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The Calibration Method for Free-Discontinuity Problems on Vector-Valued Maps

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The calibration method is a classical minimality criterion, which has been recently adapted to functionals with free discontinuities by Alberti, Bouchitté, Dal Maso. In this paper we present a further generalization of this theory to functionals defined on vector-valued maps.

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1. Introduction

Many variational problems arising from several branches of applied analysis lead to consider minimum problems for functionals of the form

$$F(u) = \int_\Omega f(x, u, \nabla u)\, dx + \int_{S_u} \psi(x, u^-, u^+, \nu_u)\, d\mathcal{H}^{n-1}, \qquad (1)$$

where Ω is a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary, $f : \Omega\times\mathbb{R}^N\times\mathbb{R}^{nN} \to [0, +\infty]$, and $\psi : \Omega\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{n-1} \to [0, +\infty]$ are Borel functions, $\mathbb{S}^{n-1} := \{v \in \mathbb{R}^n : |v| = 1\}$, $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure, and the unknown function u belongs to the space $[SBV(\Omega)]^N$ of special functions of bounded variation in Ω; ∇u denotes the approximate gradient of u, S_u is the set of essential discontinuity points of u, ν_u is the approximate unit normal vector to S_u, and u^-, u^+ the approximate limits of u on the two sides of S_u.

Following a terminology by De Giorgi, variational problems of this form are denoted by *free-discontinuity problems*. A typical example is provided by the so-called Mumford-Shah functional, introduced in [14] in the context of image segmentation, which can be written as

$$MS(u) := \int_\Omega |\nabla u|^2 dx + \alpha\mathcal{H}^{n-1}(S_u) + \beta \int_\Omega |u - g|^2\, dx,$$

where g is a function in $[L^\infty(\Omega)]^N$, and $\alpha > 0$ and $\beta \geq 0$ are constants.

One of the main features of functionals of the form (1) is that they are in general not convex; therefore, all conditions which can be obtained by infinitesimal variations are necessary for minimality, but in general not sufficient.

In [2] Alberti, Bouchitté, Dal Maso have proposed a sufficient condition for minimality,

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based on the *calibration method*, which leads to many applications (see [2], [6], [11], [12], [13]). In the form presented in [2], this minimality criterion is limited to scalar functions.

The purpose of this paper is to develop a similar theory for functionals on vector-valued maps.

In order to describe the basic idea behind the calibration method, let us focus our attention on *Dirichlet minimizers* of F, that is minimizers with prescribed boundary values. Given a candidate u, if we are able to construct a functional G which is invariant on the class of functions having the same boundary values and satisfies

$$G(u) = F(u), \qquad \text{and} \qquad G(v) \le F(v) \quad \text{for every admissible } v, \tag{2}$$

then u is a Dirichlet minimizer of F. Indeed, if such a functional exists, for every v with the same boundary values as u we have

$$F(u) = G(u) = G(v) \le F(v).$$

In [2] the rôle of G is carried out by the flux of a suitable divergence-free vectorfield $\phi : \Omega \times \mathbb{R} \to \mathbb{R}^{n+1}$ through the complete graph of v, which is defined as the union of the usual graph of v and of all segments joining $(x, v^-(x))$ and $(x, v^+(x))$ with x ranging in S_v. Since ϕ is divergence-free, the flux is clearly invariant with respect to the boundary values, while suitable further conditions on ϕ guarantee (2) (see Remark 3.6).

In this paper this theory is generalized to the vectorial case by considering a different kind of invariant functional: the calibration is no longer a vectorfield, but a pair of functions $(\mathcal{S}, \mathcal{S}_0)$, where $\mathcal{S} : \Omega \times \mathbb{R}^N \to \mathbb{R}^n$ is suitably regular (more precisely, globally Lipschitz and piecewise C^1), while $\mathcal{S}_0$ belongs to $L^1(\Omega)$; the comparison functional for F is given by

$$-\int_{\partial\Omega} \langle \mathcal{S}(x, v), \nu_{\partial\Omega} \rangle \, d\mathcal{H}^{n-1} + \int_\Omega \mathcal{S}_0(x)\, dx, \tag{3}$$

where $\nu_{\partial\Omega}$ is the inner unit normal to $\partial\Omega$. It is clear that the functional (3) is constant on the functions having the same values at $\partial\Omega$. Moreover, by the divergence theorem we can rewrite (3) as

$$\int_\Omega d\mu_v + \int_\Omega \mathcal{S}_0(x)\, dx,$$

where μ_v is the divergence (in the sense of distributions) of the composite function $\mathcal{S}(\cdot, v(\cdot))$. A generalized version of the chain rule in BV (which is proved in Section 2) implies that

$$\mu_v = ([\mathrm{div}_x \mathcal{S}](x, v) + \langle (D_y \mathcal{S}(x, v))^\tau, \nabla v \rangle)\, \mathcal{L}^n + \langle \mathcal{S}(x, v^+) - \mathcal{S}(x, v^-), \nu_v \rangle\, \mathcal{H}^{n-1} \lfloor S_v,$$

where $[\mathrm{div}_x \mathcal{S}]$ denotes the divergence of $\mathcal{S}$ with respect to the variable $x \in \Omega$, and $(D_y \mathcal{S})^\tau$ the transpose of the Jacobian matrix of $\mathcal{S}$ with respect to the variable $y \in \mathbb{R}^N$. Therefore the functional (3) turns out to be equal to

$$\int_\Omega ([\mathrm{div}_x \mathcal{S}](x, v) + \langle (D_y \mathcal{S}(x, v))^\tau, \nabla v \rangle + \mathcal{S}_0(x))\, dx + \int_{S_v} \langle \mathcal{S}(x, v^+) - \mathcal{S}(x, v^-), \nu_v \rangle \, d\mathcal{H}^{n-1}.$$

By comparing this expression with the functional (1), we find pointwise conditions on $\mathcal{S}_0$, $\mathcal{S}$, and the derivatives of $\mathcal{S}$, which guarantee (2), and then the Dirichlet minimality of a given u. For a precise statement of these conditions see Lemma 3.2 and Lemma 3.3.

This formulation is related to classical field theory for multiple integrals of the form

$$F_0(u) = \int_\Omega f(x, u, \nabla u)\, dx.$$

In this framework a sufficient condition for the minimality of a candidate $u \in [C^1(\Omega)]^N$ is obtained by comparing F_0 with the integral of a null-lagrangian of divergence type, which is constructed starting from a suitably defined slope field $\mathcal{P}$, called *Weyl field*, and a function $\mathcal{S} \in [C^2(\Omega \times \mathbb{R}^N)]^n$, the *eikonal map* associated with $\mathcal{P}$ (cf., e.g., [9]). In Section 4 we prove that, under suitable assumptions on f and ψ, whenever a Weyl field exists for a function $u \in [C^1(\Omega)]^N$ (so that u is a Dirichlet minimizer for F_0), then there exists a calibration for u with respect to the functional F (which is given by the eikonal map $\mathcal{S}$ and by $\mathcal{S}_0 \equiv 0$), so u is also a Dirichlet minimizer for F among SBV functions.

Some examples and applications are presented in Section 5. In Examples 5.1, 5.3, 5.4, and 5.5 we deal with minimizers of the Mumford-Shah functional, and we generalize some results proved in [2] for the scalar case. A purely vectorial example is given by Example 5.2, where we study the minimality of continuous solutions of the Euler equations for a functional arising in fracture mechanics, which can be defined only on maps from $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^n$.

Finally, we point out that, as mentioned in [2], one could try to generalize the calibration theory from the scalar case to the vectorial one by replacing divergence-free vectorfields by closed n-forms on $\Omega \times \mathbb{R}^N$, acting on the graphs of the functions v, viewed as suitably defined surfaces in $\Omega \times \mathbb{R}^N$. This could lead to the idea that our choice of writing the calibration in terms of the pair $(\mathcal{S}, \mathcal{S}_0)$ is somehow restrictive. This is not at all the case, since the existence of a calibration expressed via differential forms implies the existence of a calibration expressed in terms of a pair $(\mathcal{S}, \mathcal{S}_0)$, as shown in Section 6.

The plan of the paper is the following: in Section 2 we fix the notation and we recall some basic results from BV functions theory; in Section 3 we present the calibration method for functionals of the form (1) on vector-valued maps; Section 4 is devoted to the link between calibration theory and classical field theory; Section 5 contains some examples and applications; finally, in Section 6 we reformulate the theory of calibrations in terms of differential forms and show that this formulation does not lead to new results.

2. Notation and preliminary results

In this section we fix the notation and we recall some results from BV functions theory.

Given $x, y \in \mathbb{R}^n$, we denote their scalar product by $\langle x, y \rangle$, and the euclidean norm of x by $|x|$. Given $a, b \in \mathbb{R}$, the maximum and the minimum of $\{a, b\}$ are denoted by $a \vee b$ and $a \wedge b$, respectively.

In the following Ω is a fixed bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary, $\nu_{\partial\Omega}$ is its inner unit normal, while U is a closed subset of $\overline{\Omega} \times \mathbb{R}^N$. The letter x usually denotes the variable in Ω (or $\mathbb{R}^n$), while y or z is the variable in $\mathbb{R}^N$.

We say that a function $u : \Omega \to \mathbb{R}^N$ has *bounded variation in* Ω, and we write $u \in [BV(\Omega)]^N$, if u belongs to $[L^1(\Omega)]^N$ and its distributional derivative Du is a finite Radon $\mathbb{R}^{nN}$-valued measure in Ω.

Since Ω has Lipschitz boundary, we can speak about the *trace* of u on $\partial\Omega$, which belongs to $L^1(\partial\Omega, \mathcal{H}^{n-1})$ and will be still denoted by u.

The *singular set* S_u of u is defined as the set of all points where u does not admit an approximate limit; S_u is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable, that is, it can be covered, up to an $\mathcal{H}^{n-1}$-negligible set, by countably many C^1-hypersurfaces. Moreover, at $\mathcal{H}^{n-1}$-a.e. $x_0 \in S_u$ there exists a triplet $(u^+(x_0), u^-(x_0), \nu_u(x_0)) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^{n-1}$ such that $u^+(x_0) \neq u^-(x_0)$, $\nu_u(x_0)$ is normal to S_u in an approximate sense, and

$$\lim_{r\to 0^+} \frac{1}{\mathcal{L}^n(B_r^\pm(x_0))} \int_{B_r^\pm(x_0)} |u(x) - u^\pm(x_0)| \, dx = 0,$$

where $B_r^\pm(x_0)$ is the intersection of the ball of radius r centred at x_0 with the half-plane $\{x \in \mathbb{R}^n : \pm\langle x - x_0, \nu_u(x_0)\rangle \geq 0\}$. The triplet $(u^+(x_0), u^-(x_0), \nu_u(x_0))$ is uniquely determined up to a permutation of $(u^+(x_0), u^-(x_0))$ and a change of sign of $\nu_u(x_0)$.

The measure Du can be decomposed as

$$Du = D^a u + D^j u + D^c u,$$

where $D^a u$ is the absolutely continuous part with respect to $\mathcal{L}^n$, $D^j u$ is the *jump part* and satisfies $D^j u = (u^+ - u^-) \otimes \nu_u \mathcal{H}^{n-1} \lfloor S_u$, and finally $D^c u$ is the so-called *Cantor part*. The density of $D^a u$ with respect to $\mathcal{L}^n$ is denoted by ∇u and agrees with the *approximate gradient* of u. Moreover, we call $D^a u + D^c u$ the *diffuse part* of the derivative of u and we denote it by $\tilde{D}u$.

We say that a function $u : \Omega \to \mathbb{R}^N$ is a *special function of bounded variation*, and we write $u \in [SBV(\Omega)]^N$, if $u \in [BV(\Omega)]^N$ and $D^c u = 0$.

Finally, we define as *graph of* u the set $G_u := \{(x, \tilde{u}(x)) : \ x \in \Omega \setminus S_u\}$, where $\tilde{u}$ is the precise representative of u (which is defined everywhere on $\Omega \setminus S_u$ by definition of S_u).

For more details on BV functions theory see [4].

We conclude this section with the proof of a generalized chain rule in BV. If $u \in [BV(\Omega)]^N$ and $\mathcal{S}$ is a Lipschitz continuous function from $\mathbb{R}^N$ into $\mathbb{R}^M$, it is known that $\mathcal{S} \circ u$ belongs to $[BV(\Omega)]^M$. When in addition $\mathcal{S} \in [C^1(\mathbb{R}^N)]^M$, the following chain rule formula can be written:

$$\begin{aligned} \tilde{D}(\mathcal{S} \circ u) &= D\mathcal{S}(\tilde{u}(x))\tilde{D}u(x) \quad \text{on } \Omega \setminus S_u, \\ D^j(\mathcal{S} \circ u) &= [\mathcal{S}(u^+) - \mathcal{S}(u^-)] \otimes \nu_u \mathcal{H}^{n-1} \lfloor S_u, \end{aligned} \tag{4}$$

(see Theorem 3.96 in [4]). Following an idea by [15], we generalize formula (4) to the case of a function $\mathcal{S}$, which may depend also on the variable x and is only piecewise C^1 in the sense of the following definition.

Definition 2.1. We say that a Lipschitz continuous function $\mathcal{S} : U \to \mathbb{R}^M$ is *piecewise* C^1 if $\mathcal{S}$ can be written as

$$\mathcal{S}(x, y) = \sum_{\alpha \in A} \mathcal{S}^\alpha(x, y) 1_{U^\alpha}(x, y), \tag{5}$$

where $(U^\alpha)_{\alpha\in A}$ is a finite family of pairwise disjoint Borel sets such that $\cup_{\alpha\in A} U^\alpha = U$, and $(\mathcal{S}^\alpha)_{\alpha\in A}$ is a family of Lipschitz continuous functions belonging to $[C^1(\overline{\Omega}\times\mathbb{R}^N)]^M$.

