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On the Topology of Generalized Semi-Infinite Optimization

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This *survey article* reflects the topological and inverse *behaviour* of *generalized semi-infinite optimization* problems $\mathcal{P}(f,h,g,u,v)$, and presents the *analytical methods*. These differentiable problems admit an infinite set $Y(x)$ of inequality constraints y which depends on the state x. We extend investigations from *Weber* [77] based on research of *Guddat, Jongen, Rückmann, Twilt* and others. Under suitable assumptions on boundedness and qualifying conditions on lower y-stage and upper x-stage, we present *manifold, continuity* and global *stability* properties of the feasible set $M[h,g,u,v]$ and corresponding *structural stability* properties of $\mathcal{P}(f,h,g,u,v)$, referring to slight data perturbations. Hereby, the character of our investigation is essentially specialized by the linear independence constraint qualification locally imposed on $Y(x)$. The achieved results are important for algorithm design and convergence. Two extensions refer to unboundedness and nondifferentiable max-min-type objective functions. In the course of explanation, the perturbational approach gives rise to study *inverse problems* of *reconstruction.* We trace them into *optimal control* of ordinary differential equations, and indicate related investigations in heating processes, continuum mechanics and discrete tomography. Throughout the article, we realize discrete-combinatorial aspects and methods.

Keywords: Generalized semi-infinite optimization, constraint qualification, structural stability, inverse problem, reconstruction, nondifferentiability, optimal control, directed graph, discrete tomography

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1. Introduction

Generalized semi-infinite ($\mathcal{GSI}$) problems have the form

$$\mathcal{P}(f,h,g,u,v)\begin{cases} \text{Minimize} \quad f(x) \ \text{ on } \ M[h,g], \quad \text{where} \\ M[h,g] \ := \ \{\, x \in I\!R^n \mid \ h_i(x)=0 \ \ (i \in I), \\ \qquad\qquad\qquad\qquad\qquad g(x,y) \geq 0 \ \ (y \in Y(x))\,\}. \end{cases}$$

The semi-*infinite* character comes from the perhaps infinite number of elements of $Y(= Y(x))$ ([16], [54]), while the *generalized* character comes from the x-dependence of $Y(\cdot)$. We suppose these index sets to be finitely constrained ($\mathcal{F}$):

$$Y(x) \ = \ M_{\mathcal{F}}[u(x,\cdot),v(x,\cdot)] := \{\, y \in I\!R^q \mid u_k(x,y)=0 \ (k \in K), \\ v_\ell(x,y) \geq 0 \ (\ell \in L)\} \ \ (x \in I\!R^n).$$

Under suitable assumptions, the following fields of problems from science, engineering and control lead to generalized semi-infinite ($\mathcal{GSI}$) optimization: ∘ optimizing the layout of a special assembly line, ∘ maneuverability of a robot, ∘ time minimal heating of a

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ball of an homogeneous material, ∘ approximation of a thermo-couple characteristic in chemical engineering, ∘ robust optimization, ∘ structure and stability in optimal control of ordinary differential equations. For motivation and references see, e.g., [77], [80]. In future, $\mathcal{GSI}$ applications may also be expected in optimal experimental design ([12]).

Notation: $h = (h_i)_{i\in I}$, $u = (u_k)_{k\in K}$, $v = (v_\ell)_{\ell\in L}$, where $h_i : I\!R^n \to I\!R$, $i \in I := \{1,\dots,m\}$, $u_k : I\!R^n \times I\!R^q \to I\!R$, $k \in K := \{1,\dots,r\}$, $v_\ell : I\!R^n \times I\!R^q \to I\!R$, $\ell \in L := \{1,\dots,s\}$ $(m < n; r < q)$. Let $f : I\!R^n \to I\!R$, $g : I\!R^n \times I\!R^q \to I\!R$, $h_i\,(i \in I)$, $u_k\,(k \in K)$, $v_\ell\,(\ell \in L)$ be continuously differentiable (C^1). By $Df(x)$, $D^T f(x)$ we denote the row-(column) vector of the first-order partial derivatives $\frac{\partial}{\partial x_\kappa} f(x)$, and $D_x g(x,y)$, $D_y g(x,y)$ consist of $\frac{\partial}{\partial x_\kappa} g(x,y)$ and $\frac{\partial}{\partial y_\sigma} g(x,y)$. Let $\mathcal{U} \subset I\!R^n$, $M[h,g] \cap \mathcal{U} \neq \emptyset$, be some *bounded, open* set.

Assumption $\mathbf{A}_{\mathcal{U}}$: $\cup_{x\in\overline{\mathcal{U}}} Y(x)$ is *bounded* (hence, by continuity, compact).

In *generalized* semi-infinite optimization, the feasible set $M[h,g]$ need not be closed ([33]). The following assumption, however, ensures closedness:

Assumption $\mathbf{B}_{\mathcal{U}}$: For all $x \in \overline{\mathcal{U}}$, the **linear independence constraint qualification (LICQ)** is fulfilled for $M_{\mathcal{F}}[u(x,\cdot), v(x,\cdot)]$, i.e., linear independence of

$$D_y u_k(\overline{x},\overline{y}),\ k \in K, \quad D_y v_\ell(\overline{x},\overline{y}),\ \ell \in L_0(\overline{x},\overline{y})$$

(considered as a family), where $L_0(\overline{x},\overline{y}) := \{\, \ell \in L \mid v_\ell(\overline{x},\overline{y}) = 0 \,\}$ consists of active indices. We shall realize strong Assumption $\mathrm{B}_{\mathcal{U}}$ to be a central condition of this article, but also a structural frontier overcome by recent research.

Under both assumptions we start our continuity and stability research. Using differential topology ([25], [29]), they admit local linearization of $Y(x)$ $(x \in \overline{\mathcal{U}})$ by finitely many C^1-diffeomorphisms $\phi_x^j : \mathcal{V}^j \to S^j$ $(j \in J)$ in such a way that the image sets Z^j of indices are *x-independent* squares (in a linear subspace). Herewith, $\mathcal{P}(f,h,g,u,v)$ becomes locally (in $\overline{\mathcal{U}}$) equivalently expressed as an **ordinary** *semi-infinite* optimization problem $\mathcal{P}_{\mathcal{OSI}}(f,h,g^0,u^0,v^0)$, where $M_{\mathcal{OSI}}[h,g^0] \cap \overline{\mathcal{U}} = M[h,g] \cap \overline{\mathcal{U}}$, f being unaffected ([75], [77]).

On the upper stage of variable x, we shall use a constraint qualification, too. This *cq* geometrically means the existence of an (at $M[h] = h^{-1}(\{0\})$) tangential, "inwardly" pointing direction at x:

Definition. We say that the **extended Mangasarian-Fromovitz constraint qualification (EMFCQ)** is fulfilled at a given $\overline{x} \in M[h,g]$, if the conditions $EMF_{1,2}$ are satisfied:

EMF_1. $Dh_i(\overline{x})$, $i \in I$, are linearly independent.
EMF_2. There exists an "*EMF-vector*" $\zeta \in I\!R^n$ such that

$$\begin{aligned} Dh_i(\overline{x})\,\zeta &= 0\,, && \text{for all } i \in I,\\ D_x g_j^0(\overline{x},z)\,\zeta &> 0\,, && \text{for all } z \in I\!R^q,\ j \in J, \text{ with } (\phi_{\overline{x}}^j)^{-1}(z) \in Y_0(\overline{x}), \end{aligned}$$

where $Y_0(\overline{x}) := \{\, y \in Y(\overline{x}) \mid g(\overline{x},y) = 0 \,\}$ consists of *active* indices. **EMFCQ** is said to be fulfilled for $M[h,g]$ on $\overline{\mathcal{U}}$, if EMFCQ is fulfilled for all $x \in M[h,g] \cap \overline{\mathcal{U}}$.

For further information and versions of EMFCQ see [23], [29], [33], [35], [49], [65], but also [11] ([27]).

Let a local minimizer $\hat{x}$ of $\mathcal{P}(f,h,g,u,v)$ be given and EMFCQ be fulfilled at $\hat{x}$. Then, we can state the existence of *Lagrange multipliers* λ_i, μ_κ such that the conditions

$$Df(\hat{x}) = \sum_{i\in I} \lambda_i Dh_i(\hat{x}) + \sum_{\kappa\in\{1,\ldots,\hat{\kappa}\}} \mu_\kappa D_x g^0_{j^\kappa}(\hat{x}, z^\kappa),$$
$$\mu_\kappa \geq 0 \ \ (\kappa \in \{1,\ldots,\hat{\kappa}\})$$

are satisfied, referring to *ordinary* semi-infinite ($\mathcal{OSI}$) data ([23], [75], [77]). Now, we call $\hat{x}$ a *$\mathcal{G}$-$\mathcal{O}$ Kuhn-Tucker point*. Here, the points $z^\kappa \in Z^{j^\kappa}$ are suitable active indices. Below, $Z^j_0(x)$ stands for the set of $z \in Z^j$ being *active* for $g^0_j(x,\cdot)$. Referring to *all* the given $\mathcal{GSI}$ data, a further evaluation yields the following **Kuhn-Tucker conditions** with corresponding *Lagrange multipliers* λ_i, μ_κ, $\alpha_{\kappa,k}$, $\beta_{\kappa,\ell}$ ([75], [77]):

$$\mathrm{KT}_1.\ Df(\hat{x}) = \sum_{i\in I} \lambda_i Dh_i(\hat{x}) + \sum_{\kappa\in\{1,\ldots,\hat{\kappa}\}} \mu_\kappa D_x g(\hat{x}, y^\kappa) - \\ - \sum_{k\in K} \alpha_{\kappa,k} D_x u_k(\hat{x}, y^\kappa) - \sum_{\substack{\ell\in L_0(\hat{x},y^\kappa)\\ \kappa\in\{1,\ldots,\hat{\kappa}\}}} \beta_{\kappa,\ell} D_x v_\ell(\hat{x}, y^\kappa),$$

$$\mathrm{KT}_2.\ \mu_\kappa,\ \beta_{\kappa,\ell} \geq 0 \ \ (\ell \in L_0(\hat{x}, y^\kappa),\ \kappa \in \{1,\ldots,\hat{\kappa}\}).$$

Again, the $y^\kappa \in Y_0(\hat{x})$ are active. Now, we call $\hat{x}$ a *$\mathcal{G}$ Kuhn-Tucker point*. Under general assumptions, the **necessary optimality condition** $\mathrm{KT}_{1,2}$ was for the first time proved by Jongen, Rückmann and Stein ([33]). Note, that the linear combination KT_1 contains the derivatives of *all* the defining functions. The foregoing conditions can also be stated as growth (angular) conditions over tangent cones (see [42], [75], [77]), estimating scalar products against 0. They give rise to deduce first-order **sufficient** *optimality conditions* (for further information see [65]). In fact, let LICQ be satisfied at a given point $\hat{x}$ as an element of $M[h]$, and $M[h] \cap \overline{\mathcal{U}}$ be *star-shaped* with *star point* $\hat{x}$. Moreover, $g^0_j(\cdot, z)$ ($z \in Z^j$, $j \in J$) be *quasi-concave* and f be *pseudo-convex* on $M[h] \cap \overline{\mathcal{U}}$. This means the following implications for all $x \in M[h] \cap \overline{\mathcal{U}}$ ([24], [42]):

$$g^0_j(x,z) \geq g^0_j(\hat{x},z) \quad \Longrightarrow \quad D_x g^0_j(\hat{x},z)\,(x-\hat{x}) \geq 0,$$
$$Df(\hat{x})\,(x-\hat{x}) \geq 0 \quad \Longrightarrow \quad f(x) \geq f(\hat{x}).$$

Then, $\hat{x}$ turns out to be a local minimizer of $\mathcal{P}(f,h,g,u,v)$ ([75], [77]; cf. [38]). Concerning structural frontiers in ($\mathcal{F}$) *nonconvex* optimization see [37]. After this introduction of basic conditions, we make the following convention for the ease of presentation. In fact, as the theoretical treatment of the equality constraint functions is merely technical (cf. [18], [56], [71], [77]), we may delete them (cf. also Figure 2 for indication of perturbed sets $(n-1)$-dimensional manifolds $M[h]$):

Convention. Until the end of Subsection 4.1 we assume: $I = \emptyset$, $K = \emptyset$.

