Journal of Convex Analysis
Volume 9 (2002), No. 1, 291–294

Variational Sum and Kato's Conjecture

Toka Diagana
Howard University, Department of Mathematics,
2441 6th Street, N.W, Washington, DC 20059, USA
tdiagana@howard.edu

Received July 6, 2001
Revised manuscript received September 20, 2001

Our aim in this paper is to compare the variational sum recently introduced by Attouch, Baillon and Théra and the concept of generalized sum. Under appropriate hypotheses, we show that these sums coincide and that the variational sum is a maximal monotone operator.

Keywords: Algebraic sum, sum form, variational sum, Kato's condition, maximal monotone operators

1991 Mathematics Subject Classification: 47B44, 47H05, 47A07, 11E05

1. Introduction

Recently a new concept of sum of maximal monotone operators A and B has been introduced by Attouch, Baillon and Théra (see [1]). This sum is called "variational sum" of A and B and will be denoted in this paper by $A +_{\mathrm{v}} B$. The algebraic sum of two operators is not always well adapted to problems arising in mathematical analysis. The variational sum coincides with the algebraic sum when the latter happens to be a maximal monotone operator. One obtains this sum by using Yosida's approximates of A and B, denoted by A_λ and B_μ, respectively, and by considering their algebraic sum given by $C_{\lambda,\mu} = A_\lambda + B_\mu$. Thus, the variational sum of A and B is defined as the limit of $C_{\lambda,\mu}$ in the graph sense of Kuratowski-Painlevé in a neighborhood of the origin $(0, 0)$. The main idea here is to compare two concepts of sum, the variational sum and the sum form of A and B, in connection with Kato's condition (see [6], [7]). Under appropriate hypotheses, we prove that the variational sum is a maximal monotone operator and coincides with the sum form. This last fact generalizes Nisipeanu's result. Indeed, Nisipeanu proved that if A and B are supposed to be selfadjoint operators, then their varaitional sum is a maximal monotone operator (see [9]).

Let us recall some notions related to the concept of variational sum and Kato's condition. In this paper we will assume that H is a real Hilbert space.

Definition 1.1. Let A be a linear maximal operator. A is said to verify Kato's condition (Kato's conjecture) if,

$$D(A^{\frac{1}{2}}) = D(\Phi) = D(A^{*\frac{1}{2}}) \tag{1}$$

where Φ is the sesquilinear form associated to A given by Kato's representation theorem as (see [6], [7]),

$$\Phi(u, v) =< A^{\frac{1}{2}}u, A^{*\frac{1}{2}}v > \text{ for all } u, v \in D(A^{\frac{1}{2}}). \tag{2}$$

ISSN 0944-6532 / $ 2.50 © Heldermann Verlag

Let us denote by I the set given by $I = \{(\lambda, \mu) \in R^2 \ : \ \lambda, \mu \geq 0 \text{ and } \lambda + \mu \neq 0\}$ and for any $(\lambda, \mu) \in I$ we consider the pointwise sum $C_{\lambda,\mu}$. Note that since $\lambda + \mu \neq 0$, at least one of the two operators A_λ or B_μ is lipschitz continuous and consequently $C_{\lambda,\mu}$ is a maximal monotone operator. In the sequel, we denote by f the filter of all the pointed neighbordhoods of the origin in I and by $graph{-}\liminf_f C_{\lambda,\mu}$ (respectively $graph{-}\limsup_f C_{\lambda,\mu}$) the lower limit(respectively upper limit) of the family $\{graphC_{\lambda,\mu} : (\lambda, \mu) \in I\}$ along the filter f with respect to the box topology on $H \times H$.

• $(x, y) \in graph{-}\liminf_f C_{\lambda,\mu}$, whenever for every neighborhood $V_{(x,y)}$ of (x, y), there exists $F \in f$ such that for $(\lambda, \mu) \in F$, $graphC_{\lambda,\mu} \cap V_{(x,y)} \neq \phi$.

• $(x, y) \in graph{-}\limsup_f C_{\lambda,\mu}$, if for every neighborhood $V_{(x,y)}$ of (x, y) and every $F \in f$ there exists $(\lambda, \mu) \in F$ such that $graphC_{\lambda,\mu} \cap V_{(x,y)} \neq \phi$.