Lemma 2.2. *Let $\mathcal{S} : U \to \mathbb{R}^M$ be a Lipschitz continuous function, piecewise C^1 in the sense of Definition 2.1, and satisfying (5), and let $u \in [BV(\Omega)]^N$ be such that $G_u \subset U$. Then, $v := \mathcal{S}(\cdot, u(\cdot))$ belongs to $[BV(\Omega)]^M$ and*

$$\begin{aligned} \tilde{D}v &= \sum_{\alpha\in A} 1_{U^\alpha}(x,\tilde{u})(D_x\mathcal{S}^\alpha(x,\tilde{u})\mathcal{L}^n + D_y\mathcal{S}^\alpha(x,\tilde{u})\tilde{D}u) \quad \text{on } \Omega\setminus S_u, && (6)\\ D^j v &= [\mathcal{S}(x,u^+) - \mathcal{S}(x,u^-)]\otimes \nu_u \mathcal{H}^{n-1}\lfloor S_u. && (7)\end{aligned}$$

Proof. Since the function $\mathcal{S}$ can be extended to a Lipschitz function on the whole $\overline{\Omega}\times\mathbb{R}^N$, by Theorem 3.101 in [4] we have that the function $v = \mathcal{S}(\cdot, u(\cdot))$ belongs to $[BV(\Omega)]^M$ and formula (7) holds true.

Since $\mathcal{S}^\alpha$ is globally Lipschitz and of class C^1 on $\overline{\Omega}\times\mathbb{R}^N$, by Theorem 3.96 in [4] the function $v^\alpha := \mathcal{S}^\alpha(\cdot, u(\cdot))$ belongs to $[BV(\Omega)]^M$ and the diffuse part of its derivative satisfies the following equality:

$$\tilde{D}v^\alpha = D_x\mathcal{S}^\alpha(x,\tilde{u})\mathcal{L}^n + D_y\mathcal{S}^\alpha(x,\tilde{u})\tilde{D}u. \tag{8}$$

Consider now the set

$$E^\alpha := \{x\in\Omega\setminus S_u :\ \tilde{v}(x) = \tilde{v}^\alpha(x)\}.$$

Since v and v^α are both BV functions and their jump sets are both contained in S_u, by the locality property of the derivative of a BV function (see Remark 3.93 in [4]) it follows that $Dv\lfloor E^\alpha = Dv^\alpha\lfloor E^\alpha$. Since $E^\alpha \subset \Omega\setminus S_u$, the previous equality can be rewritten as

$$\tilde{D}v\lfloor E^\alpha = \tilde{D}v^\alpha\lfloor E^\alpha. \tag{9}$$

If we define

$$P^\alpha := \{x\in\Omega\setminus S_u :\ (x,\tilde{u}(x))\in U^\alpha\},$$

since $P^\alpha \subset E^\alpha$, by (9) and (8) we can conclude that

$$\tilde{D}v\lfloor P^\alpha = \tilde{D}v^\alpha\lfloor P^\alpha = D_x\mathcal{S}^\alpha(x,\tilde{u})\mathcal{L}^n\lfloor P^\alpha + D_y\mathcal{S}^\alpha(x,\tilde{u})\tilde{D}u\lfloor P^\alpha,$$

which immediately gives formula (6). □

3. Calibrations for functionals on vector-valued maps

In this section we develop the theory of calibrations for functionals depending on vector-valued maps $u\in[SBV(\Omega)]^N$ and of the following form:

$$F(u) = \int_\Omega f(x,u,\nabla u)\,dx + \int_{S_u} \psi(x,u^-,u^+,\nu_u)\,d\mathcal{H}^{n-1}, \tag{10}$$

where $f : \Omega\times\mathbb{R}^N\times\mathbb{R}^{nN} \to [0,+\infty]$, and $\psi : \Omega\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{n-1} \to [0,+\infty]$ are Borel functions, and ψ satisfies the condition $\psi(x,y,z,\nu) = \psi(x,z,y,-\nu)$.

In the sequel we will refer to the following definition of minimizers of F.

Definition 3.1. An *absolute minimizer of (10) in* Ω is a function $u \in [SBV(\Omega)]^N$ such that $F(u) \le F(v)$ for all $v \in [SBV(\Omega)]^N$, while a *Dirichlet minimizer in* Ω is a function $u \in [SBV(\Omega)]^N$ such that $F(u) \le F(v)$ for all $v \in [SBV(\Omega)]^N$ with the same trace on $\partial\Omega$ as u. A function $u \in [SBV(\Omega)]^N$ is a *U-minimizer* if the graph of u is contained in U and $F(u) \le F(v)$ for all $v \in [SBV(\Omega)]^N$ whose graph is contained in U, while u is a *Dirichlet U-minimizer* if we add the requirement that the competing functions v have the same trace on $\partial\Omega$ as u.

Before proving the key lemma about calibrations, we fix some further notation.

Given two functions $\mathcal{S} : U \to \mathbb{R}^n$, and $u : \Omega \to \mathbb{R}^N$, we will denote the divergence of the composite function $\mathcal{S}(\cdot, u(\cdot))$ by $\mathrm{div}_x[\mathcal{S}(x, u(x))]$, while the divergence of $\mathcal{S}$ with respect to the variable x computed at the point $(x, u(x))$ by $[\mathrm{div}_x\mathcal{S}](x, u(x))$. The Jacobian matrix of $\mathcal{S}$ with respect to y will be denoted by $D_y\mathcal{S}$ and its transpose by $(D_y\mathcal{S})^\tau$. Note that if $\mathcal{S}$ and u are sufficiently regular,

$$\mathrm{div}_x[\mathcal{S}(x, u(x))] = [\mathrm{div}_x\mathcal{S}](x, u) + \langle (D_y\mathcal{S}(x, u))^\tau, \nabla u\rangle.$$

We call f^* and $\partial^-_\xi f$ the convex conjugate and the subdifferential of f with respect to the last variable. We recall that, if g is a function from $\mathbb{R}^{nN}$ into $[0, +\infty]$, the subdifferential of g at the point $\xi \in \mathbb{R}^{nN}$ is defined as the set of all matrices $\eta \in \mathbb{R}^{nN}$ such that $g(\xi) + \langle \eta, \zeta - \xi\rangle \le g(\zeta)$ for every $\zeta \in \mathbb{R}^{nN}$. It is well known that $\langle \xi, \eta\rangle - g^*(\eta) \le g(\xi)$ for every $\xi, \eta \in \mathbb{R}^{nN}$, and the equality holds if and only if $\eta \in \partial^-_\xi g(\xi)$. Moreover, if g is convex and differentiable, then $\partial^-_\xi g(\xi) = \{\partial_\xi g(\xi)\}$. Using these properties, we can prove the following lemma.

Lemma 3.2. *Let F be the functional defined in (10). Let $\mathcal{S} \in [C^1(\overline{\Omega}\times\mathbb{R}^N)]^n$ be Lipschitz continuous and let $\mathcal{S}_0 \in L^1(\Omega)$. Assume that the following conditions are satisfied:*

- *(a) $[\mathrm{div}_x\mathcal{S}](x, y) + \mathcal{S}_0(x) \le -f^*(x, y, (D_y\mathcal{S}(x, y))^\tau)$ for $\mathcal{L}^n$-a.e. $x \in \Omega$ and for every y with $(x, y) \in U$;*
- *(b) $\langle \mathcal{S}(x, z) - \mathcal{S}(x, y), \nu\rangle \le \psi(x, y, z, \nu)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Omega$, for every $\nu \in \mathbb{S}^{n-1}$, and for every y, z with $(x, y) \in U$, $(x, z) \in U$.*

Then for every $u \in [SBV(\Omega)]^N$ such that $G_u \subset U$ we have that $\mathrm{div}_x[\mathcal{S}(\cdot, u(\cdot))]$ is a Radon measure on Ω, which will be denoted as μ_u, and

$$F(u) \ge \int_\Omega d\mu_u + \int_\Omega \mathcal{S}_0(x)\, dx. \tag{11}$$

Moreover, equality holds in (11) for a given u if and only if

- *(a') $[\mathrm{div}_x\mathcal{S}](x, u) + \mathcal{S}_0(x) = -f^*(x, u, (D_y\mathcal{S}(x, u))^\tau)$ and $(D_y\mathcal{S}(x, u))^\tau \in \partial^-_\xi f(x, u, \nabla u)$ for $\mathcal{L}^n$-a.e. $x \in \Omega$;*
- *(b') $\langle \mathcal{S}(x, u^+) - \mathcal{S}(x, u^-), \nu_u\rangle = \psi(x, u^-, u^+, \nu_u)$ for $\mathcal{H}^{n-1}$-a.e. $x \in S_u$,*

where u, $u^\pm$, ∇u, and ν_u are always computed at x.

Proof. Let $u \in [SBV(\Omega)]^N$ be such that $G_u \subset U$. By Theorem 3.96 in [4] the function $S(\cdot, u(\cdot))$ belongs to $[SBV(\Omega)]^n$, and therefore, its divergence is a Radon measure on Ω. Moreover, we have that

$$D_{x_i}[\mathcal{S}_i(x, u)] = \partial_{x_i}\mathcal{S}_i(x, u)\, \mathcal{L}^n + D_y\mathcal{S}_i(x, u)\partial_{x_i}u\, \mathcal{L}^n + [\mathcal{S}_i(x, u^+) - \mathcal{S}_i(x, u^-)](\nu_u)_i\, \mathcal{H}^{n-1}\lfloor S_u,$$

so that the measure μ_u can be written as

$$\begin{aligned}
\mu_u(x) &= \sum_{i=1}^{n} D_{x_i}[\mathcal{S}_i(x,u(x))] \\
&= [\mathrm{div}_x\mathcal{S}](x,u)\,\mathcal{L}^n + \sum_i D_y\mathcal{S}_i(x,u)\partial_{x_i}u\,\mathcal{L}^n + \sum_i[\mathcal{S}_i(x,u^+) - \mathcal{S}_i(x,u^-)](\nu_u)_i\,\mathcal{H}^{n-1}\lfloor S_u \\
&= [\mathrm{div}_x\mathcal{S}](x,u)\,\mathcal{L}^n + \langle (D_y\mathcal{S}(x,u))^\tau, \nabla u\rangle\,\mathcal{L}^n + \langle \mathcal{S}(x,u^+) - \mathcal{S}(x,u^-), \nu_u\rangle\,\mathcal{H}^{n-1}\lfloor S_u,
\end{aligned}$$

and the functional at the right-hand side of (11) has the following expression

$$\begin{aligned}
\int_\Omega d\mu_u + \int_\Omega \mathcal{S}_0(x)\,dx = \int_\Omega ([\mathrm{div}_x\mathcal{S}](x,u) + \langle (D_y\mathcal{S}(x,u))^\tau, \nabla u\rangle + \mathcal{S}_0(x))\,dx \\
+ \int_{S_u} \langle \mathcal{S}(x,u^+) - \mathcal{S}(x,u^-), \nu_u\rangle\,d\mathcal{H}^{n-1}. \quad (12)
\end{aligned}$$

Using assumption (a) we obtain that for $\mathcal{L}^n$-a.e. $x \in \Omega$

$$\begin{aligned}
[\mathrm{div}_x\mathcal{S}](x,u) + \langle (D_y\mathcal{S}(x,u))^\tau, \nabla u\rangle + \mathcal{S}_0(x) &\le -f^*(x,u,(D_y\mathcal{S}(x,u))^\tau) + \langle (D_y\mathcal{S}(x,u))^\tau, \nabla u\rangle \\
&\le f(x,u,\nabla u),
\end{aligned}$$

and consequently

$$\int_\Omega ([\mathrm{div}_x\mathcal{S}](x,u) + \langle (D_y\mathcal{S}(x,u))^\tau, \nabla u\rangle + \mathcal{S}_0(x))\,dx \le \int_\Omega f(x,u,\nabla u)\,dx. \quad (13)$$

Moreover, equality holds in (13) if and only if $(D_y\mathcal{S}(x,u))^\tau \in \partial_\xi^- f(x,u,\nabla u)$ and $[\mathrm{div}_x\mathcal{S}](x,u)$ $+\mathcal{S}_0(x) = -f^*(x,u,(D_y\mathcal{S}(x,u))^\tau)$, which is condition (a').

As for the second integral in (12), condition (b) implies that

$$\int_{S_u} \langle \mathcal{S}(x,u^+) - \mathcal{S}(x,u^-), \nu_u\rangle\,d\mathcal{H}^{n-1} \le \int_{S_u} \psi(x,u^-,u^+,\nu_u)\,d\mathcal{H}^{n-1}. \quad (14)$$

Moreover, equality holds in (14) if and only if (b') is satisfied.

The statement follows now from (12), (13), and (14). □

The assumption of C^1-regularity for $\mathcal{S}$ is often too strong for many applications. We prove now a new version of Lemma 3.2 under weaker regularity assumptions for $\mathcal{S}$.

Lemma 3.3. *Let F be the functional defined in (10). Let $\mathcal{S} : U \to \mathbb{R}^n$ be a Lipschitz continuous function, piecewise C^1 in the sense of Definition 2.1, and satisfying (5). Let $\mathcal{S}_0 \in L^1(\Omega)$. Assume that the following conditions are satisfied:*

(a) $[\mathrm{div}_x\mathcal{S}^\alpha](x,y) + \mathcal{S}_0(x) \le -f^*(x,y,(D_y\mathcal{S}^\alpha(x,y))^\tau)$ *for every $\alpha \in A$, for $\mathcal{L}^n$-a.e. $x \in \Omega$, and for every $y \in \mathbb{R}^N$ with $(x,y) \in U^\alpha$;*

(b) $\langle \mathcal{S}(x,z) - \mathcal{S}(x,y), \nu\rangle \le \psi(x,y,z,\nu)$ *for $\mathcal{H}^{n-1}$-a.e. $x \in \Omega$, for every $\nu \in \mathbb{S}^{n-1}$, and for every y,z with $(x,y) \in U$, $(x,z) \in U$.*

Then for every $u \in [SBV(\Omega)]^N$ such that $G_u \subset U$ we have that $\operatorname{div}_x[\mathcal{S}(\cdot, u(\cdot))]$ is a Radon measure on Ω, which will be denoted as μ_u, and

$$F(u) \geq \int_\Omega d\mu_u + \int_\Omega \mathcal{S}_0(x)\, dx. \tag{15}$$

Moreover, equality holds in (11) for a given u if and only if

(a') $[\operatorname{div}_x \mathcal{S}^\alpha](x,u) + \mathcal{S}_0(x) = -f^*(x,u,(D_y\mathcal{S}^\alpha(x,u))^\tau)$ and $(D_y\mathcal{S}^\alpha(x,u))^\tau \in \partial_\xi^- f(x,u,\nabla u)$ for every $\alpha \in A$, for $\mathcal{L}^n$-a.e. $x \in \Omega$ such that $(x,u(x)) \in U^\alpha$:

(b') $\langle \mathcal{S}(x,u^+) - \mathcal{S}(x,u^-), \nu_u \rangle = \psi(x,u^-,u^+,\nu_u)$ for $\mathcal{H}^{n-1}$-a.e. $x \in S_u$,

where u, $u^\pm$, ∇u, and ν_u are always computed at x.

