Variational Sum and Kato’s Conjecture

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Our aim in this paper is to compare the variational sum recently introduced by Attouch, Baillon and Théra and the concept of generalized sum. Under appropriate hypotheses, we show that these sums coincide and that the variational sum is a maximal monotone operator.

**Keywords:** Algebraic sum, sum form, variational sum, Kato’s condition, maximal monotone operators

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1. Introduction

Recently a new concept of sum of maximal monotone operators $A$ and $B$ has been introduced by Attouch, Baillon and Théra (see [1]). This sum is called "variational sum" of $A$ and $B$ and will be denoted in this paper by $A +_v B$. The algebraic sum of two operators is not always well adapted to problems arising in mathematical analysis. The variational sum coincides with the algebraic sum when the latter happens to be a maximal monotone operator. One obtains this sum by using Yosida’s approximates of $A$ and $B$, denoted by $A_\lambda$ and $B_\mu$, respectively, and by considering their algebraic sum given by $C_{\lambda,\mu} = A_\lambda + B_\mu$. Thus, the variational sum of $A$ and $B$ is defined as the limit of $C_{\lambda,\mu}$ in the graph sense of Kuratowski-Painlevé in a neighborhood of the origin $(0,0)$. The main idea here is to compare two concepts of sum, the variational sum and the sum form of $A$ and $B$, in connection with Kato’s condition (see [6], [7]). Under appropriate hypotheses, we prove that the variational sum is a maximal monotone operator and coincides with the sum form. This last fact generalizes Nisipeanu’s result. Indeed, Nisipeanu proved that if $A$ and $B$ are supposed to be selfadjoint operators, then their variational sum is a maximal monotone operator (see [9]).

Let us recall some notions related to the concept of variational sum and Kato’s condition. In this paper we will assume that $H$ is a real Hilbert space.

**Definition 1.1.** Let $A$ be a linear maximal operator. $A$ is said to verify Kato’s condition (Kato’s conjecture) if,

$$D(A^{\frac{1}{2}}) = D(\Phi) = D(A^{\frac{1}{2}})$$

(1)

where $\Phi$ is the sesquilinear form associated to $A$ given by Kato’s representation theorem as (see [6], [7]),

$$\Phi(u, v) = \langle A^{\frac{1}{2}}u, A^{\frac{1}{2}}v \rangle \text{ for all } u, v \in D(A^{\frac{1}{2}}).$$

(2)
Let us denote by \( I \) the set given by \( I = \{ (\lambda, \mu) \in \mathbb{R}^2 : \lambda, \mu \geq 0 \text{ and } \lambda + \mu \neq 0 \} \) and for any \((\lambda, \mu) \in I\) we consider the pointwise sum \( C_{\lambda, \mu} \). Note that since \( \lambda + \mu \neq 0 \), at least one of the two operators \( A_1 \) or \( B_1 \) is lipschitz continuous and consequently \( C_{\lambda, \mu} \) is a maximal monotone operator. In the sequel, we denote by \( f \) the filter of all the pointed neighborhoods of the origin in \( I \) and by \( \text{graph}\-\lim\inf_f C_{\lambda, \mu} \) (respectively \( \text{graph}\-\lim\sup_f C_{\lambda, \mu} \)) the lower limit (respectively upper limit) of the family \( \{ \text{graph} C_{\lambda, \mu} : (\lambda, \mu) \in I \} \) along the filter \( f \) with respect to the box topology on \( H \times H \).

- \((x, y) \in \text{graph}\-\lim\inf_f C_{\lambda, \mu}\), whenever for every neighborhood \( V_{(x,y)} \) of \((x, y)\), there exists \( F \in f \) such that for \((\lambda, \mu) \in F\), \( \text{graph} C_{\lambda, \mu} \cap V_{(x,y)} \neq \phi \).

- \((x, y) \in \text{graph}\-\lim\sup_f C_{\lambda, \mu}\), if for every neighborhood \( V_{(x,y)} \) of \((x, y)\) and every \( F \in f \) there exists \((\lambda, \mu) \in F\) such that \( \text{graph} C_{\lambda, \mu} \cap V_{(x,y)} \neq \phi \).

The set \( \{ C_{\lambda, \mu} : (\lambda, \mu) \in I \} \) is declared convergent to \( C \) and we denote \( \text{gph}\-\lim_f C_{\lambda, \mu} = C \), if

\[
\text{gph}\-\lim_f \sup C_{\lambda, \mu} = \text{gph}\-\lim_f \inf C_{\lambda, \mu} = C
\]

Thus, the variational sum of two maximal monotone operators \( A \) and \( B \) is defined as:

\[
(A + B)_v := \text{gph}\-\lim_f (A + B)_v
\]

Since the box topology on \( H \times H \) is first countable and the filter \( f \) is countably based, it amounts to saying that \((x, y) \in \text{graph}(A + B)_v\), if and only if, for every sequences \( \lambda_n, \mu_n \in I \) with \( \lim \lambda_n = \lim \mu_n = 0 \), there exists \((x_n, y_n)\) such that:

\[
\lim x_n = x, \lim y_n = y \text{ and } y_n \in y_n \in (A_{\lambda_n} + B_{\mu_n})x_n
\]

Equivalently, in terms of resolvents this means that, for any \( y \in R[I + (A + B)_v] \) (in particular by Minty’s theorem whenever \((A + B)_v\) is maximal monotone for every \( y \in H \)), the family of solutions of

\[
y \in u_{\lambda, \mu} + A_{\lambda} u_{\lambda, \mu} + B_{\mu} u_{\lambda, \mu}
\]

converges for \((\lambda, \mu) \in I\) to the solution \( u \) of

\[
y \in u + (A + B)_v u.
\]

**Definition 1.2.** An operator \( A \) is said to be strictly maximal monotone if \( A \) is strictly monotone and maximal.

