (iii) are verified, so (x, x^*) belongs to ∂f. This will be the case, e.g., whenever the indices ν run through $\mathbb{N}$, i.e. whenever $(x_\nu, x^*_\nu)_\nu$ is actually a sequence, as follows from the uniform boundedness theorem, or whenever f is Lipschitz near x, since for all $\bar{x}$ close to x the sets $\partial f(\bar{x})$ will then be contained in some common ball. We derive that ∂f is always sequentially $\|\,.\,\| \times w^*$-closed, and that it is topologically $\|\,.\,\| \times w^*$-closed at each point x where f is continuous.

Remark 2.2. Property (iii-c), which expresses some control on the behavior of the net $(x^*_\nu, x_\nu)_\nu$ via the duality pairing $\langle .\,, .\rangle$, plays an important role in nonlinear analysis: it appears for example in the theory of monotone-type operators and in the study of partial differential equations (see Brézis [5], Lions [15, Chap. 2], Brézis-Nirenberg-Stampacchia [6], Browder [7], Attouch-Baillon-Théra [1]).

2.2. Closures of sums of subdifferentials lying in the subdifferential of the sum

In this subsection, we introduce several kinds of closure operations on the sum of convex subdifferentials that preserve the inclusion of this set in the subdifferential of the sum. The preceding considerations which led to the sequence of inclusions (8) are extended to this setting. To express the closure operations locally, we shall use the same lim sup type notations as in the case of a single function.

Given functions $f_i : X \to \mathbb{R} \cup \{\infty\}$, nets $(x_{i,\nu}, x^*_{i,\nu})_\nu \subset X \times X^*$, $i = 1, \ldots, k$, and a topology τ^* over X^*, we consider the assertions:

(i) $x_{i,\nu} \longrightarrow x, \quad f_i(x_{i,\nu}) \longrightarrow f_i(x), \quad \forall\, i;$

(ii)$_{\tau^*}$ $x^*_{1,\nu} + \ldots + x^*_{k,\nu} \xrightarrow{\tau^*} x^*;$

(iii) $\mathrm{diam}(x_{1,\nu}, \ldots, x_{k,\nu})\, \|x^*_{i,\nu}\| \to 0 \;\;\&\;\; \langle x^*_{1,\nu} + \ldots + x^*_{k,\nu}, x_{i,\nu} - x\rangle \to 0, \;\; \forall\, i;$

and the following variants:

(i′) $x_{i,\nu} \longrightarrow x, \;\; \forall\, i;$

(iii-a) $\langle x^*_{i,\nu}, x_{i,\nu} - x_{1,\nu}\rangle \to 0 \;\;\&\;\; \langle x^*_{i,\nu}, x_{i,\nu} - x\rangle \to 0, \;\; \forall\, i;$

(iii-b) $\limsup_\nu \sum_{i=1}^k \langle x^*_{i,\nu}, x_{i,\nu} - x_{1,\nu}\rangle \leq 0 \;\;\&\;\; \limsup_\nu \langle \sum_{i=1}^k x^*_{i,\nu}, x_{1,\nu} - x\rangle \leq 0;$

(iii-c) $\limsup_\nu \sum_{i=1}^k \langle x^*_{i,\nu}, x_{i,\nu} - x\rangle \leq 0;$

(iii-d) $\sum_{i=1}^k f_i(x) \leq \limsup_\nu \sum_{i=1}^k \left(f_i(x_{i,\nu}) - \langle x^*_{i,\nu}, x_{i,\nu} - x\rangle\right).$

We then set:

$$\tau^*\text{–}\limsup_{x_i \xrightarrow{+} x} \sum_{i=1}^{k} \partial f_i(x_i) := \{\, x^* \in X^* \mid \exists\, (x_{i,\nu}, x^*_{i,\nu})_\nu \subset \partial f_i, \text{ with (i), (ii)}_{\tau^*}, \text{(iii)} \,\},$$

$$\tau^*\text{–}\limsup_{x_i \xrightarrow{\alpha} x} \sum_{i=1}^{k} \partial f_i(x_i) := \{\, x^* \in X^* \mid \exists\, (x_{i,\nu}, x^*_{i,\nu})_\nu \subset \partial f_i, \text{ with (i}'\text{), (ii)}_{\tau^*}, \text{(iii-}\alpha\text{)} \,\},$$

where α stands for a, b, c or d. We may also wish to consider the sequential versions of these sets, defined by replacing the nets $(x_{i,\nu}, x^*_{i,\nu})_\nu$ by sequences $(x_{i,n}, x^*_{i,n})_n$. In case $\tau^* = \|\,.\,\|_*$, it is easily seen that the topological and sequential versions coincide.

