Journal of Convex Analysis
Volume 9 (2002), No. 1, 301–307

Separation Properties for Locally Convex Cones

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Received September 12, 2001
Revised manuscript received January 14, 2002

We introduce an order relation on a locally convex cone $\mathcal{P}$ which implies the strict separation property for continuous linear functionals. Using this order we establish representations for $\mathcal{P}$ as a cone of extended real-valued functions and as a cone of convex subsets of a vector space.

Keywords: Locally convex cones, separation properties, representation

1991 Mathematics Subject Classification: 46A20, 46A22

1. Introduction

The theory of locally convex cones, as developed in [1], deals with ordered cones that are not necessarily embeddable in vector spaces. A topological structure is introduced using order theoretical concepts. We shall review some of the concepts and refer to [1] for details and proofs:

An *ordered cone* is a set $\mathcal{P}$ endowed with an addition, a non-negative scalar multiplication and a partial order such that the following hold: The addition is associative and commutative, and there is a neutral element $0 \in \mathcal{P}$. For the scalar multiplication the usual associative and distributive properties hold, that is $\alpha(\beta a) = (\alpha\beta)a$, $(\alpha+\beta)a = \alpha a + \beta a$, $\alpha(a+b) = \alpha a + \alpha b$, $1a = a$ and $0a = 0$ for all $a, b \in \mathcal{P}$ and $\alpha, \beta \geq 0$. The (partial) order is a reflexive transitive relation $\leq$ such that $a \leq b$ implies $a + c \leq b + c$ and $\alpha a \leq \alpha b$ for all $a, b, c \in \mathcal{P}$ and $\alpha \geq 0$.

A *linear functional* on a cone $\mathcal{P}$ is a mapping $\mu : \mathcal{P} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ such that $\mu(a+b) = \mu(a) + \mu(b)$ and $\mu(\alpha a) = \alpha\mu(a)$ for all $a, b \in \mathcal{P}$ and $\alpha \geq 0$. In $\overline{\mathbb{R}}$ we consider the usual algebraic operations, in particular $\alpha + \infty = +\infty$ for all $\alpha \in \overline{\mathbb{R}}$, $\alpha \cdot (+\infty) = +\infty$ for all $\alpha > 0$ and $0 \cdot (+\infty) = 0$. Linear functionals can assume only finite values at invertible elements of $\mathcal{P}$.

A *full locally convex cone* $(\mathcal{P}, \mathcal{V})$ is an ordered cone $\mathcal{P}$ that contains an *abstract neighborhood system* $\mathcal{V}$, that is a subset of positive elements which is directed downward, closed for addition and multiplication by strictly positive scalars. The elements v of $\mathcal{V}$ define *upper* resp. *lower neighborhoods* for the elements of $\mathcal{P}$ by

$$v(a) = \{\, b \in \mathcal{P} \mid b \leq a + v \,\} \quad \text{resp.} \quad (a)v = \{\, b \in \mathcal{P} \mid a \leq b + v \,\},$$

creating the *upper* resp. *lower topologies* on $\mathcal{P}$. Their common refinement is called the *symmetric topology*. All elements of $\mathcal{P}$ are supposed to be *bounded below*, that is for

ISSN 0944-6532 / \$ 2.50

every $a \in \mathcal{P}$ and $v \in \mathcal{V}$ we have $0 \leq a + \lambda v$ for some $\lambda \geq 0$. Finally, a *locally convex cone*$(\mathcal{P}, \mathcal{V})$ is a subcone of a full locally convex cone not necessarily containing the abstract neighborhood system $\mathcal{V}$. Every locally convex topological vector space is a locally convex cone in this sense, as it may be canonically embedded into a full locally convex cone (see [1], Example I.2.7). Endowed with the neighborhood system $\mathcal{W} = \{\varepsilon \in \mathbb{R} \mid \varepsilon > 0\}$, $\overline{\mathbb{R}}$ is a full locally convex cone.

