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Energy Functionals on Certain Fractal Structures

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We describe - going back to the early eighties - a few ideas occurring in some papers by physicists and mathematicians concerning the dynamical properties of certain fractal structures and we outline an effective metric theory devised to bring these ideas together, in a metric framework of pseudo-Riemannian type.

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1. The work of physicists

In the early eighties physicists discovered - by theory, experiment or simulation - that certain disordered media occurring, for example, in percolation phenomena, ferromagnetic/paramagnetic transitions, electrical depositions etc. display "strange" diffusive and vibrational properties. The anomaly appears in the main scaling exponents governing, for example, the space-time relation of the diffusive process, typically on these media of the type

$$|X_t - x|^2 \approx t^{2/D_w}.$$

The exponent D_w - called *the walk dimension* - while depending on the body it is always found to deviate from the value 2, which it takes in the case of usual Brownian motions in all Euclidean dimensions, indeed

$$D_w > 2.$$

Similarly, the vibrational modes obey an asymptotic law of the type

$$\lambda_k \approx k^{2/D_s}, \quad \text{for} \quad k >> 1,$$

where the exponent D_s - the so-called *spectral dimension* - for structures imbedded in a Euclidean space of dimension D is found in general to be

$$D_s \neq D,$$

in contrast with the value $D_s = D$ that D_s takes for Euclidean D-dimensional membranes, as established by H. Weyl in 1912, [63].

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The exponents D_w and D_s are mutually related by the so-called *Einstein relation*

$$\frac{D_w}{2} = \frac{D_f}{D_s},$$

where D_f is the *fractal dimension*, that is the Hausdorff dimension of the body.

From the more general point of view of disordered media, Euclidean D-dimensional membranes appear then to be a sort of "degenerate case", in which the three relevant dimensional parameters D_w, D_s and D_f satisfy

$$D_w = 2, \quad D_s = D_f = D.$$

It is enlightening to try to capture some of the main ideas that led physicists to elaborate simple models that could explain the fractal "anomalies" described before.

The intuition was that unusual diffusion and vibrations could be explained as due to *ramification at all scales* inside the body, and that the basic scaling exponents could be found by the method of *decimation* in solving the basic equilibrium equations at each scale.

From the point of view of mathematics, ramification - in a topological sense - had been a main issue at the beginning of last century in the study of "strange curves", with weak or none differential smoothness. In particular, W. Sierpiński, in 1915, published a paper whose title was *"Sur une courbe cantorienne dont tout point est un point de ramification"*, [59].

We owe to physicists the creative act that turned these funny topological curves into something, in some sense, alive, capable of diffusing heat or produce sounds by vibration: "...e ora parla!", as Michelangelo is quoted by the legend to exclaim, after hitting his just finished marble Moses.

In somewhat more technical words, what physicists did was to turn topological curves into *elastic bodies*, of the kind of the strings and membranes of classic physics and analysis. In the process, as we shall see below, deep mathematical new ideas came into light.

What makes the Sierpinski strings a particularly interesting model of fractal behavior are two main geometrical features of these curves. First, they exist in all Euclidean dimensions $D \geq 2$. Second, they exhibit *self-similarity* at all scales $(1/2)^k$, $k = 1, 2, \ldots$.

A curve of this family can be obtained from a D-dimensional unit simplex
$\Gamma = \{a_1, \ldots, a_{D+1}\}$, by indefinitely iterating the $D + 1$ similitudes

$$\psi_i(x) \equiv a_i + \frac{1}{2}(x - a_i), \quad i = 1, \ldots, D + 1.$$

This gives the set

$$S^\infty = \bigcup_{n=0}^{\infty} S^n,$$

where each S^n is obtained by putting together all smaller images of the initial simplex under the iterated action of the maps $\psi_{i_1} \circ \ldots \circ \psi_{i_n}$, each index i_h taking all values of the set $I = \{1, \ldots, D + 1\}$, for $h = 1, \ldots, n$. The final curve is then the closure S of the

set S^∞. At each point of the set $S^\infty - \Gamma$, everywhere dense in S, $2D$ branches of S meet together.

In this way, a full scale of simple models of increasing ramification becomes available.

The decimation procedure is also carried on iteratively, by applying Kirchhoff laws at the stage *nth* of iteration to the graph S^n, treated now as an electric circuit. The idea consists in eliminating - in the system of equilibrium equations satisfied at each node - the lowest scale amplitudes corresponding to the mid-points of hyper-tethraedon edges.

The equilibrium conditions of the free circuit at each (interior) node p of S^n are then found to be

$$\Delta_n u(p) = 0.$$

The *pre-Laplacean* Δ_n is the graph operator

$$\Delta_n u(p) = (D+3)^n (\sum_{q \sim_n p} u(q) - \chi\{q \sim_n p\} u(p))$$

where, for each $p \in S^n$, $\{q \sim_n p\}$ denotes the set of all neighbouring points $q \in S^n$ of p - that is, the set of the $2D$ vertices of the 2 simplexes of size $(1/2)^n$ meeting at p - and $\chi\{q \sim_n p\}$ is the number of such neighbouring points, that is, $\chi\{q \sim_n p\} = 2D$.

The *interior nodes*, mentioned before, are the points of the graph distinct from the vertices of the initial simplex. The $D+1$ vertices of the initial simplex assume the role of the *boundary*, Γ, of S^n and of the curve S itself. On Γ, Dirichlet or Neumann conditions can be prescribed. In what follows, we shall assume for simplicity that all functions on S^∞, as well as on S, satisfy a homogeneous Dirichlet condition on the boundary, that is, they vanish at the points of $\Gamma = \{a_1, \ldots, a_{D+1}\}$.

The operator Δ_n, with its scaling factor $(D+3)^n$, is the main outcome of the decimation technique. This method gives also the spectral dimension of S, namely

$$D_s = 2\frac{\log(D+1)}{\log(D+3)}.$$

Note also that the walk dimension is given by the Einstein relation and is

$$D_w = \frac{\log(D+3)}{\log 2}.$$

For these results we refer *e.g.* to R. Rammal [55].

This is a rather surprising result, which shows that increasing ramification affects geometry and dynamics in quite diverse extent. In fact, the (Hausdorff) geometric dimension of S, D_f, easily calculated to be

$$D_f = \frac{\log(D+1)}{\log 2},$$

grows with D logarithmically to ∞. Also D_w grows logarithmically to ∞. On the other side, the spectral dimension D_s remains bounded and strictly less than 2 as $D \to \infty$. Therefore, the spectral dimension of S - as a function of the imbedding dimension D - interpolates, in a logarithmic scale, the spectral dimension $D_s = 1$ of 1-dimensional

strings and that, $D_s = 2$, of 2-dimensional membranes. The value 2 is reached only asymptotically, as $D \to \infty$.

