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Another Counterexample to Lower Semicontinuity in Calculus of Variations

Robert Černý
Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic
rcerny@karlin.mff.cuni.cz

Jan Malý
Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic
maly@karlin.mff.cuni.cz

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An example is shown of a functional

$$F(u) = \int_I f(u, u')\, dt$$

which is not lower semicontinuous with respect to L^1-convergence. The function f is nonnegative, continuous and strictly convex in the second variable for each $u \in \mathbb{R}^n$.

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1. Introduction

In the recent paper [3], the authors construct an example of functional of type

$$F(u) = \int_\Omega f(z, u, \nabla u)\, dz \tag{1}$$

on $W^{1,1}(\Omega; \mathbb{R}^d)$ which is not lower semicontinuous with respect to L^1-convergence, although the integrand is convex in the last variable. In this paper we present another such example, but of different nature and giving answer to other questions.

Here $\Omega \subset \mathbb{R}^N$ is an open set and $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N}$ is a nonnegative function convex in the last variable.

The lower semicontinuity questions are important in connection with searching for minimizers in the Calculus of Variations. We refer to the books [1], [2], [7], [11] for treatments, notes and bibliography.

If the integrand f (nonnegative, convex) depends only on the last variable, then the functional F is lower semicontinuous with respect to L^1-weak convergence without any additional assumptions, see [10], [9, Th. 1.8.1.]. The situation is more complicated in the general case. The following theorem is due to Serrin [10]. Let us emphasize that $d = 1$ in Serrin's theorem.

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Serrin's l.s.c. theorem. *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ be a continuous function, with $f(z, \zeta, \cdot)$ convex. Assume that f satisfies one of the following conditions:*

(a) $f(z, \zeta, \bar{\zeta}) \to \infty$ *as* $|\bar{\zeta}| \to \infty$ *for each* $(z, \zeta) \in \Omega \times \mathbb{R}$.

(b) $f(z, \zeta, \cdot)$ *is strictly convex for each* $(z, \zeta) \in \Omega \times \mathbb{R}$.

(c) *The derivatives* f_z, $f_{\bar{\zeta}}$ *and* $f_{\bar{\zeta}x}$ *exist and are continuous.*

Then F is lower semicontinuous in $W^{1,1}_{\mathrm{loc}}(\Omega)$ with respect to local convergence in L^1.

Recently the lower semicontinuity problems for convex and generalized-convex functionals were treated by Fonseca and Leoni [5], [6], [8]. They obtained a lower semicontinuity results for vector-valued case. Namely, they assume that f is a lower semicontinuous function convex in the last variable satisfying the linear coercivity assumption

$$f(x, \zeta, \overline{\zeta}) \geq c|\overline{\zeta}| - \frac{1}{c}$$

for some $c > 0$, and that for each $(x_0, u_0) \in \Omega \times \mathbb{R}^d$ there exists a continuous function g on a neighborhood U of (x_0, u_0) with values in $\mathbb{R}^{d \times N}$ such that $f(x, u, g(x, u)) \in L^\infty(U)$. This assumption on existence of such a g might look artificial, however the example in [3] shows that it cannot be dropped.

The vectorial case is much different from the scalar one and various counterexamples are useful to complete the picture. Eisen [4] constructed a counterexample to lower semicontinuity with respect to L^1-convergence with a smooth integrand. The example in [3] has a lower semicontinuous integrand verifying a "linear coercivity" assumption

$$f(z, \zeta, \bar{\zeta}) \geq c|\bar{\zeta}|, \qquad c > 0.$$

Here we present an example with a smooth integrand, such that the function f is *strictly* convex in the last variable and, again, the functional is not lower semicontinuous with respect to (strong) L^1-convergence. In [6], [8], the problem has been recalled whether the part (b) of the Serrin theorem can be generalized to vector-valued case, namely whether the functional (1) is lower semicontinuous with respect to L^1-convergence under the assumption that f is strictly convex in the last variable. Our example solves this problem negatively.

The vector-valued counterpart of the Serrin theorem (c) is disproved by our example as well, but for this already Eisen's example can be used. The vector-valued case of (a) is however valid, as shown by Fonseca and Leoni [6, Theorem 1.1].

One of the features (and perhaps the most important one) which enables the failure of lower semicontinuity in our example is the lack of coercivness. Notice however that in some other situations (scalar case, or no dependence on ζ) lower semicontinuity results can be achieved even without the coercivness assumption.

2. Example of non lower semicontinuity

Construction of f

We write $\theta = (\xi, \eta)$, $\bar{\theta} = (\bar{\xi}, \bar{\eta})$. We define the function f as

$$f(\xi, \eta, \bar{\xi}, \bar{\eta}) = \frac{\bar{\eta}^2 + \exp \bar{\xi}}{\exp |\theta|^4} = \frac{\bar{\eta}^2 + \exp \bar{\xi}}{\exp(\xi^2 + \eta^2)^2}.$$

We consider the functional

$$F(u) = \int_0^1 f(x(t), y(t), x'(t), y'(t)), \qquad u = (x, y) \in W_0^{1,1}((0,1), \mathbb{R}^2).$$

Proposition 2.1. *The function $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is smooth, nonnegative and continuous. For each $\theta \in \mathbb{R}^2$, $f(\theta, \cdot)$ is strictly convex.*