The sequence of inclusions (8) of the previous subsection is then generalized in the following way:

Proposition 2.3. *Let $f_1, \ldots, f_k : X \to \mathbb{R} \cup \{\infty\}$ be lower semicontinuous convex functions on a normed space X such that $\bigcap_{i=1}^k \operatorname{dom} f_i$ is not empty, and let τ^* be any topology on X^* lying between w^* and $\|\,.\,\|_*$. Then for every $x \in X$ one has*

$$\begin{aligned} \tau^*\text{–}\limsup_{x_i \xrightarrow{+} x} \sum_{i=1}^{k} \partial f_i(x_i) &\subset \tau^*\text{–}\limsup_{x_i \xrightarrow{a} x} \sum_{i=1}^{k} \partial f_i(x_i) \\ &\subset \tau^*\text{–}\limsup_{x_i \xrightarrow{b} x} \sum_{i=1}^{k} \partial f_i(x_i) \\ &\subset \tau^*\text{–}\limsup_{x_i \xrightarrow{c} x} \sum_{i=1}^{k} \partial f_i(x_i) \\ &\subset \tau^*\text{–}\limsup_{x_i \xrightarrow{d} x} \sum_{i=1}^{k} \partial f_i(x_i) \\ &\subset \partial \left(\sum_{i=1}^{k} f_i \right)(x). \end{aligned} \tag{9}$$

Proof. (1) All the inclusions except the first one follow easily from the following observations:

(iii-a) $\Rightarrow$ (iii-b), because (iii-a) and the equality

$$\langle x^*_{i,\nu}, x_{1,\nu} - x \rangle = \langle x^*_{i,\nu}, x_{1,\nu} - x_{i,\nu} \rangle + \langle x^*_{i,\nu}, x_{i,\nu} - x \rangle$$

yield that $\langle x^*_{i,\nu}, x_{1,\nu} - x \rangle \to 0$, hence (iii-b) holds.

(iii-b) $\Rightarrow$ (iii-c), because (iii-b) applied to

$$\sum_{i=1}^{k} \langle x^*_{i,\nu}, x_{i,\nu} - x \rangle = \sum_{i=1}^{k} \langle x^*_{i,\nu}, x_{i,\nu} - x_{1,\nu} \rangle + \left\langle \sum_{i=1}^{k} x^*_{i,\nu}, x_{1,\nu} - x \right\rangle \tag{10}$$

yields that $\limsup_\nu \sum_{i=1}^k \langle x^*_{i,\nu}, x_{i,\nu} - x \rangle \le 0$.

(iii-c) $\Rightarrow$ (iii-d), because (iii-c) and the lower semicontinuity of the f_i's at x yield that

$$\begin{aligned}\left(\sum_{i=1}^k f_i\right)(x) &\leq \liminf_\nu \sum_{i=1}^k f_i(x_{i,\nu}) \\ &\leq \limsup_\nu \sum_{i=1}^k \big(f_i(x_{i,\nu}) - \langle x^*_{i,\nu}, x_{i,\nu} - x\rangle\big) + \limsup_\nu \sum_{i=1}^k \langle x^*_{i,\nu}, x_{i,\nu} - x\rangle \\ &\leq \limsup_\nu \sum_{i=1}^k \big(f_i(x_{i,\nu}) - \langle x^*_{i,\nu}, x_{i,\nu} - x\rangle\big)\,.\end{aligned}$$

Finally, let $x \in X$ and let x^* be in the limiting set at x defined by (i′), (ii)$_{\tau^*}$ and (iii-d). Then, for every y in $\bigcap_{i=1}^k \operatorname{dom} f_i$, we have

$$\begin{aligned}\langle \sum_{i=1}^k x^*_{i,\nu}, y - x\rangle + \sum_{i=1}^k (f_i(x_{i,\nu}) - \langle x^*_{i,\nu}, x_{i,\nu} - x\rangle) &= \sum_{i=1}^k \big(\langle x^*_{i,\nu}, y - x_{i,\nu}\rangle + f_i(x_{i,\nu})\big) \\ &\leq \sum_{i=1}^k f_i(y),\end{aligned}$$

so, taking the upper limit on the left side, we find that

$$\langle x^*, y - x\rangle + \left(\sum_{i=1}^k f_i\right)(x) \leq \left(\sum_{i=1}^k f_i\right)(y),$$

showing that x^* belongs to $\partial\left(\sum_{i=1}^k f_i\right)(x)$.