Corresponding results for the cases $I \neq \emptyset$ or $K \neq \emptyset$ are the ones found straightforwardly (see [77]). Before we introduce the second-order condition of *strong stability* we state (under Assumptions $\mathrm{A}_\mathcal{U}$, $\mathrm{B}_\mathcal{U}$):

Lemma ([77])**.** *Let $\hat{x} \in M[g] \cap \overline{\mathcal{U}}$ be given, and EMFCQ be fulfilled at $\hat{x}$. Then, $\hat{x}$ is a $\mathcal{G}$-$\mathcal{O}$ Kuhn-Tucker point for $\mathcal{P}(f,g,v)$, if and only if the extended Mangasarian-Fromovitz constraint qualification on $M[(g, -f + f(\hat{x}))]$, called $\widehat{EMFCQ}$, is violated at $\hat{x}$.*

Proof. This result comes from Farkas' Lemma on infinite systems ([23]; [71], [77]). □

We prepare our introduction of strong stability of a stationary point by assuming that f, g, v are C^2 and putting for any bounded open neighbourhood $\mathcal{V} \subseteq I\!R^q$ of $\bigcup_{x \in \overline{\mathcal{U}}} Y(x)$ and any subset $\mathcal{M} \subseteq I\!R^n$:

$$\begin{aligned}
&\mathrm{norm}[(f,g,v),\mathcal{M}] \;:= \\
&\qquad \sup\Bigg\{ \sup_{x\in\mathcal{M}} \Bigg\{ |f(x)| + \sum_{i=1}^{n} |\frac{\partial f}{\partial x_i}(x)| + \sum_{\substack{i=1\\ j=1}}^{n} |\frac{\partial^2 f}{\partial x_i \partial x_j}(x)| \Bigg\}, \\
&\sup_{\substack{x\in\mathcal{M}\\ y\in\overline{\mathcal{V}}}} \; \max_{\substack{\eta\in\{g\}\cup\\ \{v_\nu|\nu\in L\}}} \Bigg\{ |\eta(x)| + \sum_{i=1}^{n} |\frac{\partial \eta}{\partial x_i}(x,y)| + \sum_{j=1}^{q} |\frac{\partial \eta}{\partial y_j}(x,y)| + \\
&\qquad + \sum_{\substack{i=1\\ j=1}}^{n} |\frac{\partial^2 \eta}{\partial x_i \partial x_j}(x)| + \sum_{i=1}^{n}\sum_{j=1}^{q} |\frac{\partial^2 \eta}{\partial x_i \partial y_j}(x)| + \sum_{\substack{i=1\\ j=1}}^{q} |\frac{\partial^2 \eta}{\partial y_i \partial y_j}(x)| \Bigg\}\Bigg\}.
\end{aligned}$$

In $\mathcal{F}$ or $\mathcal{OSI}$ optimization we replace $\overline{\mathcal{V}}$ by J, Y or disregard v, using notation $\mathrm{norm}_{\mathcal{F}}[\cdot,\cdot]$, $\mathrm{norm}_{\mathcal{OSI}}[\cdot,\cdot]$ then. By continuity stated in Section 2, the next condition is well-defined ([77]).

Definition. Suppose a point $\hat{x}^u \in M[g] \cap \mathcal{U}$ for $\mathcal{P}(f,g,v)$ (of class C^2), $\mathcal{P}_{\mathcal{OSI}}(f,g^0,v^0)$ be locally (in $\overline{\mathcal{U}}$) representing $\mathcal{P}(f,g,v)$, and $\hat{x}^u$ be a $\mathcal{G}$-$\mathcal{O}$ Kuhn-Tucker point of $\mathcal{P}(f,g,v)$. Then, we say that $\hat{x}^u$ is ($\mathcal{G}$-$\mathcal{O}$) **strongly stable**, if for some $\overline{\epsilon} > 0$ with $B(\hat{x}^u,\overline{\epsilon}) \subseteq \mathcal{U}$ and for each $\epsilon \in (0,\overline{\epsilon}]$ there is some $\delta > 0$ such that for each C^2-function $(\tilde{f},\widetilde{g^0})$ with $\mathrm{norm}_{\mathcal{OSI}}[(f-\tilde{f}, g^0-\widetilde{g^0}), B(\hat{x}^u,\epsilon)] \leq \delta$ the open ball $B(\hat{x}^u,\epsilon)$ contains an *ordinary Kuhn-Tucker point* $\hat{x}^d$ of $\mathcal{P}^*_{\mathcal{OSI}}(\tilde{f},\widetilde{g^0}) := \mathcal{P}_{\mathcal{OSI}}(\tilde{f},\widetilde{g^0},v^0)$, which is unique in $B(\hat{x}^u,\overline{\epsilon})$

Referring to a $\mathcal{G}$ Kuhn-Tucker point $\hat{x}^u$ and to $\mathrm{norm}[(f-\tilde{f}, g-\tilde{g}, v-\tilde{v}), B(\hat{x}^u,\epsilon)]$, we get the condition of ($\mathcal{G}$) **strong stability**.

Here, "u" (and "d") stands for (*un*)*disturbed*. For our preferred ($\mathcal{G}$-$\mathcal{O}$) strong stability expressed by original $\mathcal{GSI}$ data, see [77]. In Section 3, we utilize an algebraical characterization of strong stability in the way of Kojima ([40]) and Rückmann ([57]).

In the proofs of the following results, we may focus on the main underlying $\mathcal{OSI}$ ideas, and on the essential $\mathcal{GSI}$ items arising additionally to methods and results from $\mathcal{OSI}$ optimization. Note, that in virtue of our Assumptions $\mathrm{A}_{\mathcal{U}}, \mathrm{B}_{\mathcal{U}}$ the $\mathcal{OSI}$ theory applies, whereby the finitely many functions v^0_ℓ are affinely linear now. By those additional items we introduce *inverse problems* being a central line of interpretation throughout in our article.

2. Stability of the Feasible Set

The following theorems underline the importance of *EMFCQ* for concluding that $M[g,v]$ $:= M[g]$ is a topological manifold with boundary, that it behaves continuous and stable under perturbations of our defining C^1-functions. With these perturbations we remain inside of suitable open neighbourhoods of (g,v) in the sense of **strong** or **Whitney topology** C^1_S which respects asymptotic effects (for topologies C^k_S, $k \in I\!N \cup \{\infty\}$, cf. [25], [29]). We call a given $M \subseteq I\!R^n$ a **Lipschitzian manifold** (with boundary) of dimension κ, if for each $\overline{x} \in M$ there are open neighbourhoods $\mathcal{W}^1 \subseteq \mathbb{R}^n$ of $\overline{x}$, $\mathcal{W}^2 \subseteq \mathbb{R}^n$ of 0_n, and a bijective $\varphi : \mathcal{W}^1 \to \mathcal{W}^2$, $\varphi(\overline{x}) = 0_n$, with Lipschitzian continuity of φ, φ^{-1} such that φ carries $M \cap \mathcal{W}^1$ to the relatively open set $(\{0_{n-\kappa}\} \times \mathbb{R}^{\kappa}) \cap \mathcal{W}^2$ or to the set $(\{0_{n-\kappa}\} \times \{w \in \mathbb{R} \,|\, w \geq 0\} \times \mathbb{R}^{\kappa-1}) \cap \mathcal{W}^2$ with (relative) boundary. So, Lipschitzian manifolds can locally be linearized, however, without preserving "angulars" in the boundary. According to our Convention (Section 1), we shall focus on the case $\kappa = n$. In $\mathcal{F}$ optimization, that preservation is given by the stronger condition LICQ, using C^1-smooth linearizing "charts". Herewith, we find qualified versions of the following topological results for $Y(x)$, too ([28], [77]).

Manifold Theorem. ([77]) *Let EMFCQ be fulfilled in $\overline{\mathcal{U}}$ for $M[g]$. Then, there is an open neighbourhood $\mathcal{W} \subseteq \mathbb{R}^n$ of $\overline{\mathcal{U}}$ such that $M[g] \cap \mathcal{W}$ is a Lipschitzian manifold (with boundary) of dimension n (in general, $n - m$). Moreover, then we can represent the (relative) boundary:*

$$(\partial M[g]) \cap \mathcal{W} \;=\; \{\, x \in \mathcal{W} \,|\; \min_{y \in Y(x)} g(x, y) = 0 \,\}.$$

Proof. Assumption $\mathrm{B}_{\mathcal{U}}$ supplies diffeomorphisms ϕ_x^j for all x of some open neighbourhood $\mathcal{W}$ of $\overline{\mathcal{U}}$. These ϕ_x^j guarantee that the insight from [35] on $\mathcal{OSI}$ optimization can be carried over for our $\mathcal{GSI}$ problem. □

For properties **upper** and **lower semi-continuity**, **continuity** (in Hausdorff metric), **genericity** (implying density) and **transversality** (absense of tangentiality), considered for functions or sets next, we refer to [4], [25], [29], [35], [77].

Continuity Theorem. ([76], [77]) *Let EMFCQ be fulfilled in $\overline{\mathcal{U}}$ for $M[g]$. Moreover, let the closure $\overline{\mathcal{W}} \subseteq \mathbb{R}^n$ of some open set $\mathcal{W} \subseteq \mathcal{U}$ be representable as a feasible set from $\mathcal{F}$ optimization which fulfills LICQ, and let its boundary $\partial \mathcal{W}$ transversally intersect $M[g]$. Then, there is an open C^1_S-neighbourhood $\mathcal{O} \subseteq (C^1(\mathbb{R}^{n+q}, \mathbb{R}))^{s+1}$ of (g, v) such that $\mathsf{M}^{\mathcal{W}} :\; (\tilde{g}, \tilde{v}) \mapsto M[\tilde{g}, \tilde{v}] \cap \overline{\mathcal{W}}$, is upper and lower semi-continuous at all $(\tilde{g}, \tilde{v}) \in \mathcal{O}$.*
If, moreover, $\mathcal{W}$ is bounded, then $\mathcal{O}$ can be chosen so that $\mathcal{O}$ is mapped to $\mathcal{P}_c(\mathbb{R}^n)$ by $\mathsf{M}^{\mathcal{W}}$, and $\mathsf{M}^{\mathcal{W}}$ is continuous.

Proof. These assertions are consequences of the *continuous* dependence of the $\mathcal{OSI}$ functional data g^0, v^0 on the $\mathcal{GSI}$ data g, v and, then, of [35], Theorem 2.2. We apply this theorem on $M_{\mathcal{OSI}}[g^0, v^0] := M_{\mathcal{OSI}}[g^0]$. In the proof of *Genericity Theorem* below, we investigate this continuous dependence $\Psi_R :\; (\tilde{g}, \tilde{v}) \mapsto (\tilde{g}^0, \tilde{v}^0)$. □

In [77], also a global version and a version on $(\tilde{x}, \tilde{v}) \mapsto Y^{\tilde{v}}(\tilde{x})$ are presented for the previous result. The following theorem refers to the straightforward generalization **ELICQ** of LICQ which is a stronger condition than EMFCQ ([35], [71], [77]). (The double usage of $\mathcal{F}$ should not cause confusion. For a global result see [77].) We emphasize that, here, genericity is obtained in a restricted (relative) sense where, in particular, Assumption $B_{\mathcal{U}}$ is supposed. However, the important structure of the feasible set *without* this strong assumption on the lower stage where, e.g., MFCQ or no constraint qualification is satisfied, leads to the basically new phenomena (re-entrant corner points and local non-closedness, respectively). This generic structure is discussed in detail by O. Stein; see the deep research [64] written under the aspect of marginal functions and evaluating a codimension formula.

Genericity Theorem. ([77])

(a) *Let* $\mathsf{C}^\infty := (C^\infty(I\!R^n \times I\!R^q, I\!R))^{s+1}$ *be endowed with the* C^∞_S*-topology. Furthermore, let its subspace* $\mathsf{C}^\infty_{\mathsf{loc}}$ *of all* $(g,v) \in \mathsf{C}^\infty$ *with validity of Assumptions* $A_{\mathcal{U}}, B_{\mathcal{U}}$ *be endowed with the* C^∞_S*-relative topology.*
Then, there exists a generic subset $\mathcal{E} \subseteq \mathsf{C}^\infty_{\mathsf{loc}}$ *such that ELICQ is satisfied for each* $(g,v) \in \mathcal{E}$.