Physicists have been able to "construct" a full scale of strange "balalaika", sounding less and less like a string and more and more like a drum!

For more informations about the physical aspects of the fractal theory mentioned before we refer, *e.g.*, to D. Stauffer [60], Y. Gefen - A. Aharoni - B. B. Mandelbrot - S. Kirkpatrick [22], Y. Gefen - A. Aharoni - S. Alexander [21], S. Alexander - R. L. Orbach [1], R. B. Stinchcombe - C. H. Harris [61], R. Rammal [55], R. Rammal - G. Toulouse [56] and to the surveys S. Havlin [24], T. Nakayama - K. Yakubo - R. L. Orbach [51].

2. The work of mathematicians

Such intriguing analytical models could not be ignored by mathematicians. At the turning of eighties and nineties, a group of probabilists - working independently - were able to construct rigorously a diffusion process within certain model fractals.

The work was done in the late eighties, initially for the Sierpinski curve, by S. Goldstein [23], S. Kusuoka [31], M. T. Barlow and E. A. Perkins [3] and by T. Lindstrøm [37]. They considered suitably scaled random walks on the pre-fractal graphs S^n and then went to the limit in n, as the graph approaches the continuous curve. It should be noted, in this regard, that only the limit continuous set enjoys self-similarity at all scales $h = (1/2)^n$, hence only the limit fractal curve possesses the intrinsic invariance postulated in physical applications.

Out of Sierpinski models, Lindstrøm was able to describe a larger family of fractals - called by him *nested fractals* - intended to be a good mathematical model for what physicists call *finitely ramified* fractals, that is, self-similar bodies that can be disconnected by a finite number of cuts. Incidentally, a peculiar feature of Lindstrøm approach - also based on probability and the constructions of random walks - is the use of non-standard analysis. Non-standard analysis appeared indeed to him as the natural tool to get a continuous diffusion out of the approximating discrete random walks.

An analytic approach to the construction of fractal "Laplaceans" was taken by J. Kigami, [28], who used suitably scaled finite-difference schemes on the pre-fractals. Given a continuous function u on the curve, he defined $f = \Delta u$ to be the continuos function on the curve obtained as the uniform limit of suitable finite-difference approximations, provided this limit exists. Due to the restriction on its domain, this operator plays the role of a sort of analogue of the classic Laplacean on smooth functions.

We note, incidentally, that uniqueness and non-uniqueness results of "Laplaceans" on certain model fractals have been given by V. Metz, [41], R. Peirone, [53], C. Sabot, [58].

To show that both approaches - the probabilistic and the analytic one - are two sides of the same medal requires to show that there is a same *self-adjoint* operator in the role, on the one side, of the generator of the diffusion semigroup and, on the other side, in that of the Friedrichs extension of the analytic Laplacean considered by Kigami.

Here the theory of Dirichlet forms of Beurling-Deny came into play and allowed S. Kusuoka, [32], [33], and M. Fukushima - T. Shima, [20], to make the desired connection. It is perhaps worthwile to remember that random walks and finite-difference scheme,

Brownian motions and differential operators had already been brought together more or less sixty years before by R. Courant - K. O. Friedrichs - H. Lewy, in their by now famous paper [16].

Non finitely-ramified fractals are harder to deal with in rigorous mathematical terms. The typical example is, for example, the so-called *Sierpinski carpet*, which - in the plane - is obtained by a similar Cantor-like construction as for the Sierpinski curves, but with squares replacing triangles. The construction of nontrivial diffusion processes was done in this case by M. T. Barlow - R. F. Bass - J. Sherwood [4] and, in a series of papers, by M. T. Barlow - R. F. Bass, see *e.g.* [2], relying on probabilistic techniques. More analytically minded is the work carried out by S. Kusuoka - X. Y. Zhou, [34], and S. Kozlov, [29]. In the approach of these authors, certain energy estimates play the main role.

We do not attempt here to describe these mathematical results. We shall confine ourselves to outline an approach to the theory inspired to classic Riemmanian models.

3. The effective distance

We come back to the physical ideas of *ramification* and *decimation*. Having in mind classic Riemannian models, out of these physical concepts we can extract new *pseudo-Riemannian* notions which help us to better understand the strange behavior of fractals and other non-Euclidean structures, like certain homogeneous groups, [43].

Let us then go back to the Sierpinski curves and their ramifications. The curve S itself is clearly a non rectifiable set and there is no easy way to define a distance function on it. On the other hand, Riemannian geometry teaches us that a good metric, from the analytic point of view, is the one related to a gradient form and, eventually, to a differential operator of Laplace-Beltrami type.

Applied to our complicated curve, this means that a good candidate for a distance inside the curve - or, to start with, inside the pre-fractal S^n - must cope with the presence of ramifications in S^n and be associated with a gradient form for functions on S^n.

Note that, as a set, S^n can be identified with $I^n = \{1, \dots, D+1\}^n$ and S^n is a graph under the relation $\sim_n$. We recall that $p \sim_n q$ if p and q are two vertices of a same symplex of size $(1/2)^n$ occurring in S^n. Therefore, by connecting every two neighbouring points $p \sim_n q$ of S^n with a straight line segment, we get the union of the edges of all $(D+1)^n$ small simplexes, each one of size $(1/2)^n$, obtained at the step n of the iteration. Moreover, each node $p \in S^n - \Gamma$ is now at the crossing of $2D$ straight-line segments of length $(1/2)^n$.

For fixed n, the piece-wise straight-line curve so obtained exhibits ramification only at the scale $(1/2)^n$. In this sense, it is a poor approximation of the limit curve S constructed before.

We make S^n into a better approximation of the limit curve, by filling each small simplex of size $(1/2)^n$ - constructed before - not only with its edges, but with the whole portion of the limit curve which it encompasses. If we connect two vertices p and q of one of these simplexes - hence two contiguous nodes of S^n - with a generic path along the curve inside that simplex, we now meet points of ramifications at all scales $(1/2)^{n+k}$, $k = 1, 2, \dots$.

Note that we reach the same result if, instead of filling in whole portions of the limit curve, we fill each initial simplex of size $(1/2)^n$ only with the union of the edges of all

simplexes of smaller size $(1/2)^{n+k}$, $k = 1, 2, \ldots$, which originate from it by similarity.

Now we turn this fully ramified S^n into an *effective graph*, by defining on it a suitable *effective gradient* which takes ramification into account.

To achieve this, we introduce a parameter

$$\delta > 0$$

intended to be a sort of *effective measurement* of the ramification existing in our graph.

More precisely, we postulate that ramification - for every $n \geq 1$ - affects the initial Euclidean distance $|p - q| = (1/2)^n$ of two neighbouring point of S^n, by changing it into the distance

$$d_n(p, q) = |p - q|^\delta d_0(\xi, \eta) = (1/2)^{n\delta} d_0(\xi, \eta).$$

Here $d_0(\xi, \eta)$ is a given function assigned on $\Gamma \times \Gamma$ and ξ, η are points of Γ carried into p and q, respectively, by one of the maps $\psi_{i_1} \circ \ldots \circ \psi_{i_n}$.