(2) It remains to prove the first inclusion, so let $x \in X$ and let x^* be in the limiting set at x defined by (i), (ii)$_{\tau^*}$ and (iii). We first show that this forces x to be in $\bigcap_{i=1}^k \operatorname{dom} f_i$. Indeed, it follows from the decomposition (10) that

$$\sum_{i=1}^k \langle x^*_{i,\nu}, x_{i,\nu} - x\rangle \leq \operatorname{diam}(x_{1,\nu}, \dots, x_{k,\nu}) \sum_{i=1}^k \|x^*_{i,\nu}\| + \left\langle \sum_{i=1}^k x^*_{i,\nu}, x_{1,\nu} - x \right\rangle,$$

hence, from (iii), we conclude that $\limsup_\nu \sum_{i=1}^k \langle x^*_{i,\nu}, x_{i,\nu} - x\rangle \leq 0$, i.e. (iii-c) holds. Consequently, as seen above, (iii-d) also holds, hence x^* belongs to $\partial\left(\sum_{i=1}^k f_i\right)(x)$, proving that $\sum_{i=1}^k f_i(x)$ is finite.

Now, to prove that x^* belongs to the set defined by (iii-a), it is enough to show that

$$\langle x^*_{j,\nu}, x_{j,\nu} - x\rangle \longrightarrow 0, \quad \forall\, j = 1, \dots, k,$$

because (iii-a) will then be clearly satisfied. So let j in $\{1, \dots, k\}$ be fixed. Since

$$\begin{aligned}\sum_{\substack{i=1\\ i\neq j}}^k \langle x^*_{i,\nu}, x - x_{j,\nu}\rangle &= \sum_{\substack{i=1\\ i\neq j}}^k \langle x^*_{i,\nu}, x - x_{i,\nu}\rangle + \sum_{\substack{i=1\\ i\neq j}}^k \langle x^*_{i,\nu}, x_{i,\nu} - x_{j,\nu}\rangle \\ &\leq \sum_{\substack{i=1\\ i\neq j}}^k (f_i(x) - f_i(x_{i,\nu})) + \operatorname{diam}(x_{1,\nu}, \dots, x_{k,\nu}) \sum_{\substack{i=1\\ i\neq j}}^k \|x^*_{i,\nu}\|,\end{aligned}$$

it follows from (i) and (iii) that

$$\limsup_{\nu} \sum_{\substack{i=1 \\ i \neq j}}^{k} \langle x^*_{i,\nu}, x - x_{j,\nu} \rangle \leq 0.$$

Using (i), (iii) and the above inequality, we then derive that

$$\begin{aligned} 0 &= \liminf_{\nu} \left\langle \sum_{i=1}^{k} x^*_{i,\nu}, x - x_{j,\nu} \right\rangle \\ &\leq \liminf_{\nu} \langle x^*_{j,\nu}, x - x_{j,\nu} \rangle + \limsup_{\nu} \sum_{\substack{i=1 \\ i \neq j}}^{k} \langle x^*_{i,\nu}, x - x_{j,\nu} \rangle \\ &\leq \liminf_{\nu} \langle x^*_{j,\nu}, x - x_{j,\nu} \rangle \\ &\leq \limsup_{\nu} \langle x^*_{j,\nu}, x - x_{j,\nu} \rangle \\ &\leq \limsup_{\nu} (f_j(x) - f_j(x_{j,\nu})) = 0, \end{aligned}$$

showing that $\langle x^*_{j,\nu}, x - x_{j,\nu} \rangle \to 0$. The proof of the proposition is therefore complete. □

