Journal of Convex Analysis
Volume 9 (2002), No. 1, 245–257

Nonlinear Energy Forms and Lipschitz Spaces on the Koch Curve

Raffaela Capitanelli
Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate,
Via A. Scarpa 16, 00161 Roma, Italy
raffaela.capitanelli@uniroma1.it

Maria Rosaria Lancia
Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate,
Via A. Scarpa 16, 00161 Roma, Italy
lancia@dmmm.uniroma1.it

Received September 10, 2001
Revised manuscript received October 23, 2001

We consider the nonlinear convex energy forms $\mathcal{E}^{(p)}$ on the Koch curve K and we prove that the corresponding domains coincide with the spaces $Lip_{\alpha,D_f}(p,\infty,K)$. Then we give a precise interpretation of the smoothness index α in terms of the structural constants of the fractal.

Keywords: Nonlinear convex energy forms, fractals, Lipschitz spaces

1. Introduction

The first fractal examples of nonlinear energy forms

$$\mathcal{E}^{(p)},\ p > 1$$

have been recently constructed on the Koch curve in [2], where the properties of the domains $D_{\mathcal{E}^{(p)}}$ of these nonlinear energy forms are studied and it is shown that they can be considered to be the analogous of the usual Sobolev spaces $W^{1,p}$.

In the quadratic case $p = 2$, constructions of this type are standard and have been done for various fractals K like the Sierpinski gasket, the Koch curve and more general simple nested fractals: in these cases, the energy functionals $\mathcal{E}^{(2)}$ are Dirichlet forms, whose domains $D_{\mathcal{E}^{(2)}}$ are the fractal analogue of the Sobolev space $W^{1,2}$.

Always in the case $p = 2$, these spaces $W^{1,2}(K) \equiv D_{\mathcal{E}^{(2)}}(K)$ have been put in relation with the theory of Lipschitz spaces $Lip_{\alpha,D_f}(2,\infty,K)$; in particular, in Jonsson [5] first and later also in Lancia and Vivaldi [9], Paluba [13] and Kumagai [8], for various examples of fractals K the following characterization has been given

$$W^{1,2}(K) = Lip_{\alpha,D_f}(2,\infty,K), \tag{1}$$

where D_f is the fractal (Hausdorff) dimension of K and $\alpha > 0$ is a suitable parameter depending on K. We note that the spaces $Lip_{\alpha,D_f}(2,\infty,K)$ belong to a more general scale of spaces $Lip_{\alpha,D_f}(p,\infty,K)$, where $p \geq 1$ is an additional parameter, which will be defined in Section 2.

ISSN 0944-6532 / \$ 2.50

In [9] an intrinsic approach to the Lipschitz spaces has been considered, which has the advantage of making more explicit the dependence of α on the fractal K. This approach is based on a change of metric within the fractal K — introduced by Mosco in [10], [11] —

$$d(x,y) = |x-y|^\delta, \ x,y \in K,$$

where

$$\delta = \frac{log(N\rho)}{2logl} \tag{2}$$

here N is the number of similitudes of K, l^{-1} is the contraction factor of the similitudes and ρ is the scaling factor of the energy $\mathcal{E}^{(2)}$. By using this intrinsic metric d, in [9] a one parameter scale of spaces

$$\mathcal{L}ip_\nu(K)$$

is then constructed in a similar way as the spaces $Lip_{\alpha,D_f}(2,\infty,K)$ (see Section 5). Now the parameter ν is given by

$$\nu = \frac{D_f}{\delta}.$$

If K is the Koch curve, the following characterizations have been proved

$$W^{1,2}(K) = Lip_{\alpha,D_f}(2,\infty,K) = \mathcal{L}ip_\nu(K),$$

where now $\alpha = \frac{log4}{log3}$, $D_f = \frac{log4}{log3}$ and $\nu = \frac{D_f}{\delta} = 1$ since now $\delta = \frac{log4}{log3}$. Notice that $\alpha = \delta$: this provides a characterization of α in terms of the intrinsic metric, hence of the energy scaling factor ρ. This interpretation of α is conjectured in [9] to hold for more general classes of fractals K and it follows from the results of [5], [13], [8] concerning the spaces $Lip_{\alpha,D_f}(2,\infty,K)$ and for the various fractals K considered in these papers.

Let us now come back to the nonlinear energy forms $\mathcal{E}^{(p)}$, $p > 1$, on the Koch curve K, mentioned at the beginning. The aim of the present paper is to extend the results of [9], where $p = 2$, to this more general case $p > 1$. We will first prove the following characterization

$$W^{1,p}(K) = Lip_{\alpha,D_f}(p,\infty,K), \text{ for every } p > 1$$

where $\alpha = \frac{log4}{log3}$ as in the case $p = 2$. We shall then construct the spaces $\mathcal{L}ip_\nu^{(p)}(K)$ for every $p > 1$, where $\nu = \frac{D_f}{\delta} = 1$ as in the case $p = 2$. We then prove the intrinsic characterization

$$W^{1,p}(K) = \mathcal{L}ip_1^{(p)}(K), \text{ for every } p > 1. \tag{3}$$

We point out that in the present case the following identity $\delta = \frac{log(N\rho_p)}{p\, logl}$ is satisfied for every $p > 1$, where $\rho_p = 4^{p-1}$ is the scaling factor of $\mathcal{E}^{(p)}$ (see Section 4), $N = 4$ and $l = 3$. This shows in particular that δ is independent from p. It would be interesting to check whether the previous expression of δ in terms of the energy factor ρ_p keeps true on other fractals like for example the Sierpinski gasket; however, no forms of the type of $\mathcal{E}^{(p)}$ have been yet constructed in this case.