The matrix d_0 is supposed to express the *effective distance* of two points of the initial simplex Γ, filled in with the whole S, or S^∞. In the present case, by obvious symmetry reasons, we choose $d_0(\xi, \eta)$ to be independent of the pair ξ, η, hence d_0 to be a constant, which, for simplicity, we put equal to 1 in the following.

Having incorporated the metric effect of ramification into the - by now unknown - parameter δ, hence in the effective distance, we are now in a position to define the effective gradient

$$\bigtriangledown_n u \cdot \bigtriangledown_n v$$

on S^n.

If u and v are arbitrary functions on S^n, for every fixed $p \in S^n$ we put

$$\bigtriangledown_n u \cdot \bigtriangledown_n v(p) = \sum_{q \sim_n p} \frac{u(p) - u(q)}{d_n(p, q)} \frac{v(p) - v(q)}{d_n(p, q)},$$

where, as before, the sum is extended to all q which are neighbours of p in S^n. Therefore,

$$\bigtriangledown_n u \cdot \bigtriangledown_n v(p) = 4^{n\delta} \sum_{q \sim_n p} (u(p) - u(q))(v(p) - v(q))$$

which shows more explicitely that δ is a sort of "control parameter" of the (effective) gradient on S^n.

The main point of our pseudo-Riemannian approach consists in choosing δ so that this gradient can be legitimated - in a convenient sense to be better specified - as a gradient of pseudo-Riemannian type.

The pseudo-Riemannian nature of the gradient - in our picture - should consist in that: distance, volume and energy on the graph must be in mutual relationship - all changes done - as on a Riemannian manifold.

On a Riemannian manifold, distance, volume and energy descend all from a same Riemannian metric tensor. Here - for the Sierpinski strings - we simply demand that:

The square distance scales as the Lagrangean form

$$\mathcal{L}_n(u,v) := \bigtriangledown_n u \cdot \bigtriangledown_n v \ \mu_n$$

under the action of the basic similarities.

Above,

$$\mu_n = \sum_{p \in S^n} (D+1)^{-n} \ \delta_{\{p\}}$$

is the volume measure of S^n, $\delta_{\{p\}}$ being a unit Dirac mass at the point p.

More explicitely, we require that the identities

$$d_n^2(p,q) = \sum_{i=1}^{D+1} \rho d_{n+1}^2(\psi_i(p), \psi_i(q))$$

as well as

$$\mathcal{L}_n(u,v) = \sum_{i=1}^{D+1} \rho \mathcal{L}_{n-1}(u \circ \psi_i, v \circ \psi_i),$$

expressing (self-similar)invariance of distance and energy, both hold with the *same* scaling factor $\rho > 0$.

The problem now consists in determining ρ and, in this way, finding the good value of δ. Since this is the crucial step, we shall do it in three equivalent ways: *decimation, Gauss principle, spectral gap.*

Decimation

By integrating the Lagrangean $\mathcal{L}_n(u,v)$ over S^n, we get the *total energy*

$$E_n(u,v) = \int \ d\mathcal{L}_n(u,v).$$

The Lagrangean is easily calculated:

$$\mathcal{L}_n(u,v)(dx) = 4^{n\delta}(D+1)^{-n} \sum_{p \in S^n} \sum_{q \sim_n x} (u(x)-u(q))(v(x)-v(q)) \ \delta_{\{p\}}(dx).$$

Therefore,

$$E_n(u,v) = 4^{n\delta}(D+1)^{-n} \sum_{p \in S^n} \sum_{q \sim_n p} (u(p)-u(q))(v(p)-v(q))$$

This identifies the scaling factor of the Lagrangean, hence of the total energy, as

$$\rho = 4^{\delta}(D+1)^{-1},$$

and gives δ as a function of ρ

$$\delta = \frac{\log((D+1)\rho)}{\log 4}.$$

By integrating by parts and taking boundary conditions - say, $u = 0$ on Γ - into account, we find

$$E_n(u,v) = -\int (\Delta_n u)v \; d\mu_n = \; -\sum_{p\in S^n} (\Delta_n u)(p)v(p)(D+1)^{-n}$$

where

$$(\Delta_n u)(p) := 4^{n\delta}(\sum_{q\sim p} u(q) - \chi\{q \sim_n p\}u(p)).$$

Here is where decimation comes into the picture. The operator Δ_n occurring in the previous relation takes the role - in our approach - of an (effective) Laplace-Beltrami operator on the graph S^n. On the other hand - according to the physical evidence described before - this operator must coincide with the pre-Laplacean Δ_n obtained by decimation. This requires the two scaling factors $4^{n\delta}$ and $(D+3)^n$ to be equal, that is, we must have

$$\delta = \frac{\log(D+3)}{\log 4},$$

hence

$$\rho = \frac{D+3}{D+1}.$$

Note that this identifies D_w as

$$D_w = 2\delta.$$

We can summarize what we have done sofar, by saying that the pre-Laplacean obtained by decimation can be interpreted to be a bona fide effective Laplace-Beltrami operator on the graph S^n, provided we introduce a suitable (effective) metric inside the graph.

The Gauss principle

The integrated version of the decimation principle is that the equilibrium potential minimizes the total energy of the circuit.

This can be expressed in the form of a *Gauss principle*, requiring the equilibrium solution to be given at a point by a suitable average of neighbouring points, see also K. J. Falconer - J. Hu [18]. This principle can be equivalently stated as a *harmonic variational principle* as follows:

The energy $E_n(u,u)$, obtained by minimizing $E_n(v,v)$ on S^n over the values of v in $S^n - S^{n-1}$ given the values of $v = u$ on S^{n-1}, must be equal to the energy $E_{n-1}(u,u)$.

It can be seen that, in order to calculate the scaling factor ρ, it suffices to apply this principle at the first step of the iteration, that is with $n = 1$, when $S^{n-1} = S^0 = \Gamma$. This requires solving a simple quadratic minimization problem and gives the same value of ρ as before. This is the usual way in which ρ is determined, for example in M. Fukushima - T. Shima [20], S. Kozlov [29], see also R. Capitanelli [13] for more details on this point.

The spectral gap

An interesting method is the one suggested by S. Kusuoka - X. Y. Zhou in [34], presented here in a slightly different form, the crucial notion being now that of spectral gap. By *spectral gap* of S^n we mean the number

$$\lambda_1^{\mathrm{NEUM}} = \frac{1}{c_P(n)},$$

where $c_P(n)$ denotes the best constant in the *Poincaré inequality*

$$\int_{S^n} |u - \bar{u}_n|^2 d\mu_n \leq c_P(n) \int_{S^n} \bigtriangledown_n u \cdot \bigtriangledown_n u d\mu_n,$$

where

$$\bar{u}_n = \mu_n(S^n)^{-1} \int_{S^n} u d\mu_n.$$

It is clear that the choice of δ in the gradient form affects the best Poincaré constant $c_P(n)$, hence the eigenvalue $\lambda_1^{\mathrm{NEUM}}(n)$. In other words, in this picture, δ is the *control parameter* of the spectral gap.