Remark 2.4. Various notions of convergence, associated to combinations of the above assertions, have been used by Attouch, Baillon and Théra [1], Thibault [19, 20, 21] and Penot [16] to solve the same kind of problem (see in particular [21, Theorem 1]). All the limiting sets considered by these authors lie between the smallest set in the sequence of inclusions (9), namely

$$\|\,.\,\|_*\text{–}\limsup_{x_i \xrightarrow{+} x} \sum_{i=1}^{k} \partial f_i(x_i),$$

and the biggest one, namely

$$w^*\text{–}\limsup_{x_i \xrightarrow{d} x} \sum_{i=1}^{k} \partial f_i(x_i).$$

The sequential version of this last set, for $k = 2$, X reflexive and x in $\operatorname{dom} f_1 \cap \operatorname{dom} f_2$, is considered in [20, Theorem 2.1].

2.3. Case of strong closure of sums of subdifferentials

In view of inclusion (6) of subsection 2.1, the graph of the subdifferential of a convex lower semicontinuous function is closed in $X \times X^*$ for the strong topology $\|\,.\,\| \times \|\,.\,\|_*$. In this subsection, we are interested in extending this result to the case of a sum of two subdifferentials. Of course, given that the sum of two closed sets is not closed in general, an additional condition is required to establish the result. Roughly speaking, this condition asserts that the domains of the functions overlap sufficiently.

Proposition 2.5. *Let $f_1, f_2 : X \to \mathbb{R} \cup \{\infty\}$ be convex lower semicontinuous functions on a Banach space X such that* $0 \in \operatorname{cor}(\operatorname{dom} f_1 - \operatorname{dom} f_2)$. *Then for every $x \in X$ one has*

$$\partial f_1(x) + \partial f_2(x) = \|\,.\,\|_*\text{–}\limsup_{x_i \xrightarrow{+} x} (\partial f_1(x_1) + \partial f_2(x_2)).$$

Proof. It is evident that the first set is always contained in the second one. Let us show that the reverse inclusion holds under the qualification condition $0 \in \operatorname{cor}(\operatorname{dom} f_1 - \operatorname{dom} f_2)$. Let then $(x_{i,n}, x^*_{i,n})_n$ be sequences in ∂f_i, $i = 1, 2$, and let $x^* \in X^*$ verifying (i), (ii)$_s$ and (iii) (where (ii)$_s$ denotes (ii)$_{\tau^*}$ for $\tau^* = \|\,.\,\|_*$). In view of our condition, for every v in X there exist $t > 0$ and v_i in $\operatorname{dom} f_i$, $i = 1, 2$, such that $tv = v_1 - v_2$, so

$$\begin{aligned} t\langle x^*_{1,n}, v\rangle &= \langle x^*_{1,n}, v_1 - x_{1,n}\rangle + \langle x^*_{1,n}, x_{1,n} - v_2\rangle \\ &\le f_1(v_1) - f_1(x_{1,n}) + \langle x^*_{2,n}, v_2 - x_{2,n}\rangle + \langle x^*_{2,n}, x_{2,n} - x_{1,n}\rangle \\ &\qquad + \langle x^*_{1,n} + x^*_{2,n}, x_{1,n} - v_2\rangle \\ &\le f_1(v_1) - f_1(x_{1,n}) + f_2(v_2) - f_2(x_{2,n}) + \langle x^*_{2,n}, x_{2,n} - x_{1,n}\rangle \\ &\qquad + \langle x^*_{1,n} + x^*_{2,n}, x_{1,n} - v_2\rangle, \end{aligned}$$

from which it follows, thanks to (i), (ii)$_s$ and (iii), that

$$t \limsup_n \langle x^*_{1,n}, v\rangle \le f_1(v_1) - f_1(x) + f_2(v_2) - f_2(x) + \langle x^*, x - v_2\rangle.$$

We derive that the sequence $(x^*_{1,n})_n$ is w^*-bounded, hence, by the Banach-Steinhaus theorem, it is (norm) bounded. Consequently, according to the Banach-Alaoglu theorem, $(x^*_{1,n})_n$ has a bounded subnet $(x^*_{1,\nu})_\nu$ that w^*-converges to some point x^*_1 in X^*. The convergence in (ii)$_s$ then implies that the net $x^*_{2,\nu} := (x^*_{1,\nu} + x^*_{2,\nu}) - x^*_{1,\nu}$ w^*-converges to $x^*_2 := x^* - x^*_1$ and is bounded.