Proof. Let $u \in [SBV(\Omega)]^N$ be such that $G_u \subset U$. By Lemma 2.2 the function $S(\cdot, u(\cdot))$ belongs to $[SBV(\Omega)]^n$, and therefore, its divergence is a Radon measure on Ω. By (6) and (7) we have that the measure μ_u can be written as

$$\begin{aligned}\mu_u(x) = \sum_{\alpha\in A} 1_{U^\alpha}(x,u)[\operatorname{div}_x \mathcal{S}^\alpha](x,u)\, \mathcal{L}^n + \sum_{\alpha \in A} 1_{U^\alpha}(x,u)\langle (D_y\mathcal{S}^\alpha(x,u))^\tau, \nabla u\rangle\, \mathcal{L}^n \\ + \langle \mathcal{S}(x,u^+) - \mathcal{S}(x,u^-), \nu_u\rangle\, \mathcal{H}^{n-1} \lfloor S_u.\end{aligned}$$

The proof of Lemma 3.2 can be now repeated simply replacing $[\operatorname{div}_x \mathcal{S}]$ with $\sum_{\alpha\in A} 1_{U^\alpha}[\operatorname{div}_x \mathcal{S}^\alpha]$, and $D_y\mathcal{S}$ with $\sum_{\alpha\in A} 1_{U^\alpha} D_y \mathcal{S}^\alpha$. □

Definition 3.4. We say that a pair of functions $(\mathcal{S}, \mathcal{S}_0)$ is a *calibration for $u \in [SBV(\Omega)]^N$ on U with respect to the functional (10)* if $\mathcal{S} : U \to \mathbb{R}^n$ is a Lipschitz continuous function, piecewise C^1 in the sense of Definition 2.1, $\mathcal{S}_0 \in L^1(\Omega)$, and they satisfy assumptions (a), (b), (a'), and (b') in Lemma 3.3.

We can now prove the main result of this section.

Theorem 3.5. *Let u be a function in $[SBV(\Omega)]^N$ whose graph is contained in U. Assume that there exists a calibration $(\mathcal{S}, \mathcal{S}_0)$ for u on U with respect to the functional (10). Then u is a Dirichlet U-minimizer of F. If, in addition, the normal component of $\mathcal{S}$ at $\partial U \cap (\partial\Omega \times \mathbb{R}^N)$ does not depend on y, namely for $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega$ there exists a constant $a(x) \in \mathbb{R}$ such that*

$$\langle \mathcal{S}(x,y), \nu_{\partial\Omega}(x)\rangle = a(x) \qquad \text{for every } y \text{ such that } (x,y) \in U, \tag{16}$$

then u is also an absolute U-minimizer of F.

Proof. Let v be a function in $[SBV(\Omega)]^N$ such that $v = u$ on $\partial\Omega$ and $G_v \subset U$. Then the definition of the measure μ_v and the divergence theorem imply that

$$\int_\Omega d\mu_v = -\int_{\partial\Omega} \langle \mathcal{S}(x,v), \nu_{\partial\Omega}\rangle\, d\mathcal{H}^{n-1}.$$

If v has the same trace on $\partial\Omega$ as u, from this identity it follows that

$$\int_\Omega d\mu_v = \int_\Omega d\mu_u, \tag{17}$$

and by applying Lemma 3.3 we obtain

$$F(v) \geq \int_\Omega d\mu_v + \int_\Omega \mathcal{S}_0(x)\,dx = \int_\Omega d\mu_u + \int_\Omega \mathcal{S}_0(x)\,dx = F(u).$$

We have thus proved that u is a Dirichlet U-minimizer of F.

If we assume, in addition, that (16) holds true, then $\int_\Omega d\mu_v = -\int_\Omega a\,d\mathcal{H}^{n-1}$ for every $v \in [SBV(\Omega)]^N$ whose graph is contained in U; so, the equality (17) is fulfilled even if the traces of u and v on $\partial\Omega$ differ. This proves that u is an absolute U-minimizer of F. $\square$

Remark 3.6. It is natural to wonder what is the link in the case $N = 1$ between our vectorial theory and the calibration method for the scalar case, developed in [2], which involves a divergence-free vectorfield ϕ (see [2, Theorem 3.10]).

Let $N = 1$. Let us suppose that $(\mathcal{S}, \mathcal{S}_0)$ is a calibration for u and assume furthermore that $\mathcal{S}$ is globally C^1. Take the vector field $\phi = (\phi^x, \phi^y) : U \to \mathbb{R}^n \times \mathbb{R}$ defined by $\phi^x(x,y) := \partial_y \mathcal{S}(x,y)$ and $\phi^y(x,y) := -[\mathrm{div}_x \mathcal{S}](x,y) - \mathcal{S}_0(x)$. Then ϕ satisfies all the assumptions of Theorem 3.10 in [2]. Indeed, by Remark 2.3 in [2] ϕ is approximately regular on U in the sense of Definition 2.1 in [2]; moreover, ϕ satisfies all the following conditions:

(A) $\phi^y(x,y) \geq f^*(x,y,\phi^x(x,y))$ for $\mathcal{L}^n$-a.e. $x \in \Omega$ and for every $y \in \mathbb{R}$ with $(x,y) \in U$;

(A') $\phi^x(x,u) \in \partial_\xi^- f(x,u,\nabla u)$ and $\phi^y(x,u) = f^*(x,u,\phi^x(x,u))$ for $\mathcal{L}^n$-a.e. $x \in \Omega$;

(B) $\displaystyle\int_{y_1}^{y_2} \langle \phi^x(x,y), \nu \rangle\,dy \leq \psi(x,y_1,y_2,\nu)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Omega$, every $\nu \in \mathbb{S}^{n-1}$, and every $y_1 < y_2$ such that $(x,y_1) \in U$, $(x,y_2) \in U$;

(B') $\displaystyle\int_{u^-}^{u^+} \langle \phi^x(x,y), \nu_u \rangle\,dy = \psi(x,u^-,u^+,\nu_u)$ for $\mathcal{H}^{n-1}$-a.e. $x \in S_u$;

(C) ϕ is divergence-free in the sense of distributions in U.

Conditions (A) and (A') directly follow from (a) and (a'), respectively. By definition of ϕ we have that

$$\int_{y_1}^{y_2} \phi^x(x,y)\,dy = \mathcal{S}(x,y_2) - \mathcal{S}(x,y_1),$$

so that conditions (b) and (b') imply (B) and (B'), respectively. If $\mathcal{S}$ is C^2 and $\mathcal{S}_0$ is C^1, then it is trivial that ϕ is C^1 and $\mathrm{div}\phi = 0$; in the general case, condition (C) can be obtained by an approximation argument.

Analogously it is easy to see that, if ϕ is a bounded Lipschitz C^1-vectorfield satisfying (A), (B), (A'), (B'), and (C), then we can construct a calibration $(\mathcal{S}, \mathcal{S}_0)$. Take indeed

$$\mathcal{S}(x,y) := \int_{\tau(x)}^y \phi^x(x,t)\,dt \qquad \text{and} \qquad \mathcal{S}_0(x) := \langle \phi^x(x,\tau(x)), \nabla\tau(x) \rangle - \phi^y(x,\tau(x)),$$

where τ is any smooth function satisfying $(x,\tau(x)) \in U$ for every $x \in \overline{\Omega}$.

4. An application related to classical field theory

We recall now some classical results from field theory for multiple integrals of the form

$$F_0(u) = \int_\Omega f(x,u,\nabla u)\,dx, \tag{18}$$

where $u \in [C^1(\overline{\Omega})]^N$ and $f \in C^2(\overline{\Omega}\times\mathbb{R}^N\times\mathbb{R}^{nN})$.

We will call *extremals* of F_0 or f*-extremals* the solutions u of class C^2 of the Euler equations for the integral F_0, i.e.

$$\sum_{i=1}^{n} D_{x_i}[\partial_{\xi_{ij}} f(x,u(x),\nabla u(x))] - \partial_{u_j} f(x,u(x),\nabla u(x)) = 0, \qquad 1 \le j \le N. \tag{19}$$

In the classical field theory for multiple integrals several sufficient conditions for the minimality of an f-extremal have been proposed. Among the others, we recall Weyl field theory, which is strictly related to the calibration theory for vector-valued functionals and ensures that a given f-extremal u is in fact a minimizer of F_0 among all functions of class C^1, with the same boundary values as u and whose graph is contained in a suitable neighbourhood of the graph of u. It consists in the construction of a suitable slope field $\mathcal{P}$, called Weyl field, and of a smooth function $\mathcal{S}$, called the eikonal map associated with the field, satisfying the system of equations (20) – (21). This set of conditions arises from the comparison of F_0 with an invariant functional of divergence type, which is nothing but the functional

$$\int_\Omega \mathrm{div}_x[\mathcal{S}(x,v)]\, dx,$$

where $\mathcal{S}$ is the eikonal map (see, e.g. [9, Chapter 7, Section 4]).

We will show via calibrations that, if a Weyl field exists for an f-extremal u (and then there exists a neighbourhood U of the graph of u such that u minimizes F_0 among C^1-functions with the same boundary values as u and with graph contained in U), then u is also a Dirichlet U-minimizer of the functional (10) in the sense of Definition 3.1, provided U is a sufficiently small neighbourhood of G_u and the function ψ satisfies the estimate (23); moreover, if $\mathcal{S}$ is the eikonal map associated with the Weyl field, then the pair $(\mathcal{S}, \mathcal{S}_0)$ with $\mathcal{S}_0 \equiv 0$ is a calibration for u on U.

Definition 4.1. Let U be a closed domain in $\overline{\Omega}\times\mathbb{R}^N$. A mapping $p: U \to U\times\mathbb{R}^{nN}$ is called a *slope field* on U if it is of class C^1 and of the form

$$p(x,y) = (x,y,\mathcal{P}(x,y)) \qquad \text{for every } (x,y) \in U;$$

we denote $\mathcal{P}(x,y) = (\mathcal{P}_{ij}(x,y))$ as the *slope function* of the field p. We say that *a map* $u \in [C^1(\overline{\Omega})]^N$ *fits the slope field* p if $G_u \subset U$ and

$$\partial_{x_i} u_j(x) = \mathcal{P}_{ij}(x,u(x)) \qquad \text{for every } x \in \overline{\Omega}.$$

Finally, a slope field p is said to be a *Weyl field* if there is a map $\mathcal{S} \in [C^2(U)]^n$ such that $\{\mathcal{S}, \mathcal{P}\}$ solves the Weyl equations:

$$[\mathrm{div}_x \mathcal{S}](x,y) = f(x,y,\mathcal{P}(x,y)) - \langle \mathcal{P}(x,y), \partial_\xi f(x,y,\mathcal{P}(x,y))\rangle, \tag{20}$$

$$\partial_{y_j}\mathcal{S}_i(x,y) = \partial_{\xi_{ij}} f(x,y,\mathcal{P}(x,y)). \tag{21}$$

The function $\mathcal{S}$ is called the *eikonal map* associated with p.

The main results in Weyl field theory can be stated as follows. For a proof we refer to [9].

Theorem 4.2.

(1) *Assume that the function f satisfies*

$$f(x,y,\xi) - f(x,y,\eta) - \langle \xi - \eta, \partial_\xi f(x,y,\eta)\rangle \geq 0$$

for every $(x,y) \in U$ and $\xi, \eta \in \mathbb{R}^{nN}$, and let $u \in [C^2(\overline{\Omega})]^N$ fit a Weyl field $p : U \to U \times \mathbb{R}^{nN}$ with the eikonal map $\mathcal{S} : U \to \mathbb{R}^n$. Then u is a minimizer of F_0 among all $v \in [C^1(\overline{\Omega})]^N$ such that $v|_{\partial\Omega} = u|_{\partial\Omega}$ and $G_v \subset U$; in particular, u is an f-extremal. Moreover, if there is a constant $\mu > 0$ such that

$$\sum_{i,j,h,k} \partial^2_{\xi_{ij}\xi_{hk}} f(x,y,\xi)\zeta_{ij}\zeta_{hk} \geq \mu|\zeta|^2 \qquad \forall (x,y) \in \overline{\Omega}\times\mathbb{R}^N,\ \xi,\zeta \in \mathbb{R}^{nN}, \tag{22}$$

then u is a strict minimizer of F_0 in the same class.

(2) *Vice-versa, if f satisfies the strict convexity condition (22), then every f-extremal fits at least locally a Weyl field and is therefore locally minimizing F_0. In other words, for every $x_0 \in \Omega$ there exist $\varepsilon > 0$ and an open neighbourhood A of x_0 such that u minimizes F_0 among all $v \in [C^1(\overline{A})]^N$ such that $v|_{\partial A} = u|_{\partial A}$ and $G_v \subset \{(x,y) \in \overline{A}\times\mathbb{R}^N :\ |y - u(x_0)| \leq \varepsilon\}$.*

Let us now state and prove a similar result for free-discontinuity problems.

Theorem 4.3. *Let $f : \overline{\Omega}\times\mathbb{R}^N\times\mathbb{R}^{nN} \to [0,+\infty]$ be a function of class C^2 satisfying (22) and let $\psi : \Omega\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{n-1} \to [0,+\infty]$ be a Borel function satisfying*

$$\psi(x,y,z,\nu) \geq c\,\theta(|y-z|), \tag{23}$$

where c is a positive constant, while θ is such that $\lim_{t\to 0^+} \frac{\theta(t)}{t} = +\infty$. Let u be an f-extremal. Then for every $x_0 \in \Omega$ there exist $\varepsilon > 0$, an open neighbourhood A (with Lipschitz boundary) of x_0, and a pair $(\mathcal{S}, \mathcal{S}_0)$ such that $(\mathcal{S}, \mathcal{S}_0)$ is a calibration for u with respect to the functional (10) on the set

$$U := \{(x,y) \in \overline{A}\times\mathbb{R}^N :\ |y - u(x_0)| \leq \varepsilon\}; \tag{24}$$

therefore u is a Dirichlet U-minimizer of the functional (10).