By *spectral gap control*, we now mean that $\lambda_1^{\mathrm{NEUM}}(n)$ stays bounded and away from 0, as $n \to \infty$. That is, we look for two constants c_1, c_2, such that

$$0 \ < \ c_1 \ \leq \ \lambda_1^{\mathrm{NEUM}}(n) \leq c_2 \ < \infty$$

for every n.

This non-triviality of the spectral gap is indeed the fundamental property which really turns our geometric curve into an elastic body. In fact, if $\lambda_1^{\mathrm{NEUM}}(n)$ vanishes as $n \to \infty$, then the Poincaré constant $c_P(n)$ grows up to ∞. This means that, in the left side of Poincaré inequality, we can achieve infinite displacements with, at the right hand side, vanishing energy: the continuous curve exhibits no resistance at all to deformation under external forces, hence it has trivial elasticity. On the other hand, if $\lambda_1^{\mathrm{NEUM}}(n)$ grows up to ∞, then $c_P(n)$ vanishes and we get no displacement at all at the left hand side of the inequality, notwithstanding how big the energy at the right hand side of the inequality is: the curve stays stiff like a stone, again no meaningful elasticity is inherited in the limit.

It is easily seen that if a value of the parameter δ exists, such that both uniform bounds on $\lambda_1^{\mathrm{NEUM}}(n)$ hold, then this value is unique. In fact, if δ^* is such a value and we replace it by $\delta < \delta^*$ in the expression of the effective gradient, then this would affect $\lambda_1^{\mathrm{NEUM}}(n)$ by a factor $4^{n(\delta-\delta^*)}$, leading to a vanishing spectral gap. Similarly, the uniform upper bound $\lambda_1^{\mathrm{NEUM}}(n)$ would be violated, if we take $\delta > \delta^*$.

It can be seen by direct calculations that the good value of δ, assuring both uniform bounds of $\lambda_1^{\mathrm{NEUM}}(n)$ from below and from above, is - for the Sierpinski strings - the same value

$$\delta = \frac{\log(D+3)}{2\log 2}$$

obtained before.

It should be noted that, in order to fulfill the condition on the spectral gap, no exact calculations are required in principle, but only *estimates* of energy type. This brings this topic in connection with *homogenization theory* and introduces flexibility in the strict

self-similar approach represented by decimation and other similar techniques, see also S. Kozlov [29] in this respect.

We consider the control of spectral gap - as specified by the uniform estimates above - to be the appropriate tool, in an effective pseudo-Riemannian approach, to build up a *structural theory* of fractal bodies, stable under energy perturbations in the sense of homogenization. That is, a theory aimed at establishing *uniform estimates* for a whole class of operators with comparable Lagrangean forms, as in the classic theory of uniformly elliptic operators of De Giorgi-Nash-Moser type.

The main lines of such a general theory emerge from the example of the Sierpinski strings, as we shall see below. Before, we shall better describe the analytic objects - *variational fractals* - obtained by the previous construction in the limit as $n \to \infty$.

4. Variational fractals

When applied to the Sierpinski strings, the analytic theory we have in mind refers not to the graphs S^n considered up to now, but to the (nonrectifiable) curve S itself. This curve has been constructed in Section 1 as the limit of S^n as $n \to \infty$. As already noted, it is S - not S^n - the self-similar set under the action of the mappings ψ_i, $i = 1, \dots, D+1$, that is,

$$S = \bigcup_{i=1}^{D+1} \psi_i(S),$$

therefore it is S the natural candidate for an invariant theory.

S is a compact subset of R^D and the Hausdorff dimension of S, as already mentioned in Section 2, is

$$D_f = \frac{\log(D+1)}{\log 2}.$$

The natural "volume measure" in S is the restriction to S of the D_f- dimensional Hausdorff measure of R^D, normalized to have total mass 1, that is, the measure

$$\mu = (H^{D_f}(S))^{-1}\ H^{D_f}|S.$$

It is easy to see that μ is the weak limit in S of the discrete measures μ_n - supported on S^n - as $n \to \infty$. The measure μ is also self-similar, this property being expressed by the identity

$$\mu(F) = \frac{1}{D+1} \sum_{i=1}^{D+1} \mu(\psi_i^{-1}(F))$$

for every Borel set F in S.

We can thus construct the Hilbert space $L^2(S, \mu)$ on S, which of course plays an important role in the whole theory.

More problematic is taking the limit of the graph Lagrangeans $\mathcal{L}_n$ as $n \to \infty$. Indeed, the pointwise values $\mathcal{L}_n(u, v)(p)$ at a fixed node $p \in S^\infty$ are all well defined if n is large enough, however they oscillate wildly and do not converge as $n \to \infty$ for nontrivial functions u and v.

What helps here is that the discrete measures $\mathcal{L}_n(u,v)$ do indeed converge *weakly* to a measure $\mathcal{L}(u,v)$ on S as $n \to \infty$, for a *dense* family $\mathcal{C}$ of (continuous) functions u and v in $L^2(S,\mu)$. Moreover, as a function of u and v, the measure $\mathcal{L}(u,v)$ inherits from the discrete $\mathcal{L}_n(u,v)$ the nature of a gradient map. In fact, the following *chain rule*

$$\mathcal{L}(g(u),v) \;=\; g'(u)\;\mathcal{L}(u,v)$$

is satisfied in the measure sense, for every real function $g \in C^1(R,R)$ and for all functions $u, v \in \mathcal{C}$. We shall call such a measure-valued map a $Lagrangean form$.

The Lagrangean form on S turns out to be *closable* in the space $L^2(S,\mu)$. This means that the abstract completion of $\mathcal{C}$ in the norm

$$||u|| = \;(\int u^2\; d\mu \;+\; \int d\mathcal{L}(u,u))^{1/2},$$

is *injected* in the space $L^2(S,\mu)$.

The Lagrangean $\mathcal{L}$ is also self-similar. In fact, it satisfies the identity

$$\mathcal{L}(u,v) = \sum_{i=1}^{N}\; \rho\mathcal{L}(u\circ\psi_i, v\circ\psi_i)$$

for every $u, v \in \mathcal{C}$, that follows from the iterative analogous similarity relation of the discrete $\mathcal{L}_n$. The parameter ρ is the one already calculated in Section 3. For more details on the construction of $\mathcal{L}$ see for example S. Kusuoka [33], or R. Capitanelli [11].