The *polar* v° of a neighborhood $v \in \mathcal{V}$ consists of all linear functionals μ on a locally convex cone $(\mathcal{P}, \mathcal{V})$ satisfying $\mu(a) \leq \mu(b) + 1$, whenever $a \leq b + v$ for $a, b \in \mathcal{P}$. The union of all polars of neighborhoods forms the *dual cone* $\mathcal{P}^*$ of $\mathcal{P}$. The functionals belonging to $\mathcal{P}^*$ are said to be *(uniformly) continuous*. Continuity requires that μ is monotone, and for a full cone $\mathcal{P}$ it means just that $\mu(v) \leq 1$ holds for some $v \in \mathcal{V}$ in addition. We endow $\mathcal{P}^*$ with the topology $w(\mathcal{P}^*, \mathcal{P})$ of pointwise convergence of the elements of $\mathcal{P}$, considered as functions on $\mathcal{P}^*$ with values in $\overline{\mathbb{R}}$ with its usual topology. The polar v° of a neighborhood $v \in \mathcal{V}$ is seen to be $w(\mathcal{P}^*, \mathcal{P})$-compact and convex ([1], Theorem II.2.4). Hahn-Banach type extension and separation theorems for locally convex cones were established in [1] and [2]. Theorem II.2.9 from [1] (a more general version is Theorem 4.1 from [2]) states that for a subcone $(\mathcal{Q}, \mathcal{V})$ of $(\mathcal{P}, \mathcal{V})$ every linear functional in $v^\circ \subset \mathcal{Q}^*$ extends to an element of $v^\circ \subset \mathcal{P}^*$.

While all elements of a locally convex cone are bounded below, they need not be bounded above. An element $a \in \mathcal{P}$ is *bounded (above)* if for every $v \in \mathcal{V}$ there is $\lambda \geq 0$ such that $a \leq \lambda v$. The set of all bounded elements forms a subcone and even a face in $\mathcal{P}$. All invertible elements of $\mathcal{P}$ are bounded, and continuous linear functionals take only finite values at bounded elements.

In Section 2 we define and compare several order relations on a locally convex cone. Section 3 contains our main separation results, showing that the weak local and global preorders and their induced neighborhoods are determined by continuous linear functionals. Separation properties of this type are instrumental in the context of approximation theory in locally convex cones, where duality theory is used in the investigation of sequences of continuous linear operators (see Chapter IV in[1]). Our Example 2.4 demonstrates that the given order of a locally convex cone does in general not support these properties. In Section 4 we establish representations for a locally convex cone as a cone of $\overline{\mathbb{R}}$-valued functions and as a cone of convex subsets of a locally convex topological vector space, respectively.

2. Global and Local Preorders

Throughout the following let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone. We define the *(global) preorder* $\prec$ for elements $a, b \in \mathcal{P}$ by

$$a \prec b \qquad \text{if} \qquad a \leq b + v \quad \text{for all} \quad v \in \mathcal{V}.$$

The properties of an order relation on a cone are easily verified, and $a \leq b$ obviously implies $a \prec b$. The *weak (global) preorder* $\preccurlyeq$ is defined by

$$a \preccurlyeq b \qquad \text{if} \qquad a \leq \gamma b + v$$

for every $v \in \mathcal{V}$ and $\varepsilon > 0$ with some $1 \leq \gamma \leq 1 + \varepsilon$. Obviously $a \prec b$ implies $a \preccurlyeq b$. It is again straightforward to verify the required properties for an order relation: For transitivity, let $a \preccurlyeq b$ and $b \preccurlyeq c$. Given $v \in \mathcal{V}$ and $0 < \varepsilon \leq 1$, there are $1 \leq \gamma, \gamma' \leq 1+\varepsilon/3$ such that