Proof. Let u be an f-extremal. By the second part of Theorem 4.2 for every $x_0 \in \Omega$ there exist $\varepsilon > 0$ and an open neighbourhood A (with Lipschitz boundary) of x_0 such that u fits a Weyl fiel in the set (24). Denote the Weyl field by $p(x,y) = (x,y,\mathcal{P}(x,y))$ and the eikonal map associated with p by $\mathcal{S}$.

We claim that, if we take $\mathcal{S}_0(x) := 0$ for every $x \in \Omega$, then the pair $(\mathcal{S}, \mathcal{S}_0)$ is a calibration for u on U with respect to the functional F defined in (10), provided ε is sufficiently small. Let us prove it. Since f is convex, for every $\eta \in \mathbb{R}^{nN}$ we have that

$$f(x,y,\eta) - \langle \eta, \partial_\xi f(x,y,\eta)\rangle = -f^*(x,y,\partial_\xi f(x,y,\eta));$$

this fact, jointly with (20), implies that

$$\begin{aligned} [\mathrm{div}_x \mathcal{S}](x,y) &= -f^*(x,y,\partial_\xi f(x,y,\mathcal{P}(x,y))) \\ &= -f^*(x,y,(D_y\mathcal{S}(x,y))^\tau), \end{aligned} \tag{25}$$

where the second equality follows from (21). Therefore, condition (a) is satisfied.

Condition (a') follows from (25) and (21), using the fact that u fits the field $\mathcal{P}$, hence $\mathcal{P}(x, u(x)) = \nabla u(x)$ for every $x \in \overline{\Omega}$.

If we call L the L^∞-norm of the Jacobian matrix of $\mathcal{S}$ on U, then we have that

$$\langle \mathcal{S}(x,z) - \mathcal{S}(x,y), \nu\rangle \leq L\,|z-y| \tag{26}$$

for every $x \in \overline{\Omega}$, $y, z \in \mathbb{R}^N$ such that $(x,y) \in U$, $(x,z) \in U$, and $\nu \in \mathbb{S}^{n-1}$. By the assumption on the function θ there exists $\delta > 0$ such that $\theta(t) \geq Lt/c$ for every $t \in (0,\delta)$; then from (23) it follows that

$$\psi(x,y,z,\nu) \geq L|y-z| \qquad \text{for } |y-z| < \delta. \tag{27}$$

Taking $\varepsilon < \delta/2$, from (26) and (27) we have that condition (b) is satisfied.

Since $S_u = \emptyset$, condition (b') is trivial.

The conclusion follows now from Theorem 3.5. □

As made precise in the next proposition, when the function f depends only on the variables x, ξ, we are able to prove the minimality of an f-extremal u on the whole domain $\overline{\Omega}$ and to give an estimate of the width ε of the neighbourhood of G_u where the minimality holds.

Proposition 4.4. *In addition to the assumptions of Theorem 4.3, suppose that $f = f(x,\xi)$. Let u be an f-extremal. For every $(x,y) \in \overline{\Omega}\times\mathbb{R}^N$ define*

$$\mathcal{S}(x,y) := [\partial_\xi f(x, \nabla u(x))]^\tau (y - u(x)) + \sigma(x), \tag{28}$$

where $\sigma : \overline{\Omega} \to \mathbb{R}^n$ is a solution of the equation $\mathrm{div}\sigma = f(x, \nabla u)$. *Then the pair $(\mathcal{S}, \mathcal{S}_0)$ with $\mathcal{S}_0 \equiv 0$ is a calibration for u with respect to the functional (10) on the set*

$$U := \{(x,y) \in \overline{\Omega}\times\mathbb{R}^N : \ |y - u(x)| \leq \varepsilon(x)\}, \tag{29}$$

where

$$\varepsilon(x) < \frac{1}{2}\inf\left\{t > 0 : \ c\,\frac{\theta(t)}{t} < |\partial_\xi f(x, \nabla u(x))|\right\}, \tag{30}$$

and c, θ are the quantities appearing in (23). Therefore u is a Dirichlet U-minimizer of the functional (10).

Proof. Note that by the assumption on θ, the infimum in (30) is strictly positive for every $x \in \overline{\Omega}$.

Let us prove that $(\mathcal{S}, \mathcal{S}_0)$ satisfies all the conditions in Lemma 3.2.

By direct computations we have that $D_y\mathcal{S}(x,y) = [\partial_\xi f(x, \nabla u)]^\tau$; using the Euler equations (19), the definition of σ, and the convexity of f, we find out that

$$\begin{aligned}
[\mathrm{div}_x\mathcal{S}](x,y) &= \sum_{ij} D_{x_i}(\partial_{\xi_{ji}} f(x, \nabla u))(y_j - u_j) - \langle[\partial_\xi f(x,\nabla u)]^\tau, \nabla u\rangle + \mathrm{div}\sigma \\
&= -\langle[\partial_\xi f(x,\nabla u)]^\tau, \nabla u\rangle + f(x, \nabla u) \\
&= -f^*(x, [\partial_\xi f(x,\nabla u)]^\tau).
\end{aligned}$$

Conditions (a) and (a') are therefore satisfied.

By the definition of $\mathcal{S}$ we obtain

$$|\mathcal{S}(x,z)-\mathcal{S}(x,y)| \le |\partial_\xi f(x,\nabla u(x))|\cdot|z-y|;$$

since $|z-y|\le 2\varepsilon(x)$, (30) implies that

$$|\partial_\xi f(x,\nabla u(x))|\cdot|z-y| \le c\,\theta(|z-y|);$$

so condition (b) follows now from (23).

Condition (b') is trivial since S_u is empty. This concludes the proof.

We notice that the thesis can be proved also in the following way: if we define $\mathcal{P}(x,y) := \nabla u(x)$ for every $(x,y)\in\overline{\Omega}\times\mathbb{R}^N$, it is easy to see that the field $p(x,y) := (x,y,\mathcal{P}(x,y))$ is a Weyl field, $\mathcal{S}$ is the eikonal map associated with p, and u fits p. Then we can follow the proof of Theorem 4.3: the check of condition (a), (a'), (b') remains the same, while the estimate on the size of $\varepsilon(x)$ is given by a more careful proof of condition (b). □

Remark 4.5. When the functional (10) satisfies some special further conditions, it is enough to prove the Dirichlet minimality of a given u on a neighbourhood of its graph to conclude that u is in fact a Dirichlet minimizer on the whole cylinder $\overline{\Omega}\times\mathbb{R}$, reducing the domain Ω if needed. For istance, in addition to the assumptions of Proposition 4.4, suppose that the two following conditions are satisfied:

(1) $f(x,\xi)\ge f(x,(I-e_j\otimes e_j)\,\xi)$ for every $x\in\Omega$, $\xi\in\mathbb{R}^{nN}$, $j=1,\dots,N$, where $\{e_1,\dots,e_N\}$ is the canonical basis of $\mathbb{R}^N$;

(2) $\psi(x,y,z,\nu)\ge\psi(x,T_a^b(y),T_a^b(z),\nu)$ for every $(x,y)\in\Omega\times\mathbb{R}^N$, $\xi\in\mathbb{R}^{nN}$, $\nu\in\mathbb{S}^{n-1}$, $a,b\in\mathbb{R}^N$, where we have set

$$T_a^b:\mathbb{R}^N\to\mathbb{R}^N,\qquad (T_a^b)_j(y) := (y_j\wedge a_j)\vee b_j.$$

If u is an f-extremal, then by Proposition 4.4 we know that u is a Dirichlet U-minimizer of F, where U is the set (29). We want to show that for every $x_0\in\Omega$ there exists an open neighbourhood A (with Lipschitz boundary) of x_0 such that u is a Dirichlet minimizer of F in A.

First of all, we can find an open neighbourhood A (with Lipschitz boundary) of x_0 and two vectors $m,M\in\mathbb{R}^N$ such that $|M-m|<\varepsilon(x)$ for every $x\in\overline{A}$ and

$$m_j\le u_j(x)\le M_j\qquad \forall x\in\overline{A},\ 1\le j\le N. \tag{31}$$

Let v be a function in $[SBV(A)]^N$ with the same trace on ∂A as u and define $\hat{v} := T_m^M(v)$, which still belongs to $[SBV(A)]^N$. Note that $\nabla\hat{v}_j = 1_{\{m_j<v_j<M_j\}}\nabla v_j$ for every j, so that, if we call $J_0(x)$ the set of all indexes j such that $v_j(x)\notin(m_j,M_j)$, the matrix $\nabla\hat{v}(x)$ can be written as

$$\nabla\hat{v}(x) = \nabla v(x) - \sum_{j\in J_0}(e_j\otimes e_j)\,\nabla v(x).$$

By using iteratively condition (1), we obtain that $f(x,\nabla\hat{v})\le f(x,\nabla v)$, which implies

$$\int_A f(x,\nabla\hat{v})\,dx \le \int_A f(x,\nabla v)\,dx. \tag{32}$$

Since $S_{\hat{v}} \subset S_v$, and $\hat{v}^- = T_m^M(v^-)$, $\hat{v}^+ = T_m^M(v^+)$ on $S_{\hat{v}}$, by condition (2) we obtain

$$\int_{S_{\hat{v}}\cap A} \psi(x, \hat{v}^-, \hat{v}^+, \nu_{\hat{v}})\, d\mathcal{H}^{n-1} \le \int_{S_v \cap A} \psi(x, v^-, v^+, \nu_v)\, d\mathcal{H}^{n-1}. \tag{33}$$

On the other hand, by (31) the function $\hat{v}$ has the same trace on ∂A as u, and its graph is contained in the set

$$\{(x,y) \in \overline{A}{\times}\mathbb{R}^N : \ |y - u(x)| \le \varepsilon(x)\}.$$

Since u is a Dirichlet minimizer on this set, we have that

$$\int_A f(x, \nabla u)\, dx \le \int_A f(x, \nabla \hat{v})\, dx + \int_{S_{\hat{v}}} \psi(x, \hat{v}^-, \hat{v}^+, \nu_{\hat{v}})\, d\mathcal{H}^{n-1}. \tag{34}$$

Therefore by (32), (33), and (34), u is a Dirichlet minimizer of F in A.

The same result can be achieved by calibration: indeed, we can extend the function $\mathcal{S}$ in (28) to the whole $\overline{\Omega}{\times}\mathbb{R}^N$ simply by taking $\hat{\mathcal{S}}(x,y) := \mathcal{S}(x, T_m^M(y))$; it is easy to see that assumptions (1) – (2) guarantee that the pair $(\hat{\mathcal{S}}, \mathcal{S}_0)$ provides a calibration for u on $\overline{A}{\times}\mathbb{R}^N$.

We conclude the remark with some comments on conditions (1) – (2). Condition (1) ensures that the functional decreases when any row of the matrix ∇u is annihilated, which is what occurs when a component of u is truncated. For istance, (1) is fulfilled for $f(\xi) = \sum_{ij} \varphi_{ij}(\xi_{ij})$ where φ_{ij} are convex and positive, and $\varphi_{ij}(0) = 0$. As for condition (2), note that it is satisfied whenever ψ depends on y, z only through the distance $|z - y|$.

5. Some further applications

In this section we present some examples and applications. In Examples 5.1, 5.3, 5.4, and 5.5 we deal with minimizers of the Mumford-Shah functional, and we generalize some results proved in [2] for the scalar case. Example 5.2 is a purely vectorial example, since it involves a functional arising in fracture mechanics which can be defined only on maps from $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^n$.

Example 5.1. Let $u : \Omega \to \mathbb{R}^N$ be a harmonic function. It is well known that u is an extremal of the functional $\int_\Omega |\nabla u|^2$, and a Dirichlet minimizer of it. We can prove via calibrations that u is a Dirichlet minimizer also of the homogeneous Mumford-Shah functional

$$MS_0(u) = \int_\Omega |\nabla u|^2 dx + \alpha \mathcal{H}^{n-1}(S_u), \tag{35}$$

if the following condition is satisfied:

$$\operatorname*{osc}_\Omega u \cdot \sup_\Omega |\nabla u| \le \alpha, \tag{36}$$

where osc u denotes the modulus of the vector in $\mathbb{R}^N$ whose components are the oscillations of the components of u. When (36) is not fulfilled, u is still a Dirichlet U-minimizer of the functional MS_0, where

$$U := \left\{ (x,y) \in \overline{\Omega}{\times}\mathbb{R}^N : \ |y - u(x)| \le \frac{\alpha}{4|\nabla u(x)|} \right\}. \tag{37}$$

This second result directly follows from Proposition 4.4, where $f(\xi) = |\xi|^2$ and $\psi \equiv \alpha$. Moreover, a calibration is given by $(\mathcal{S}, \mathcal{S}_0)$ with $\mathcal{S}_0 \equiv 0$ and

$$\mathcal{S}(x, y) = 2[\nabla u(x)]^\tau (y - u(x)) + \sigma(x),$$

where $\sigma : \Omega \to \mathbb{R}^n$ is a solution of the equation $\mathrm{div}\sigma = |\nabla u|^2$. Since u is harmonic in Ω, it is easy to see that we can take $\sigma(x) := [\nabla u(x)]^\tau u(x)$, so that

$$\mathcal{S}(x, y) = 2[\nabla u(x)]^\tau \left(y - \frac{u(x)}{2} \right). \tag{38}$$

As for the Dirichlet minimality of u, we can show that, under the assumption (36), the calibration $(\mathcal{S}, \mathcal{S}_0)$ can be extended to the whole $\overline{\Omega} \times \mathbb{R}^N$, applying a similar argument to the one used in Remark 4.5.