As shown by Kusuoka, [32], however, the measure $\mathcal{L}(u,v)$ - for nontrivial u and v - is *singular* with respect to m: this makes life difficult!

In analogy with what we have done on the graphs, we now define the effective distance

$$d(x,y) := \;|x-y|^{\delta}$$

for $x, y \in S$, by choosing δ such that d^2 enjoys the same invariant scaling as $\mathcal{L}$, that is

$$d^2(x,y) = \sum_{i=1}^{N} \rho d^2(\psi_i(x),\psi_i(y).$$

It is easy to check that this gives a unique value of δ, the same as in Section 3.

An important property, which mutually relies S, μ and $\mathcal{L}$, follows from the estimates leading to the spectral gap control of Section 3. Namely, the following global Poincaré inequality holds

$$\int_S\; |u-\bar{u}|^2\; d\mu \;\leq\; c_P\; \int_S d\mathcal{L}(u,u)$$

for every $u, v \in \mathcal{C}$, $\bar{u} = \mu(S)^{-1}\int_S u d\mu$.

This inequality is a strong form of global *irreducibility* of $\mathcal{L}$, because it implies, in particular, that if $u \in \mathcal{C}$ is such that $\mathcal{L}(u,u) = 0$ on S, then u is a constant. The physical meaning of this property is the one already described in Section 3.

A triple

$$(S, \mu, \mathcal{L})$$

with the above properties, has been called a *variational fractal* in [44], [45]. This is a class of self-similar fractals for which a variational theory of Laplace and heat equation can be carried on, as we shall now briefly describe.

By integration of $\mathcal{L}(u, v)$ over S we get the *energy form*

$$E(u, v) = \int d\mathcal{L}(u, v)$$

on the domain $\mathcal{C}$. The closure of E - taken initially with domain $\mathcal{C}$ - in the norm $||u||$ is a closed form in $L^2(S, \mu)$, with dense domain denoted D_E. We keep the notation E also for the closed form.

We shall also denote D_E by H^1, in analogy with the classic theory, in which $\mathcal{C} = \mathcal{C}^\infty$, $\mu = dx$, $\mathcal{L}(u, v) = \bigtriangledown u \cdot \bigtriangledown v \, dx$, and the domain of the closed form $E(u, v) = \int \bigtriangledown u \cdot \bigtriangledown v \, dx$ is the usual Sobolev space H^1. Similarly, by H^1_0 we denote the closure in H^1 of all functions of $\mathcal{C}$ which vanish on Γ.

By the general theory of forms, there exists then a *self-adjoint operator* Δ^{DIR} in $L^2(S, \mu)$, with domain $D_{\Delta^{DIR}}$ dense in H^1_0, such that

$$E(u, v) = - \int (\Delta^{DIR} u) \; v \; d\mu$$

for every $u \in D_{\Delta^{DIR}}$ and every $v \in H^1_0$. This is the *Laplace operator* in S, with Dirichlet boundary conditions. Similarly defined is the Laplace operator with Neumann boundary conditions Δ^{NEUM}, obtained by replacing H^1_0 with H^1 in the preceding identity.

By relying on the spectral theory of self-adjoint operators, we can also construct the semigroup

$$P_t = e^{\Delta^{DIR} t}, \quad t > 0.$$

Thus we have all the ingredients to develop a theory of Laplace and heat equations and related Green function $g(x, y)$ and heat kernel $p_t(x, y)$.

We do not intend to develop further the theory of these equations here. We refer to the papers mentioned in Section 2 and also to [45], [46]. Instead, we prefer to describe how self-similar invariance can be turned into metric invariance and, consequently, how a purely metric fractal theory - that of *metric fractals* - can be extracted from the previous example of a variational fractal.

5. Metric fractals

The variational fractal $S = (S, \mu, \mathcal{L})$ constructed in Section 4 enjoys self-similar invariance with respect to the mappings $\psi_1, \ldots, \psi_{D+1}$ generating the Sierpinski string under consideration.

We recall that S is a quasi-metric space endowed with the quasi-distance d. We also recall that d^2 also enjoys self-similar invariance, with the same factor as $\mathcal{L}$.

Clearly, two concentric balls - in the quasi-metric d - are not invariant under the action of the mappings ψ_i. However, it can be proved that there exists a constant $M \geq 1$, independent of x and R, such that each ball $B_R \equiv B_R(x)$ can be approximated by the union of at most M sets

$$S_{i_1,\dots,i_k} := \psi_{i_1} \circ \cdots \circ \psi_{i_k}(S),$$

whose diameter - in the metric d - is of order R, *i.e.*, $diam S_{i_1,\dots,i_k} \approx R$.

To simplify notation, above - and in the following - we write $A \approx B$, where A and B are two positive quantities, to mean that $c_1 A \leq B \leq c_2 A$, for some constants c_1 and c_2, independent of x and R and of all functions possibly involved in A and B, depending only on the structural constants of X.

As shown in [44], [45], by using this approximation and the self-similarity relations of Section 4, it is possible to derive the following inequalities:

$$\frac{m(B_r)}{m(B_R)} \approx (\frac{r}{R})^{\nu}$$

$$\int_{B_R} |u - \bar{u}|^2 dm \leq c_P R^2 \int_{B_{qR}} d\mathcal{L}(u,u)$$

$$cap(B_R, B_{2qR}) \leq c_C \frac{m(B_R)}{R^2},$$

where $m = \mu$, which hold for every $0 < p \leq R < R_0$ and every $B_R = B_R(x)$ and where $\bar{u}$ is the mean-value of $u \in \mathcal{C}$ in $B_R, q \geq 1$.

We have single out these inequalities because they capture the main dynamic features of our fractals. Indeed, as shown below, they bring into light the pseudo-Riemannian nature of the effective metric.

We call *metric fractal* any triple $(X, m, \mathcal{L})$ with the following properties:

X is a quasi-metric space, endowed with a quasi distance d, that is, X is a topological space and d a quasi-distance on X (*i.e.*, d is a function on $X \times X$ that has the properties of a distance except for the triangle inequality, satisfied up to a mutiplicative constant $c_T \geq 1$), such that the (quasi-) balls $B_R(x)$, $R > 0$, of d form a basis of neighborhoods at each point $x \in X$;

m is a positive measure supported on X;

$\mathcal{L}$ is a measure-valued Lagrangean form on a dense subalgebra $\mathcal{C}$ of the space of bounded continuous functions on X, closable in $L^2(X, m)$,

d, m and $\mathcal{L}$ being mutually related by the family of inequalities on balls listed above, where now ν, $q \geq 1$, c_P and c_C are any given positive constants, independent of $x \in X$ and of R, for $0 < r \leq R < R_0$, $R_0 \in (0, \infty]$, and of $u \in \mathcal{C}$.

Before proceeding, let us make a few comments on these inequalities.