$$a \leq \gamma b + v/2 \qquad \text{and} \qquad b \leq \gamma' c + v/2,$$

hence $a \leq \gamma\gamma' c + (1/2)(1+\gamma)v \leq \gamma\gamma' c + v$. Since $1 \leq \gamma\gamma' \leq (1+\varepsilon/3)^2 \leq 1+\varepsilon$, this shows $a \preccurlyeq c$. Compatibility with the multiplication by positive scalars is obvious. For compatibility with the addition, let $a \preccurlyeq b$ and $c \in \mathcal{P}$. Given $v \in \mathcal{V}$ and $\varepsilon > o$, there is $\lambda \geq 0$ such that $0 \leq c + \lambda v$, as c is bounded below. There is $1 \leq \gamma \leq 1 + \min\{\varepsilon,\, 1/(2\lambda)\}$ such that $a \leq \gamma b + v/2$. Thus

$$a + c \leq \gamma b + v/2 + c + (\gamma - 1)(c + \lambda v) \leq \gamma(b+c) + v.$$

This shows $a + c \preccurlyeq b + c$. We may also use these preorders in a full cone containing $\mathcal{P}$, hence involving the elements of $\mathcal{P}$ as well as the neighborhoods.

Lemma 2.1. *For $a, b \in \mathcal{P}$ we have $a \preccurlyeq b$ if and only if $a \preccurlyeq b + v$ for all $v \in \mathcal{V}$.*

Proof. Assume that $a \preccurlyeq (b+v)$ for all $v \in \mathcal{V}$. Given $v \in \mathcal{V}$ and $0 < \varepsilon \leq 1$, we have $a \preccurlyeq (b + v/4)$, hence $a \leq \gamma(b + v/4) + v/2$ for some $1 \leq \gamma \leq 1+\varepsilon$. As $\gamma \leq 2$, this shows $a \leq \gamma b + v$, and indeed $a \preccurlyeq b$. $\square$

Given a neighborhood $v \in \mathcal{V}$ we define the corresponding *local preorders* $\prec_v$ and $\preccurlyeq_v$ for elements $a, b \in \mathcal{P}$ by

$$\begin{array}{llll} a \prec_v b & \text{if} & a \prec b + \varepsilon v & \text{for all} \quad \varepsilon > 0. \\ a \preccurlyeq_v b & \text{if} & a \preccurlyeq b + \varepsilon v & \text{for all} \quad \varepsilon > 0. \end{array}$$

The required properties for an order relation follow from the corresponding global relations. Note that the local and global preorders coincide if the neighborhood system $\mathcal{V}$ consists of the multiples of a single element. The local preorders may be directly expressed using the given order as follows:

Lemma 2.2. *For elements $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$*

(a) $a \prec_v b$ *if and only if* $a \leq b + \varepsilon v$ *for all* $\varepsilon > 0$.
(b) $a \preccurlyeq_v b$ *if and only if* $a \leq \gamma b + \varepsilon v$ *for every* $\varepsilon > 0$ *with some* $1 \leq \gamma \leq 1+\varepsilon$.

Proof. (a) If $a \prec_v b$, then $a \prec (b+\varepsilon v)$ holds for all $\varepsilon > 0$, hence $a \leq (b+\varepsilon v)+\varepsilon v = b+2\varepsilon v$. If on the other hand, $a \leq b + \varepsilon v$ holds for all $\varepsilon > 0$, this implies $a \prec b + \varepsilon v$, hence $a \prec_v b$. For (b), let $a \preccurlyeq_v b$ and $0 < \varepsilon \leq 1$. Then $a \preccurlyeq \big(b+(\varepsilon/4)v\big)$, hence $a \leq \gamma\big(b+(\varepsilon/4)v\big)+(\varepsilon/2)v$ for some $1 \leq \gamma \leq 1+\varepsilon$. As $\gamma \leq 2$, this shows $a \leq \gamma b + \varepsilon v$, as claimed. For the converse, suppose that for every $\varepsilon > 0$ there exists such a γ as stated. Given $\varepsilon > 0$, we shall show that $a \preccurlyeq (b+\varepsilon v)$. For this, let $w \in \mathcal{V}$ and $\varepsilon' > 0$. For $\varepsilon'' = \min\{\varepsilon, \varepsilon'\}$ choose $1 \leq \gamma \leq 1+\varepsilon''$ such that $a \leq \gamma b + \varepsilon'' v$. Then $a \leq \gamma(b + \varepsilon v) + w$. This shows $a \preccurlyeq (b + \varepsilon v)$. $\square$