We recall that, in the case of the functional (35), conditions (a), (a'), (b), and (b') in Lemma 3.3 become

(a) $[\mathrm{div}_x \mathcal{S}^\gamma](x, y) + \mathcal{S}_0(x) \le -\frac{1}{4}|D_y \mathcal{S}^\gamma(x, y)|^2$ for every $\gamma \in A$, for $\mathcal{L}^n$-a.e. $x \in \Omega$, and for every $y \in \mathbb{R}^N$ with $(x, y) \in U^\gamma$;

(a') $[\mathrm{div}_x \mathcal{S}^\gamma](x, u) + \mathcal{S}_0(x) = -|\nabla u(x)|^2$ and $(D_y \mathcal{S}^\gamma(x, u))^\tau = 2\nabla u(x)$ for every $\gamma \in A$, and for $\mathcal{L}^n$-a.e. $x \in \Omega$ such that $(x, u(x)) \in U^\gamma$;

(b) $|\mathcal{S}(x, z) - \mathcal{S}(x, y)| \le \alpha$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Omega$ and for every $y, z \in \mathbb{R}^N$ such that $(x, y) \in U$, $(x, z) \in U$;

(b') $\mathcal{S}(x, u^+) - \mathcal{S}(x, u^-) = \alpha\nu_u$ for $\mathcal{H}^{n-1}$-a.e. $x \in S_u$,

where $\mathcal{S}(x, y) = \sum_{\gamma \in A} \mathcal{S}^\gamma(x, y) 1_{U^\gamma}(x, y)$.

Let m_j and M_j be the infimum and the supremum of u_j in Ω, respectively (then $\mathrm{osc}\, u_j = M_j - m_j$). Let T be the function from $\mathbb{R}^N$ into $\mathbb{R}^N$ defined as $T_j(y) = (y_j \vee m_j/2) \wedge M_j/2$. Define

$$\hat{\mathcal{S}}(x, y) := 2[\nabla u(x)]^\tau \, T\left(y - \frac{u(x)}{2} \right).$$

It is easy to see that $(\hat{\mathcal{S}}, \mathcal{S}_0)$ satisfies conditions (a) and (a'). Condition (b') is trivial. Finally, for every $y, z \in \mathbb{R}^N$ we have

$$|\hat{\mathcal{S}}(x, z) - \hat{\mathcal{S}}(x, y)| \le 2|\nabla u(x)| \cdot |T(z - u/2) - T(y - u/2)|. \tag{39}$$

Since $T_j(z - u/2)$ and $T_j(y - u/2)$ belong to the interval $[m_j/2, M_j/2]$ for every $1 \le j \le N$, we deduce that $|T(z - u/2) - T(y - u/2)| \le |M - m|/2$; so, condition (b) follows from (39) and (36).

These two minimality results generalize those obtained in [1] for scalar harmonic functions. Note that the minimality of u can be proved by applying the scalar argument to each component u_j, but this provides a more restrictive condition on the size of the domains where the minimality holds. Indeed, by the scalar result in [1], since u_j is harmonic for every j, if

$$\operatorname*{osc}_{\Omega} u_j \cdot \sup_{\Omega} |\nabla u_j| \le \frac{\alpha}{N} \qquad 1 \le j \le N, \tag{40}$$

then

$$\int_\Omega |\nabla u_j|^2 dx \le \int_\Omega |\nabla v_j|^2 dx + \frac{\alpha}{N}\mathcal{H}^{n-1}(S_{v_j})$$

for every $v_j \in SBV(\Omega)$ with the same boundary values as u_j; summing over j, we obtain the Dirichlet minimality of u in Ω. On the other hand, it is easy to see that condition (40) is stronger than (36). Analogous remarks hold for the Dirichlet minimality of u in a neighbourhood of its graph.

Example 5.2. In this example we consider a functional related to Griffith and Barenblatt theories of fracture mechanics of the form

$$H(u) := \mu \int_\Omega |e(u)|^2 dx + \frac{\lambda}{2} \int_\Omega (\mathrm{div} u)^2\, dx + \int_{S_u} \theta(|u^+ - u^-|)\, d\mathcal{H}^{n-1}$$

where u is a function from $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^n$, $e(u)$ denotes the symmetrized gradient of u, θ is a positive function satisfying $\lim_{t\to 0^+} \theta(t)/t = +\infty$, and μ, λ are real parameters. In the context of fracture mechanics, Ω is a reference configuration of an elastic body, possibly subject to fracture, and u parameterizes its displacement; the bulk term represents the energy relative to the elastic deformation outside the fracture, while the surface integral is the energy needed to produce the crack.

The functional H is clearly of the form (10) with $f(\xi) = \mu\,|(\xi^\tau + \xi)/2|^2 + \frac{\lambda}{2}(\mathrm{tr}\,\xi)^2$ and $\psi(y,z) = \theta(|z-y|)$. However, since the bulk term in H involves only the symmetric part of the matrix ∇u, the appropriate setting for the minimum problem for H is not exactly the space $[SBV(\Omega)]^n$, but the space $SBD(\Omega)$ of special functions with bounded deformation (for a complete overview on the properties of this space see [3]). Even if the calibration method has been developed only for SBV functions, we can actually prove by calibration that, if u is an f-extremal, i.e. $u \in [C^1(\overline{\Omega})]^n \cap [C^2(\Omega)]^n$ and u solves the equation

$$\mu\Delta u + (\mu + \lambda)\nabla(\mathrm{div} u) = 0 \qquad \text{on } \Omega, \tag{41}$$

then u minimizes H among all functions $v \in SBD(\Omega)$ with the same trace on $\partial\Omega$ as u, and whose graph is contained in the set

$$U := \{(x,y) \in \overline{\Omega}\times\mathbb{R}^n :\ |y - u(x)| \le \varepsilon(x)\},$$

where

$$\varepsilon(x) < \frac{1}{2} \inf\left\{ t > 0 :\ \frac{\theta(t)}{t} < |2\mu e(u)(x) + \lambda \mathrm{div} u(x) I| \right\}.$$

Indeed, since $\partial_{\xi_{ij}} f(\xi) = \mu(\xi_{ji}+\xi_{ij}) + \lambda(\mathrm{tr}\,\xi)\delta_{ij}$, Proposition 4.4 implies that u is a Dirichlet U-minimizer of H in the class $[SBV(\Omega)]^n$ and a calibration is given by $(\mathcal{S}, \mathcal{S}_0)$ with $\mathcal{S}_0 \equiv 0$ and

$$\mathcal{S}(x,y) = [2\mu e(u)(x) + \lambda \mathrm{div} u(x) I] \left(y - \frac{u(x)}{2} \right); \tag{42}$$

this last fact follows from formula (28) where we have taken $\sigma(x) := [\mu e(u)(x) + \frac{\lambda}{2}\mathrm{div} u(x) I]\, u(x)$, which is a solution of $\mathrm{div}\sigma = f(\nabla u)$ thanks to (41).

On the other hand, we can show that the pair $(\mathcal{S}, \mathcal{S}_0)$ provides a calibration also in the space $SBD(\Omega)$ in the following sense: consider the functional

$$H_1(v) := -\int_{\partial\Omega} \langle \mathcal{S}(x,v), \nu_{\partial\Omega}\rangle\, d\mathcal{H}^{n-1},$$

which is the same used as comparison functional in the proof of Theorem 3.5; then, H_1 is well defined on $SBD(\Omega)$, is invariant on SBD functions having the same trace on $\partial\Omega$, and satisfies the equality $H_1(u) = H(u)$ and the inequality $H_1(v) \leq H(v)$ for every $v \in SBD(\Omega)$. This implies that u is a Dirichlet minimizer of the functional H in the class of SBD functions.

Let us prove the properties of H_1 stated above. If we set for simplicity of notation $A(x) := 2\mu e(u)(x) + \lambda \mathrm{div} u(x) I$, by (42) the functional H_1 can be rewritten as

$$H_1(v) = -\frac{1}{2}\int_{\partial\Omega} \langle A(2v-u), \nu_{\partial\Omega}\rangle \, d\mathcal{H}^{n-1},$$

whence it is clear that it is well defined on $SBD(\Omega)$ and invariant on the class of functions in $SBD(\Omega)$ having the same trace on $\partial\Omega$. By the generalized Green's formula in $SBD(\Omega)$ we have that

$$\begin{aligned} -\frac{1}{2}\int_{\partial\Omega} \langle A(2v-u), \nu_{\partial\Omega}\rangle \, d\mathcal{H}^{n-1} &= \frac{1}{2}\int_{\Omega} \langle 2v-u, \mathrm{div} A\rangle \, dx + \frac{1}{2}\int_{\Omega} A \, d(2Ev - Eu) \\ &= \frac{1}{2}\int_{\Omega} \langle A, 2e(v) - e(u)\rangle \, dx + \int_{S_v} \langle A(v^+ - v^-), \nu_v\rangle \, d\mathcal{H}^{n-1}, \quad (43) \end{aligned}$$

where the last equality follows by the fact that $\mathrm{div} A = 0$, by the decomposition theorem for the measures Ev, Eu and by the remark that $S_u = \emptyset$. Using the definition of the matrix A and (43) it is easy to see that

$$H_1(u) = \frac{1}{2}\int_{\Omega} \langle A, e(u)\rangle \, dx = H(u), \tag{44}$$

while, using also the elementary inequality $2(\xi, \eta) \leq |\xi|^2 + |\eta|^2$ for every $\xi, \eta \in \mathbb{R}^{n^2}$, we obtain

$$\begin{aligned} \int_{\Omega} \langle A, e(v)\rangle \, dx &= 2\mu \int_{\Omega} \langle e(u), e(v)\rangle \, dx + \lambda \int_{\Omega} \mathrm{div} u \, \mathrm{div} v \, dx \\ &\leq \mu \int_{\Omega} |e(v)|^2 dx + \frac{\lambda}{2}\int_{\Omega} (\mathrm{div} v)^2 dx + H(u). \quad (45) \end{aligned}$$

Since the graph of v is contained in U, we have that $\langle A(v^+ - v^-), \nu_v\rangle \leq \theta(|v^+ - v^-|)$ $\mathcal{H}^{n-1}$-a.e. on S_v, so that

$$\int_{S_v} \langle A(v^+ - v^-), \nu_v\rangle \, d\mathcal{H}^{n-1} \leq \int_{S_v} \theta(|v^+ - v^-|) \, d\mathcal{H}^{n-1}. \tag{46}$$

By (43), (44), (45), and (46), we deduce that $H_1(v) \leq H(v)$ for every $v \in SBD(\Omega)$ whose graph is contained in U.

We conclude this example by noticing that the existence of a weak solution in $[W^{1,2}(\Omega)]^n$ for the Dirichlet boundary value problem associated with the equation (41) is guaranteed if $\mu > 0$ and $2\mu + 3\lambda > 0$; moreover, the additional requirements of regularity for u are always satisfied in any open subset $\Omega' \subset\subset \Omega$ (see [5]).

Example 5.3. Let Ω be a product of the form $(0,a)\times V$, where V is a regular domain in $\mathbb{R}^{n-1}$, and let u be the step function defined as $u(x) := 0$ for $0 < x_1 < c$, and $u(x) = h$ for $c < x_1 < a$, where $c \in (0,a)$ and $h \in \mathbb{R}^N$, $h \neq 0$. Then, u is a Dirichlet minimizer of the Mumford-Shah functional (35) in Ω if $|h|^2 \geq a\alpha$.

This result generalizes Example 4.12 in [1], where u is a scalar step function.

We prove the statement by calibration. Let $\{e_1, \dots, e_n\}$ be the canonical basis of $\mathbb{R}^n$. A calibration for u is given by the pair $(\mathcal{S}, \mathcal{S}_0)$ with $\mathcal{S}_0 \equiv 0$ and

$$\mathcal{S}(x,y) := \begin{cases} 0 & \text{if } \langle y, \frac{h}{|h|}\rangle \leq \frac{\lambda}{2}\langle x, e_1\rangle, \\ 2\lambda\left(\langle y, \frac{h}{|h|}\rangle - \frac{\lambda}{2}\langle x, e_1\rangle\right) e_1 & \text{if } \frac{\lambda}{2}\langle x, e_1\rangle \leq \langle y, \frac{h}{|h|}\rangle \leq \frac{\lambda}{2}\langle x, e_1\rangle + \frac{\lambda}{2}a, \\ a\lambda^2 e_1 & \text{if } \langle y, \frac{h}{|h|}\rangle \geq \frac{\lambda}{2}\langle x, e_1\rangle + \frac{\lambda}{2}a, \end{cases} \tag{47}$$

where $\lambda := \sqrt{\alpha/a}$. Some direct computations show that

$$|D_y\mathcal{S}(x,y)|^2 = \begin{cases} 4\lambda^2 & \text{if } \frac{\lambda}{2}\langle x, e_1\rangle \leq \langle y, \frac{h}{|h|}\rangle \leq \frac{\lambda}{2}\langle x, e_1\rangle + \frac{\lambda}{2}a, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathrm{div}\mathcal{S}(x,y) = \begin{cases} -\lambda^2 & \text{if } \frac{\lambda}{2}\langle x, e_1\rangle \leq \langle y, \frac{h}{|h|}\rangle \leq \frac{\lambda}{2}\langle x, e_1\rangle + \frac{\lambda}{2}a, \\ 0 & \text{otherwise,} \end{cases}$$

so that condition (a) is trivially satisfied, while condition (a') is true if $|h| \geq \frac{\lambda}{2}x_1 + \frac{\lambda}{2}a$ for every $x_1 \in [c,a)$, which is guaranteed by the assumption $|h|^2 \geq a\alpha$.

One easily checks that the vector $\mathcal{S}(x,z) - \mathcal{S}(x,y)$ can always be written as μe_1 with $|\mu| \leq \alpha$ (μ depending on x, y, z), so that condition (b) is fulfilled. As for condition (b'), since $|h| \geq \frac{\lambda}{2}(c+a)$ by the assumption $|h|^2 \geq a\alpha$, we have that $\mathcal{S}(x,h) - \mathcal{S}(x,0) = a\lambda^2 e_1 - 0 = \alpha e_1$ for every $x \in S_u$.