**Theorem 1.3.** ([6]) Let \( A \) and \( B \) be linear operators, which are strictly maximal monotone and satisfy Kato’s condition. Assume that \( D(A^\frac{1}{2}) \cap D(B^\frac{1}{2}) \) is dense in \( H \), then there exists a unique maximal monotone operator \( A \oplus B \) such that

\[
D((A \oplus B)^\frac{1}{2}) = D(A^\frac{1}{2}) \cap D(B^\frac{1}{2}) = D((A \oplus B)^{1/2}),
\]

where \( A \oplus B \) is the sum form of \( A \) and \( B \) given by the sum of sesquilinear forms associated to \( A \) and \( B \).

**Theorem 1.4.** ([9]) Let \( A \) and \( B \) be linear operators which are self-adjoint and monotone such that \( D(A^\frac{1}{2}) \cap D(B^\frac{1}{2}) \) is dense in \( H \). Then the variational sum of \( A \) and \( B \) is a maximal monotone operator and

\[
(A + B)_v = A \oplus B
\]
2. Sum form of maximal operators

In this part we will focus on a generalization of Theorem 1.4 above. Here we will suppose that \( A \) and \( B \) are both linear operators and strictly maximal monotone.

**Theorem 2.1.** Let \( A \) and \( B \) be linear operators which are strictly maximal monotone linear and verify both Kato’s condition. Suppose that \( D(A^{1/2}) \cap D(B^{1/2}) \) is dense in \( H \). Then the variational sum of \( A \) and \( B \) is a strictly maximal monotone operator verifying Kato’s condition and coincides with the sum form, that is,

\[
(A + B)_v = A \oplus B
\]

**Proof.** Let \( w \in H \). For any \((\lambda, \mu) \in I\) there exists a solution \( u_{\lambda, \mu} \) of the equation

\[
u_{\lambda, \mu} + A_{\lambda} u_{\lambda, \mu} + B_{\mu} u_{\lambda, \mu} = w
\]

The operator \([I + (A_{\lambda} + B_{\mu})]^{-1}\) is nonexpansive by the fact that \( A_{\lambda} + B_{\mu} \) is a maximal monotone operator, and hence \( \|u_{\lambda, \mu}\| \) is bounded. By equation (11) it follows that

\[
\|u_{\lambda, \mu}\|^2 + < A_{\lambda}^{1/2} u_{\lambda, \mu}, A_{\lambda}^{1/2} u_{\lambda, \mu} > + < B_{\mu}^{1/2} u_{\lambda, \mu}, B_{\mu}^{1/2} u_{\lambda, \mu} > = < w, u_{\lambda, \mu} > \quad (12)
\]

By using equations (11) and (12) we may show that the following weak limit exists, that is,

\[
w - \lim_{n} A_{\lambda_n}^{1/2} u_{\lambda_n, \mu_n} = A^{1/2} u, \quad \text{where} \quad u = w - \lim_{n} u_{\lambda_n, \mu_n}.
\]

We have the same result for its conjugate. We also have

\[
w - \lim_{n} B_{\mu_n} u_{\lambda_n, \mu_n} = B^{1/2} u.
\]

By a similar reasoning we show that strong convergence of the above limits occur. Let \( v \in D(A^{1/2}) \cap D(B^{1/2}) \), then (11) yields

\[
< u_{\lambda_n, \mu_n}, v > + < A_{\lambda_n}^{1/2} u_{\lambda_n, \mu_n}, A_{\lambda_n}^{1/2} v > + < B_{\mu_n}^{1/2} u_{\lambda_n, \mu_n}, B_{\mu_n}^{1/2} v > = < w, v > .
\]

Considering the limits of both sides in (13) it follows that

\[
< u, v > + < A^{1/2} u, A^{1/2} v > + < B^{1/2} u, B^{1/2} v > = < w, v > .
\]

Thus,

\[
< u, v > + < (A \oplus B) u, v > = < w, v > .
\]

Since \( D(A^{1/2}) \cap D(B^{1/2}) \) is dense in \( H \), using (15) it follows that

\[
u + (A \oplus B) u = w.
\]

Thus \( A \oplus B = (A + B)_v \) and Kato’s condition is verified according to Theorem 1.3 and by the fact that \( (A + B)_v \) is a strictly maximal monotone operator.

**Remark.** According to Theorem 1.3 and the fact that

\[
A \oplus B = (A + B)_v
\]

it follows that Kato’s condition is satisfied, that is,

\[
D((A + B)^{1/2}) = D((A + B)^{1/2}) = D(A^{1/2}) \cap D(B^{1/2}).
\]
References