An additional result of the present paper is the proof that the functions $u \in W^{1,p}(K)$ are Hölder-continuous on K (with respect to the Euclidean distance) with Hölder exponent $\beta_e = (1-\frac{1}{p})\frac{log4}{log3}$. We point out that the space of Hölder continuous functions on the Koch curve — with Hölder exponent $\beta_e > 1$ — does not consist only of constant functions (see Corollary 4.2 and [6]).

In this paper we deduce the Hölder continuity following two different approaches. One is abstract and is based on the use of the embedding theorems in the Besov spaces (see [7]). The other one relies on the Hölderianity results of the spaces $W^{1,p}(K)$ — as a Morrey embedding — proved in [3] and on the intrinsic characterization (3).

2. The Lipschitz spaces $Lip_{\alpha,D_f}(p,q,K)$

In this section we recall the definition of the Lipschitz spaces introduced by Jonsson in [5].

Let $B_e(x,r)$ denote the closed Euclidean ball with center $x \in \mathbb{R}^D$ and radius r. According to [7], we first recall the definition of D_f-set.

Definition 2.1. A closed non empty subset $F \subset \mathbb{R}^D$ is a D_f*-set* $(0 < D_f \leq D)$ if there exists a Borel measure μ in $\mathbb{R}^D$ with $supp\mu = F$, such that for some positive constants $c_1 = c_1(F)$ and $c_2 = c_2(F)$:

$$c_1 r^{D_f} \leq \mu(B_e(x,r)) \leq c_2 r^{D_f} \text{ for } x \in F,\ 0 < r \leq 1. \tag{4}$$

Such a μ is called a D_f-measure on F.

If F is a D_f-set, then the restriction to F of the D_f-dimensional Hausdorff measure of $\mathbb{R}^D$ is a D_f-measure on F and thus the Hausdorff dimension of F is D_f. For details and proofs see [7].

Let $F \subset \mathbb{R}^D$ be a D_f-set, $0 < D_f \leq D$, let μ be the D_f-measure on F.

Definition 2.2. Let $c_0 > 0$, $\alpha > 0$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, $Lip_{\alpha,D_f}(p,q,F)$ is the space of those functions f such that

$$f \in L^p(F,\mu) : \|\{a_h\}\|_{l_q} = \left(\sum_{h=0}^{\infty} a_h^q\right)^{1/q} < \infty \tag{5}$$

where for each $h \in \mathbb{N}$

$$a_h = \left(3^{h\alpha p + hD_f} \int\int_{|x-y|<c_0 3^{-h}} |f(x) - f(y)|^p d\mu(x) d\mu(y)\right)^{1/p}. \tag{6}$$

The norm in $Lip_{\alpha,D_f}(p,q,F)$ is defined as

$$|||f||| := \|f\|_{p,\mu} + \|\{a_h\}\|_{l_q} \tag{7}$$

where $\|f\|_{p,\mu}$ denotes the norm of f in $L^p(F,\mu)$.

In Jonsson's notations these spaces are denoted by $Lip(\alpha,p,q,F)$; we modified this notation in $Lip_{\alpha,D_f}(p,q,F)$ to put in evidence also the dependence on the fractal dimension. Moreover, the constant 3 in Jonsson's definition is replaced by the constant 2 : this clearly gives equivalent spaces with equivalent norms.

By Proposition 1 in [5] and Proposition 5 page 213 in [7], the following proposition is proved.

Proposition 2.3. *The space $Lip_{\alpha,D_f}(p,q,F)$ is continuously embedded in $Lip_{\gamma,D_f}(\infty,\infty,F)$, where $\gamma = \alpha - D_f/p$.*

Note that the functions f of $Lip_{\gamma,D_f}(\infty,\infty,F)$ are bounded functions on F such that

$$\text{there exists}\, c > 0 : |f(x) - f(y)| \leq c|x-y|^{\gamma},\ x, y \in F. \tag{8}$$

We will denote the continuous functions on F satisfying (8) by $C^{0,\gamma}(F, |\cdot|)$ where $|\cdot|$ denotes the Euclidean distance.

3. The nonlinear energy form on the Koch curve

In this section we introduce the nonlinear energy form on the Koch curve, whose construction has been developed by one of the authors in [2]. We start by recalling the construction of the unit Koch curve K. Let us denote by A and B the points $(0,0)$ and $(1,0)$. Let $\Psi = \{\psi_i, i = 1, ..., 4\}$ denote the set of the $N = 4$ contractive similitudes $\psi_i : \mathbb{C} \to \mathbb{C}$, with contraction factor $l^{-1} = \frac{1}{3}$ given by $\psi_1 = \frac{z}{3}$, $\psi_2 = \frac{z}{3}e^{i\frac{\pi}{3}} + \frac{1}{3}$, $\psi_3 = \frac{z}{3}e^{-i\frac{\pi}{3}} + \frac{1}{2} + i\frac{\sqrt{3}}{6}$, $\psi_4 = \frac{z+2}{3}$.