We shall observe that the preorders $\prec$ and $\preccurlyeq$ coincide if $\mathcal{P}$ contains sufficiently many bounded elements. In this vein, a locally convex cone $(\mathcal{P}, \mathcal{V})$ is said to be *tightly covered by its bounded elements* (see [1], II.2.13) if for all $v \in \mathcal{V}$ and $a, b \in \mathcal{P}$ such that $a \not\leq b + v$ there is a bounded element $a' \in \mathcal{P}$ such that $a' \leq a$ but $a' \not\leq b + v$.

Proposition 2.3. *If the locally convex cone $(\mathcal{P}, \mathcal{V})$ is tightly covered by its bounded elements, then the preorders $\prec$ and $\preccurlyeq$ (and therefore also the corresponding local preorders) coincide.*

Proof. Following Lemma 2.1 we only have to show that for $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$ the relations $a \prec b + v$ and $a \preccurlyeq b + v$ coincide. Always, $a \prec b + v$ implies $a \preccurlyeq b + v$. For the converse, let $a \preccurlyeq b + v$ and assume that contrary to our claim $a \not\prec b + v$, that is $a \not\leq b + v + w$ for some $w \in \mathcal{V}$. By our assumption then there is a bounded element $a' \leq a$ such that $a' \not\leq b + v + w$. We have $a' \leq \lambda w$ for some $\lambda > 0$. There is $1 \leq \gamma \leq 1 + 1/(2\lambda)$ such that $a \leq \gamma(b + v) + w/2$. Thus $a' \leq \gamma(b + v) + w/2$ and $(1/\gamma)a' \leq (b + v) + w/2$, hence

$$a' = \frac{1}{\gamma}a' + \frac{\gamma - 1}{\gamma}a' \leq (b + v) + w/2 + \frac{\gamma - 1}{\gamma}\lambda w \leq (b + v) + w,$$

contradicting the above. Thus $a \prec b + v$ holds as claimed. $\square$

Example 2.4. Let $\mathcal{P}$ be the cone whose elements are the real intervals $[0, \alpha]$ for $\alpha \geq 0$ and $[0, \beta)$ for $\beta > 0$, endowed with the usual addition of subsets of $\mathbb{R}$, and the set inclusion as order. Let $\mathcal{V} = \{0\}$. Then $a \leq b + 0$ if and only if $a \subset b$ for $a, b \in \mathcal{P}$. The orders $\prec_0$ and $\preccurlyeq_0$ do not coincide in this example, since we have $[0, 1] \subset \gamma[0, 1)$ for all $\gamma > 1$, hence $[0, 1] \preccurlyeq_0 [0, 1)$, but not $[0, 1] \prec_0 [0, 1)$.

The choice for $\mathcal{V} = \{0\}$ while permissible seems peculiar. However, even in standard examples a non-zero neighborhood system may act in such a way on certain subcones: Let $\mathcal{P}$ be the cone of all non-empty convex subsets of $\mathbb{R}^2$, endowed with the standard operations and set inclusion as order, and the neighborhood system $\mathcal{V}$ consisting of all non-zero multiples of the Euclidean unit ball. On the subcone $\mathcal{Q}$ of $\mathcal{P}$ consisting of all positive multiples of the set $\{(x, y) \in \mathbb{R}^2 \mid x^2 \leq y\}$ all neighborhoods act as zero.

3. Separation Properties

We shall show that the weak local and global preorders on a locally convex cone are determined by the elements of the dual cone.

Theorem 3.1. *For elements $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$*

(a) $a \preccurlyeq_v b$ *if and only if* $\mu(a) \leq \mu(b)$ *for all* $\mu \in v^\circ$.
(b) $a \preccurlyeq b$ *if and only if* $\mu(a) \leq \mu(b)$ *for all* $\mu \in \mathcal{P}^*$.