Thus, the nets $(x_{i,\nu}, x^*_{i,\nu})_\nu \subset \partial f_i$, $i = 1, 2$, converge to (x, x^*_i) w.r.t. $\|\,.\,\| \times w^*$, with $(x^*_{i,\nu})_\nu$ being bounded and $x^* = x^*_1 + x^*_2$. Since ∂f_i is closed for this convergence (see Remark 2.1), (x, x^*_i) belongs to ∂f_i, for $i = 1, 2$. This completes the proof. □

3. Fuzzy sum rules

This section is concerned with inclusion (3). As already mentioned in the introduction, this inclusion can be obtained as a consequence of a more general theory belonging to nonsmooth analysis. We briefly recall the definitions and results relevant to the present problem and refer to [11, 12, 13] for more details.

A family $\{f_1, \ldots, f_k\}$ of lower semicontinuous functions on a Banach space X is said to be *decouplable at* $x \in X$ if the following decoupling condition

$$r_{B_\lambda(x)}(f_1 + \ldots + f_k) = r_{B_\lambda(x)}(f_1, \ldots, f_k) \tag{DC}$$

is verified for any small $\lambda > 0$, where

$$r_{B_\lambda(x)}(f_1 + \ldots + f_k) := \lim_{\delta \searrow 0} \inf\{(f_1 + \ldots + f_k)(y) \mid y \in B_{\lambda+\delta}(x)\},$$

and

$$\begin{aligned} r_{B_\lambda(x)}(f_1, \ldots, f_k) := \lim_{\delta \searrow 0} \inf\{&f_1(x_1) + \ldots + f_k(x_k) \mid \operatorname{diam}(x_1, \ldots, x_k) \le \delta, \\ &x_i \in B_{\lambda+\delta}(x),\ \ i = 1, \ldots, k\}. \end{aligned}$$

We say for short that $\{f_1, \ldots, f_k, X^*\}$ is decouplable at x whenever the family $\{f_1, \ldots, f_k, x^*\}$ is decouplable at x for every x^* in X^*.

The decoupling condition (DC) is weaker than the conditions used so far to establish fuzzy sum rules for subdifferentials. We gather the relevant special cases in the following proposition proved in [12]:

Proposition 3.1. (a) *If all the functions $f_1, \dots, f_k$ are τ-lower semicontinuous and at least one of them is τ-inf-compact on some ball centered at x (where τ is any vector topology on X lying between the weak and the norm topologies), then the family $\{f_1, \dots, f_k\}$ is decouplable at x;*

(b) *If the family $\{f_1, \dots, f_k + \varphi\}$ is decouplable at x and φ is uniformly continuous near x, then the family $\{f_1, \dots, f_k, \varphi\}$ is decouplable at x; consequently, if all but at most one of the functions $f_1, \dots, f_k$ are uniformly continuous near x, then the family $\{f_1, \dots, f_k\}$ is decouplable at x;*

(c) *If the inf-convolution $f_{B_\lambda(x)} \nabla g^-_{B_\lambda(x)}$ is lower semicontinuous at 0 for any small $\lambda > 0$, then the family $\{f, g\}$ is decouplable at x.*

When the functions are convex, it is possible to make a link between the decoupling condition (DC) and the qualification condition of Robinson-Rockafellar [17, 18], $0 \in \operatorname{cor}(\operatorname{dom} f - \operatorname{dom} g)$. This is the object of the following proposition (compare with Proposition 2.1 (b) of [14]):

Proposition 3.2. *Let $f, g : X \to \mathbb{R} \cup \{\infty\}$ be lower semicontinuous convex functions on a Banach space X such that $\operatorname{dom} f \cap \operatorname{dom} g$ is not empty. The following assertions are equivalent:*

(i) $0 \in \operatorname{cor}(\operatorname{dom} f - \operatorname{dom} g)$*;*
(ii) $f \nabla g^-$ *is continuous at* 0*;*
(iii) $f_{B_\lambda(x)} \nabla g^-_{B_\lambda(x)}$ *is continuous at* 0 *for all* $x \in \operatorname{dom} f \cap \operatorname{dom} g$ *and* $\lambda > 0$*.*

Consequently, if one of the conditions (i)–(iii) *holds, then the family $\{f, g, X^*\}$ is decouplable at any point x of* $\operatorname{dom} f \cap \operatorname{dom} g$.