Let $V_0 = \{A, B\}$, we define, for arbitrary m-tuples of indices $i_1, ..., i_m \in \{1, ..., 4\}$, $\psi_{i_1...i_m} := \psi_{i_1} \circ ... \circ \psi_{i_m}$, $V_{i_1...i_m} := \psi_{i_1...i_m}(V_0)$ and

$$V_m = \bigcup_{i_1,...,i_m=1}^{4} V_{i_1...i_m}.$$

Let $V_\star = \cup_{m\geq 0} V_m$, $K = \bar{V_\star}$ the closure in $\mathbb{R}^2$. By $\#(V_m)$ we denote the number of points in V_m (it can be checked that $\#(V_m) = 4^m + 1$).

Let K_0 denote the unit segment whose endpoints are A, B, $K_{i_1...i_m} := \psi_{i_1...i_m}(K_0)$ and $V(K_{i_1...i_m}) := V_{i_1...i_m}$. For $n > 0$, we denote

$$F_n := \{K_{i_1...i_n}, i_1, ..., i_n = 1, ..., 4\}. \tag{9}$$

Fixed $i_1, ..., i_m = 1, ..., 4$ and $n \in \mathbb{N}$, we define the sets

$$\tilde{K}^{(n)}_{i_1...i_m} := \{K_{i_1...i_m j_1...j_n} : j_1, ..., j_n = 1, ..., 4\}$$

and

$$\tilde{K}_{i_1...i_m} := \psi_{i_1...i_m}(K).$$

On the Koch curve K there exists an invariant measure μ (see [4]) which is given, after normalization, by the restriction to K of the D_f- dimensional Hausdorff measure of $\mathbb{R}^2$ normalized, that is

$$\mu = (H^{D_f}(K))^{-1} H^{D_f}(\cdot)\angle K, \tag{10}$$

where $D_f = \frac{log 4}{log 3}$.

The measure μ has the property that there exists two positive constants c_1, c_2 such that

$$c_1 r^{D_f} \leq \mu(B_e(x,r)) \leq c_2 r^{D_f},\ \forall x \in K,$$

that is to say that the Koch curve is a D_f-set (see Definition 2.1).

In the following we will consider the sequence of measures μ_m defined as

$$\mu_m = \sum_{M \in F_m} \sum_{s \in V(M)} \frac{\delta(s)}{2 \cdot 4^m}, \tag{11}$$

where $\delta(s)$ is the Dirac measure at the point s.

In [9] it is proved the following lemma.

Lemma 3.1. *The sequence μ_m converges weakly to μ, for $m \to \infty$, where μ is the measure defined in (10).*

For $f : V_\star \to \mathbb{R}$, we define for $p > 1$

$$\mathcal{E}_m^{(p)}(f, f) = \frac{1}{p} 4^{(p-1)m} \sum_{M \in F_m} \sum_{r,s \in V(M)} |f(r) - f(s)|^p. \tag{12}$$

It is shown in [2] that the sequence $\mathcal{E}_m^{(p)}(f, f)$ is non-decreasing, and by defining for $f : V_\star \to \mathbb{R}$

$$\mathcal{E}^{(p)}(f, f) = \lim_{m \to \infty} \mathcal{E}_m^{(p)}(f, f), \tag{13}$$

the set

$$D_\star^{(p)} = \{f : V_\star \to \mathbb{R} : \mathcal{E}^{(p)}(f, f) < \infty\} \tag{14}$$

does not degenerate to a space containing only constant functions. As proved in [2], each $f \in D_\star^{(p)}$ can be uniquely extended in $C(K)$. We denote this extension on K still by f and by

$$D_{\mathcal{E}^{(p)}} = \{f \in C(K) : \mathcal{E}^{(p)}(f, f) < \infty\}, \tag{15}$$

where $\mathcal{E}^{(p)}(f, f) := \mathcal{E}^{(p)}(f|_{V^\star}, f|_{V^\star})$. Hence $D_{\mathcal{E}^{(p)}} \subset C(K) \subset L^p(K, \mu)$. Moreover, $(\mathcal{E}^{(p)}, D_{\mathcal{E}^{(p)}})$ is a non-negative energy functional in $L^p(K, \mu)$ and the following result holds (see [2]).

Theorem 3.2.

i) $D_{\mathcal{E}^{(p)}}$ *is complete in the norm* $(\|f\|_{p,\mu}^p + \mathcal{E}^{(p)}(f, f))^{1/p}$.

ii) $D_{\mathcal{E}^{(p)}}$ *is dense in* $L^p(K, \mu)$.

iii) $D_{\mathcal{E}^{(q)}} \subset D_{\mathcal{E}^{(p)}}$, *for* $1 < p \le q < \infty$.

In analogy with the classical case, and for a better understanding, from now on we denote by $W^{1,p}(K)$ the domain $D_{\mathcal{E}^{(p)}}$ of the nonlinear energy functional $\mathcal{E}^{(p)}$, or simply of the energy form $\mathcal{E}^{(p)}$.