Proof. Part (a) is a special case of part (b) if we use the neighborhood system $\widetilde{\mathcal{V}} = \{\lambda v \mid \lambda > 0\}$ instead of $\mathcal{V}$. The dual cone then consists of all positive multiples of the functionals in v°.

For (b), let $a \preccurlyeq b$ and let $\mu \in \mathcal{P}^*$, that is $\mu \in v^\circ$ for some $v \in \mathcal{V}$. Given $\varepsilon > 0$ there is $1 \leq \gamma \leq 1 + \varepsilon$ such that $a \leq \gamma b + \varepsilon v$, hence $\mu(a) \leq \gamma\mu(b) + \varepsilon$. This shows $\mu(a) \leq \mu(b.)$ The proof of the converse implication will however require some Hahn-Banach type arguments that had been established in [2]. For a given element $b \in \mathcal{P}$ and $\beta \in \mathbb{R}$ consider the sublinear functional p on $\mathcal{P}$ defined as $p(x) = \lambda\beta$ if $x = \lambda b$, and $p(x) = +\infty$ else, together with the superlinear functional $q(0) = 0$ and $q(x) = -\infty$ for $x \neq 0$. Following Theorem 3.1 in [2] there is a linear functional $\mu \in \mathcal{P}^*$ such that $q \leq \mu \leq p$ if and only if we can find a neighborhood $v \in \mathcal{V}$ such that $q(x) \leq p(x) + 1$ whenever $x \leq y + v$ for $x, y \in \mathcal{P}$; that is in our particular case $0 \le \lambda\beta + 1$ whenever $0 \le \lambda b + v$. If $0 \prec b$, then there are arbitrarily large such $\lambda \ge 0$, so we have to require that $\beta \ge 0$. Otherwise, for any $\beta \in \mathbb{R}$ we can find such a neighborhood. We shall use Theorem 5.1 from [2], which describes the range of all linear functionals $\mu \in \mathcal{P}^*$ such that $q \le \mu \le p$ on a fixed element $a \in \mathcal{P}$. If there is at least one such μ, then

$$\sup_{\substack{\mu\in\mathcal{P}^* \\ q\le\mu\le p}} \mu(a) \;=\; \sup_{v\in\mathcal{V}} \inf \{p(x) - q(y) \mid x, y \in \mathcal{P},\;\; q(y) \in \mathbb{R},\;\; a + y \le x + v\}.$$

With the particular insertions for p and q from above we obtain

$$\sup_{\substack{\mu\in\mathcal{P}^* \\ q\le\mu\le p}} \mu(a) \;=\; \sup_{v\in\mathcal{V}} \inf \{\lambda\beta \mid \lambda \ge 0,\;\; a \le \lambda b + v\}.$$

Now let us assume that $\mu(a) \le \mu(b)$ holds for all $\mu \in \mathcal{P}^*$. As $q \le \mu \le p$ implies that $\mu(b) \le \beta$, this yields

$$\sup_{v\in\mathcal{V}} \inf \{\lambda\beta \mid \lambda \ge 0,\;\; a \le \lambda b + v\} \;\le\; \beta$$

for all admissible values of β. We shall derive $a \preccurlyeq b$ from this. Let $v \in \mathcal{V}$ and $\varepsilon > 0$. We choose $\beta = 1$ in the above and observe that there is $\lambda \ge 0$ such that

$$a \le \lambda b + \frac{\varepsilon}{2} v \qquad \text{and} \qquad \lambda \le 1 + \varepsilon.$$