Proof. We first show that assertions (i) and (ii) are equivalent. Note that $(f \nabla g^-)(0) \in [-\infty, +\infty[$ because $\operatorname{dom} f \cap \operatorname{dom} g$ is supposed to be non empty. Evidently, only (i) implies (ii) needs a proof, so let us assume that (i) holds. To prove (ii), it suffices to show that $f \nabla g^-$ is bounded above on a neighborhood of 0. Let $r \in \mathbb{R}$ such that $(f \nabla g^-)(0) < r$, and then choose $\bar{x} \in X$ and $\varepsilon > 0$ such that $f(-\bar{x}) + g(\bar{x}) + 2\varepsilon < r$. Put $C = \{ x \in X \mid f(x) \le f(-\bar{x}) + \varepsilon \}$ and $D = \{ x \in X \mid g(x) \le g(\bar{x}) + \varepsilon \}$. Standard convexity arguments show that 0 lies in $\operatorname{cor}(C + D)$. We may therefore invoke Robinson-Rockafellar's theorem on the continuity of marginal convex lower semicontinuous functions [17, Cor. 1] to derive that $\delta_C \nabla \delta_D$ is continuous at 0, showing that 0 in fact belongs to $\operatorname{int}(C + D)$. Since $(f \nabla g^-)(x) < r$ for every x in $\operatorname{int}(C + D)$, we conclude that $f \nabla g^-$ is continuous at 0, proving that (ii) holds.

Applying the above equivalence to $f_{B_\lambda(x)}$ and $g_{B_\lambda(x)}$ yields that (iii) is equivalent to

$$0 \in \operatorname{cor}\left(\operatorname{dom} f_{B_\lambda(x)} - \operatorname{dom} g_{B_\lambda(x)}\right), \quad \forall x \in \operatorname{dom} f \cap \operatorname{dom} g, \quad \forall \lambda > 0,$$

that is,

$$0 \in \operatorname{cor}\left(\operatorname{dom} f \cap B_\lambda(x) - \operatorname{dom} g \cap B_\lambda(x)\right), \ \forall x \in \operatorname{dom} f \cap \operatorname{dom} g, \ \forall \lambda > 0.$$

Since this latter assertion is obviously equivalent to (i), the proof of the first statement is complete.

Let us now prove the second statement. If $0 \in \operatorname{cor}(\operatorname{dom} f - \operatorname{dom} g)$, then $0 \in \operatorname{cor}(\operatorname{dom} f - \operatorname{dom}(g + x^*))$ for every $x^* \in X^*$, so $f_{B_\lambda(x)} \nabla (g+x^*)^-_{B_\lambda(x)}$ is continuous at 0 for all x^* in X^*, x in $\operatorname{dom} f \cap \operatorname{dom} g$ and $\lambda > 0$. This implies that $\{f, g + x^*\}$ is decouplable at x by Proposition 3.1 (c) and so the family $\{f, g, X^*\}$ is decouplable at x by Proposition 3.1 (b). $\square$

The following fuzzy sum rules are proved in [13]: *If $f_1, \ldots, f_k : X \to \mathbb{R} \cup \{\infty\}$ are lower semicontinuous functions on a Banach space X, if ∂ is any subdifferential and if X is ∂-regular, then for every $x \in X$ one has*

$$\begin{cases} \partial^F \left(\sum_{i=1}^k f_i\right)(x) \subset \|.\|_*\text{–}\limsup\limits_{x_i \xrightarrow{+} x} \sum_{i=1}^k \partial f_i(x_i) \\ \textit{provided } \{f_1, \ldots, f_k, X^*\} \textit{ is decouplable at } x, \end{cases}$$

and

$$\partial^F \left(\sum_{i=1}^k f_i\right)(x) \subset w^*\text{–}\limsup\limits_{x_i \xrightarrow{+} x} \sum_{i=1}^k \partial f_i(x_i),$$

where ∂^F denotes the Fréchet subdifferential.