(b) *Let* $\mathsf{C}^1 := (C^1(I\!R^n \times I\!R^q, I\!R))^{s+1}$ *be endowed with the* C^1_S*-topology. Furthermore, let its subspace* $\mathsf{C}^1_{\mathsf{loc}}$ *of all* $(g,v) \in \mathsf{C}^1$ *with validity of* $A_{\mathcal{U}}, B_{\mathcal{U}}$ *be endowed with the* C^1_S*-relative topology.*
Then, there exists an open and dense subset $\mathcal{F} \subseteq \mathsf{C}^1_{\mathsf{loc}}$ *such that EMFCQ is satisfied for each* $(g,v) \in \mathcal{F}$. *The set* $\mathcal{F}$ *can just be defined by the fulfillment of EMFCQ.*

Outline of proof. The first insight on the desired <u>*subset* $\mathcal{E}$</u> of C^∞-functions follows from the $\mathcal{OSI}$ result [35], Theorem 2.4, that applies Multi-Jet Transversality Theorem ([25], [29]) and additional reflections. For that theorem our v^0 is kept *fixed*, focussing topological interest on g^0; here, the role of some constant set $\mathcal{Y}$ is played by the union of the sets Z^j $(j \in J)$. Without loss of generality, J consists of a singleton. Now, we can state that there is a generic set $\mathcal{E}^{\mathcal{O}}$ of $\mathcal{OSI}$ data functions g^0, which (by definition of genericity) is the intersection of countably many open and dense subsets $\mathcal{E}^{\mathcal{O},\nu}$ $(\nu \in I\!N)$.

However, for the back-tracing of the $\mathcal{OSI}$ genericity (or, below, openess and density) to $\mathcal{GSI}$ optimization, we utilize that the problem representation is *continuous.* In fact, by *Implicit Function Theorem in Banach Spaces* ([29], [47]), the inserted local coordinate transformations continuously depend on $(\tilde{g}, \tilde{v})$. Let us regard this continuous dependence (representation) as a function Ψ_R locally mapping $(\tilde{g}, \tilde{v}) \in \mathsf{C}^\infty$ into the space of all C^∞-functions $(\tilde{g}^0, \tilde{v}^0)$. Using Ψ_R we find $\mathcal{E}$ as the intersection of the countably many <u>open</u> sets $\mathcal{E}^\nu := \Psi_R^{-1}(\mathcal{E}^{\mathcal{O},\nu})$ $(\nu \in I\!N)$.
Next, let $(g,v) \in \mathsf{C}^\infty_{\mathsf{loc}}$ be given. After sufficiently slight perturbations this function still remains in $\mathsf{C}^\infty_{\mathsf{loc}}$. Let also some $\nu \in I\!N$ be given. In the $\mathcal{OSI}$ problem, however, we consider *separate* (*de-coupled*) perturbations $g^0_j \to \widetilde{g^0_j}$ $(j \in J)$ (before we really turn to one single inequality j). Therefore, the "problem representation" Ψ_R is *not surjective.* Actually, as for some $x \in \overline{\mathcal{U}}$ and two (or more) different $j^1, j^2 \in J$ the sets $(\phi_x^{j^1})^{-1}(Z_0^{j^2}(x))$, $(\phi_x^{j^2})^{-1}(Z_0^{j^2}(x))$ might have a nonempty intersection, these perturbations cannot always be traced back to a perturbation $g \to \tilde{g}$ of the given $\mathcal{GSI}$ problem. The following perturbational technique, however, will help to get rid with such a difficulty, and it finally guides us to the asserted density.

By definition of ϕ_x^j $(j \in J)$ (linearization) the implicitly disturbed sets $\widetilde{Z^j}$ can be chosen as Z^j. Moreover, because of the locally finite covering structure underlying Ψ_R, no difficulty arises. In view of that locally "fixed" v^0, we delete v^0 in the definition of Ψ_R. So, we get a mapping called Ψ_R^*. First of all, we add to g one j-independent, arbitrarily C^∞_S-small positive function g in an arbitrarily small neighbourhood of the compact set $\bigcup_{x \in M[g] \cap \overline{\mathcal{U}}} (\phi_x^{j^1})^{-1}(Z_0^{j^1}(x)) \cap (\phi_x^{j^2})^{-1}(Z_0^{j^2}(x))$, making active indices y inactive there. Then, $g^* := g + \mathsf{g}$ is a globally defined C^∞-function. Now, for each $\nu \in I\!N$ we find a (componentwise) arbitrarily C^∞_S-close approximation $(\widetilde{g^{\nu 0}}, \widetilde{v^{\nu 0}}) \in \mathcal{E}^{\mathcal{O},\nu}$ of (g^0, v^0), where the approximation $\widetilde{g^{\nu 0}}$ coincides with $g^{*0} := \Psi_R^*(g^*, v)$ in $\cup_{j \in J} Z^j$. Here, we may

choose the C^1-function $\widetilde{v^{\nu 0}} := v^0$. Hence, that perturbed function $\widetilde{g^{\nu 0}}$ is continuously back-tracable under ${\Psi_R^*}^{-1}$ to one C^∞-function $\tilde{g}^\nu$, i.e., $\{(\tilde{g}^\nu, v)\} = {\Psi_R^*}^{-1}(\{\widetilde{g^{\nu 0}}\})$. So we are in a position to state, that (g, v) can arbitrarily well be C_S^∞-approximated by $(\tilde{g}, \tilde{v}) := (\tilde{g}^\nu, v) \in \mathcal{E}^\nu$. This means that $\mathcal{E}^\nu$ is *dense*, too. Altogether, we have shown that $\mathcal{E}$ is **generic**.
Preparation: This (relative) genericity implies (relative) density ([29]), because of the "C_S^∞*-openess*" of both LICQ and (y-) boundedness. Now, we use the implication of EMFCQ by ELICQ, and the C_S^1-density of $C^\infty(I\!R^k, I\!R)$ in $C^1(I\!R^k, I\!R)\,(k \in I\!N)$. Moreover, we take account of our preparation and of the perturbational "C_S^1-openess" of EMFCQ. □

We underline "$\mathcal{F}$" or "$\mathcal{GSI}$ *open*" *properties*: LICQ and EMFCQ remain preserved under sufficiently slight data perturbation.

Next, we refer to the *same* underlying dimensions n, q in x- or y-space, and the number s of functions v_ℓ. Two feasible sets $M[g^1, v^1]$, $M[g^2, v^2]$ are called **(topologically) equivalent**, notation: $M[g^1, v^1] \sim_M M[g^2, v^2]$, if there is a homeomorphism $\varphi_M : I\!R^n \to I\!R^n$ such that

$$\varphi_M(M[g^1, v^1]) = M[g^2, v^2].$$

The given feasible set $M[g]\,(= M[g, v])$ is called **(topologically) stable**, if there is an open C_S^1-neighbourhood $\mathcal{O}$ of (g, v) such that for each $(\tilde{g}, \tilde{v}) \in \mathcal{O}$ we have $M[g, v] \sim_M M[\tilde{g}, \tilde{v}]$ (see [18], [35], [71], [77]). Let us make the *boundedness* (hence, compactness) assumption that $M[g]$ lies in $\mathcal{U}^0$.

Stability Theorem. ([76], [77]) *The feasible set $M[g] \subset \mathcal{U}$ is topologically stable, if and only if EMFCQ is fulfilled for $M[g]$.*

Proof. We trace back to the $\mathcal{OSI}$ situation again, given by [35], Theorem 2.3, now. As being the case in the proof of *Genericity Theorem*, technical items arise (furthermore, if $I \neq \emptyset$). These difficulties can be overcome: In Section 3 we prove *Characterization Theorem* on the lower level sets of the whole $\mathcal{GSI}$ optimization problem; that theorem implies our *Stability Theorem.* We note that under our overall boundedness assumptions, $M[g]$ is a lower level set of $\mathcal{P}(f, g, v)$ for a sufficiently high f-level. Already to point out the essential ideas for the **sufficiency part**, "$\Longleftarrow$", proved in a *constructive* way, and for the **necessity part**, "$\Longrightarrow$", proved in an *indirect* way, we look at Figures 2.1, 2.2, respectively. For both parts differential topology and Morse theory are helpful. While for the *necessity part* some algebraic topology ([28], [63]) is essential to evaluate unstable situations by a finite counting argument, for the *sufficiency part* flows ([2]) are important. In the latter part, we homeomorphically map the feasible set $M[g]$ onto the feasible set $M[\tilde{g}]$ by steering the boundary $\partial M[g]$ onto $\partial M[\tilde{g}]$ along an EMF-vector field. Herewith, we have constructed a suitable transformation φ_M. □

Remark. The previous result exploits transversality by applying *Implicit Function Theorem.* While in the necessity part this *inverse* aspect consisted of *suitable* perturbations, in the sufficiency part we locally had *"full"* perturbations. The same will be observed in Section 3, where the main result again identifies *topological* and analytical conditions. The analytical ones have an algebraical or *discrete-combinatorial* nature.

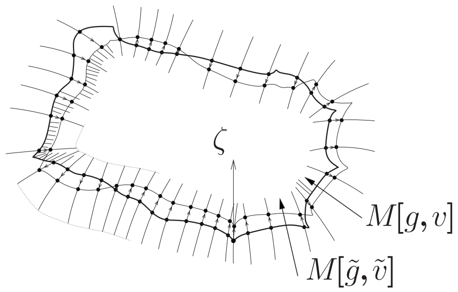


Figure 2.1: Proof of sufficiency part, Stability Theorem

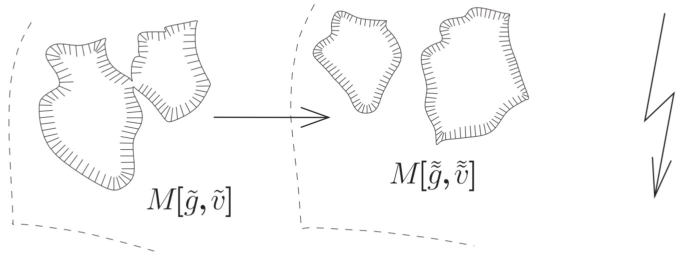


Figure 2.2: Proof of necessity part, Stability Theorem

3. Structural Stability and its Characterization

3.1. Structural Stability of the Problem

Under Assumptions $\mathrm{A}_{\mathcal{U}}$, $\mathrm{B}_{\mathcal{U}}$, we still refer to the bounded set $M[g]$, but additionally take f into consideration. The *structure* of the entire problem $\mathcal{P}(f,g,v)$ is established by all the lower level sets

$$L^\tau(f,g,v) \ := \ \{\, x \in I\!R^n \mid x \in M[g,v], \ \ f(x) \le \tau \,\} \quad (\tau \in I\!R).$$

We observe this structure under data perturbation and define *structural stability*. Here, *descent* has to be preserved, if the level varies. Let us still assume that the defining functions are C^2. Then, this global stability can essentially be *characterized* by EMFCQ of $M[g]$ and by strong stability of all considered stationary points.

Two problems $\mathcal{P}(f^1,g^1,v^1)$, $\mathcal{P}(f^2,g^2,v^2)$ (with defining C^2-functions) are called **structurally equivalent**:

$$\mathcal{P}(f^1,g^1,v^1) \ \sim_{\mathcal{P}} \ \mathcal{P}(f^2,g^2,v^2),$$

if there are continuous functions $\varphi_{\mathcal{P}} : I\!R \times I\!R^n \to I\!R^n$ and $\psi : I\!R \to I\!R$ with the three properties $\mathcal{E}_{1,2,3}$ (see Figure 3.1):

E$_1$. $\varphi_{\mathcal{P},\tau} : I\!R^n \to I\!R^n$ is a homeomorphism, where $\varphi_{\mathcal{P},\tau}(x) := \varphi_{\mathcal{P}}(\tau,x)$, for every $\tau \in I\!R$.
E$_2$. $\psi : I\!R \to I\!R$ is a monotonically increasing homeomorphism.
E$_3$. $\varphi_{\mathcal{P},\tau}(L^\tau(f^1,g^1,v^1)) \ = \ L^{\psi(\tau)}(f^2,g^2,v^2)$ for all $\tau \in I\!R$.