We notice that the minimality of u can be proved by applying the scalar result to one component of u. Take, indeed, $j \in \{1, \dots, N\}$ such that $h_j \neq 0$; we know that if $h_j^2 \geq a\alpha$, then

$$\alpha\mathcal{H}^{n-1}(S_{u_j}) \leq \int_\Omega |\nabla v_j|^2 dx + \alpha\mathcal{H}^{n-1}(S_{v_j}),$$

for every $v \in SBV(\Omega)$ with the same boundary values as u. Now, the left-hand side coincides with $MS_0(u)$, while the right-hand side is less than or equal to $MS_0(v)$, since $S_{v_j} \subset S_v$. So, the Dirichlet minimality of u is shown, but under the stronger condition $h_j^2 \geq a\alpha$.

Actually, since the Mumford-Shah functional is invariant by rotation (and then u is a Dirichlet minimizer if and only if $R \circ u$ is a Dirichlet minimizer, where R is any rotation in $\mathbb{R}^N$), the scalar result can be exploited in a more efficient way. Let R be a rotation in $\mathbb{R}^N$ transforming the vector $h/|h|$ in e_1 and let $\hat{u} := R \circ u$. Applying the argument above to the first component of $\hat{u}$, we have that $\hat{u}$ is a Dirichlet minimizer of MS_0 if $|h|^2 \geq a\alpha$, which is the same condition we have found via vectorial calibrations theory. We also note that the calibration (47) can be obtained starting from the vectorfield which calibrates $\hat{u}_1$ simply replacing the one-dimensional vertical variable by the component of the vector y along $h/|h|$ and following the instructions of Remark 3.6.

Example 5.4. Let $\Omega := B(0,r)$ be the open ball in $\mathbb{R}^2$ centred at the origin with radius r, and let (A_1, A_2, A_3) be the partition of Ω defined as follows:

$$A_i := \left\{ x = (\rho\cos\theta, \rho\sin\theta) : \; 0 \le \rho < r, \; \frac{2}{3}\pi(i-1) \le \theta < \frac{2}{3}\pi i \right\}.$$

Let $u \in [SBV(\Omega)]^N$ be the function defined as $u := a_i$ in each A_i, where a_1, a_2, a_3 are three distinct vectors in $\mathbb{R}^N$. In [2, Example 4.14] it is proved that, when $N = 1$, u is a Dirichlet minimizer of the Mumford-Shah functional (35) if the values a_i are sufficiently far apart, more precisely if

$$\min\{|a_1 - a_2|, |a_2 - a_3|, |a_3 - a_1|\} \ge \sqrt{2\alpha r}. \tag{48}$$

This result can be generalized to the vectorial case $N > 1$, where beside condition (48) we require that

$$\max\{|a_1 - a_2|, |a_2 - a_3|, |a_3 - a_1|\} \ge \sqrt{(2+\sqrt{3})\alpha r}. \tag{49}$$

Note that when $N = 1$ condition (49) is implied by (48): indeed, without loss of generality we can assume that $a_1 \le a_2 \le a_3$, so that the maximum in (49) is $a_3 - a_1$; then by (48) we obtain

$$a_3 - a_1 = (a_3 - a_2) + (a_2 - a_1) \ge 2\sqrt{2\alpha r} > \sqrt{(2+\sqrt{3})\alpha r}.$$

We prove the statement by calibration. For every i, j we call S_{ij} the interface between A_i and A_j, which is oriented by the normal ν_{ij} pointing from A_i to A_j and we suppose that the maximum in (49) is given by $|a_1 - a_2|$. Let $\mathcal{S}_0 \equiv 0$ and

$$\mathcal{S}(x,y) := [\sigma_1(x,y) \vee 0]\,\nu_{31} + [\sigma_2(x,y) \vee 0]\,\nu_{32},$$

where

$$\sigma_1(x,y) := \alpha - \frac{|y-a_1|^2}{r - \langle \nu_{31}, x\rangle}, \qquad \sigma_2(x,y) := \alpha - \frac{|y-a_2|^2}{r - \langle \nu_{32}, x\rangle}.$$

For any $r' < r$ the function $\mathcal{S}$ is Lipschitz in $\overline{B(0,r')} \times \mathbb{R}^N$. By direct computations we have that

$$\begin{aligned} |D_y\mathcal{S}(x,y)|^2 = 4\frac{|y-a_1|^2}{(r - \langle \nu_{31}, x\rangle)^2} 1_{\{\sigma_1 > 0\}} + 4\frac{|y-a_2|^2}{(r - \langle \nu_{32}, x\rangle)^2} 1_{\{\sigma_2 > 0\}} \\ + 4\frac{\langle y - a_1, y - a_2\rangle}{(r - \langle \nu_{31}, x\rangle)(r - \langle \nu_{32}, x\rangle)} 1_{\{\sigma_1 > 0,\, \sigma_2 > 0\}}, \end{aligned} \tag{50}$$

while

$$[\mathrm{div}_x\mathcal{S}](x,y) = -\frac{|y-a_1|^2}{(r - \langle \nu_{31}, x\rangle)^2} 1_{\{\sigma_1 > 0\}} - \frac{|y-a_2|^2}{(r - \langle \nu_{32}, x\rangle)^2} 1_{\{\sigma_2 > 0\}}. \tag{51}$$

Condition (a) is therefore fulfilled if and only if $\langle y - a_1, y - a_2\rangle \le 0$ for every y such that there exists $x \in B(0,r')$ so that $\sigma_1(x,y) > 0$ and $\sigma_2(x,y) > 0$. Taking into account the definition of σ_1, σ_2, this is equivalent to require the following: if y belongs to the intersection of the ball centred at a_1 with radius $(r - \langle \nu_{31}, x\rangle)$ and the ball centred at a_2

with radius $(r - \langle \nu_{32}, x\rangle)$, then the angle spanned by the two vectors $y - a_1$ and $y - a_2$ is greater or equal to $\pi/2$. Some elementary geometric considerations show that this is guaranteed if

$$|a_1 - a_2|^2 \geq \alpha(2r - \langle \nu_{31}, x\rangle - \langle \nu_{32}, x\rangle) \qquad \forall x \in B(0, r'),$$

which is implied by condition (49).

From (48) it follows that $\sigma_2(x, a_1) \leq 0$, so that by (50) and (51) we have $|D_y\mathcal{S}(x, a_1)|^2 = 0$ and $[\mathrm{div}_x\mathcal{S}](x, a_1) = 0$. Since (48) implies analogously that $\sigma_1(x, a_2) \leq 0$, and $\sigma_1(x, a_3) \leq 0$, $\sigma_2(x, a_3) \leq 0$, we deduce that condition (a') is satisfied.

Let $(x, y), (x, z) \in \overline{B(0, r')} \times \mathbb{R}^N$. If neither (x, y) nor (x, z) belongs to $\{\sigma_1 > 0, \sigma_2 > 0\}$, then it is easy to check that the vector $\mathcal{S}(x, z) - \mathcal{S}(x, y)$ can be written as a linear combination $\lambda_1\nu_{31} - \lambda_2\nu_{32}$ with either $\lambda_1, \lambda_2 \in [0, \alpha]$ or $\lambda_1, \lambda_2 \in [-\alpha, 0]$ (depending on x, y, z); since ν_{31} and $-\nu_{32}$ span an angle equal to $2\pi/3$, the modulus of $\mathcal{S}(x, z) - \mathcal{S}(x, y)$ is in this case less than or equal to α. If $(x, y) \in \{\sigma_1 > 0, \sigma_2 > 0\}$, only two cases can occur: either $\mathcal{S}(x, z) - \mathcal{S}(x, y)$ is a linear combination of ν_{31} and $-\nu_{32}$ of the same kind as before (so, the same conclusion holds), or $\mathcal{S}(x, z) - \mathcal{S}(x, y)$ can be written as $\mu_1\nu_{31} + \mu_2\nu_{32}$ with $\mu_i \in [0, \sigma_i(x, y)]$ (depending on x, y, z). In this second case, we obtain

$$|\mathcal{S}(x, z) - \mathcal{S}(x, y)|^2 \leq \sigma_1^2(x, y) + \sigma_2^2(x, y) + \sigma_1(x, y)\sigma_2(x, y) \leq (\sigma_1(x, y) + \sigma_2(x, y))^2.$$

It is easy to see that, under condition (49), $\sigma_1(x, y) + \sigma_2(x, y) \leq \alpha$ for every $(x, y) \in \{\sigma_1 > 0, \sigma_2 > 0\}$, so that (b) is always satisfied.

Finally, using (48) we have that $\mathcal{S}(x, a_2) - \mathcal{S}(x, a_1) = \alpha\nu_{32} - \alpha\nu_{31} = \alpha\nu_{12}$ for every $x \in S_{12}$, $\mathcal{S}(x, a_3) - \mathcal{S}(x, a_2) = -\nu_{32} = \nu_{23}$ for every $x \in S_{23}$, while $\mathcal{S}(x, a_1) - \mathcal{S}(x, a_3) = \nu_{31}$ for every $x \in S_{31}$; so, we can conclude that (b') holds true for every $x \in S_u$.

We have thus proved that under conditions (48) – (49), u is a Dirichlet minimizer of MS_0 in $B(0, r')$ for every $r' < r$. By an approximation argument this implies the Dirichlet minimality of u in the whole $B(0, r)$.

As in the previous example, the minimality of u can be proved by using the scalar result in [2]: indeed, even if S_{u_j} is strictly contained in S_u for every j, one can always find a rotation R in $\mathbb{R}^N$ tranforming the range of u in a set of three vectors which differ each other for the same component and apply the scalar result to this component. This procedure leads to the following condition: u is a Dirichlet minimizer if

$$\max_{v \in \mathbb{R}^N, |v|=1} \min\{|\langle a_1 - a_2, v\rangle|, |\langle a_2 - a_3, v\rangle|, |\langle a_3 - a_1, v\rangle|\} \geq \sqrt{2\alpha r},$$

which is always more restrictive than (48) – (49), except when the vectors $a_i - a_j$ are collinear.

Example 5.5. In this example we deal with the complete Mumford-Shah functional

$$MS(u) := \int_\Omega |\nabla u|^2 dx + \alpha\mathcal{H}^{n-1}(S_u) + \beta\int_\Omega |u - g|^2\, dx \tag{52}$$

where $\Omega \subset \mathbb{R}^2$, g is a given function in $[L^\infty(\Omega)]^N$, and α, β are positive parameters.

Let $\{\Gamma_i\}_{i\in I}$ be a finite family of simple and connected curves of class C^2 such that for every i Γ_i is either a closed curve contained in Ω or it orthogonally meets $\partial\Omega$. Suppose also that $\Gamma_i \cap \Gamma_h = \emptyset$ if $i \neq h$. If g is a piecewise constant function, whose discontinuity set coincides with $\cup_{i\in I}\Gamma_i$, then for large values of β the function g itself is an absolute minimizer of (52).

We prove the statement by calibration. We recall that conditions (a), (a'), (b), and (b') in Lemma 3.3 read for the functional (52) as

(a) $[\mathrm{div}_x\mathcal{S}^\gamma](x,y) + \mathcal{S}_0(x) \le -\frac{1}{4}|D_y\mathcal{S}^\gamma(x,y)|^2 + \beta|y-g(x)|^2$ for every $\gamma\in A$, for $\mathcal{L}^2$-a.e. $x\in\Omega$, and for every $y\in\mathbb{R}^N$ with $(x,y)\in U^\gamma$;

(a') $[\mathrm{div}_x\mathcal{S}^\gamma](x,u) + \mathcal{S}_0(x) = -|\nabla u(x)|^2 + \beta|u-g|^2$ and $(D_y\mathcal{S}^\gamma(x,u))^\tau = 2\nabla u(x)$ for every $\gamma\in A$, and for $\mathcal{L}^2$-a.e. $x\in\Omega$ such that $(x,u(x))\in U^\gamma$;

(b) $|\mathcal{S}(x,z)-\mathcal{S}(x,y)|\le\alpha$ for $\mathcal{H}^1$-a.e. $x\in\Omega$ and for every $y,z\in\mathbb{R}^N$ such that $(x,y),(x,z)\in U$;

(b') $\mathcal{S}(x,u^+)-\mathcal{S}(x,u^-) = \alpha\nu_u$ for $\mathcal{H}^1$-a.e. $x\in S_u$,

where $\mathcal{S}(x,y) = \sum_{\gamma\in A}\mathcal{S}^\gamma(x,y)1_{U^\gamma}(x,y)$.

Let $\{E_j\}_{j\in J}$ be the partition of Ω generated by the family of curves $\{\Gamma_i\}_{i\in I}$. Then the function g can be written as

$$g(x) = \sum_{j\in J} a_j 1_{E_j}(x),$$

where $a_j\in\mathbb{R}^N$ and $a_j\neq a_k$ if $j\neq k$. For $j<k$ we call S_{jk} the interface between E_j and E_k, oriented by the normal ν_{jk} pointing from E_j to E_k (in other words, S_{jk} is the set of all $x\in S_g$ such that $g^-(x)=a_j$ and $g^+(x)=a_k$). In this way we have simply relabelled the curves Γ_i.

For every $j<k$ we can construct a C^1-vectorfield $\psi_{jk}:\overline{\Omega}\to\mathbb{R}^n$ such that it agrees with ν_{jk} on S_{jk}, is supported on a neighbourhood of S_{jk}, is tangent to the boundary of Ω, and $|\psi_{jk}|\le 1$ everywhere. Since the curves S_{jk} are disjoint, the functions ψ_{jk} can be constructed in such a way that their supports are still disjoint; moreover, if S_{jk} is a closed curve, we can also assume that the support of ψ_{jk} is relatively compact in Ω. Finally, for every $j<k$ we define the functions $\lambda_{jk}:\mathbb{R}^N\to\mathbb{R}$ as

$$\lambda_{jk}(y) := \sigma\left(\frac{\langle y-a_j, a_k-a_j\rangle}{|a_k-a_j|^2}\right),$$

where $\sigma:\mathbb{R}\to[0,\alpha]$ is a non decreasing function of class C^2 such that $\sigma(t) := \frac{1}{3}\alpha t^3$ for $t\in[0,1/8]$, $\sigma(t) := \alpha + \frac{1}{3}\alpha(t-1)^3$ for $t\in[7/8,1]$, $\sigma'(t)\in[0,2\alpha]$ for every t, and $|\sigma''(t)|\le 16\alpha$ for every t.