In the above result, we can get free from any assumption on the Banach space X if we take for ∂, e.g., the Clarke subdifferential, since any Banach space is ∂-regular for this subdifferential (for more details on these questions see [14, 8, 12]). Now, the Fréchet and Clarke subdifferentials coincide on the class of proper lower semicontinuous convex functions with the subdifferential of convex analysis, so, for this class, the above result immediately simplifies to:

Proposition 3.3. *Let $f_1, \ldots, f_k : X \to \mathbb{R} \cup \{\infty\}$ be lower semicontinuous convex functions on a Banach space X such that $\bigcap_{i=1}^k \operatorname{dom} f_i$ is not empty. Then for every $x \in X$ one has*

$$\begin{cases} \partial \left(\sum_{i=1}^k f_i\right)(x) \subset \|\cdot\|_*\text{–}\limsup\limits_{x_i \xrightarrow{+} x} \sum_{i=1}^k \partial f_i(x_i) \\ \textit{provided } \{f_1, \ldots, f_k, X^*\} \textit{ is decouplable at } x, \end{cases} \tag{11}$$

and

$$\partial \left(\sum_{i=1}^k f_i\right)(x) \subset w^*\text{–}\limsup\limits_{x_i \xrightarrow{+} x} \sum_{i=1}^k \partial f_i(x_i). \tag{12}$$

Remark 3.4. The main tool for proving the above inclusions is Ekeland's variational principle, or, in the convex case, Brøndsted-Rockafellar's theorem. A direct proof of (special cases of) Proposition 3.3 can be found in Thibault [20], for inclusion (11) in a reflexive Banach space X, and in Thibault [19, 21] and Penot [16] for inclusion (12).

4. Sum formulas

4.1. Weak fuzzy sum formula

The first theorem proposes a weak fuzzy formula for the subdifferential of the sum of lower semicontinuous convex functions, valid on any Banach space X without qualification condition.

Theorem 4.1. *Let $f_1, \ldots, f_k : X \to \mathbb{R} \cup \{\infty\}$ be lower semicontinuous convex functions on a Banach space X such that $\bigcap_{i=1}^k \operatorname{dom} f_i$ is not empty. Then for every $x \in X$ one has*

$$\partial(\sum_{i=1}^k f_i)(x) = w^*\text{–}\limsup_{x_i \xrightarrow{+} x} \sum_{i=1}^k \partial f_i(x_i). \tag{13}$$

Proof. The first set is contained in the second one by inclusion (12) of Proposition 3.3. The second set is contained in the first one by inclusion (9) of Proposition 2.3. □

Remark 4.2. By combining Theorem 4.1 with the sequence of inclusions (9), we can get various formulas, from the most precise to the less restricting ones. We thus recover and complete the results by Thibault [19, Theorem 3] and Penot [16, Theorem 2.3], and partially the result of Thibault [21, Theorem 1], who deals with more elaborated formulas mixing sum and composition. We point out that in these papers the formulas are established at points x lying in $\bigcap_{i=1}^k \operatorname{dom} f_i$, whereas the formulas we obtain via Theorem 4.1 and Proposition 2.3 are proved to be valid at any point of X. Numerous applications, notably the Hiriart-Urruty and Phelps formula and the Rockafellar theorems on integration and on maximal monotonicity of the convex subdifferential, are presented in [21], thus demonstrating the relevance of such fuzzy formulas without any qualification condition.

4.2. Strong fuzzy sum formula

The next theorem proposes a strong fuzzy formula for the subdifferential of the sum of lower semicontinuous convex functions, also valid on any Banach space X, but this time with a qualification condition.

Theorem 4.3. *Let $f_1, \ldots, f_k : X \to \mathbb{R} \cup \{\infty\}$ be lower semicontinuous convex functions on a Banach space X such that $\bigcap_{i=1}^k \operatorname{dom} f_i$ is not empty. Then for every $x \in X$ one has*

$$\partial(\sum_{i=1}^k f_i)(x) = \|.\|_*\text{–}\limsup_{x_i \xrightarrow{+} x} \sum_{i=1}^k \partial f_i(x_i) \tag{14}$$

provided the family $\{f_1, \ldots, f_k, X^*\}$ is decouplable at x.