If $1 \le \lambda$, this settles our claim. Otherwise we proceed distinguishing 2 cases. If $0 \prec b$, then $0 \le b + (\varepsilon/2)v$. Thus $a \le \lambda b + (\varepsilon/2)v + (1-\lambda)\big(b + (\varepsilon/2)v\big) \le b + \varepsilon v$, satisfying our requirement. If $0 \not\prec b$, then we may use the above inequality for $\beta = -1$ as well. There is $\rho > 0$ such that $0 \le b + \rho v$. We find $\lambda' \ge 0$ such that

$$a \le \lambda' b + \frac{\varepsilon}{2} v \qquad \text{and} \qquad -\lambda' \le -1 + \frac{\varepsilon}{2\rho},$$

that is $\lambda' \ge 1 - \varepsilon/(2\rho)$. Next we choose $0 \le \alpha \le 1$ such that $1 - \varepsilon/(2\rho) \le \lambda'' < 1$ holds for $\lambda'' = \alpha\lambda + (1-\alpha)\lambda'$. (Recall that we are considering the case that $\lambda < 1$, therefore such a choice of α is possible.) Then $a \le \lambda'' b + (\varepsilon/2)v$ holds as well. This shows $a \le \lambda'' b + (\varepsilon/2)v + (1-\lambda'')\big(b + \rho v\big) \le b + \varepsilon v$, again satisfying our requirement. □

Theorem 3.2. *For elements $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$ we have $a \preccurlyeq b + v$ if and only if $\mu(a) \le \mu(b) + 1$ for all $\mu \in v^\circ$.*

Proof. We shall use Theorem 3.1(a) in a full cone containing $\mathcal{P}$ and $\mathcal{V}$. If $a \preccurlyeq (b+v)$, then $\mu(a) \le \mu(b+v) = \mu(b) + \mu(v) \le \mu(b) + 1$ for all $\mu \in v^\circ$ by Theorem 3.1 (a). On the other hand, if $a \not\preccurlyeq (b+v)$, following 3.1(b) there is $\mu \in \mathcal{P}^*$ such that $\mu(a) > \mu(b+v) = \mu(b) + \mu(v)$. This shows $\mu(v) < +\infty$ in particular. If $\mu(v) = 0$, then $\lambda\mu \in v^\circ$ for all $\lambda \ge 0$, and we may choose λ such that $(\lambda\mu)(a) > (\lambda\mu)(b) + 1$. If $\mu(v) > 0$, we set $\lambda = 1/\mu(v)$ and have $\lambda\mu \in v^\circ$, and again $(\lambda\mu)(a) > (\lambda\mu)(b) + 1$. □

The statement of Theorem 3.2, if it applies to the preorder $\prec$ as well, is called the *strict separation property* for $(\mathcal{P}, \mathcal{V})$ (see [1], II.2.12).

Corollary 3.3. *A locally convex cone $(\mathcal{P},\mathcal{V})$ has the strict separation property if and only if the preorders $\prec$ and $\preccurlyeq$ coincide.*

Proof. If the preorders $\prec$ and $\preccurlyeq$ coincide, then our claim follows from Theorem 3.2. If on the other hand $\mathcal{P}$ has the strict separation property and if $a \preccurlyeq b+v$ for $a,b \in \mathcal{P}$ and $v \in \mathcal{V}$, then $\mu(a) \le \mu(b)+1$ for all $\mu \in v^\circ$, hence $a \prec b+v$ as well. Lemma 2.1 will complete our argument. □

If we replace the given order of $(\mathcal{P},\mathcal{V})$ with the preorder $\prec$ or the weak preorder $\preccurlyeq$, respectively, then $(\mathcal{P},\mathcal{V})$ becomes again a locally convex cone. The dual cone $\mathcal{P}^*$ of $\mathcal{P}$ and the polars of neighborhoods remain unchanged. Theorem 3.2 demonstrates that $\preccurlyeq$ is in fact the coarsest order on $(\mathcal{P},\mathcal{V})$ that generates this duality.