We note that the form $\mathcal{E}_m^{(p)}(f, f)$ in (12) can be also written as

$$\mathcal{E}_m^{(p)}(f, f) = \frac{1}{p} \sum_{i_1, \dots, i_m = 1}^{4} \sum_{\xi, \eta \in V_o} \frac{|f(\psi_{i_1 \dots i_m}(\xi)) - f(\psi_{i_1 \dots i_m}(\eta))|^p}{4^{-mp}} 4^{-m}. \tag{16}$$

4. The main result

The main result of this paper is the following Theorem 4.1. Since, as mentioned above, the Koch curve K in $\mathbb{R}^2$ is a D_f-set with $D_f = \frac{log4}{log3} > 1$, the space $Lip_{\alpha, D_f}(p, q, K)$ is well-defined.

Throughout the paper by the same letter c we will denote different constants.

Theorem 4.1. *Let $p > 1$. Let K denote the Koch curve, $W^{1,p}(K)$ the domain of the associated nonlinear energy form $\mathcal{E}^{(p)}$ then*

$$W^{1,p}(K) = Lip_{\delta,D_f}(p,\infty,K), \tag{17}$$

where $\delta = \frac{log4}{log3}$ $(\delta = D_f)$, with equivalent norms.

We note that the smoothness index δ does not depend on p.

Proof. We start by proving the embedding of $W^{1,p}(K)$ in $Lip_{\delta,D_f}(p,\infty,K)$. For every fixed $h \in \mathbb{N}$, consider the set F_h as defined in (9). Fixed the h-tuple $i_1, ..., i_h = 1, ..., 4$ consider the element $K_{i_1...i_h}$ in F_h. We set for brevity $M := K_{i_1...i_h}$, $V(M) := V(K_{i_1...i_h})$, $\tilde{M} := \tilde{K}_{i_1...i_h}$ and, for every fixed $n \in \mathbb{N}$, $\tilde{M}^{(n)} := \tilde{K}^{(n)}_{i_1...i_h}$. By $M^\star$ we denote the set whose elements are M and all those segments from F_h that have a point in common with M, that is $M^\star := \{M_1, M_2, M_3\}$ where $M_3 := M$; moreover, we set $\tilde{M}^\star := \{\tilde{M}_1, \tilde{M}_2, \tilde{M}_3\}$. Let us choose c_0 in (6) so small that if $x \in \tilde{M}, M \in F_h, y \in K$ with $|x - y| < c_0 3^{-h}$ then $y \in \tilde{M}^\star$.

We set

$$I_h = 3^{ph\delta + hD_f} \int\int_{|x-y|<c_0 3^{-h}} |f(x) - f(y)|^p d\mu_m(x) d\mu_m(y); \tag{18}$$

we will prove that for $m \geq h$

$$I_h \leq c\mathcal{E}^{(p)}(f,f)$$

and then letting $m \to \infty$ one can pass to the limit in (18) (see Theorem 7.7.10 in [1]) thus obtaining $W^{1,p}(K) \subset Lip_{\delta,D_f}(p,\infty,K)$ where $\delta = log4/log3$. Let us consider I_h. It turns out that

$$\begin{aligned} I_h &\leq 3^{ph\delta + hD_f} \sum_{M \in F_h} \int_{x \in \tilde{M}} \int_{y \in \tilde{M}^\star} |f(x) - f(y)|^p d\mu_m(x) d\mu_m(y) = \\ &= 3^{ph\delta + hD_f} \sum_{M \in F_h} \frac{1}{4 \cdot 4^{2m}} \sum_{x \in \tilde{M} \cap V_m} \sum_{y \in \tilde{M}^\star \cap V_m} |f(x) - f(y)|^p. \end{aligned} \tag{19}$$

If $x \in \tilde{M}$ and $y \in \tilde{M}_i$, $i = 1, 2, 3$, we have that

$$|f(x) - f(y)|^p \leq 2^{p-1}|f(x) - f(x_0)|^p + 2^{p-1}|f(x_0) - f(y)|^p$$

where x_0 is the common vertex of M and $\tilde{M}_i$ if $i = 1, 2$, otherwise x_0 coincides with anyone of the two endpoints of M. Therefore

$$\begin{aligned} &\sum_{x \in \tilde{M} \cap V_m} \sum_{y \in \tilde{M}^\star \cap V_m} |f(x) - f(y)|^p \leq \\ &\quad \leq 2^{p-1} \sum_{i=1}^{3} \#(V_{m-h}) \{ \sum_{y \in \tilde{M}_i \cap V_m} |f(y) - f(x_0)|^p + \sum_{x \in \tilde{M} \cap V_m} |f(x) - f(x_0)|^p \} \end{aligned}$$

so that (19) becomes

$$I_h \leq c\,3^{ph\delta+hD_f} \sum_{M\in F_h} \frac{\#(V_{m-h})}{4\cdot 4^{2m}} \left(\sum_{x\in\tilde{M}\cap V_m} \sum_{x_0\in V(M)} |f(x)-f(x_0)|^p \right). \tag{20}$$

Let $M \in F_h$. Every $x \in \tilde{M} \cap V_m$ can be written as

$$x = \psi_{i_1\ldots i_h i_{h+1}\ldots i_m}(\xi)$$

for some ξ in V_0 and every $x_0 \in V(M)$ can be written as

$$x_0 = \psi_{i_1\ldots i_h}(\eta)$$

for some η in V_0.