The set $\{C_{\lambda,\mu} : (\lambda, \mu) \in I\}$ is declared convergent to C and we denote $gph{-}\lim_f C_{\lambda,\mu} = C$, if

$$gph{-}\limsup_f C_{\lambda,\mu} = gph{-}\liminf_f C_{\lambda,\mu} = C \tag{3}$$

Thus, the variational sum of two maximal monotone operators A and B is defined as:

$$(A + B)_v := gph{-}\liminf_f (A_\lambda + B_\mu) \tag{4}$$

Since the box topology on $H \times H$ is first countable and the filter f is countably based, it amounts to saying that $(x, y) \in graph(A + B)_v$, if and only if, for every sequences $\lambda_n, \mu_n \in I$ with $\lim \lambda_n = \lim \mu_n = 0$, there exists (x_n, y_n) such that:

$$\lim x_n = x, \ \lim y_n = y \text{ and } y_n \in y_n \in (A_{\lambda_n} + B_{\mu_n})x_n \tag{5}$$

Equivalently, in terms of resolvents this means that, for any $y \in R[I + (A + B)_v]$ (in particular by Minty's theorem whenever $(A+B)_v$ is maximal monotone for every $y \in H$), the family of solutions of

$$y \in u_{\lambda,\mu} + A_\lambda u_{\lambda,\mu} + B_\mu u_{\lambda,\mu} \tag{6}$$

converges for $(\lambda, \mu) \in I$ to the solution u of

$$y \in u + (A + B)_v u. \tag{7}$$

Definition 1.2. An operator A is said to be strictly maximal monotone if A is strictly monotone and maximal.

Theorem 1.3. ([6]) *Let A and B be linear operators, which are strictly maximal monotone and satisfy Kato's condition. Assume that $D(A^{\frac{1}{2}}) \cap D(B^{\frac{1}{2}})$ is dense in H, then there exists a unique maximal monotone operator $A \oplus B$ such that*

$$D((A \oplus B)^{\frac{1}{2}} = D(A^{\frac{1}{2}}) \cap D(B^{\frac{1}{2}}) = D((A \oplus B)^{*\frac{1}{2}}), \tag{8}$$

where $A \oplus B$ is the sum form of A and B given by the sum of sesquilinear forms associated to A and B.

Theorem 1.4. ([9]) *Let A and B be linear operators which are self-adoint and monotone such that $D(A^{\frac{1}{2}}) \cap D(B^{\frac{1}{2}})$ is dense in H. Then the variational sum of A and B is a maximal monotone operator and,*

$$(A + B)_v = A \oplus B \tag{9}$$

2. Sum form of maximal operators

In this part we will focus on a generalization of Theorem 1.4 above. Here we will suppose that A and B are both linear operators and strictly maximal monotone.

Theorem 2.1. *Let A and B be linear operators which are strictly maximal monotone linear and verify both Kato's condition. Suppose that $D(A^{\frac{1}{2}}) \cap D(B^{\frac{1}{2}})$ is dense in H. Then the variational sum of A and B is a strictly maximal monotone operator verifying Kato's condition and coincides with the sum form,that is,*

$$(A + B)_v = A \oplus B \tag{10}$$

Proof. Let $w \in H$. For any $(\lambda, \mu) \in I$ there exists a solution $u_{\lambda,\mu}$ of the equation

$$u_{\lambda,\mu} + A_\lambda u_{\lambda,\mu} + B_\mu u_{\lambda,\mu} = w \tag{11}$$

The operator $[I + (A_\lambda + B_\mu)]^{-1}$ is nonexpansive by the fact that $A_\lambda + B_\mu$ is a maximal monotone operator, and hence $\|u_{\lambda,\mu}\|$ is bounded. By equation (11) it follows that

$$\|u_{\lambda,\mu}\|^2 + < A_\lambda^{\frac{1}{2}} u_{\lambda,\mu}, A_\lambda^{*\frac{1}{2}} u_{\lambda,\mu} > + < B_\mu^{\frac{1}{2}} u_{\lambda,\mu}, B_\mu^{*\frac{1}{2}} u_{\lambda,\mu} > = < w, u_{\lambda,\mu} > \tag{12}$$