The inequality satisfied by m says that the volume in X has *polynomial growth*.

The Poincaré inequality is a weak *lower bound* for the spectral gap, that is, for the first nontrivial eigenvalue $\lambda_1^{\mathrm{NEUM}}(B_R)$ on the ball B_R. In fact, if $q = 1$, then by Raleigh-Ritz formula we find

$$\lambda_1^{\mathrm{NEUM}}(B_R) = \mathrm{m}in \quad \frac{\int_{B_R} d\mathcal{L}(u,u)}{\int_{B_R} |u - \bar{u}|^2 dm} \geq c_P^{-1} R^{-2}$$

hence

$$c_P^{-1} \leq R^2 \lambda_1^{\mathrm{NEUM}}(B_R).$$

The capacity inequality gives a *upper bound* for the first eigenvalue $\lambda_1^{DIR}(B_R)$ of the Dirichlet problem with vanishing boundary condition on the boundary of B_R. In fact, again by Raleigh-Ritz formula, if Φ is the capacitary potential of B_R in B_{2qR}, we have

$$m(B_R) = \int_{B_R} \Phi^2 dm$$

$$\leq \frac{\int_{B_{2qR}} \Phi^2 dm}{\int_{B_{2qR}} d\mathcal{L}(\Phi,\Phi)} \int_{B_{2qR}} d\mathcal{L}(\Phi,\Phi)$$

$$\leq (\lambda_1^{DIR}(B_{2qR}))^{-1} cap(B_R, B_{2qR})$$

$$\leq c_C (\lambda_1^{DIR}(B_{2qR}))^{-1} \frac{m(B_R)}{R^2}$$

therefore

$$R^2 \lambda_1^{DIR}(B_R) \leq 4q^2 c_C.$$

These spectral bounds, together with the polynomial growth of the volume, can be seen as the characteristic features of metric fractals. Note that the above set of inequalities is stable under changes to equivalent quasi-distances, measures and Lagrangeans. In this sense, the inequalities express "effective" properties of X.

Before mentioning some of the results that hold on metric fractals, let us give further examples which better show the pseudo-Riemannian character of these structures.

A first classic example of a metric fractal is the one arising in the obvious way from any second order uniformly elliptic operator in divergence form, with bounded measurable coefficients, as in the theory of De Giorgi, Nash and Moser.

Another classic example is given by X a complete D-dimensional Riemannian manifold with Ricci curvature bounded from below. Then, if $k_0 \geq 0$ is the constant occurring in the lower bound $Ric \geq -k_0 g$ of the curvature, *Bishop-Gromov comparison theorem* gives the bound

$$|B_{2R}| \leq 2^D \sqrt{(D-1)k_0} 2R |B_R|$$

see J. Cheeger - M. Gromov - M. Taylor [15], while *Buser inequality* gives

$$\int_{B_R} |u - \bar{u}_{B_R}| dvol \ \leq cexp(\sqrt{k_0}R) R \int_{B_R} |\bigtriangledown u| dvol$$

c a constant depending only on D and $|B| = vol(B)$, see P. Buser [10].

A semi-classic example is the one of certain stratified homogeneous Lie groups, like the *Heisenberg groups*. The simplest of these Lie groups is R^3 with the vector fields

$$Y_1 = \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_3}$$

$$Y_1 = \frac{\partial}{\partial x_2} - 2x_1 \frac{\partial}{\partial x_3},$$

that satisfy a Hörmander condition of length $L = 2$: the vector fields $\{Y_1, Y_2, [Y_1, Y_2]\}$ span the whole R^3 at every point. The distance d here is the one, $d = d^{OPT}$, given by geometric optics along the fields, namely $d^{OPT}(x, \cdot)$ is the maximal subsolution - vanishing at x - of the *eiconal equation*

$$|Y\Phi|^2 \equiv |Y_1\Phi|^2 + |Y_2\Phi|^2 = 1,$$

that is,

$$d^{OPT}(x, y) = sup\{\Phi(x) - \Phi(y) : |Y\Phi|^2 \leq 1,\ \Phi \in C_0^\infty\}.$$

Then, the following inequalities hold, due to A. Nagel - E. M. Stein - S. Wainger [50], L. P. Rothschild - E. M. Stein [57], C. L. Fefferman - D. H. Phong [19], D. Jerison [25]:

$$d^{OPT}(x, y) \leq c|x - y|^\delta$$

where now $\delta = 1/L = 1/2$;

$$|B_{2R}| \leq c|B_R|;$$

$$\int_{B_R} |u - \bar{u}|^2 dx \leq cR^2 \int_{B_R} |Yu|^2 dx.$$

Note that now δ is *smaller* than 1, to compare with the previous fractal case, in which $\delta > 1$: fractals and degenerate subelliptic operators of Hörmander type show opposite intrinsic metric behaviour! It is in some sense surprising that a common *effective* metric theory can be developed for both.

In this regard, let us also notice that the theory applies also to *uniformly subelliptic* operators with possibly discontinuous coefficients. In the previous Heisenberg case, for example, these operators are obtained by replacing Y_hY_k with $Y_h\alpha_{h,k}Y_k$ - where $\alpha_{h,k}$ is a uniformly positive-definite 2×2 matrix of bounded measurable functions on R^3 - hence by replacing the (smooth) operator $\sum_{h,k} Y_hY_k$ with the operator $\sum_{h,k} Y_h\alpha_{h,k}Y_k$ whose coefficients are discontinuous, see [42].

Let us now mention some of the main results that can be obtained in the general framework of metric fractals.

Inequalities of Morrey-Sobolev type, in which ν plays the role of critical exponent, and inequalities of John-Nirenberg type when $\nu = 2$, as well as Nash inequalities for all dimension $\nu > 0$, see M. Biroli - U. Mosco [7], [8], J. Malý - U. Mosco [38], R. Capitanelli [12].

Moreover, Harnack inequalities of De Giorgi-Nash-Moser type in the case $\nu < 2$, which, by the way, includes the fractal example of Sierpinski strings, [49].

Harnack inequalities in the case $\nu > 2$ have been obtained in [9] - in a different but essentially equivalent framework - under the additional assumption that the Lagrangean measure $\mathcal{L}(u, v)$ admits a density with respect to the underlying volume measure m. This assumption, however, rules out the fractal case - as we have seen in the example of Sierpinski curves.

The study of Harnack inequality in the general case, certainly one of the main objective of the theory, is the object of current research. The main difficulty relies in the fact that cut-off functions of finite energy are not easily available in the present context.