4. Representations for Locally Convex Cones

We shall establish a representation for a locally convex cone $(\mathcal{P},\mathcal{V})$ as a cone of continuous $\overline{\mathbb{R}}$-valued functions on some topological space X. Using the preceding separation results, this representation is straightforward. Let $X=\mathcal{P}^*$, endowed with the topology $w(\mathcal{P}^*,\mathcal{P})$, and let $\widetilde{\mathcal{P}}$ denote the cone of all continuous $\overline{\mathbb{R}}$-valued functions on X, endowed with the pointwise order and algebraic operations. Corresponding to the neighborhoods $v \in \mathcal{V}$ we define neighborhoods $\tilde{v}$ for functions $f,g \in \widetilde{\mathcal{P}}$ by

$$f \le g+\tilde{v} \qquad \text{if} \qquad f(x) \le g(x)+1 \quad \text{for all} \quad x \in v^\circ.$$

Then $w(\mathcal{P}^*,\mathcal{P})$-compactness of the sets $v^\circ \subset X$ guarantees that the continuous functions $f \in \widetilde{\mathcal{P}}$ are bounded below on v°, hence $0 \le f+\lambda\tilde{v}$ for some $\lambda \ge 0$. With the neighborhood system $\widetilde{\mathcal{V}} = \left\{ \sum_{i=1}^n \lambda_i \tilde{v}_i \mid \lambda_i > 0,\ v_i \in \mathcal{V},\ n \in \mathbb{N}\right\}$, thus $(\widetilde{\mathcal{P}},\widetilde{\mathcal{V}})$ becomes a locally convex cone. We consider the canonical embedding

$$a \mapsto f_a : \mathcal{P} \to \widetilde{\mathcal{P}},$$

where $f_a(x)=x(a)$ for all $x \in X$. This embedding is linear and preserves the weak global preorder by Theorem 3.1(b), that is $a \preccurlyeq b$ for $a,b \in \mathcal{P}$ if and only if $f_a \le f_b$. It also preserves the neighborhoods with respect to this order, that is $a \preccurlyeq b+v$ holds for $a,b \in \mathcal{P}$ and $v \in \mathcal{V}$ if and only if $f_a \le f_b+\tilde{v}$. The latter is a consequence of Theorem 3.2.

We may use the same procedure to represent $(\mathcal{P},\mathcal{V})$ as a cone of convex subsets of a locally convex topological vector space. Let $\widetilde{\mathcal{P}}$ and $\widetilde{\mathcal{V}}$ be as above, and let E be the vector space of all continuous real-valued functions on X. A neighborhood basis for $0 \in E$ is given by the convex sets $V=\{f \in E \mid |f(x)| \le 1 \text{ for all } x \in v^\circ\}$ corresponding to the neighborhoods $v \in \mathcal{V}$. Now let $\overline{\mathcal{P}}$ be the cone of all non-empty convex subset of E, endowed with the usual addition and multiplication by scalars and the set inclusion as order. With the neighborhood system $\overline{\mathcal{V}} = \left\{ \sum_{i=1}^n \lambda_i V_i \mid \lambda_i > 0,\ v_i \in \mathcal{V}\right\}$, then $(\overline{\mathcal{P}},\overline{\mathcal{V}})$ is a full locally convex cone. We consider the embedding

$$a \mapsto A : \mathcal{P} \to \overline{\mathcal{P}},$$

where $A=\{f \in E \mid f \le f_a\}$. This embedding is linear (a consequence of the Riesz decomposition property for functions in E), and preserves the weak global preorder, as $a \preccurlyeq b+v$ means that $f_a \le f_b+\tilde{v}$ which in turn holds if and only if $A \subset B+V$. We summarize:

Theorem 4.1. *Every locally convex cone $(\mathcal{P}, \mathcal{V})$ may be embedded into*

(i) *a locally convex cone of continuous $\overline{\mathbb{R}}$-valued functions on some topological space X, endowed with the pointwise order and operations and the topology of uniform convergence on a family of compact subsets of X.*

(ii) *a locally convex cone of convex subsets of a locally convex topological vector space, endowed with the usual addition and multiplication by scalars, the set inclusion as order and the neighborhoods inherited from the vector space.*

These embeddings are linear and preserve the weak global preorder and the neighborhoods.

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