Now we set

$$\mathcal{S}(x,y) := \sum_{(j,k):j<k}\lambda_{jk}(y)\,\psi_{jk}(x), \qquad \mathcal{S}_0(x) := -\alpha\sum_{(j,k):j<k}\mathrm{div}\psi_{jk}(x)1_{E_k}(x),$$

and we claim that the pair $(\mathcal{S},\mathcal{S}_0)$ is a calibration for g when β is large enough.

First of all, independently of the choice of σ, the function $\mathcal{S}$ has vanishing normal component on $\partial\Omega$ because of the choice of ψ_{jk}, so that condition (16) of Theorem 3.5 is satisfied.

Using the fact that the supports of the functions ψ_{jk} are disjoint, and that $|\psi_{jk}| \leq 1$, while λ_{jk} takes values only on $[0,\alpha]$, it is easy to see that condition (b) is fulfilled.

Since S_g is the union of the disjoint curves $\{S_{jk}\}_{j<k}$, for every $x \in S_g$ there exists one and only one pair (j,k) with $j<k$ such that $x \in S_{jk}$, so that

$$\mathcal{S}(x,g^+(x)) - \mathcal{S}(x,g^-(x)) = (\lambda_{jk}(a_k) - \lambda_{jk}(a_j))\,\psi_{jk}(x) = (\sigma(1)-\sigma(0))\,\nu_{jk}(x) = \alpha\nu_g(x).$$

Therefore, also condition (b') is satisfied.

By direct computations we obtain that

$$[\mathrm{div}_x\mathcal{S}](x,y) = \sum_{(j,k):j<k} \lambda_{jk}(y)\,\mathrm{div}\psi_{jk}(x),$$

while

$$D_y\mathcal{S}(x,y) = \sum_{(j,k):j<k} \sigma'\left(\frac{\langle y-a_j, a_k-a_j\rangle}{|a_k-a_j|^2}\right)\psi_{jk}(x) \otimes \frac{a_k-a_j}{|a_k-a_j|^2}.$$

If $x \in E_h$ for any $h \in J$, then

$$\begin{aligned}[\mathrm{div}_x\mathcal{S}](x,g(x)) &= [\mathrm{div}_x\mathcal{S}](x,a_h) = \sum_{j<h}\lambda_{jh}(a_h)\,\mathrm{div}\psi_{jh}(x) + \sum_{k>h}\lambda_{hk}(a_h)\,\mathrm{div}\psi_{hk}(x)\\ &= \alpha\sum_{j<h}\mathrm{div}\psi_{jh}(x),\end{aligned}$$

where the last equality follows from the fact that $\lambda_{jh}(a_h) = \sigma(1) = \alpha$, while $\lambda_{hk}(a_h) = \sigma(0) = 0$. Arguing analogously, since $\sigma'(0) = \sigma'(1) = 0$, we have that $D_y\mathcal{S}(x,g(x)) = 0$, so that, taking into account the definition of $\mathcal{S}_0$, condition (a') is satisfied.

It remains to prove condition (a). Let $(x,y) \in \overline{\Omega}\times\mathbb{R}^N$. If x does not belong to any of the supports of the functions ψ_{jk}, then $[\mathrm{div}_x\mathcal{S}](x,y) = 0$, $\mathcal{S}_0(x) = 0$, and $D_y\mathcal{S}(x,y) = 0$, so (a) is trivially satisfied. If x belongs to the support of ψ_{jk} for any $j<k$, then

$$[\mathrm{div}_x\mathcal{S}](x,y) = \lambda_{jk}(y)\,\mathrm{div}\psi_{jk}(x), \qquad \mathcal{S}_0(x) = -\alpha\,\mathrm{div}\psi_{jk}(x)1_{E_k}(x),$$

$$D_y\mathcal{S}(x,y) = \sigma'\left(\frac{\langle y-a_j, a_k-a_j\rangle}{|a_k-a_j|^2}\right)\psi_{jk}(x)\otimes\frac{a_k-a_j}{|a_k-a_j|^2};$$

if we write the vector $y-a_j$ as the sum $v+t(a_k-a_j)$ where $v \in \mathbb{R}^N$ is orthogonal to a_k-a_j, and $t \in \mathbb{R}$, condition (a) turns to be equivalent to

$$\mathrm{div}\psi_{jk}(x)(\sigma(t) - \alpha 1_{E_k}(x)) \leq -\frac{1}{4}|\psi_{jk}(x)|^2|\sigma'(t)|^2 + \beta|v+t(a_k-a_j)+a_j-g(x)|^2. \quad (53)$$

Since we are assuming that x is in the support of ψ_{jk}, x belongs either to E_j or to E_k. When $x \in E_j$, inequality (53) reduces to

$$\mathrm{div}\psi_{jk}(x)\,\sigma(t) \leq -\frac{1}{4}|\psi_{jk}(x)|^2|\sigma'(t)|^2 + \beta|v|^2 + \beta|a_k-a_j|^2t^2,$$

which is implied by

$$\operatorname{div}\psi_{jk}(x)\,\sigma(t) \le -\frac{1}{4}|\psi_{jk}(x)|^2|\sigma'(t)|^2 + \beta|a_k - a_j|^2 t^2. \tag{54}$$

So, let us prove (54) for every $t \in \mathbb{R}$ and $x \in E_j$. Since in (54) the equality holds for $t = 0$, it is enough to show the following inequality

$$\operatorname{div}\psi_{jk}(x)\,\sigma'(t) < -\frac{1}{4}|\psi_{jk}(x)|^2 2\sigma'(t)\,\sigma''(t) + 2\beta|a_k - a_j|^2 t \qquad \text{for } t > 0, \tag{55}$$

and the opposite inequality for $t < 0$. Since $\sigma' \equiv 0$ for $t > 1$, inequality (55) is trivially satisfied for $t > 1$. For $0 < t \le 1$, (55) follows immediately from

$$-\|\operatorname{div}\psi_{jk}\|_\infty \sigma'(t) > \frac{1}{2}\sigma'(t)|\sigma''(t)| - 2\beta|a_k - a^j|^2 t,$$

which is satisfied (taking into account the structure of the function σ) for

$$\beta|a_k - a_j|^2 > 8\alpha\|\operatorname{div}\psi_{jk}\|_\infty + 64\alpha^2.$$

The same condition implies also the opposite inequality for $t < 0$. Moreover, the same argument can be applied in the case $x \in E_k$.

In conclusion, condition (a) is fulfilled for $\beta > \beta_0$, where β_0 is defined by

$$\beta_0 := \max_{(j,k):j<k} \frac{1}{|a_k - a_j|^2}\left(8\alpha\|\operatorname{div}\psi_{jk}\|_\infty + 64\alpha^2\right). \tag{56}$$

We conclude this example by noticing that this result generalizes Example 5.5 in [2], where g is the characteristic function of a regular set. As in the previous examples, the vectorial statement can be proved by applying the scalar result to one suitable component of g, but this leads to a worse estimate on β_0.

6. Calibrations in terms of closed differential forms

In this section we develop the theory of calibrations in terms of differential forms. The scalar method presented in [2] involves a divergence-free vectorfield on $\Omega\times\mathbb{R}$ (and its flux through the complete graph of the maps u), which is now replaced by a closed n-form on $\Omega\times\mathbb{R}^N$, acting on the graphs of the maps u, viewed as suitably defined n-surfaces in $\Omega\times\mathbb{R}^N$.

As we will see, this formulation is indeed not preferable to the one described in Section 3, since it leads to the same kind of conditions, requiring a greater technical effort.

For simplicity we restrict our discussion to piecewise smooth functions $u \in [SBV(\Omega)]^N$ in the sense of the following definition.

Definition 6.1. We say that a function $u \in [SBV(\Omega)]^N$ is *piecewise smooth*, and we write $u \in \mathcal{A}(\Omega)$, if the following conditions are satisfied: up to an $\mathcal{H}^{n-1}$-negligible set, S_u is a finite union of pairwise disjoint $(n-1)$-dimensional boundaryless C^1-manifolds of $\mathbb{R}^n$; u is C^1 on $\Omega \setminus S_u$ up to S_u, that is $u \in [C^1(\Omega \setminus S_u)]^N$ and there exist the limits of u and ∇u on both sides of (the regular part of) S_u.

For $u \in \mathcal{A}(\Omega)$ we define the n-surfaces

$$\Sigma_u := \{(x,y) \in \Omega\times\mathbb{R}^N : \ x \in S_u \ \text{ and } \ \exists\, t \in [0,1] \ \text{ such that } \ y = tu^+(x) + (1-t)u^-(x)\},$$

$$\Gamma_u := G_u \cup \Sigma_u,$$

where G_u is the graph of u on $\Omega \setminus S_u$. Using notation from [10], let us consider an n-form

$$\omega : \Omega\times\mathbb{R}^N \to \wedge^n\mathbb{R}^{n+N},$$

$$\omega(x,y) = \sum_{|\alpha|+|\beta|=n} \omega_{\alpha\beta}(x,y)\, dx^\alpha \wedge dy^\beta,$$

whose coefficients $\omega_{\alpha\beta}$ are of class C^1, and for $u \in \mathcal{A}(\Omega)$ the following functional

$$\int_{\Gamma_u} \omega, \tag{57}$$

where the orientation of Γ_u will be defined later in a precise way.

If ω is a closed form, then the functional (57) is constant on the functions u which take the same values on $\partial\Omega$. Moreover, if F is the functional (10), and if

$$\begin{array}{rlcll} & \displaystyle\int_{\Gamma_v} \omega & \leq & F(v) & \quad \text{for every } v \in \mathcal{A}(\Omega), \\ \text{and} & & & & \\ & \displaystyle\int_{\Gamma_u} \omega & = & F(u) & \quad \text{for a given } u \in \mathcal{A}(\Omega), \end{array} \tag{58}$$

then u is a Dirichlet minimizer of F in the class $\mathcal{A}(\Omega)$.

Let us now look for pointwise conditions on the coefficients of the form ω which guarantee (58).

By definition we have that

$$\int_{\Gamma_u} \omega = \int_{G_u} \omega + \int_{\Sigma_u} \omega. \tag{59}$$

On G_u we consider the natural orientation given by the parameterization $x \in \Omega \setminus S_u \mapsto (x, u(x))$, so that

$$\int_{G_u} \omega = \sum_{|\alpha|+|\beta|=n} \int_\Omega \omega_{\alpha\beta}(x, u(x))\mu_{\alpha\beta}(x)\, dx, \tag{60}$$

where

$$\mu_{\alpha\beta}(x) := \epsilon(\alpha) \det\left(\frac{\partial u_\beta}{\partial x_{\hat\alpha}}(x)\right).$$

In the previous formula $\hat\alpha$ denotes the increasing complement of α in $\{1,\ldots,n\}$, $\epsilon(\alpha)$ is the sign of permutation of $(1,\ldots,n)$ into $(\alpha,\hat\alpha)$, and $\frac{\partial u_\beta}{\partial x_{\hat\alpha}}$ is the $|\beta|\times|\beta|$ matrix $\frac{\partial u_{\beta_i}}{\partial x_{\hat\alpha_j}}$.

On Σ_u we consider the orientation given by the following parameterization: since $u \in \mathcal{A}(\Omega)$, without loss of generality, we may assume that S_u is an $(n-1)$-dimensional C^1-manifold of $\mathbb{R}^n$ without boundary and that S_u can be covered by just one parameter patch

$\gamma : S \to S_u$, where S is an $(n-1)$-dimensional domain (the general case can be easily obtained by summing over the C^1-pieces). Assume that γ yields ν_u as orientation, that is the vector

$$\eta(\gamma(\sigma)) := \sum_{i=1}^{n} (-1)^{n-i} \det\left(\frac{d\gamma_{\hat{\imath}}}{d\sigma}(\sigma)\right) e_i$$

(where $\{e_1, \dots, e_n\}$ is the canonical basis of $\mathbb{R}^n$) satisfies

$$\frac{\eta(\gamma(\sigma))}{|\eta(\gamma(\sigma))|} = \nu_u(\gamma(\sigma)) \qquad \forall \sigma \in S.$$