We now construct a chain of points p_j with $j = h, ..., m+1$ such that

$$p_j = \begin{cases} x_0, & \text{if } j = h \\ \psi_{i_1\ldots i_h i_{h+1}\ldots i_j}(\xi) \in \tilde{M}\cap V_j,\ \xi \in V_0, & \text{if } j = h+1, ..., m \\ x, & \text{if } j = m+1. \end{cases}$$

Note that p_j and p_{j-1} belong to V_j for $j = h+1, ..., m+1$; it can occurr that p_j and p_{j-1} are not the two endpoints of a same segment belonging to F_j. In the worst case, for each pair (p_j, p_{j-1}), we introduce two more points $p_{j,3}$ and $p_{j,2}$, belonging to V_j, in order to obtain a chain $p_{j-1} = p_{j,4}, p_{j,3}, p_{j,2}, p_{j,1} = p_j$ such that the pair $(p_{j,i}, p_{j,i-1})$ for $i = 2, 3, 4$ are the two endpoints of a segment belonging to F_j. Once we fix the point x_0 and we let $x \in \tilde{M} \cap V_m$, the pair $(p_{j,i}, p_{j,i-1})$, $i = 1, 2, 3, 4$, $j = h, ..., m+1$ can occur at most $3\cdot 4^{m-j}$ times. Recall that the number of the segments of F_m generated by a segment $M_j \in F_j$ is 4^{m-j}. By the inequalities

$$\begin{aligned} &|f(p_j) - f(p_{j-1})|^p \leq \\ &\leq 2^{p-1}|f(p_{j,1}) - f(p_{j,2})|^p + 4^{p-1}|f(p_{j,2}) - f(p_{j,3})|^p + 4^{p-1}|f(p_{j,3}) - f(p_{j,4})|^p \end{aligned} \tag{21}$$

and

$$|f(x) - f(x_0)|^p \leq \sum_{j=h+1}^{m+1} (2^{p-1})^{j-h-\gamma_j} |f(p_j) - f(p_{j-1})|^p$$

where $\gamma_j = 1$ if $j = m+1$ or $\gamma_j = 0$ else, the term in the brackets in the right-hand side of (20) becomes

$$\sum_{x\in\tilde{M}\cap V_m} \sum_{x_0\in V(M)} |f(x)-f(x_0)|^p \leq c \sum_{j=h+1}^{m} \sum_{M_j\in\tilde{M}^{(j-h)}} (2^{p-1})^{j-h} 4^{m-j} \sum_{r,s\in V(M_j)} |f(r)-f(s)|^p.$$

Thus (20) becomes

$$I_h \leq c\,3^{ph\delta+hD_f} \sum_{M\in F_h} \frac{\#(V_{m-h})}{4\cdot 4^{2m}} \cdot \left[\sum_{j=h+1}^{m} \sum_{M_j\in\tilde{M}^{(j-h)}} (2^{p-1})^{j-h} 4^{m-j} \sum_{r,s\in V(M_j)} |f(r)-f(s)|^p \right], \tag{22}$$

taking into account that $\mathcal{E}_j^{(p)}(f,f)$ is a non-decreasing sequence we obtain

$$\sum_{M_j\in\tilde{M}^{(j-h)}}\sum_{r,s\in V(M_j)}|f(r)-f(s)|^p\le(4^{p-1})^{-j}\mathcal{E}^{(p)}(f,f)$$

so that we have the following estimate

$$\begin{aligned} I_h &\le c\,3^{ph\delta+hD_f}\frac{4^{2m-h}}{4\cdot 4^{2m}}\cdot\sum_{j=h+1}^{m}(2^{p-1})^{j-h-2j}2^{-2j}\mathcal{E}^{(p)}(f,f)\le\\ &\le c\,3^{ph\delta+hD_f}4^{-h}(2^{p-1})^{-h}2^{-(p+1)h}\mathcal{E}^{(p)}(f,f)\le\\ &\le c\,3^{ph\delta+hD_f}2^{-2(p+1)h}\mathcal{E}^{(p)}(f,f); \end{aligned}\tag{23}$$

by plugging the value of $\delta=log4/log3$ and $D_f=log4/log3$ we get $3^{2p\delta+hD_f}2^{-2(p+1)h}=1$ (it could be any other constant), therefore we have that

$$I_h\le c\,\mathcal{E}^{(p)}(f,f)$$

and this proves the first inclusion.

We now have to prove that $Lip_{\delta,D_f}(p,\infty,K)\subset W^{1,p}(K)$.

We will assume that $f\in Lip_{\delta,D_f}(p,\infty,K)$ and we will estimate, for every fixed $m\in\mathbb{N}$,

$$\mathcal{E}_m^{(p)}(f,f)=\frac{1}{p}(4^{p-1})^m\sum_{M\in F_m}\sum_{r,s\in V(M)}|f(r)-f(s)|^p.\tag{24}$$

By integrating the inequality $|f(r)-f(s)|^p\le 2^{p-1}|f(r)-f(x)|^p+2^{p-1}|f(x)-f(s)|^p$ over $x\in M$, we obtain

$$|f(r)-f(s)|^p\le\frac{2^{p-1}}{\mu(M)}\left(\int_M|f(r)-f(x)|^pd\mu(x)+\int_M|f(x)-f(s)|^pd\mu(x)\right),\tag{25}$$

by substituting (25) in (24) and taking into account that $\#(V(M))=2$ it follows

$$\mathcal{E}_m^{(p)}(f,f)\le\frac{1}{p}(4^{p-1})^m\cdot 2\cdot 2^{p-1}\sum_{M\in F_m}\sum_{r\in V(M)}\frac{1}{\mu(M)}\int_M|f(r)-f(x)|^pd\mu(x).\tag{26}$$