When Harnack inequalities are at hand, then estimates of fundamental solutions can also be obtained. For example, the Green function $g_{B_R}(x_0, x)$ for the Dirichlet problem in a ball B_R can be estimated on the boundary of a smaller concentric ball B_r by

$$g_{B_R}(x_0, x)|_{x \in \partial B_r} \approx \frac{1}{2-\nu}(R^{2-\nu} - r^{2-\nu})$$

if $\nu \neq 2$, and by

$$g_{B_R}(x_0, x)|_{x \in \partial B_r} \approx \log(\frac{R}{r}),$$

if $\nu = 2$. These estimates reproduce the Euclidean behavior, showing the *universality* of the effective metric theory presented here.

6. Remarks and conclusions

We conclude by mentioning a few other results, in the spirit of partial differential equations and convex analysis.

Semilinear equations of the type

$$-\Delta u - \lambda u + f(u) = 0$$

have been studied on variational fractals by M. Biroli - S. Tersian [6], K. J. Falconer [17], K. J. Falconer - J. Hu [18], M. Matzeu [40], the critical exponent being $(\nu + 2)/(\nu - 2)$, in analogy with the Euclidean case.

Transmission problems accross a fractal layer Σ - *e.g.*, the Koch curve - in a plane domain Ω have been recently studied by M. R. Lancia [35] and M.R. Lancia - M. A. Vivaldi [36]. These are problems of the kind considered by H. Pham Huy - E. Sanchez Palencia in the 70's, [54], involving a *second order* transmission condition, of the type

$$-\Delta u = f \quad in \quad \Omega - \Sigma, \qquad \frac{\partial u}{\partial n^+} - \frac{\partial u}{\partial n^-} = \Delta_\Sigma u,$$

where Δ is the usual Laplace operator in R^2, Δ_Σ is the fractal Laplacean on the Koch curve - constructed in a similar way as on the Sierpinski strings - and ∂n^+, ∂n^- denote the normal derivatives at the opposite sides of Σ in Ω. This study relies on the recent theory of *Besov-* and *Lipschitz spaces* on possibly very irregular sets, due to A. Jonsson, H. Wallin and H. Triebel, [26], [27], [62]. Similar problems have been also studied by T. Lindstrøm, [37], and T. Kumagai, [30], without, however, making the transmission condition explicit.

Convex energy functionals - and related Sobolev spaces $W^{1,p}$ and p-Laplaceans - have been constructed by R. Capitanelli, [12] on certain model fractals like the Koch curve and then compared with Besov and Lipschitz spaces by M. R. Lancia - R. Capitanelli in [14].

Finally, certain aspects of fractal theory, like existence of general *selfsimilar measures* in *quasi-metric spaces* and dimensionality problems for *affine fractals*, have been studied by M. V. Marchi [39] - where a construction of a fractal set in the Heisenberg group is also given - and by M. P. Bernardi - C. Bondioli [5], see also [48].

As a conclusion, we can say that various classic, semiclassic and fractal analytic theories of general - possibly intrinsically non-Euclidean and fractal - "elastic bodies" can be built on a common metric background, named above a *metric fractal*. This "primitive" structure is made of *quasi-distances*, *doubling measures* and *measure-valued Lagrangeans*, related by the fundamental property loosely stated by saying that Lagrangeans control mean oscillations at all metric scales.

The main analytic "mechanisms" generating these structures are *Lie group invariance*, from the one side, and *self-similar invariance* on the other side. In both cases, the non-Euclidean nature of the structure is incorporated in the intrinsic effective metric - typically with $\delta < 1$ in the first case, $\delta > 1$ in the fractal case - this way leading to "universal" scaling laws.

The theory is "effective", in the sense that - as in homogenization - all estimates hold uniformly for equivalent metrics and energy forms.

References

[1] S. Alexander, R. Orbach: Densities of states on fractals: “fractions”, J. Physique Lett. 43 (1982) L625.

[2] M. T. Barlow, R. F. Bass: The construction of Brownian motion on the Sierpinski carpet, Ann. Inst. Henri Poincaré, Probab. Stat. 25 (1989) 225–257.

[3] M. T. Barlow, E. A. Perkins: Brownian motion on the Sierpinski gasket, Probab. Theory. Related Fields 79 (1988) 543–623.

[4] M. T. Barlow, R. F. Bass, J. Sherwood: Resistance and spectral dimension of Sierpinski carpets, J. Phys. A, Math. Gen. 23 (1990) L253.

[5] M. P. Bernardi, C. Bondioli: On some dimension problems for self–affine fractals, Zeitschr. Anal. Anw. 18 (1999) 733–751.

[6] M. Biroli, S. Tersian: On the existence of nontrivial solutions to a semilinear equations relative to a Dirichlet form, Rend. Ist. Lomb. Scienze e Lettere 131 (1997) 151–168.

[7] M. Biroli, U. Mosco: Sobolev and isoperimetric inequalities for Dirichlet forms on homogeneous spaces, Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat., Rend. Lincei, Mat. Appl. 6 (1995) 37–44.

[8] M. Biroli, U. Mosco, Sobolev inequalities on homogeneous spaces, Potential Anal. 4 (1995) 311–324.

[9] M. Biroli, U. Mosco: A Saint-Venant type principle for Dirichlet forms on discontinuous media, Ann. Mat. Pura Appl. 169 (1995) 125–181.

[10] P. Buser: A note on the isoperimetric constant, Ann. Sci. Ecole Norm. Sup. 15 (1982) 213–230.

[11] R. Capitanelli: Lagrangians on homogeneous spaces, PhD Thesis, Univ. Roma "La Sapienza".

[12] R. Capitanelli: Functional inequalities for measure valued Lagrangeans on homogeneous spaces, Univ. Roma "La Sapienza", 2001 (preprint).

[13] R. Capitanelli: Nonlinear energy forms on certain fractal curves, J. Convex Analysis, to appear.

[14] R. Capitanelli, M. R. Lancia: Nonlinear energy forms and Lipschitz spaces on the Koch curve, J. Convex Analysis 9 (2002) 245–257.

[15] J. Cheeger, M. Gromov, M. Taylor: Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds, J. Differential Geom. 17 (1982) 15–33.

[16] R. Courant, K. O. Friedrichs, H. Lewy: Über partielle Differenzengleichungen der mathematischen Physik, Math. Ann. 100 (1928) 32–74.

[17] K. J. Falconer: Semilinear PDEs on self-similar fractals, Comm. Math. Phys. 206 (1999) 235–245.

[18] K. J. Falconer, J. Hu: Non-linear elliptical equations on the Sierpinski gasket, Journal of Math. Analysis and Appl. 240 (1999) 552–573.

[19] C. L. Fefferman, D. H. Phong: Subelliptic eigenvalue problems, in: Conference on Harmonic Analysis in Honor of Antoni Zygmund (Chicago, 1981) 590–606, Wadsworth (1983).

[20] M. Fukushima, T. Shima: On a spectral analysis for the Sierpinski gasket, Potential Anal. 1 (1992) 1–35.

[21] Y. Gefen, A. Aharony, S. Alexander: Anomalous diffusion on percolating clusters, Phys. Rev. Lett. 50 (1983) L 77.