We consider as parameterization of Σ_u the function $\varphi = (\varphi^x, \varphi^y) : S \times [0,1] \to \Omega \times \mathbb{R}^N$ defined as $\varphi^x(\sigma, t) := \gamma(\sigma)$, $\varphi^y(\sigma, t) := tu^+(\gamma(\sigma)) + (1-t)u^-(\gamma(\sigma))$ for every $(\sigma, t) \in S \times [0,1]$, so that the second integral in (59) is given by

$$\int_{\Sigma_u} \omega = \sum_{|\alpha|+|\beta|=n} \int_0^1 \int_S \omega_{\alpha\beta}(\varphi(\sigma,t)) \det\left(\frac{\partial \varphi_{\alpha\beta}}{\partial(\sigma,t)}(\sigma,t)\right) d\sigma dt, \tag{61}$$

where $\varphi_{\alpha\beta} = (\varphi^x_{\alpha_1}, \dots, \varphi^x_{\alpha_p}, \varphi^y_{\beta_1}, \dots, \varphi^y_{\beta_q})$ for $|\alpha| = p$ and $|\beta| = q = n - p$. By direct computations one can find that

$$\det\left(\frac{\partial \varphi_{\hat{0}0}}{\partial(\sigma,t)}\right) = 0,$$

while for every $1 \le i \le n$, $1 \le j \le N$

$$\det\left(\frac{\partial \varphi_{\hat{\imath}j}}{\partial(\sigma,t)}\right) = (u_j^+ - u_j^-) \det\left(\frac{d\gamma_{\hat{\imath}}}{d\sigma}\right) = (-1)^{n-i}(u_j^+ - u_j^-)(\nu_u)_i |\eta|,$$

where all the functions at the right-hand side are computed at $\gamma(\sigma)$. Finally, by straightforward computations, if we set $a := \hat{\alpha}$, for $|a| = |\beta| = q \ge 2$ it results that

$$\det\left(\frac{\partial \varphi_{\alpha\beta}}{\partial(\sigma,t)}\right) =$$
$$= \sum_{m,k=1}^{q} \epsilon(\alpha, a_{\widehat{k}})(-1)^{n-q+m-a_k}(u^+_{\beta_m} - u^-_{\beta_m}) \det\left(\frac{\partial(tu^+ + (1-t)u^-)_{\beta_{\widehat{m}}}}{\partial x_{a_{\widehat{k}}}}\right)(\nu_u)_{a_k}|\eta|,$$

where $\beta_{\widehat{m}}$, $a_{\widehat{k}}$ are the increasing complement of β_m in $\{\beta_1, \dots, \beta_q\}$ and of a_k in $\{a_1, \dots, a_q\}$, respectively, while $\epsilon(\alpha, a_{\widehat{k}})$ is the sign of permutation of $(\alpha, a_{\widehat{k}})$ in $\widehat{a_k}$; again all the functions at the right-hand side are computed at $\gamma(\sigma)$. Set $w^t := tu^+ + (1-t)u^-$ and substitute all the above expressions in formula (61); since $|\eta|\, d\sigma$ is the area element of the manifold S_u parameterized by γ, we obtain

$$\begin{aligned}
\int_{\Sigma_u} \omega \;&=\; \sum_{i,j} \int_0^1 \int_{S_u} (-1)^{n-i} \omega_{\hat{\imath}j}(x, w^t)(u_j^+ - u_j^-)(\nu_u)_i \, d\mathcal{H}^{n-1} dt \\
&+ \sum_{\substack{|a|=|\beta|=q \\ q\ge 2}} \int_0^1 \int_{S_u} \omega_{\hat{a}\beta}(x, w^t) \sum_{m,k=1}^{q} \epsilon(\alpha, a_{\widehat{k}})(-1)^{n-q+m-a_k}(u^+_{\beta_m} - u^-_{\beta_m}) \det\left(\frac{\partial w^t_{\beta_{\widehat{m}}}}{\partial x_{a_{\widehat{k}}}}\right)(\nu_u)_{a_k} \, d\mathcal{H}^{n-1} dt \\
&=: \int_{S_u} g_\omega(x, u^-, u^+, \nabla u^-, \nabla u^+, \nu_u) \, d\mathcal{H}^{n-1},
\end{aligned} \tag{62}$$

where the last equality follows from changing the order of integration and calling g_ω the integrand with respect to $\mathcal{H}^{n-1}$. Now we wonder what kind of conditions on $\omega_{\alpha\beta}$ guarantees that

$$g_\omega(x, u^-, u^+, \nabla u^-, \nabla u^+, \nu_u) \le \psi(x, u^-, u^+, \nu_u) \qquad \text{on } S_u \tag{63}$$

for every admissible u. The answer is given by the following proposition.

Proposition 6.2. *Inequality (63) holds true for every $u \in \mathcal{A}(\Omega)$ if and only if the following conditions are satisfied:*

(b_0) $\omega_{\alpha\beta} \equiv 0$ *for every α, β such that $|\beta| \ge 2$, $|\alpha| + |\beta| = n$;*

(b_1) $\displaystyle\sum_{i,j} \int_0^1 (-1)^{n-i} \omega_{\hat{i}j}(x, tz+(1-t)y)(z_j-y_j)\nu_i\, dt \le \psi(x, y, z, \nu)$ *for every $x \in \Omega$, for every $y, z \in \mathbb{R}^N$, and for every $\nu \in \mathbb{S}^{n-1}$.*

Moreover, the equality holds for a given u if and only if

(b_2) $\displaystyle\sum_{i,j} \int_0^1 (-1)^{n-i} \omega_{\hat{i}j}(x, tu^+ + (1-t)u^-)(u_j^+ - u_j^-)(\nu_u)_i\, dt = \psi(x, u^-, u^+, \nu_u)$ *for every $x \in S_u$.*

Proof. Let $(x, y) \in \Omega \times \mathbb{R}^N$, and let us prove that $\omega_{\alpha\beta}(x, y) = 0$ for $|\hat{\alpha}| = |\beta| = 2$. By renumbering the coordinates of x and y, we may suppose that $\beta = (1, 2)$ and $a = \hat{\alpha} = (1, 2)$. Given $C \in \mathbb{R}$, we can construct $u \in \mathcal{A}(\Omega)$ such that $x \in S_u$, $\nabla u^-(x) = \nabla u^+(x)$ (hence $\nabla w^t(x) = \nabla u^-(x)$ for every $t \in [0, 1]$), and $\partial_{x_i} w_j^t(x) = 0$ for every $(i, j) \ne (1, 1)$ and $\partial_{x_1} w_1^t(x) = C$. With this choice we have that

$$\begin{aligned} g_\omega(x, u^-, u^+, \nabla u^-, \nabla u^+, \nu_u) = {} & \sum_{i,j} \int_0^1 (-1)^{n-i} \omega_{\hat{i}j}(x, w^t)(u_j^+ - u_j^-)(\nu_u)_i\, dt \\ & + C \sum_{i \ne 1, j \ne 1} \int_0^1 (-1)^i \omega_{(\widehat{1,i})(1,j)}(x, w^t)(u_j^+ - u_j^-)(\nu_u)_i\, dt. \end{aligned}$$

Since the value of C is arbitrary and independent of $u^-(x), u^+(x), \nu_u(x)$, inequality (63) implies that

$$\sum_{i \ne 1, j \ne 1} \int_0^1 (-1)^i \omega_{(\widehat{1,i})(1,j)}(x, w^t)(u_j^+ - u_j^-)(\nu_u)_i\, dt = 0 \tag{64}$$

whatever are the values of $u^-(x), u^+(x), \nu_u(x)$. Choosing $\nu_u(x)$ such that $(\nu_u(x))_i = 0$ for every $i \ne 2$, $(\nu_u(x))_2 = 1$, we have that (64) is equivalent to

$$\sum_{j \ne 1} \int_0^1 \omega_{(\widehat{1,2})(1,j)}(x, w^t)(u_j^+ - u_j^-)\, dt = 0 \tag{65}$$

whatever are the values of $u^-(x), u^+(x)$. Choosing $u^-(x) = y$, while $u_j^+(x) = y_j$ for every $j \ne 2$, $u_2^+(x) = y_2 + c$ with $c \ne 0$, we obtain that (65) is equivalent to

$$c \int_0^1 \omega_{(\widehat{1,2})(1,2)}(x, y_1, y_2 + ct, y_3, \ldots, y_N)\, dt = 0 \tag{66}$$

for every $c \neq 0$. By a change of variables, (66) can be rewritten as

$$\int_{y_2}^{y_2+c} \omega_{(\widehat{1,2})(1,2)}(x, y_1, s, y_3, \dots, y_N)\, ds = 0. \tag{67}$$

Since (67) has to be true for every $c \neq 0$, this implies that $\omega_{(\widehat{1,2})(1,2)}(x, y) = 0$.

Using the fact that the coefficients $\omega_{\alpha\beta} \equiv 0$ for every $|\beta| = 2$, we can repeat the same proof to show that $\omega_{\alpha\beta} \equiv 0$ for every $|\hat{\alpha}| = |\beta| = 3$, and so on.

We have thus proved that (63) implies condition (b_0). At this point, it is trivial that (63) implies also condition (b_1), and that the equality holds in (63) for a given u if and only if also (b_2) is satisfied. □

Summarizing, if conditions (b_0) and (b_1) hold true, by Proposition 6.2 inequality (63) is satisfied, hence by (62) we have that

$$\int_{\Sigma_u} \omega \leq \int_{S_u} \psi(x, u^-, u^+, \nu_u)\, d\mathcal{H}^{n-1} \tag{68}$$

for every $u \in \mathcal{A}(\Omega)$, while the equality holds in (68) for a given u if and only if also (b_2) is verified.

Assuming that ω satisfies condition (b_0), formula (60) reduces to

$$\begin{aligned} \int_{G_u} \omega &= \int_\Omega \Big(\omega_{\hat{0}0}(x, u(x)) + \sum_{i,j} (-1)^{n-i} \omega_{\hat{i}j}(x, u(x)) \partial_{x_i} u_j(x) \Big)\, dx \\ &= \int_\Omega \left(\omega_{\hat{0}0}(x, u(x)) + \langle A_\omega(x, u(x)), \nabla u(x) \rangle \right) dx, \end{aligned}$$

where in the last equality $(A_\omega(x, y))_{ji} := (-1)^{n-i} \omega_{\hat{i}j}(x, y)$. It is easy to see that, if we require the following condition:

(a_1) $\omega_{\hat{0}0}(x, y) \leq -f^*(x, y, A_\omega(x, y))$ for $\mathcal{L}^n$-a.e. $x \in \Omega$ and every $y \in \mathbb{R}^N$,

then

$$\int_{G_u} \omega \leq \int_\Omega f(x, u, \nabla u)\, dx$$

for every $u \in \mathcal{A}(\Omega)$; moreover, the equality holds for a given u if and only if

(a_2) $(A_\omega)_{ij}(x, u(x)) \in \partial_{\xi_{ij}} f(x, u(x), \nabla u(x))$ and $\omega_{\hat{0}0}(x, u(x)) = -f^*(x, u(x), A_\omega(x, u(x)))$ for $\mathcal{L}^n$-a.e. $x \in \Omega$.

Therefore by (59) we can conclude that (58) is guaranteed if conditions (a_1), (a_2), (b_0), (b_1), and (b_2) are satisfied. In other words, we have proved the following theorem.

Theorem 6.3. *Let u be a function in $\mathcal{A}(\Omega)$. Assume that there exists a closed n-differential form $\omega : \Omega \times \mathbb{R}^N \to \wedge^n \mathbb{R}^{n+N}$ with coefficient of class C^1 and satisfying condition (a_1), (a_2), (b_0), (b_1), and (b_2). Then u is a Dirichlet minimizer of the functional (10) in the class $\mathcal{A}(\Omega)$.*

We conclude this section by proving that, if $u \in \mathcal{A}(\Omega)$ and there exists a differential form ω which calibrates u in the sense of Theorem 6.3, then there exists a calibration $(\mathcal{S}, \mathcal{S}_0)$ for u in the sense of Definition 3.4.

Proposition 6.4. *Let u be a function in $\mathcal{A}(\Omega)$ and let $\omega : \Omega\times\mathbb{R}^N \to \wedge^n\mathbb{R}^{n+N}$ be a closed n-differential form satisfying all the assumptions of Theorem 6.3. Then there exists a calibration $(\mathcal{S}, \mathcal{S}_0)$ for u, with $\mathcal{S} \in [C^2(\Omega\times\mathbb{R}^N)]^n$ and $\mathcal{S}_0 \in C^1(\Omega)$.*

Proof. First of all, we notice that from condition (b_0) it follows that

$$\omega(x, y) = \omega_{\hat{0}0}(x, y)\, dx + \sum_{i,j} \omega_{\hat{\imath}j}(x, y)\, dx^{\hat{\imath}} \wedge dy^j.$$

Since ω is a closed form, by computing explicitly the exterior derivative of ω, we obtain that the coefficients $\omega_{\hat{0}0}, \omega_{\hat{\imath}j}$ satisfy the two following equations:

$$\sum_{i=1}^n (-1)^{n-i} \frac{\partial \omega_{\hat{\imath}j}}{\partial x_i}(x, y) - \frac{\partial \omega_{\hat{0}0}}{\partial y_j}(x, y) = 0 \qquad 1 \le j \le N, \tag{69}$$

$$(-1)^{n-i} \frac{\partial \omega_{\hat{\imath}j}}{\partial y_k}(x, y) = (-1)^{n-i} \frac{\partial \omega_{\hat{\imath}k}}{\partial y_j}(x, y) \qquad 1 \le i \le n,\ 1 \le j, k \le N. \tag{70}$$

The last condition is equivalent to require that for every i the vector $((-1)^{n-i}\omega_{\hat{\imath}j}(x, y))_{j=1,\dots,N}$ is the gradient with respect to y of a function of class C^2; more precisely, there exists a function $\mathcal{S} \in [C^2(\Omega\times\mathbb{R}^N)]^n$ such that

$$\partial_{y_j}\mathcal{S}_i(x, y) = (-1)^{n-i}\omega_{\hat{\imath}j}(x, y) \qquad 1 \le i \le n,\ 1 \le j \le N. \tag{71}$$

Equation (69) can be therefore rewritten as

$$0 = \sum_{i=1}^n \frac{\partial^2 \mathcal{S}_i}{\partial x_i \partial y_j}(x, y) - \frac{\partial \omega_{\hat{0}0}}{\partial y_j}(x, y) = \partial_{y_j}\left[\sum_{i=1}^n \partial_{x_i}\mathcal{S}_i(x, y) - \omega_{\hat{0}0}(x, y)\right] \qquad 1 \le j \le N,$$

and then there exists a function $\mathcal{S}_0 : \Omega \to \mathbb{R}$ of class C^1 such that $\omega_{\hat{0}0}(x, y) = [\mathrm{div}_x\mathcal{S}](x, y) + \mathcal{S}_0(x)$. By substituting this equality and (71) in conditions (a_1) and (a_2), we directly obtain that the pair $(\mathcal{S}, \mathcal{S}_0)$ satisfies conditions (a) and (a') of Lemma 3.2. Since the left-hand side in (b_1) can be rewritten as

$$\begin{aligned}
&\sum_{i,j} \int_0^1 (-1)^{n+i}\omega_{\hat{\imath}j}(x, tz + (1-t)y)(z_j - y_j)\nu_i\, dt \\
&\quad = \sum_{i,j} \int_0^1 \partial_{y_j}\mathcal{S}_i(x, tu^+ + (1-t)u^-)(u_j^+ - u_j^-)(\nu_u)_i\, dt \\
&\quad = \sum_{i=1}^n \int_0^1 \frac{d}{dt}[\mathcal{S}_i(x, tu^+ + (1-t)u^-)](\nu_u)_i\, dt \\
&\quad = \sum_{i=1}^n [\mathcal{S}_i(x, u^+) - \mathcal{S}_i(x, u^-)](\nu_u)_i \\
&\quad = \langle \mathcal{S}(x, u^+) - \mathcal{S}(x, u^-), \nu_u\rangle,
\end{aligned}$$

condition (b_1) implies that the function $\mathcal{S}$ satisfies condition (b) of Lemma 3.2, and in the same way (b_2) implies (b'). $\square$

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