Considering the first problem as *undisturbed* and the second one as slightly *disturbed*, we arrive at *structural stability* ([17], [32], [36], [71], [77]; cf. also [2], [8], [29], [61]):

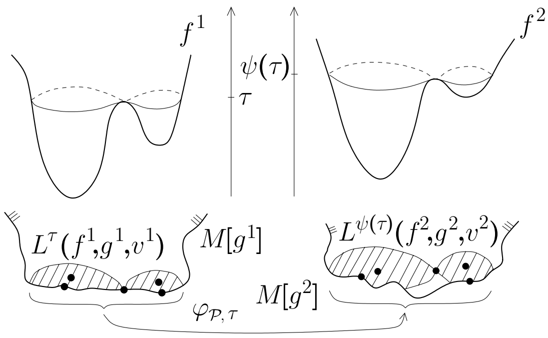


Figure 3.1: Structural equivalence (under bird's-eye view below)

$\mathcal{P}(f,g,v)$ (with defining C^2-functions) is called **structurally stable**, if there exists a C^2_S-neighbourhood $\mathcal{O}$ of (f,g,v) such that for each $(\tilde{f},\tilde{g},\tilde{v}) \in \mathcal{O}$

$$\mathcal{P}(f,g,v) \sim_{\mathcal{P}} \mathcal{P}(\tilde{f},\tilde{g},\tilde{v}).$$

3.2. Characterization Theorem

Under Assumptions $A_{\mathcal{U}}$, $B_{\mathcal{U}}$ we state:

Characterization Theorem (or Structural Stability Theorem; [77]). *Let $M[g] \subset \mathcal{U}$ hold for problem $\mathcal{P}(f,g,v)$ (with defining C^2-functions).*
Then, $\mathcal{P}(f,g,v)$ is structurally stable, if and only if the three conditions $\mathcal{C}_{1,2,3}$ are fulfilled:

C$_1$. *EMFCQ holds for $M[g]$.*
C$_2$. *All the $\mathcal{G}$-$\mathcal{O}$ Kuhn-Tucker points $\overline{x}$ of $\mathcal{P}(f,g,v)$ are ($\mathcal{G}$-$\mathcal{O}$) strongly stable.*
C$_3$. *For each two different $\mathcal{G}$-$\mathcal{O}$ Kuhn-Tucker points $\overline{x}^1 \neq \overline{x}^2$ of $\mathcal{P}(f,g,v)$ the corresponding critical values are different* (separate), *too: $f(\overline{x}^1) \neq f(\overline{x}^2)$.*

In this main result, we could also make a further assumption, excluding certain inequality constraints z from the relative boundary ∂Z^j $(j \in J)$. Then we could identify the $\mathcal{G}$-$\mathcal{O}$ Kuhn-Tucker points by some $\mathcal{G}$ Kuhn-Tucker points. However, for validity of Characterization Theorem, this is *not* necessary ([77]).

3.3. Proof of Characterization Theorem

Preparations.

For preparation, let us recall the proof of *Genericity Theorem*, taking into account the parametrical dependences on the defining data $(\tilde{g},\tilde{v})$. Now, we make again applications of *Implicit Function Theorem in Banach spaces*, such that, in particular, we state a *continuous dependence* of $(\tilde{g}^0,\tilde{v}^0)$ on $(\tilde{g},\tilde{v})$. Consequently, small perturbations on the data of $\mathcal{P}(f,g,v)$ cause slight perturbations on the data of $\mathcal{P}_{\mathcal{OSI}}(f,g^0,v^0)$. The *inverse problem* arises: *Can small perturbations of the $\mathcal{OSI}$ data be reconstructed under the problem representation from slight perturbations of the given $\mathcal{GSI}$ problem ?* We give a conditionally positive answer. However, this answer will be fitting for the *perturbational* argumentations on Characterization Theorem:

Item 1. For representing $\mathcal{OSI}$ problem(s), $\tilde{v}^0$ is of special linearly affine form and, under sufficiently small perturbations of the $\mathcal{GSI}$ problem, we may treat them as *fixed*. Hence, besides the perturbations $f \to \tilde{f}$, for $\mathcal{P}_{\mathcal{OSI}}(f, g^0, v^0)$ we are concerned with $g^0 \to \widetilde{g^0}$ only. We therefore introduce the simplifying notation $\mathcal{P}^*_{\mathcal{OSI}}(f, g^0) := \mathcal{P}_{\mathcal{OSI}}(f, g^0, v^0)$.

Item 2. Subsequently, we mainly perform local perturbations for $\mathcal{P}^*_{\mathcal{OSI}}(f, g^0)$. Hereby, we treat the finitely many functions g^0_j $(j \in J)$ separately in small *disjoint* open sets $\mathcal{V}^*_j$ $(j \in J)$ such that their perturbations $g^0_j \to \widetilde{g^0_j}$ can be reconstructed by one single C^2-function $\tilde{g}$ (given below). Therefore, we would need the perturbationally stable

Assumption F*. For all $j^1, j^2 \in J$, $j^1 \neq j^2$, we have

$$\bigcup_{x \in M[g] \cap \overline{\mathcal{U}^0}} \left((\phi^{j^1}_x)^{-1}(Z^{j^1}_0(x)) \cap (\phi^{j^2}_x)^{-1}(Z^{j^2}_0(x)) \right) = \emptyset .$$

For the well-definedness (possibility) of this hardly controllable assumption we take into account, that for any $x \in M[g] \cap \overline{\mathcal{U}^0}$ the sets $Z^{j^\kappa}_0(x)$ merely consist of *active* inequality constraints z. Herewith, they are *sub*sets of Z^{j^κ} $(\kappa \in \{1, 2\})$. While by definition for some preimages $(\phi^{j^1}_x)^{-1}(Z^{j^1})$ and $(\phi^{j^2}_x)^{-1}(Z^{j^2})$ an overlapping must exist, their subsets $(\phi^{j^1}_x)^{-1}(Z^{j^1}_0(x))$ and $(\phi^{j^2}_x)^{-1}(Z^{j^2}_0(x))$ need not necessarily intersect. In fact, here we are back in the original coordinates of y where each element of the active subset $Y_0(x)$ may be lying *outside* of $(\phi^{j^1}_x)^{-1}(Z^{j^1}) \cap (\phi^{j^2}_x)^{-1}(Z^{j^2})$, i.e., $(\phi^{j^1}_x)^{-1}(Z^{j^1}_0(x)) \cap (\phi^{j^2}_x)^{-1}(Z^{j^2}_0(x))$ is empty.

We are going to exploit the condition of Assumption F* *after* perturbations. However, if we may *suitably* choose our perturbed functions $\widetilde{g^0}$, then Assumption F* is naturally fulfilled (after perturbation), and we need not make it in the unperturbed situation. Now, under problem representation and joined by v, this function $\tilde{g}$ generates $\widetilde{g^0_j}$ locally in $\mathcal{V}^*_j$ $(j \in J)$. Then, for each $j \in J$, small perturbational (global) effects outside of $\mathcal{V}^*_j$ $(j \in J)$ have no influence. They can be ignored. The announced function is

$$\tilde{g}(x, y) := \begin{cases} \widetilde{g^0_j}(x, \phi^j_x(y)), & \text{if } y \in (\phi^j_x)^{-1}(Z^j) \text{ and } (x, \phi^j_x(y)) \in \mathcal{V}^*_j, \ j \in J \\ g(x, y), & \text{else.} \end{cases}$$

Item 3. Below we must consider a certain global perturbation of $\mathcal{P}^*_{\mathcal{OSI}}(f, g^0)$ to receive C^∞-data or, finally, some (global) "open and dense" property. Therefore, we apply on the one hand the perturbation technique from the proof of Genericity Theorem. On the other hand, whenever it is possible to turn from the $\mathcal{GSI}$ problem to an $\mathcal{OSI}$ (or $\mathcal{F}$) one, then we are back in the situation of Item 2 in order to perform local perturbations.

For our proof of Characterization Theorem, the *algebraical characterization of* ($\mathcal{G}$-$\mathcal{O}$) *strong stability* for a $\mathcal{G}$-$\mathcal{O}$ Kuhn-Tucker point $\overline{x}$ is important. It was given by Rückmann ([57]) for $\mathcal{OSI}$ optimization and extended in [77] for our $\mathcal{GSI}$ one. Here, we assume EMFCQ at $\overline{x}$. That sophisticated characterization refers to (restricted) Hessians of Lagrange functions, and it bases on a case study referring to the *reduction ansatz*. This **RA** demands strong stability in the sense of $\mathcal{F}$ optimization ([40]) for the local minimizers of the problem from the lower (y-) stage. Herewith, RA admits local representation of $\mathcal{P}(f, g, v)$ around $\hat{x}$ by Implicit Function Theorem ([57], [77]; see [22], [82]). These cases are:

Case I: ELICQ and RA are fulfilled at $\hat{x}$.
Case II: EMFCQ – but *not* ELICQ – and *GRA* are fulfilled at $\hat{x}$.
Case III: EMFCQ – but *not* GRA – is fulfilled at $\hat{x}$.

In any case, we can also classify the *type* of the strongly stable stationary point $\overline{x}$: While in case I a saddle point, a local minimizer or local maximizer is detected by the "stationary index" of $\hat{x}$ (a topological invariant), in cases II, III we have a strict local minimizer throughout ([77]; cf. [41], [57], [71]).

Proof of Characterization Theorem.

Sufficiency Part.

Let $\mathcal{C}_{1,2,3}$ be satisfied. We equivalently represent $\mathcal{P}(f,g,v)$ by $\mathcal{P}_{\mathcal{OSI}}(f,g^0,v^0)$, and straightforwardly interpret $\mathrm{C}_{1,2,3}$ as $\mathcal{OSI}$ conditions $\mathrm{C}_{\mathcal{OSI}\,1,2,3}$. These are: ($\mathcal{OSI}$) constraint qualification EMFCQ, strong stability of all Kuhn-Tucker points in the sense of $\mathcal{OSI}$ optimization, and separateness of the values of these $\mathcal{OSI}$ stationary points. Under slight perturbations of the $\mathcal{GSI}$ data, v^0 does not (and need not) vary. Now, we are prepared for $\mathcal{OSI}$ explanations and, finally, $\mathcal{F}$ constructions from [32], [36], [71] in our $\mathcal{GSI}$ context. We briefly repeat main ideas of construction. In [36], [71], detailed information on the techniques can be found together with illustrations.
An easy counterexample shows that the separateness condition C_3 is not generally avoidable for establishing structural stability (see [29], [71]). Here, two connected sets, say: (arcwise) components, would have to be mapped onto one connected component, contradicting homeomorphy. A similar reasoning made for another counterexample shows that, in general, the τ- (level-) dependence of the intended homeomorphisms also cannot be avoided. Moreover, each $\mathcal{G}$-$\mathcal{O}$ Kuhn-Tucker point $\hat{x}^u$ has to be mapped to the corresponding stationary point $\hat{x}^d$ of the slightly perturbed problem $\mathcal{P}(\tilde{f},\tilde{g},\tilde{v})$. Finally, we conclude from the overall boundedness assumption, from EMFCQ and strong stability, that the number of $\mathcal{G}$-$\mathcal{O}$ Kuhn-Tucker points is *finite*: $\hat{x}^u_\sigma$ $(\sigma \in \{1,\ldots,\sigma^0\})$ (cf. [36], [71], [77]).