[22] Y. Gefen, A. Aharony, B. B. Mandelbrot, S. Kirkpatrick: Solvable fractal family, and its relation to the backbone at percolation, Phys. Rev. Lett. 47 (1981) L 1771.

[23] S. Goldstein: Random walks and diffusions on fractals, in: Percolation Theory and Ergodic Theory of Infinite Particle Systems (Minneapolis, 1984–1985), IMA Vol. Math. Appl. 8, Springer (1987) 121–129.

[24] S. Havlin, Ben Avraham: Diffusions in disordered media, Adv. Phys. 36 (1987) 695–799.

[25] D. Jerison: The Poincaré inequality for vector fields satysfying Hörmander's condition, Duke Math. J. 53 (1986) 503–523.

[26] A. Jonsson, H. Wallin: The dual of Besov spaces on fractals, Studia Math. 112 (1995) 285–300.

[27] A. Jonsson, H. Wallin: Function Spaces on Subsets of $\mathbb{R}^n$, Harwood Acad. Publ. (1984).

[28] J. Kigami: Harmonic calculus on p.c.f. selfsimilar sets, Trans. Amer. Math. Soc. 335 (1993) 721–755.

[29] S. M. Kozlov: Harmonization and homogenization on fractals, Comm. Math. Phys. 153 (1993) 339–357.

[30] T. Kumagai: Browian motion penetrating fractals: an application of the trace theorem of Besov spaces, J. Funct. Anal. 170 (2000) 69–92.

[31] S. Kusuoka: A diffusion process on a fractal, in: Probabilistic Methods in Mathematical Physics (Katata/Kyoto, 1985), Academic Press (1987) 251–274.

[32] S. Kusuoka: Dirichlet forms on fractals and product of random matrices, Publ. Res. Inst. Math. Sci. 25 (1989) 659–680.

[33] S. Kusuoka: Diffusion Processes in Nested Fractals, Lect. Notes in Math. 1567, Springer (1993).

[34] S. Kusuoka, X. Y. Zhou: Dirichlet forms on fractals: Poincaré constant and resistance, Prob. Th. Rel. Fields 93 (1992) 169–196.

[35] M. R. Lancia: A transmission problem with a fractal interface, Z. Anal. Anwend., to appear.

[36] M. R. Lancia, M. A. Vivaldi: On the regularity of the solutions for transmission problems, Adv. Math. Sci. Appl., to appear.

[37] T. Lindstrøm: Brownian motion penetrating the Sierpinski gasket, in: Asymptotic Problems in Probability Theory, Stochastic Models and Diffusions on Fractals, Longman Scientific (1993) 248–278.

[38] J. Malý, U. Mosco: Remarks on measure-valued Lagrangians on homogeneous spaces, Ricerche Mat. 48, suppl. (1999) 217–231.

[39] M. V. Marchi: Self–similarity in spaces of hogemenous type, Adv. Math. Sci. App. 9 (1999) 851–870.

[40] M. Matzeu: Mountain Pass and Linking Type Solutions for Semilinear Dirichlet Forms, in: Recent Trends in Nonlinear Analysis, Prog. Nonlinear Differential Equations Appl. 40, Birkhäuser (2000) 217–231.

[41] V. Metz: How many diffusions exist on the Viscek snowflake, Acta Appl. Math. 32 (1993) 227–241.

[42] U. Mosco: Uniformly subelliptic operators with measurable coefficients, Atti Sem. Mat. Fis. Univ. Modena 43 (1995) 209–224.

[43] U. Mosco: Invariant field metrics and dynamical scalings on fractals, Phys. Rev. Letters 79, 21 (1997) 4067–4070.

[44] U. Mosco: Variational metrics on self-similar fractals, C. R. Acad. Sci. Paris 321 (1995) 715–720.

[45] U. Mosco: Variational fractals, Ann. Scuola Norm. Sup. Pisa, Special Volume in Memory of E. De Giorgi 4 (1997) 683-712.

[46] U. Mosco: Dirichlet forms and self-similarity, in: New Directions in Dirichlet Forms, AMS/IP Stud. Adv. Math. 8, Amer. Math. Soc. (1998) 117–155.

[47] U. Mosco: Lagrangian metrics and fractal dynamics, in: Fractal Geometry and Stochastics II, Univ. Greifswald, Birkhäuser, Prog. Probab. 46 (2000) 269–283.

[48] U. Mosco: Self-similar measures in quasi-metric spaces, Progress in Nonlinear Diff. Eq. and Appl. 40, Birkhäuser (2000) 233–248.

[49] U. Mosco: Harnack inequalities on some recurrent metric fractals, J. Steklov Math. Inst., Moscow, to appear.

[50] A. Nagel, E. M. Stein, S. Wainger: Balls and metrics defined by vector fields. I: Basic properties, Acta Math. 155 (1985) 103–147.

[51] T. Nakayama, K. Yakubo, R. L. Orbach: Dynamical properties of fractal networks: scaling, numerical simulations, and physical realizations, Rev. Mod. Phys. 66 (1994) L381.

[52] H. Osada: Self-similar diffusion on a class of infinitely ramified fractals, J. Math. Soc. Japan 47 (1995) 591–616.

[53] R. Peirone: Convergence and uniqueness problems for Dirichlet forms on fractals, Boll. Unione Mat. Ital., Sez B, Artic. Ric. Mat. 3 (2000) 431–460.

[54] H. Pham Huy, E. Sanchez-Palencia: Phénomènes des transmission à travers des couches minces de conductivité élevée, J. Math. Anal. Appl. 47 (1974) 284–309.

[55] R. Rammal: Spectrum of harmonic exitations on fractals, J. Physique 45 (1984) 191–206.

[56] R. Rammal, G. Toulose: Random walks on fractal structures and percolation cluster, J. Physique Lett. 44 (1983) L13–L22.

[57] L. P. Rothschild, E. M. Stein: Hypoelliptic differential operators and nilpotent groups, Acta Math. 137 (1976) 247–320.

[58] C. Sabot: Existence et unicité de la diffusion sur un ensemble fractal, C. R. Acad. Sci. Paris 321 (1995) 1053–1059.

[59] W. Sierpiński: Sur une courbe dont tout point est un point de ramification, C. R. Acad. Sci. Paris 160 (1915) 302–306.

[60] D. Stauffer: Introduction to Percolation Theory, Taylor and Francis (1985).

[61] R. B. Stinchcombe, C. H. Harris: Dynamic scaling near the percolation threshold in the diluted Heisenberg chain, J. Phys. A 16 (1983) 4083–4095.

[62] H. Triebel: Fractals and Spectra, Monographs in Mathematics 91, Birkhäuser (1997).

[63] H. Weyl: Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung), Math. Ann. 71 (1912) 441–479.