Let $M\in F_m$ and $r\in V(M)$, we denote by $M_m=M$ and by M_j with $j>m$ the segment $M_j\in\tilde{M}^{(j-m)}$ in F_j such that has r as an endpoint. We set $S_i=M_{m+i},\ i\ge 0$. Take $x_i\in S_i\cap K,\ i=0,1,...,$ then we have with $h\ge 1$ to be chosen later:

$$\begin{aligned} |f(r)-f(x_0)|^p &\le 2^{p-1}|f(r)-f(x_h)|^p+2^{p-1}|f(x_h)-f(x_0)|^p\le\\ &\le 2^{p-1}|f(r)-f(x_h)|^p+2^{p-1}\sum_{i=1}^{h}(2^{p-1})^i|f(x_i)-f(x_{i-1})|^p. \end{aligned}\tag{27}$$

By integrating over $\Pi_{i=0}^h S_i$ with respect to the product-measure and dividing by $\Pi_{i=0}^h \mu(S_i)$ we obtain

$$\begin{aligned} &\frac{1}{\mu(S_0)}\int_{S_0}|f(r)-f(x_0)|^p d\mu(x_0) \le \\ &\le \frac{2^{p-1}}{\mu(S_h)}\int_{S_h}|f(r)-f(x_h)|^p d\mu(x_h) + \\ &+2^{p-1}\sum_{i=1}^h (2^{p-1})^i \frac{1}{\mu(S_i)\mu(S_{i-1})}\int_{S_i}\int_{S_{i-1}}|f(x_i)-f(x_{i-1})|^p d\mu(x_{i-1})d\mu(x_i). \end{aligned} \tag{28}$$

If $x_i \in S_i$ and $x_{i-1} \in S_{i-1}$, then $|x_i - x_{i-1}| \le 3^{-(m+(i-1))} = 3 \cdot 3^{-(m+i)}$. It follows that the integration domain of the double integral, the set $S_i \times S_{i-1}$, is a subset of the set $\{(x_i, x_{i-1}) : x_i \in S_i, |x_i - x_{i-1}| \le 3\cdot 3^{-(m+i)}\}$. Thus, for every fixed i, we have the following upper estimate for the sum of the double integrals over all $r \in V(M)$ and $M \in F_m$:

$$\int\int_{|x_i-x_{i-1}|\le 3\cdot 3^{-(m+i)}} |f(x_i)-f(x_{i-1})|^p d\mu(x_{i-1})d\mu(x_i). \tag{29}$$

The integral in (29), taking into account that $f \in Lip_{\delta,D_f}(p,\infty,K)$, can be estimated by $3^{-(m+i)(p\delta+D_f)}|||f|||^p c_1$, where c_1 is a constant depending on c_0.

Hence, by substituting (29) in (28) and then in (26) and taking into account that $\mu(S_i) = 4^{-(m+i)}$, we get

$$\begin{aligned} &\mathcal{E}_m^{(p)}(f,f) \le \\ &\le \frac{1}{p}(4^{p-1})^m \cdot 2^p \sum_{M\in F_m}\sum_{r\in V(M)} \frac{2^{p-1}}{\mu(S_h)}\int_{S_h}|f(r)-f(x_h)|^p d\mu(x_h) + \\ &+\frac{1}{p}\cdot(4^{p-1})^m 2^p 2^{p-1}\sum_{i=1}^h (2^{p-1})^i 4^{2(m+i)} 3^{-(m+i)(p\delta+D_f)}|||f|||^p c_1. \end{aligned} \tag{30}$$

After some straightforward calculations and taking into account that $3^{D_f} = 4$ and $3^{p\delta} = 4^p$, the second term in the right-hand side of (30) can be written as

$$\frac{1}{p}\cdot(4^{p-1})^m 2^p 2^{p-1}\sum_{i=1}^h (2^{p-1})^i 4^{2(m+i)} 3^{-(m+i)(p\delta+D_f)}|||f|||^p c_1 \le c\sum_{i=1}^h (2^{-p+1})^i |||f|||^p c_1 \le c|||f|||^p \tag{31}$$

Now we have to estimate the first term in the right-hand side of (30).

We can now apply the result of Proposition 2.3 to our case. The estimate (8) ensures that if $x_h \in S_h$ then

$$|f(r)-f(x_h)| \le c|||f|||\cdot|r-x_h|^{D_f(1-1/p)} \le c|||f|||3^{-(m+h)D_f(1-1/p)}.$$

Therefore the last estimate together with $\#(F_m) = 4^m$ and $3^{-D_f} = 4^{-1}$ yield the following estimate for the first term:

$$\begin{aligned} &\frac{1}{p}(4^{p-1})^m \cdot 2^p \sum_{M\in F_m}\sum_{p\in V(M)} \frac{2^{p-1}}{\mu(S_h)}\int_{S_h}|f(r)-f(x_h)|^p d\mu(x_h) \le \\ &\le c4^m \frac{1}{4^{h(p-1)}}|||f|||^p \le c|||f|||^p, \end{aligned} \tag{32}$$

where the last estimate can be obtained by choosing $h = [m/(p-1)] + 1$. This completes the proof of the theorem. □