By using equations (11) and (12) we may show that the following weak limit exists, that is,

$$w - \lim A_{\lambda_n}^{\frac{1}{2}} u_{\lambda_n,\mu_n} = A^{\frac{1}{2}} u, \quad \text{where} \;\; u = w - \lim u_{\lambda_n,\mu_n}.$$

We have the same result for its conjugate. We also have

$$w - \lim B_{\mu_n} u_{\lambda_n,\mu_n} = B^{\frac{1}{2}} u.$$

By a similar reasoning we show that strong convergence of the above limits occur. Let $v \in D(A^{\frac{1}{2}}) \cap D(B^{\frac{1}{2}})$, then (11) yields

$$< u_{\lambda_n,\mu_n}, v > + < A_{\lambda_n}^{\frac{1}{2}} u_{\lambda_n,\mu_n}, A_{\lambda_n}^{*\frac{1}{2}} v > + < B_{\mu_n}^{\frac{1}{2}} u_{\lambda_n,\mu_n}, B_{\mu_n}^{*\frac{1}{2}} v > = < w, v > . \tag{13}$$

Considering the limits of both sides in (13) it follows that

$$< u, v > + < A^{\frac{1}{2}} u, A^{*\frac{1}{2}} v > + < B^{\frac{1}{2}} u, B^{*\frac{1}{2}} v > = < w, v > \tag{14}$$

Thus,

$$< u, v > + < (A \oplus B) u, v > = < w, v > . \tag{15}$$

Since $D(A^{\frac{1}{2}}) \cap D(B^{\frac{1}{2}})$ is dense in H, using (15) it follows that

$$u + (A \oplus B) u = w. \tag{16}$$

Thus $A \oplus B = (A + B)_v$ and Kato's condition is verified according to Theorem 1.3 and by the fact that $(A + B)_v$ is a strictly maximal monotone operator.

Remark. According to Theorem 1.3 and the fact that

$$A \oplus B = (A + B)_v \tag{17}$$

it follows that Kato's condition is satisfied, that is,

$$D((A + B)_v^{\frac{1}{2}}) = D((A + B)_v^{*\frac{1}{2}}) = D(A^{\frac{1}{2}}) \cap D(B^{\frac{1}{2}}). \tag{18}$$

References

[1] H. Attouch, J.-B. Baillon, M. Théra: Variational sum of maximal monotone operators, J. Convex Anal. 1 (1994) 1–29.

[2] H. Attouch: On the maximality of the sum of two maximal monotone operators, Nonlinear Anal. 5 (1981) 143–147.

[3] H. Brézis: Opérateurs Maximaux Monotone et Semi-Groupes de Contractions dans les Espaces de Hilbert, North-Holland (1973).

[4] H. Brézis, M.-G. Crandall, A. Pazy: Perturbations of nonlinear maximal monotone sets in Banach spaces, Comm. Pure and Appl. Math. 23 (1970) 123–144.

[5] P. Clément, P. Egberts: On the sum of two maximal monotone operators, Differential Integral Equations 3 (1990) 1127–1138.

[6] T. Diagana: Sommes d'opérateurs et conjecture de Kato-McIntosh, C. R. Acad. Sci. Paris, Sér. I, Math. 330 (2000) 461–464.

[7] T. Kato: Perturbation Theory of Linear Operators, New York (1966).

[8] E. Krauss: A representation of maximal monotone operators by saddle functions, Rev. Roumaine. Math. Pures. Appl. 30 (1985) 823–837.

[9] M. Nisipeanu: Somme Variationnelle d'Opérateurs et Applications, Thèse de Spécialité, Université de Limoges (1997).

[10] J.-P. Revalski, M. Théra: Variational and extended sums of monotone operators, in: Ill-posed Variational Problems and Regularization Techniques, M. Théra, R. Tichatschke (eds.), Lecture Notes in Economics and Mathematical Systems 477, Springer-Verlag (1999) 229–246.

[11] J.-P. Revalski, M. Théra: Enlargements and sums of monotone operators, to appear in: Nonlinear Anal. Theory Methods and Applications (2000).

[12] R.-T. Rockafellar: Convex Analysis, Princeton University Press (1970).

[13] R.-T. Rockafellar: Monotone operators associated with saddle functions and minimax problems, Proc. Symp. Pure. Math. AMS 18 (1970) 241–250.