We start by dynamically constructing the *level shift* ψ. In fact, we integrate a C^∞-vector field such that each critical value $f(\hat{x}^u_\sigma)$ gets shifted in $I\!R$ to the corresponding critical value $\tilde{f}(\hat{x}^d_\sigma)$ $(\sigma \in \{1,\ldots,\sigma^0\})$. Now, we may think $\underline{\psi = Id_{I\!R}}$, referring to $f \circ \psi$ otherwise. There are disjoint open neighbourhoods $B(\hat{x}^u_\sigma,\epsilon)$ (balls) around $\hat{x}^u_\sigma$ such that the smaller neighbourhoods $B(\hat{x}^u_\sigma,\frac{\epsilon}{2})$ contain $\hat{x}^d_\sigma$ $(\sigma \in \{1,\ldots,\sigma^0\})$. We *assume* that the unperturbed and the perturbed lower level sets *coincide* in *all* the sets $B(\hat{x}^u_\sigma,\epsilon) \setminus \overline{B(\hat{x}^u_\sigma,\frac{\epsilon}{2})}$ $(\sigma \in \{1,\ldots,\sigma^0\})$. This assumption will not restrict generality.
Based on the foregoing reduction of ψ and the previous assumption, we go on constructing $\varphi_{\mathcal{P},\tau}$ $(\tau \in I\!R^n)$ in a local-global way. Firstly, we realize which undisturbed sets have to be homeomorphically mapped onto which corresponding sets from the disturbed situation (*mapping task*). We distinguish three situations given by levels $\tau < \overline{\tau}$, $\tau = \overline{\tau}$, or $\tau > \overline{\tau}$. Herewith, we learn that some area from outside of the feasible set possibly has to be "carried in". Moreover, outside of the stationary points, the intersections of the level sets with the boundaries are *transversal*. Our further construction will be raised on these intersections (*fundamental domains*).
Outside of $B(\hat{x}^u_\sigma,\epsilon)$ $(\sigma \in \{1,\ldots,\sigma^0\})$, we use *EMF-technique* indicated in the sufficiency part on *Stability Theorem*. Here, we use our Lemma from Section 1, and apply this dy-

namical $\widehat{EMF}$*-technique* on $L^{\tau}_{\mathcal{OSI}}(f, g^0)$ $(= L^{\tau}(f, g, v))$ and on $L^{\tau}_{\mathcal{OSI}}(\tilde{f}, \tilde{g}^0)$ (see Figure 3.2(II) later). By differential geometry, this **global construction** is glued together in $\cup_{\sigma=1}^{\sigma^0}(B(\hat{x}^u_\sigma, \epsilon) \setminus \overline{B(\hat{x}^u_\sigma, \frac{\epsilon}{2})})$ with the **local construction** sketched next. We may refer to one unperturbed stationary point $\hat{x}^u (= \hat{x}^u_\sigma) \in \{\hat{x}^u_1, \ldots, \hat{x}^u_{\sigma^0}\}$ and corresponding perturbed point $\hat{x}^d$. Now, we are <u>**inside**</u> of $B(\hat{x}^u, \epsilon)$. We restrict to $n \in \{2, 3\}$, because higher dimensions can be **reduced** to those small dimensions by successive hyperplane intersection.

Case 1. $\hat{x}^u$ *is lying in the* <u>*interior*</u> $M_{\mathcal{OSI}}[g^0] (= M[g, v])$ *:*
Then, $\hat{x}^d$, being sufficiently slightly perturbed, lies in the interior of $M_{\mathcal{OSI}}[\tilde{g}^0]$. Both stationary points are *nondegenerate* ([28]), and for each τ we transform the τ-levels around $\hat{x}^u$ onto the local τ-levels at $\hat{x}^d$. In fact, this Morse theoretical local construction can be made by a C^1-diffeomorphism ([36], [71]).

Case 2. $\hat{x}^u$ *is placed on the* <u>*boundary*</u> *of* $M_{\mathcal{OSI}}[g^0]$ *:*
Then, $\hat{x}^d$ may lie on the boundary or in the interior of $M_{\mathcal{OSI}}[\tilde{g}^0]$. Without loss of generality we assume the boundary case. Actually, using an **implantation** of a suitable level structure we turn from stationary points at the boundary to *fictive* stationary points in the interior. This level structure is locally given by *fictive* objective functions $\hat{f}^u$ and $\hat{f}^d$. For performing this implantation of $\hat{f}^u$, $\hat{f}^d$ we need precise knowledge of the configurations around the boundary points $\hat{x}^u, \hat{x}^d$. These configurations are characterizable by the position (relative to the boundary) of cones or balls, together with the growth behaviours of f, $\tilde{f}$ there. We have two conical types and one radial type, governed by strong stability (under EMFCQ; [36], [71], [77]). See, e.g., Figure 3.2(I). We arrive back in *case 1* (interior position) by means of fictive interior problems, **extrapolating** the "characteristic" of $\hat{x}^u$, $\hat{x}^d$ and implanting fictive stationary points $\hat{x}^u_{fic}$, $\hat{x}^d_{fic}$ with their local level structures. Herewith, for all $\tau \in I\!R$ the mapping task is fulfilled in case 2, too. The delicate dynamical and topological techniques (and substeps) exhibited in Figure 3.2(I) are due to the local construction in case 2 (see [36], [71]).

Necessity Part

Let $\mathcal{P}(f, g, v)$ be structurally stable. Our proof of $\mathcal{C}_{\mathcal{GSI}\,1,2,3}$ is indirect. Assuming one of the first two regularity conditions or the third technical condition to be violated always contradicts structural stability (see Figure 3.3). Based on our assumptions, we carry over the proof the $\mathcal{OSI}$ necessity part from [32] into our $\mathcal{GSI}$ setting. Many details of argumentations are Morse theoretical ([17], [35], [36], [71], [77]). To avoid loss of differentiability, we assume that all data are C^∞ ([17]). This smoothness can be achieved by fine perturbations of all $\mathcal{OSI}$ data and, by tracing them back, of all $\mathcal{GSI}$ ones.

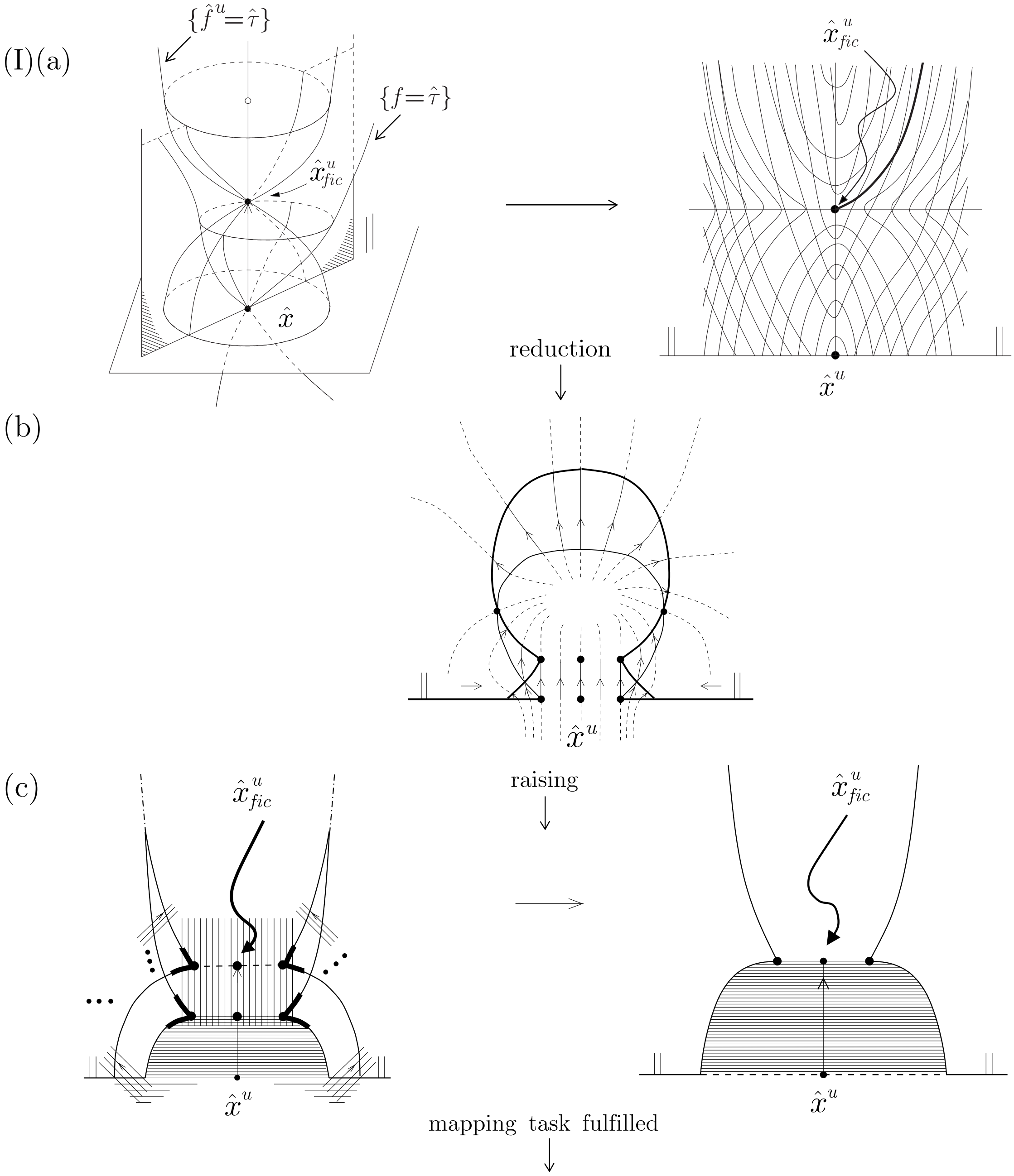
Here, we make the inequalities of different indices $\overline{z}^{\varrho^1} \neq \overline{z}^{\varrho^2}$ independent from each other (by small shifts).

<u>$\mathcal{C}_1$</u>: As $M[g]$ is compact, there exists the finite number $\tau^{\max} := \max\{f(x) |$
$x \in M[g]\}$. Herewith, $M[g] = L^{\tau}(f, g, v)$ $(\tau \in [\tau^{\max}, \infty))$. Moreover, we can choose perturbations slight enough such that $M[\tilde{g}]$ remains compact. Let $\tilde{\tau}^{\max}$ for each sufficiently slight perturbation $(\tilde{f}, \tilde{g}, \tilde{v})$ denote the maximal (feasible) value of $\tilde{f}$. Taking $\tau^* := \max\{\tau^{\max}, \psi^{-1}(\tilde{\tau}^{\max})\}$, the homeomorphism $\varphi_{\mathcal{P},\tau^*}$ gives topological equivalence between $M[g, v] = L^{\tau^*}(f, g, v)$ and $M[\tilde{g}, \tilde{v}] = L^{\psi(\tau^*)}(\tilde{f}, \tilde{g}, \tilde{v})$. By *Stability Theorem*, topological stability implies *EMFCQ*. In fact, by suitable perturbations any violation of

EMFCQ at a feasible point leads to compact sets $M[\tilde{g}]$, $M[\tilde{\tilde{g}}]$, satisfying ELICQ but being *not* of the same homotopy type ([18], [35], [71], [77]). When, e.g., the two sets have a *different* finite number of connected components, this must contradict topological equivalence (cf. also [28]).

$\underline{\mathcal{C}_2}$: Suppose EMFCQ, but C_2 not fulfilled: some $\mathcal{G}$-$\mathcal{O}$ point $\hat{x}^u$ be *not* ($\mathcal{G}$-$\mathcal{O}$) strongly stable.

Perturbation Lemma ([77]). Let a $\mathcal{G}$-$\mathcal{O}$ Kuhn-Tucker point $\hat{x}^u$ of $\mathcal{P}(f,g,v)$ be given, where EMFCQ is fulfilled, but ($\mathcal{G}$-$\mathcal{O}$) strong stability violated.