Proof. The first set is contained in the second one by inclusion (11) of Proposition 3.3. The second set is contained in the first one by (9). □

Corollary 4.4. *Let $f_1, \ldots, f_k : X \to \mathbb{R} \cup \{\infty\}$ be lower semicontinuous convex functions on a reflexive Banach space X such that $\bigcap_{i=1}^k \operatorname{dom} f_i$ is not empty. Then formula* (14) *is always true.*

Proof. The functions f_i are w^*-lower semicontinuous and, since X is reflexive, they are w^*-inf-compact on any ball, so, by Proposition 3.1 (a), the family $\{f_1, \ldots, f_k, X^*\}$ is decouplable on X, hence (14) is always true. $\square$

Remark 4.5. Here again, it suffices to combine Theorem 4.3 (or its corollary) with the sequence of inclusions (9) to obtain more formulas valid at any point x of X. In particular, we recover the formula of Attouch, Baillon and Théra [1, Theorem 7.3], where X is a Hilbert space, $k = 2$, x belongs to $\operatorname{dom} f_1 \cap \operatorname{dom} f_2$, and the limiting set is described via the operation $\|.\|_*\text{–}\limsup_{x_i \xrightarrow{c} x}$. We also recapture Thibault's formulas [20, Theorem 2.1], where X is a reflexive Banach space, $k = 2$, x belongs to $\operatorname{dom} f_1 \cap \operatorname{dom} f_2$, and the limiting sets considered are all lying between the smallest and the biggest sets in the sequence of inclusions (9). These two papers employ totally different methods, and present several applications to optimization problems.

4.3. Exact sum formula

The last theorem is the very classical exact sum formula under the Robinson-Rockafellar qualification condition:

Theorem 4.6. *Let $f_1, f_2 : X \to \mathbb{R} \cup \{\infty\}$ be lower semicontinuous convex functions on a Banach space X such that $0 \in \operatorname{cor}(\operatorname{dom} f_1 - \operatorname{dom} f_2)$. Then for every $x \in X$ one has*

$$\partial(f_1 + f_2)(x) = \partial f_1(x) + \partial f_2(x).$$

Proof. We already know that

$$\partial f_1(x) + \partial f_2(x) \subset \partial(f_1 + f_2)(x).$$

To prove the reverse inclusion, we break it down in the following way:

$$\partial(f_1 + f_2)(x) \subset \|\cdot\|_*\text{–}\limsup_{x_i \xrightarrow{+} x} (\partial f_1(x_1) + \partial f_2(x_2)) \subset \partial f_1(x) + \partial f_2(x).$$

The first inclusion follows from Proposition 3.3, because the condition $0 \in \operatorname{cor}(\operatorname{dom} f_1 - \operatorname{dom} f_2)$ implies that the family $\{f_1, f_2, X^*\}$ is decouplable on $\operatorname{dom} f_1 \cap \operatorname{dom} f_2$ according to Proposition 3.2. The second inclusion follows from Proposition 2.5. $\square$

Remark 4.7. The Robinson-Rockafellar qualification condition was introduced by Rockafellar [18] in the general context of conjugate duality theory of which the exact sum formula for subdifferentials is a by-product. In this seminal paper, it is proved that such a condition is sufficient for marginal convex lower semicontinuous functions to be continuous at 0 provided the Banach spaces are in duality [18, Th. 18] and consequently this condition is sufficient to ensure the strong duality relation [18, Th. 17], a Fenchel-type duality result [18, Example 11′] and the exact sum formula [18, Th. 20] in reflexive Banach spaces. Thus, Robinson's generalization [17, Cor. 1] of Rockafellar's theorem [18, Th. 18] to arbitrary Banach spaces automatically implies the validity of the above results in arbitrary Banach spaces following Rockafellar's method. For extensions and variants of Fenchel-type duality results and of the Robinson-Rockafellar condition, see, e.g., Borwein [3], Attouch-Brezis [2], Gowda-Teboulle [9].

Remark 4.8. The first proof of Theorem 4.6 outside the framework of convex duality theory is due to Thibault [19]. Our approach is sligthly different, since it is based on fuzzy sum rules and the Robinson-Rockafellar theorem via Proposition 3.2 while that of Thibault [19] (and also Penot [16]) is based on fuzzy sum formulas and the Krein-Šmulian theorem.

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