Corollary 4.2. *Let K be the Koch curve and $\delta = \frac{ln4}{ln3}$ then the space $Lip_{\delta,D_f}(p,\infty,K)$ does not consist only of constant functions. If $1 \le q \le \infty$, the space $Lip_{\alpha,D_f}(p,q,K)$ degenerates to a space containing constant functions only if $\alpha > \delta$ or $\alpha = \delta$ and $q < \infty$.*

Proof. From Theorem 4.1 and the results in Capitanelli [2] concerning the space $W^{1,p}(K)$, we find that for $\alpha = \delta$ and $q = \infty$ the space does not degenerate. On the other hand, by proceeding as in [5], we can prove that for $\alpha = \delta$ and $q < \infty$ or $\alpha > \delta$ and arbitrary q the space contains only constant functions. □

Corollary 4.3. *If $\delta = \frac{log4}{log3}$ and $p > 1$, the space $Lip_{\delta,D_f}(p,\infty,K)$ is dense in $L^p(K,\mu)$.*

Proof. The result follows from the density of $W^{1,p}(K)$ in $L^p(K,\mu)$ (see Theorem 3.2) and Theorem 4.1. □

Remark 4.4. We note that for general D_f-sets it can be an hard task to prove that these spaces are not trivial by explicitly constructing examples of functions belonging to these spaces. In fact in [6] when $\delta = \frac{log4}{log3}$, it is given an example of a non constant function belonging to a particular subspace of $Lip_{\delta,D_f}(p,\infty,K)$, without, however, characterizing all the functions in this space in terms of the space of functions of finite energy as those in $W^{1,p}(K)$.

Corollary 4.5. *Let K and $W^{1,p}(K)$ be as in Theorem 4.1. Then*

$$W^{1,p}(K) \subset C^{0,\beta_e}(K, |\cdot|),$$

where $\beta_e = D_f(1 - 1/p)$.

Proof. As in the quadratic case $p = 2$, it follows from Theorem 4.1 and Proposition 2.3. □

5. The intrinsic Lipschitz Spaces $\mathcal{L}ip_\nu^{(p)}(K)$

In analogy with the quadratic case [9], we now introduce the intrinsic Lipschitz spaces which will allow us to give a precise interpretation of the smoothness index α in terms of the structural constants of the fractal.

In this section, following [10], [11], [12], we introduce an effective intrinsic quasi-distance of the type $d(x,y) = |x-y|^\delta$, for $x, y \in K$. We first remark that the energy functional $\mathcal{E}^{(p)}$ inherits from its construction a self-similar invariance with respect to the mappings ψ_i, $i = 1, ..., 4$. More precisely, there exists a real constant $\rho_p = 4^{p-1}$ such that for every $u \in W^{1,p}(K)$ we have

$$\mathcal{E}^{(p)}(u,u) = \sum_{i=1}^{4} \rho_p \mathcal{E}^{(p)}(u \circ \psi_i, u \circ \psi_i). \tag{33}$$

The general criterion to select the value of the parameter δ is to further require that d^p obeys the same scaling on K as $\mathcal{E}^{(p)}$, that is,

$$d^p(x,y) = \sum_{i=1}^{4} \rho_p d^p(\psi_i(x), \psi_i(y))$$

for $x, y \in K$, where ρ_p is the same constant occurring in (33). This determines the constant δ as the unique solution of the identity

$$4\rho_p = l^{p\delta}. \tag{34}$$

A simple computation shows that $\delta = log4/log3$. Note that $\delta = D_f$ and that δ does not depend on p; we point out that this value of δ is the same which appears in Theorem 4.1.

From now on, we set

$$d(x, y) = |x - y|^{D_f}. \tag{35}$$

We denote the quasi-balls associated with d by $B(x, r)$, that is

$$B(x, r) := \{y \in K : d(x, y) \leq r\},\ x \in K, r > 0. \tag{36}$$

Remark 5.1. It can be proved (see [10]) that the measure μ of the intrinsic balls, where μ is the measure defined in (10), satisfies the following property:

there exist two positive constant $c, \bar{c}$, such that

$$cr^{\nu} \leq \mu(B(x, r)) \leq \bar{c}r^{\nu}, \text{for } x \in K, 0 < r \leq 1 \tag{37}$$

where $B(x, r)$ are the quasi-balls defined in (36) with $\nu = 1$.

This suggests to generalize the notion of D_f-sets given for Euclidean subsets of $\mathbb{R}^D$ to subsets of $\mathbb{R}^D$ endowed with the quasi-distance d such that (37) holds for a suitable constant ν. In this sense, the Koch curve, with its invariant metric d, becomes a $\nu-$set and the Hausdorff measure μ, defined in (10), becomes a $\nu-$measure. We note that in the intrinsic distance the contraction factor of the mappings ψ_i is now $\hat{l}^{-1} := l^{-\delta}$, l^{-1} being the contraction factor in the Euclidean metric.

This allows us to introduce the intrinsic Lipschitz spaces $\mathcal{L}ip_{\nu}^{(p)}(K)$.