(d)

(II)

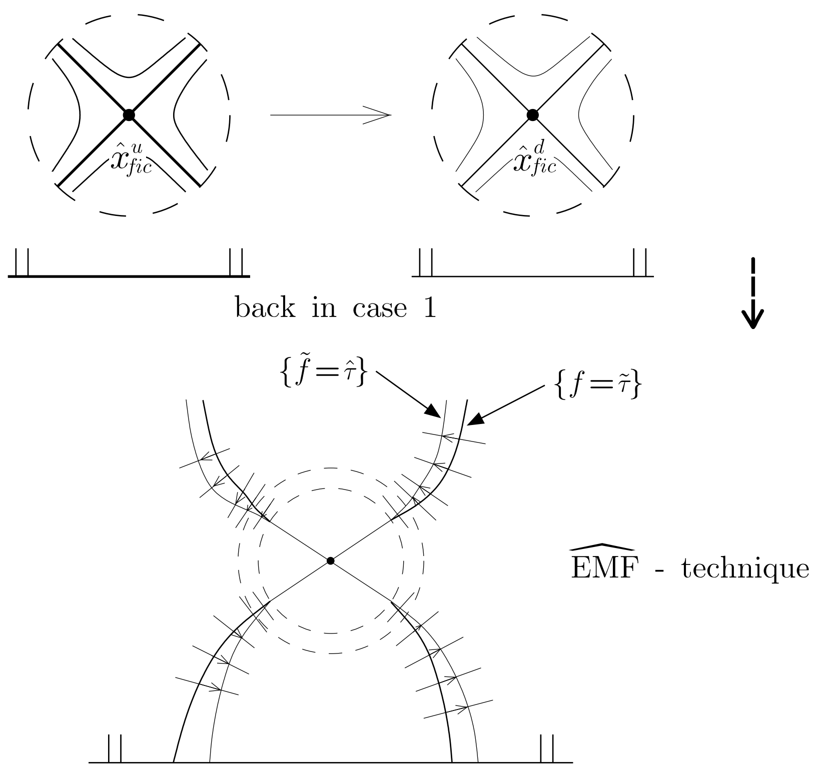


Figure 3.2: Proof of sufficiency part, Characterization Theorem

Then, for each open C^2-neighbourhood $\mathcal{O}'$ of (f,g,v) there are $(\tilde{f},\tilde{g},\tilde{v}),(\tilde{\tilde{f}},\tilde{\tilde{g}},\tilde{\tilde{v}}) \in \mathcal{O}'$ and a $k' \in I\!N$ such that:

(i) $\mathcal{P}(\tilde{f},\tilde{g},\tilde{v})$ has k' $\mathcal{G}$-$\mathcal{O}$ Kuhn-Tucker points, all being ($\mathcal{G}$-$\mathcal{O}$) strongly stable, except one (namely, $\hat{x}$).

(ii) $\mathcal{P}(\tilde{\tilde{f}},\tilde{\tilde{g}},\tilde{\tilde{v}})$ has at least $k'+1$ $\mathcal{G}$-$\mathcal{O}$ Kuhn-Tucker points, all being ($\mathcal{G}$-$\mathcal{O}$) strongly stable.

(iii) In both $\mathcal{P}(\tilde{f},\tilde{g},\tilde{v})$ and $\mathcal{P}(\tilde{\tilde{f}},\tilde{\tilde{g}},\tilde{\tilde{v}})$, EMFCQ is satisfied everywhere, and different $\mathcal{G}$-$\mathcal{O}$ Kuhn-Tucker points have different critical ($\tilde{f}$- or $\tilde{\tilde{f}}$-) values.

In $\mathcal{F}$ or $\mathcal{OSI}$ necessity parts of [17], [71] (cf. also [32]), these perturbations are realized by three steps. *Step 1* yields local isolatedness of $\hat{x}^u$ as a stationary point where, additionally, (E)LICQ is guaranteed but unstability preserved. In *step 2*, outside of the local situation, (E)MFCQ and strong stability of all (other) stationary points are established. Finally, in *step 3*, the unstable Kuhn-Tucker point $\hat{x}^u$ "splits": By this bi- (or tri-) furcation we locally get two new stationary points; they have *strongly stability.* In this $\mathcal{GSI}$ situation, we use the algebraical characterization from our preparations. Now, we introduce a topological idea: For $L^\tau(\tilde{f},\tilde{g},\tilde{v})$, $L^\tau(\tilde{\tilde{f}},\tilde{\tilde{g}},\tilde{\tilde{v}})$ we have to take into account each change of the homeomorphy type of a lower level set, when τ traverses $(-\infty,\infty)$. Based on the perturbations from above, we apply the following items on $\mathcal{P}(\tilde{f},\tilde{g},\tilde{v})$, and $\mathcal{P}(\tilde{\tilde{f}},\tilde{\tilde{g}},\tilde{\tilde{v}})$. We look at a C^2-problem $\mathcal{P}(\hat{f},\hat{g},\hat{v})$ having a compact feasible set and fulfilling EMFCQ, and we put $L^b_a(\hat{f},\hat{g},\hat{v}) := \{x \in M[\hat{g}] \,|\, a \leq \hat{f}(x) \leq b\}$ for some $a,b \in I\!R$, $a<b$.

Item 1. If $L_a^b(\hat{f},\hat{g},\hat{v})$ does not contain a stationary point, then $L^a(\hat{f},,\hat{g},\hat{v})$ and $L^b(\hat{f},\hat{g},\hat{v})$ are homeomorphic.

Item 2. Let $L_a^b(\hat{f},\hat{g},\hat{v})$ contain exactly one stationary point $\hat{x}'$. Moreover, let $a < f(\hat{x}') < b$ and $\hat{x}'$ be $(\mathcal{G}$-$\mathcal{O})$ strongly stable. Then, $L^a(\hat{f},\hat{g},\hat{v})$ and $L^b(\hat{f},\hat{g},\hat{v})$ are *not* homeomorphic.

These two items immediately result from corresponding facts on $\mathcal{P}_{\mathcal{OSI}}(\tilde{f},\tilde{g}^0,\tilde{v}^0)$, $\mathcal{P}_{\mathcal{OSI}}(\tilde{\tilde{f}},\tilde{\tilde{g}}^0,\tilde{\tilde{v}}^0)$ stated in [57]. Here, *Item 2* can be expressed with attaching κ-cells ($\kappa =$ stationary index at $\hat{x}'$; [77]). By Manifold Theorem and Lemma (Sections 1–2) we conclude for all noncritical levels τ: $L^\tau(\hat{f},\hat{g},\hat{v}) = M[(\hat{g},-\hat{f}+\tau)]$ is a *compact* topological manifold (with boundary). So, their homology spaces (over $I\!R$) are of different *finite* dimensions ([63]). As these spaces are topological invariants, the two considered lower level sets cannot be homeomorphic ([28]).

Now, we can make the following "discrete" statement on numbers of topological changes for the lower level sets: The homeomorphy type of $L^\tau(\tilde{\tilde{f}},\tilde{\tilde{g}},\tilde{\tilde{v}})$ changes (at least) at $k'+1$ times, while the homeomorphy type of $L^\tau(\tilde{f},\tilde{g},\tilde{v})$ changes (at least) at $k'-1$ times, but at most at k' times. This difference contradicts structural stability of $\mathcal{P}(f,g,v)$ (cf. [77], or see Figure 3.3).

$\mathcal{C}_3$: Let C_3 be violated, but the former two properties on EMFCQ and strong stability be satisfied. By local addition of arbitrarily small constant functions on f, we get a problem $\mathcal{P}(f^*,g,v)$ satisfying C_3. Let k^* stand for the number of critical points of $\mathcal{P}(f^*,g,v)$. Then the homeomorphy type of $L^\tau(f^*,g,v)$ changes k^* times, while the number of changes of the homeomorphy type of $L^\tau(f,g,v)$ is less than k^*. Hence, we are faced again with a situation which is incompatible with structural stability of $\mathcal{P}(f,g,v)$ (Figure 3.3). □

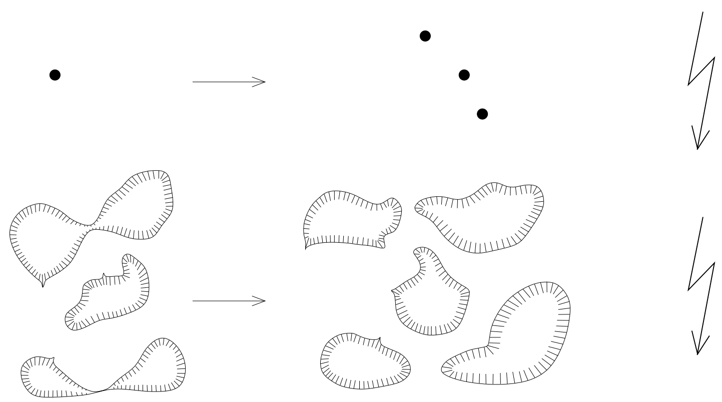


Figure 3.3: Proof of necessity part, Characterization Theorem

Our optimality conditions, topological results and techniques together prepare *iteration procedures* for treating $\mathcal{P}(f,g,v)$. For detailed explanation of the design see [51], [76], and [77], [80], where the importance of discrete-combinatorial information for transparency and rate of convergence is emphasized. Further new approaches and numerical methods are presented in [13], [46], [66], [67], [68], [69], [70].

4. Generalizations, Optimal Control and Conclusion

4.1. Generalizations

There are two lines for generalizing our topological results:

(i) $M[g]$ is *unbounded* (noncompactness),

(ii) f is of the *nondifferentiable* $\mathcal{GSI}$ *maximum-minimum-type*, i.e., the composition $f = f_p \circ f_{p-1} \circ \ldots \circ f_1$ of finitely many functions which are of max-type $f_j(x) = \max_{\varsigma \in \Upsilon^j(x)} w_j(x,\varsigma)$ or of min-type $f_j(x) = \min_{\varsigma \in \Upsilon^j(x)} w_j(x,\varsigma)$.

On *(i)*: We overcome noncompactness by turning to the family of *excised* subsets of $M[g]$. Roughly speaking, the effect of intersection is generated by subtracting lower semi-continuous functions from g ([58], [71], [77]). Herewith, we can express cuts, e.g., by cylinders or balls, by $\mathbb{R}^n$ itself or by bizarre sets. Referring to all excised sets, we get the condition of **excisional topological stability** which can actually be characterized by the overall validity of EMFCQ in the unbounded set $M[g]$. For that (*Excisional*) *Stability Theorem* see [77].

On *(ii)*: In the case of f being of *max-type*, nonsmoothness is overcome by expressing $\mathcal{P}(f,g,v)$ by minimization of x_{n+1} over the *epigraph* $E(f) := \{(x, x_{n+1}) | \, x \in M[g], \; f(x) \leq x_{n+1}\}$. From this problem in $\mathbb{R}^{n+1}$ we obtain our stationary points of $\mathcal{P}(f,g,v)$ and the appropriate condition of strong stability ([71], [72], [77]). Now, (max-) structural stability of our nondifferentiable problem can be characterized by EMFCQ, strong stability and the technical separateness condition again. This *Characterization Theorem* and the one for the *case combination* with *(i)* are demonstrated in [77].
In case of a *min-type* f, we turn to $E(-f)$ and use geometrical insights from the max-type case. Now, in our general case of finite *max-min* composition, we *un*fold nondifferentiability step by step, finally getting our *max-min structural stability* and its characterizing conditions ([79], [81]).

Remark. In *(ii)*, we treated the discrete-combinatorial nondifferentiability structure underlying f by *unfolding* or *lifting* along continuous parameters. For further examples of tracing back structures in the way *"discrete → continuous"*, or *"continuous → continuous"*, *"continuous → discrete"* and *"discrete → discrete"* cf. [79] or Subsection 4.3.

In our next subsection, we have to reconsider equality constraints (disregarded above by Convention) again.

4.2 Optimal Control of Ordinary Differential Equations

We turn to infinite dimensions by studying the following minimization problem in (x,u) ([21], [45], [53]):

$$\mathcal{P}(\ell, L, F, H, G) \quad \begin{cases} \text{Min } \mathcal{I}(x,u) := \ell(x(a), x(b)) + \int_a^b L(t, x(t), u(t))\, dt \\ (\, x \in (C^0_{\mathrm{pw}\,2}([a,b], \mathbb{R}))^n, \;\; u \in (F_{\mathrm{pw}\,2}([a,b], \mathbb{R}))^q \,), \\ \text{such that} \\ \dot{x}(t) = F(t, x(t), u(t)) \quad (\text{for almost every } t \in [a,b]), \\ (x(a), x(b)) \in M[H], \\ x(t) \in M_{\mathcal{F}}[G(t, \cdot, u(t))] \quad (\text{for almost every } t \in [a,b]), \end{cases}$$

where (L, F, G), (ℓ, H) are C^3- and C^2-functions (vector notation), respectively. Instead of referring to the larger classes of Sobolev or Lebesgue spaces, we concentrate on spaces of continuous and piecewise C^2 *states* x, and piecewise C^2 *controls*, called $C^0_{\mathrm{pw}\,2}$ and $F_{\mathrm{pw}\,2}$. For these spaces, strong topologies in Whitney's sense can be generally introduced ([77]).