Proof. All of the mentioned properties are obviously verified. □

Proposition 2.2. *There exists a sequence $\{u_n\}$ of functions from $W_0^{1,1}((0,1), \mathbb{R}^2)$ such that*

$$\|u_n\|_1 \to 0 \quad \textit{and} \quad F(u_n) \to \frac{1}{e}. \tag{2}$$

Proof. We write $u_n = (x_n, y_n)$. We construct u_n as $\frac{1}{n}$-periodic functions on $\mathbb{R}$ (formally, the elements of $W_0^{1,1}((0,1), \mathbb{R}^2)$ that we look for are then restrictions to $(0,1)$), so it is enough to determine their values on $[0, \frac{1}{n}]$. We set

$$x_n(t) = \begin{cases} -n^4\, t, & 0 \le t \le \frac{1}{n^3}, \\ -n \cos\Big(\frac{\pi n^3}{4}\Big(t - \frac{1}{n^3}\Big)\Big), & \frac{1}{n^3} \le t \le \frac{5}{n^3}, \\ n - n^3\Big(n - \frac{1}{n}\Big)\Big(t - \frac{5}{n^3}\Big), & \frac{5}{n^3} \le t \le \frac{6}{n^3}, \\ \frac{n^2}{n^2-6}\Big(\frac{1}{n} - t\Big), & \frac{6}{n^3} \le t \le \frac{1}{n}, \end{cases}$$

and

$$y_n(t) = \begin{cases} 0, & 0 \le t \le \frac{1}{n^3}, \\ n \sin\Big(\frac{\pi n^3}{4}\Big(t - \frac{1}{n^3}\Big)\Big), & \frac{1}{n^3} \le t \le \frac{5}{n^3}, \\ 0, & \frac{5}{n^3} \le t \le \frac{1}{n}. \end{cases}$$

We estimate

$$|u_n(t)| \le n, \qquad 0 < t < \frac{6}{n^3},$$
$$|u_n(t)| \le \frac{1}{n}, \qquad \frac{6}{n^3} < t < \frac{1}{n}.$$

Using periodicity, it follows

$$\int_0^1 |u_n(t)|\, dt \le n \int_0^{\frac{6}{n^3}} n\, dt + \frac{n}{n^2} \le \frac{7}{n}.$$

Hence $\|u_n\|_1 \to 0$. Now we are going to compute $\lim_n F(u_n)$. If $t \in \big(\frac{1}{n^3}, \frac{5}{n^3}\big)$, then

$$x_n'(t) \le \frac{\pi}{4} n^4 \le n^4, \qquad |y_n'(t)| \le \frac{\pi}{4} n^4 \le n^4, \qquad |x_n^2(t) + y_n^2(t)| = n^2,$$

and thus

$$f\big(x_n(t), y_n(t), x_n'(t), y_n'(t)\big) \le \frac{n^8 + \exp n^4}{\exp n^4}.$$

For a.e. $t \in (0, \frac{1}{n}) \setminus \left(\frac{1}{n^3}, \frac{5}{n^3}\right)$ we have

$$x_n'(t) \le -1, \qquad y_n'(t) = 0$$

and thus

$$f\big(x_n(t), y_n(t), x_n'(t), y_n'(t)\big) \le \exp(-1).$$

Using the periodicity of u_n we estimate

$$\begin{aligned}\int_0^1 f(x,y,x',y')\,dt &\le n \int_0^{1/n} f(x,y,x',y')\,dt \\ &\le n \int_{\frac{1}{n^3}}^{\frac{5}{n^3}} \frac{n^8 + \exp n^4}{\exp n^4}\,dt + n \int_0^{1/n} \exp(-1)\,dt \to \exp(-1).\end{aligned}$$

On the other hand, we compute

$$\begin{aligned}\int_0^1 f(x,y,x',y')\,dt &\ge n \int_{\frac{6}{n^3}}^{\frac{1}{n}} \frac{\exp\left(-\frac{n^2}{n^2-6}\right)}{\exp\left(n^{-4}\right)}\,dt \\ &\ge n\left(\frac{1}{n} - \frac{6}{n^3}\right)\exp\Big(-\frac{n^2}{n^2-6} - \frac{1}{n^4}\Big) \to \exp(-1).\end{aligned}$$

Hence

$$F(u_n) \to \exp(-1).$$

$\square$

Remark 2.3. By a mollification, for each n we can obtain an infinitely smooth $\tilde{u}_n$ such that $\|\tilde{u}_n - u_n\|_1 < \frac{1}{n}$ and $F(\tilde{u}_n) - F(u_n) < \frac{1}{n}$. For these mollified functions we also obtain $F(\tilde{u}_n) \to \exp(-1)$ and $\|\tilde{u}_n\|_1 \to 0$.

Theorem 2.4. *The functional F is not lower semicontinuous on $W_0^{1,1}((0,1), \mathbb{R}^2)$ with respect to L^1 convergence.*

Proof. We consider the sequence from Proposition 2.2 and recall that

$$u_n \to 0 \text{ in } L^1((0,1), \mathbb{R}^2).$$

We observe

$$F(0) = 1, \tag{3}$$

whereas, by Proposition 2.2,

$$\lim_n F(u_n) = \frac{1}{e}.$$

Hence

$$F(0) \not\le \liminf_n F(u_n).$$

$\square$

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