Definition 5.2. Let $c_0 > 0$, $1 \leq p \leq \infty$. Let K denote the Koch curve with intrinsic contraction factor $\hat{l}^{-1}$, $\nu = 1$. Let μ denote the ν-measure defined on K, we call $\mathcal{L}ip_{\nu}^{(p)}(K)$ the space of those functions defined as

$$f \in L^p(K, \mu) : \|\{a_h\}\|_{l_\infty} < \infty$$

where for each $h \in \mathbb{N}$

$$a_h = \left(\hat{l}^{ph+h\nu} \int\int_{d(x,y)<c_0\hat{l}^{-h}} |f(x) - f(y)|^p d\mu(x) d\mu(y)\right)^{1/p}.$$

The norm of f in $\mathcal{L}ip_{\nu}^{(p)}(K)$ is

$$|f|_{\mathcal{L}} := \|f\|_{p,\mu} + \|\{a_h\}\|_{l_\infty}$$

where $\|f\|_{p,\mu}$ denotes the norm of f in $L^p(K, \mu)$.

Note that different values of the constants c_0 and $\hat{l}$ give equivalent spaces for c_0 and $\hat{l} > 1$.

In this new setting the results of the previous section take a simpler form and Theorem 4.1 can be reformulated as in the following theorem.

Theorem 5.3. *Let K denote the Koch curve and $W^{1,p}(K)$, $p > 1$, the domain of the corresponding nonlinear energy form $\mathcal{E}^{(p)}$ then*

$$W^{1,p}(K) = \mathcal{L}ip_\nu^{(p)}(K), \tag{38}$$

where $\nu = 1$, with equivalent norms.

As shown in [3], a Morrey-type embedding holds for functions having finite p-energy (i.e. belonging to $W^{1,p}(K)$) provided that ν is strictly less than p, that is the case of the Koch curve. More precisely, by setting for $\beta > 0$, $c > 0$

$$\mathcal{C}^{0,\beta}(K,d) = \{f : K \to \mathbb{R}, f \text{continuous on } K : |f(x) - f(y)| \leq c\, d^\beta(x,y)\}$$

we can prove the next corollary.

Corollary 5.4. *Assume the above notations (hence $\nu < p$), then*

$$W^{1,p}(K) \subset \mathcal{C}^{0,\beta}(K,d) \tag{39}$$

where $\beta = 1 - \frac{\nu}{p}$.

Proof. Theorem 4.1 yields the identification $W^{1,p}(K) = \mathcal{L}ip_\nu^{(p)}(K)$; on the other hand Proposition 2.3 gives the inclusion $\mathcal{L}ip_\nu^{(p)}(K) \subset C^{0,\beta}(K,d)$ thus concluding the proof. □

We point out that the intrinsic setting allows to interpret the exponent β in Corollary 5.4 as the Morrey exponent associated to the nonlinear energy form. This interpretation fails if one considers the Euclidean Hölder exponent β_e obtained directly by the embedding theorem of Proposition 2.3.

Remark 5.5. For the quadratic case ($p = 2$), the characterization (1) has been proved for the Koch curve and for a large class of nested fractals by Kumagai and Paluba who obtained $\alpha = d_w/2$, where d_w is the walk dimension (see [13], [8]). Always in the quadratic case in [9] the parameter α is shown to coincide with the exponent δ of the intrinsic distance d for the Koch curve. As a consequence of the result of the present paper, in particular of Theorem 4.1 and Formula 34, the identity $\alpha = \delta$ remains true also in the nonlinear case for all values of p.

Acknowledgements. The authors wish to thank Prof. Umberto Mosco and Prof. Maria Agostina Vivaldi for the several stimulating discussions during the preparation of this paper.

References

[1] H. Bauer: Probability Theory and Elements of Measure Theory, Academic Press (1981).

[2] R. Capitanelli: Nonlinear energy forms on certain fractal curves, J. Nonlinear Convex Analysis, to appear.

[3] R. Capitanelli: Functional inequalities for measure-valued Lagrangians on homogeneous spaces, preprint.

[4] J. E. Hutchinson: Fractals and selfsimilarity, Indiana Univ. Math. J. 30 (1981) 713–747.

[5] A. Jonsson: Brownian motion on fractals and function spaces, Math. Z. 6 (1996) 495–504.

[6] A. Jonsson: Haar wavelets of higher order on fractals and regularity of functions, Research Reports 7, Umeå Univ. (1998).

[7] A. Jonsson, H. Wallin: Function spaces on subsets of $\mathbb{R}^n$, Math. Rep. 2, Harwood Acad. Publ. (1984).

[8] T. Kumagai: Brownian motion penetrating fractals: an application of the trace theorem of Besov spaces, J. Funct. Anal. 170 (2000) 69–92.

[9] M. R. Lancia, M. A. Vivaldi: Lipschitz spaces and Besov traces on self-similar fractals, Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. 117, XXIII (1999) 101–116.

[10] U. Mosco: Variational fractals, Special Volume in Memory of E. De Giorgi, Ann. Scuola Norm. Sup. Pisa, Cl. Sci. 4 (1997) 683–712.

[11] U. Mosco: Variational metrics on self-similar fractals, C. R. Acad. Sci. Paris, Sér. I Math. 321 (1995) 715–720.

[12] U. Mosco: Dirichlet forms and self-similarity, in: New Directions in Dirichlet Forms, J. Jost et al., AMS/IP Stud. Adv. Math. 8, Amer. Math. Soc., Providence R. I. (1998) 117–155.

[13] K. P. Paluba: Some function spaces related to the Brownian motion on simple nested fractals, Stoch. and Stoch. Rep. 67 (1999) 267–285.