Assumption (BOUND): $M[H] \subseteq I\!R^n \times I\!R^n$ and $M_{\mathcal{F}}[G] \subseteq [a, b] \times I\!R^n \times I\!R^q$, defined by the equality and inequality contraints, are *bounded*.

Assumption (LB): There exist positive functions $\alpha_0, \beta_0 \in C(I\!R^{q+1}, I\!R)$ such that (under $||\cdot||_\infty =$ maximum norm) we have *linear boundedness* of F:

$$||F(t, \mathbf{x}, \mathbf{u})||_\infty \leq \alpha_0(t, \mathbf{u})||\mathbf{x}||_\infty + \beta_0(t, \mathbf{u}) \quad ((t, \mathbf{x}, \mathbf{u}) \in I\!R^{n+q+1}).$$

We briefly present two approaches to global structure and stability of $\mathcal{P}(\ell, L, F, H, G)$ (cf. [77]). While our main Approach II is refined, Approach I is given for preparation.

Approach I: **Particular Structure.** Let u be considered as C^2 and a parameter. Then, for each fixed $u = u^*$ the optimal control problem $\mathcal{P}(\ell, L, F, H, G)$ becomes a problem $\mathcal{P}^{u^*}(\ell, L, F, H, G)$ from *calculus of variations*. The corresponding system of differential equations (on x) generates a *flow* (in $I\!R^{n+1}$; [2], [29]). Under this flow, we *trace back* the equality and inequality constraints, and the objective functional as well (cf. [73], [74], [77]). Let us mention $f^*(\mathbf{x}) = \mathcal{I}(\mathcal{Q}[\mathbf{x}](\cdot - a), u^*(\cdot))$, where $\mathcal{Q}[\mathbf{x}](0) = \mathbf{x}$, $(\mathcal{Q}[\mathbf{x}](s))(t) = \mathcal{Q}[\mathbf{x}](s + t)$ ($\mathbf{x} \in I\!R^n$; $s, t \in I\!R$). So we obtain an $\mathcal{OSI}$ problem $\mathcal{P}^{u^*}_{\mathcal{OSI}}(f^*, h^*, g^*)$ (where $Y^j = [a, b]$). Then, referring to the family of all u and to perturbations of (f^*, h^*, g^*), we get the condition of **(particular) structural stability** with its **Characterization Theorem** again (cf. Section 3; [74], and [77] where also examples are discussed). The C^2-property and parametrical treatment of u, however, are not sufficient for optimal control. So we change to

Approach II: **Composite Structure.** We evaluate the necessary optimality condition *Pontryagin's minimum principle* ([21], [53]) in the way of *Kuhn-Tucker conditions* for almost every $t \in [a, b]$, where the latter conditions refer to the "min" problems implied in the principle. Here, we have suitable multiplier vectors, (adjoint) variables, and $\mathcal{H}(t, \mathbf{x}, \mathbf{u}, \lambda, \mu) := L(t, \mathbf{x}, \mathbf{u}) - \lambda^T F(t, \mathbf{x}, \mathbf{u}) - \mu^T G(t, \mathbf{x}, \mathbf{u})$. Then, our evaluation, called **minimum principle** here ([10], [48], [50]), reads

$$\begin{aligned}
D^T_{\mathbf{u}}\mathcal{H}(t, x^0(t), u^0(t), \lambda^0(t), \mu^0(t)) &= \mathbf{0}_q, \\
\mu^0_j(t) \geq 0 \ (j \in J) \ \textit{and} \ \mu^{0^T}(t)\, G(t, x^0(t), u^0(t)) &= 0, \\
\lambda^0(a) &= -\,D^T_{\mathbf{x}_1}(\ell - \rho^{\mathbf{0}^T} H)(x^0(a), x^0(b)), \\
\lambda^0(b) &= D^T_{\mathbf{x}_2}(\ell - \rho^{\mathbf{0}^T} H)(x^0(a), x^0(b)), \\
\dot{\lambda}^0(t) &= -D^T_{\mathbf{x}}\mathcal{H}(t, x^0(t), u^0(t), \lambda^0(t), \mu^0(t)).
\end{aligned}$$

For our *causal* (composite) structure we need a condition like strong stability ([77]):

Assumption (CONT): All the ($C^0_{\mathrm{pw}\,2} \times F_{\mathrm{pw}\,2}$-) solution components (x^0, u^0) of the minimum principle depend continuously on $C^3_S \times C^2_S$-perturbations
$((L, F, G), (\ell, H)) \ \to \ ((\widetilde{L}, \widetilde{F}, \widetilde{G}), (\tilde{\ell}, \widetilde{H}))$.

In fact, we interpret the first four lines of the minimum principle as Kuhn-Tucker conditions of two families of optimization problems: $(*)\ \mathcal{P}_{\mathcal{F}}^{t,\mathbf{x},\mathbf{w}}(L, F-\mathbf{w}, G)$ and $(**)\ \mathcal{P}_{\mathcal{F}}(\lambda^0(a), \lambda^0(b), \ell, H)$, an index set $M_{pr}^{\eta_0}[F, G]$ of $(t, \mathbf{x}, \mathbf{w})$ being appropriately chosen in view of $\mathcal{P}(\ell, L, F, H, G)$. Let us only mention

$$\mathcal{P}^{t,\mathbf{x},\mathbf{w}}(L, F, G) \quad \begin{cases} \mathrm{Min}_{\mathbf{u}\in I\!R^q} \ \ L(t, \mathbf{x}, \mathbf{u}), \quad \text{where} \\ \quad \mathbf{u} \in M_{\mathcal{F}}^{t,\mathbf{x},\mathbf{w}}[F - \mathbf{w}, G], \end{cases}$$

with its KT conditions $D_{\mathbf{u}}L(t, \mathbf{x}, \overline{\mathbf{u}}) = \sum_{\kappa\in\{1,\ldots,n\}} \lambda_\kappa D_{\mathbf{u}} f_\kappa(t, \mathbf{x}, \overline{\mathbf{u}}) + \sum_{j\in J_0(t,\mathbf{x},\overline{\mathbf{u}})} \mu_j D_{\mathbf{u}} g_j(t, \mathbf{x}, \overline{\mathbf{u}})$, $\mu_j \geq 0$ ($j \in J_0(t, \mathbf{x}, \overline{\mathbf{u}})$ being active), incorporated into minimum principle, and

$$\mathcal{P}(\lambda^0(a), \lambda^0(b), \ell, H) \quad \begin{cases} \mathrm{Min} \ \ ((\lambda^{0^T}(a), -\lambda^{0^T}(b))(\cdot) + \ell)(\mathbf{x}^1, \mathbf{x}^2), \\ \qquad \text{where} \ \ (\mathbf{x}^1, \mathbf{x}^2) \in M[H], \end{cases}$$

with its KT conditions likewise. For each of these problems of the form $(*)$, $(**)$ we introduce (*composite*) *structural stability* and characterize it essentially by (E)MFCQ and strong stability (see Section 3). Analyzing $(*)$ so, we locally get implicit C^2-control functions $u_\vee(t, \mathbf{x}, \mathbf{w})$, which are Kuhn-Tucker point-valued and fulfill $u^0(t) = u_\vee(t, x^0(t), \dot{x}(t))$. Substituting $\mathbf{w} := \dot{x}(t)$ for any trajectory x of some auxiliary flow, adapted to our system of differential equations, we locally receive **core functions** $u_\vee^0(t, \mathbf{x})$. For this dimensional reduction we also have to use interpolation properties and smoothing techniques. The choices of those auxiliary or *test flows* etablish a *structural frontier* of our theory ([77]). To globalize a core such that its domain covers $[a, b]$, we admit **jumps** in $I\!R^{n+1}$ (see Figure 4.1). These jumps shall be compatible with the jumps of our variables u^0. Again we say that the piecewise globalized core functions $(\heartsuit)\ u_\vee^0$ are of class $F_{\mathrm{pw}\,2}$. Let B, B be ("boundary") sets where the jumps may or really do happen, respectively. When these sets exist as *Lipschitzian manifolds* of dimension q, and if they (by decomposition) define *piecewise structures* before or after jumps, which quantitatively remain preserved under small perturbation of (ℓ, L, F, H, G), then the core $(\heartsuit)$ is called (*composite*) *structurally stable* ([77]).

A further regularity condition, called **structural transversality**, in short: **ST**, analytically determines the boundary sets (up to a finite number of choices) and guarantees this (composite) structural stability of a core. (See also [30], [48], [50].) In one of two cases, the refined condition ST means *transversal* intersection of $u_\vee^0(\cdot, x(\cdot))$ (along trajectories x) at the boundary of the corresponding feasible set in $I\!R^q$. This implies transversality of x at the manifolds B, B, and it is an analytic condition just governing the (composite) structural stability of the considered core $(\heartsuit)\ u_\vee^0$. In fact, ST allows to record "stable" manifolds B, B by Implicit Function Theorem.
Now, inserting $u(t) = u_\vee^0(t, x(t))$ in $\mathcal{P}(\ell, L, F, H, G)$ delivers again a problem $\mathcal{P}^{u_\vee^0}(\ell, L, F, H, G)$ from calculus of variations (recall Approach I), which we also trace back under its flow $\Phi^{u_\vee^0}$. In this way we get an optimization problem with a complex underlying piecewise structure. Up to *structural frontiers* given by combinatorially more complicate index sets $Y(\mathbf{x})$ of the form $[t^1(\mathbf{x}), t^2(\mathbf{x})]$ and objective functions f of *continuous selection type* ([31]), we arrive at a $\mathcal{GSI}$ problem $(***)\ \mathcal{P}(f, h, g, v)$ with f of *max-min-type* (cf. Subsection 4.1). Here, $t^1(\mathbf{x})$ and $t^2(\mathbf{x})$ record the times of transversal intersection of codimension 1 manifolds B_1, B_2 (or $\mathsf{B}_1, \mathsf{B}_2$) by the corresponding trajectory $x(\cdot)$ which starts at the point $\mathbf{x}$. Now, the underlying max-min combinatorics of the continuous selection represents the continuity (jump) and differentiability structure of the incorporated

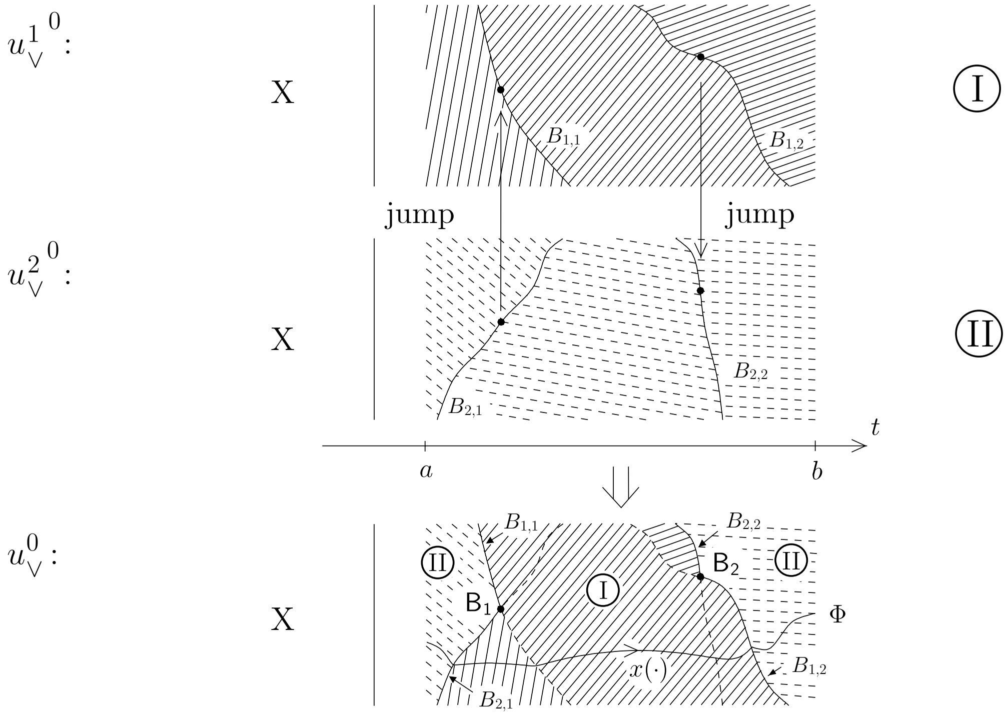


Figure 4.1: Piecewise structure and jumps of cores

cores (many technical aspects are worked out in [77]). As some kind of excision, we delete inequality constraints where they vanish identically on the manifold P bounded by B_1 and B_2. Then, we introduce this $\mathcal{GSI}$ optimization problem's condition of (*composite*) *structural stability* referring to perturbations of the original data (ℓ, L, F, H, G).
In that sense, we call $\mathcal{P}(\ell, L, F, H, G)$ **composite structurally stable** if *all* the structural elements $(*)$, $(**)$, $(***)$, $(\heartsuit)$ are (composite) structurally stable. Under our basic Assumptions (BOUND), (LB) and up to those more complex problems we state (with simplified presentation):

Characterization Theorem on Composite Structural Stability ([77]).
The problem $\mathcal{P}(\ell, L, F, H, G)$ is *composite structurally stable*, *if and only if* the conditions $\mathcal{C}_{1,2,3,4}$ are satisfied:

C_1. *(E)MFCQ* holds for all the feasible sets underlying $(*)$, $(**)$, $(***)$, $(\heartsuit)$.

C_2. All the Kuhn-Tucker points $\overline{\mathbf{u}}$, $\overline{\mathbf{x}}$ of the problems represented in $(*)$, $(**)$, $(***)$ are *strongly stable* (in $\mathcal{F}$ or $\mathcal{G}$-$\mathcal{O}$ sense).

C_3. For all optimization problems represented in $(*)$, $(**)$, $(***)$, each two different Kuhn-Tucker points have different *(separate)* critical values.

C_4. For all core functions $(\heartsuit)$ *ST* is fulfilled.

Sketch of Proof: The main lines are the same as in Subsection 3.3. The new item, given in the necessity part, "$\Longrightarrow \mathrm{C}_4$," concerns the undisturbed or disturbed piecewise structures, and it is illustrated in Figure 4.2. □

Remark. In the necessity part, we resolve the *inverse problem* of reconstructing *special* perturbations of optimization problems by special perturbations of the given optimal control problem. Example: In case of C^2-functions f, the following additive C^3-variation

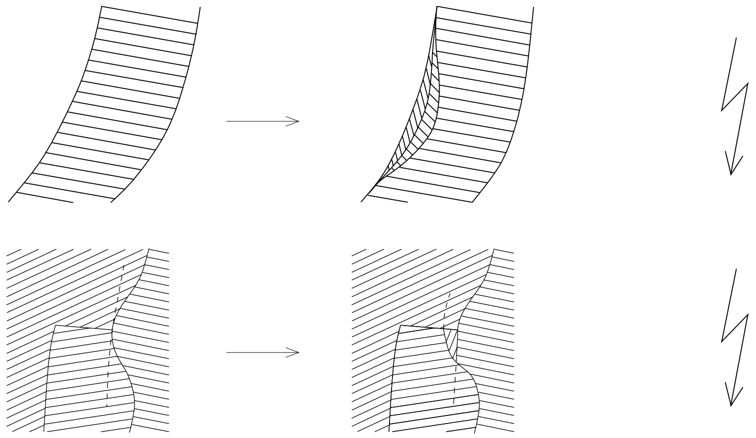


Figure 4.2: Proof of necessity part (composite structural stability)

δL (where $\widetilde{L} = L + \delta L$) reconstructs an additive C^∞-variation δf (where $\tilde{f} = f + \delta f$): $(\delta L)(t, x, u) := -D(\delta f)(x)\frac{\partial}{\partial t}\Phi^{u^0_\vee}(x, 0) + \frac{1}{b-a}(\delta f)(\Phi^{u^0_\vee}(x, b-t))$ ([77]).

For *controllability*, i.e., to come from time a to time b under given constraints of $\mathcal{P}(\ell, L, F, H, G)$, **discrete** mathematics ([9]) often turns out as a mean of investigation as follows. (For underlying finiteness and genericity considerations see [77].) Our control problem asks for a domain of the core $u^0_\vee$ (compatible with u^0) that is sufficiently large, say: tending to maximality. Provided a carefully chosen set of jumps, this *maximal domain* problem can be represented as a **maximal matching** problem in a partite graph (see, e.g., Figure 4.3). In a subset of arcs called *matching*, different elements are disjoint. Here, each partition stands for a locally defined continuous core, the directedness of the ars reflects orientation by time t. This matching problem can be solved by *Edmond's algorithm* ([34]).

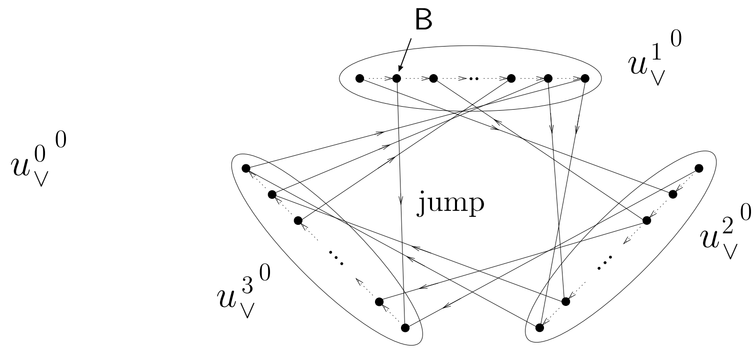


Figure 4.3: Tripartite directed graph featuring controllability problem

Inserting the global cores, arriving at an x-depending problem, we may, for example, consider the objective function as the arc length of our piecewise structured solutions $x = x^0 \in C^0_{\mathrm{pw}\,2}$ of minimum principle. Therefore, we take into account arcs between neighbouring vertices (manifolds $\mathsf{B}_1, \mathsf{B}_2$) of the same former partition such that the partite character gets lost (see, e.g., Figure 4.4, periodic constraint $x^0(a) - x^0(b) = 0$ implied). The corresponding minimization problem can be regarded as a **shortest path** problem, solvable by *Dijkstra's algorithm* ([34]).

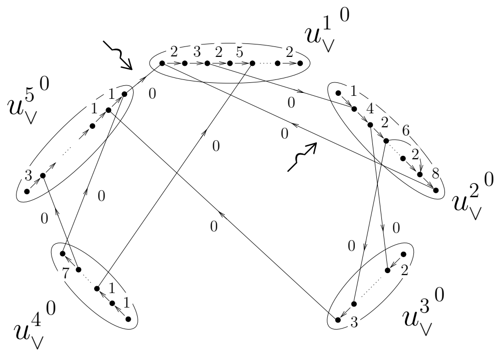


Figure 4.4: Directed graph featuring minimization of arc length

4.2. Related Problems

a) Time-Minimal Control of Heating.

Turning to optimal control of partial differential equations, we consider the example of controlling the heating (or cooling) of a ball consisting of a homogeneous material from an initial to a terminal temperature in a time ($= T$) minimal way. The initial time is 0. Our problem is governed by a heat equation, and the temperature at the ball's boundary is considered as the control ([43], [44]). Now, for every terminal time $T \geq 0$ an auxiliary *norm minimal* problem on the thermal stress tangential to the boundary turns out to have a unique solution (cf. [43]), say again: a core variable $u^0_\vee(\cdot, x)$ ($x := T$). Inserting $u^0_\vee$ into the control problem delivers a $\mathcal{GSI}$ optimization problem with $y := t$, $Y(x) := [0, x]$. Here, however, differentiability can get lost by solving the heat equation. Therefore, any iteration procedure of the control problem based on Sections 3–4 (e.g., using *discretization*) should stepwise be accompanied by smooth approximation of the functional data. (See [44]; cf. also [39].) For numerical evaluations and relations to the problem of global warming see [52].

b) Problems from Transportation and Aerodynamics.

Consider the transportation of some rigid body (e.g., a piano) from an initial position p to a terminal position q, remaining inside of a given feasible set. To be precise, all the positions are lying in group SE_3. Supposing the body and the feasible set to be polynomially defined as *semi-algebraic sets*, then we learn from algebraic geometry that we can make a *cell decompostion* of the feasible set. Representing each cell by a vertex and representing any two neighbouring cells (dimensional difference = 1) by an edge, we get a *connectedness graph*. So, the problem of finding a continuous path between p and q has become a discrete one of finding a path in this graph, e.g., by Dijkstra's algorithm again. (See [59], [60].) Sometimes such paths are called road maps. They could also be central paths from interior point methods in *semidefinite programming*, being closely related with our $\mathcal{SI}$ optimization.
Let us carry over the cell decomposition techniques to find a cell complex which reflects the different air regions around some airplane. Now, optimizing the architecture or the flight of the airplane (being a subject of hard research in continuum mechanics) can be supported by optimizing the corresponding connectedness graph. In the dynamical case of flight, we could make a colouring of the vertices in such a way that, for example, turbulent structure is *red*, laminar structure is *blue*, etc.. Then, we locally change the colours in time

just as being desired in the way of our discretized *time-minimal control of our heating* (*or cooling*) from *a*). The relation to $\mathcal{GSI}$ optimization is given by locally transferring our $\mathcal{GSI}$ discrete-algorithmic methods from time-minimal thermo-control. Such research is initiated with Yu. Shokin ([62], [78]). For further relations to continuum mechanics see [3] concerning sensitivity, and see [6] concerning thermo-control of premature infants.

c) Discrete Tomography.

A modern *inverse problem* from tomography comes from the microscopic VLSI design. Given a "nonconvex" set of atoms located in on a chip. We want to measure the number and distribution of the atoms (represented as balls in a regular grid) by "shooting" parallel electronic beams and recording the reverse "X-rays" at hyperplanes. How many directions of beams – "flows" in the sense of Subsection 4.2 – do we need, how to choose them? As a basic reference we cite [15], and we sketch three lines of present and future research ([79]):

(i) *Wavelets* detect roughness in layer structures on chips ([20]).

(ii) Representing any supposed atoms cluster within the grid box as a single word over $\{0, 1\}$, the theory of *linear codes* helps by error detection and correction. Here, we look for Hamming codes or ask for the extent of *cyclicity* ([26]). This line is also followed by A. Alpers, P. Gritzmann et al.. Concerning their first results on stability and instability in discrete tomography see [1]. For instability (distortion) of, e.g., spherical codes, cf. [19], for binary control in discrete tomography reconstruction, cf. [7].

(iii) Further *invariance* and *equivariance* information about the atoms distribution could be extracted by using *optimal experimental design* from statistics ([12], [14]). Referring to all $n \times n$ "moment matrices" C, lying in some convex (e.g., $\mathcal{GSI}$ defined) cone $\mathcal{C}$ with semidefinite elements, and the given "measurement space" $\mathcal{X}$, invariance and equivariance mean $f(QCQ^T) = f(C)\ \forall\, Q \in \mathcal{Q}, C \in \mathcal{C}$, and $F(G(x)) = Q_G F(x)\ \forall\, G \in \mathcal{G}, x \in \mathcal{X}$, respectively. Here, f is some suitable criterion on $\mathcal{C}$ to be minimized, F stands for the measurements of atom distribution, while $\mathcal{Q}$ is a compact group of regular $Q \in \mathbb{R}^{n\times n}$ (e.g., a suitable Q_G) and $\mathcal{G}$ is a group of bijective $G : \mathcal{X} \to \mathcal{X}$. This leads to dimensional reduction of our measurements (recall also Figure 3.2 (I) (a)). In [5], this algebraical model has already been utilized for optimizing elasticity of crystals. Finally, semidefinite and nonlinear integer programming problems have to be resolved. The semidefinite ones with their feasibilty being defined by the cone $\mathcal{C}$, are closely related to our semi-infinite problems.

4.4 Conclusion.

In this survey article, we studied the *topology* of generalized semi-infinite optimization problems under basic assumptions. We were concerned with global structural stability. Its characterization and extension to nondifferentiability and optimal control problems referred us to *inverse problems*. Here, we noted the importance of *discrete* intrinsic information for transparency, convergence and stability. The fruitful meeting of these three aspects gave a number of impulses for new projects.

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