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# Stability of Geometric and Harmonic Functional Means

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Geometric and Harmonic means of two convex functionals have been recently constructed. The aim of this work is to study the stability of these operations. The development of this theory is very useful in many contexts. The obtained results generalize the case of symmetric positive operators which have been studied in the literature.

*Keywords:* Convex functional, Fenchel duality, arithmetic, geometric and harmonic functional means, stability of operation means

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## 1. Introduction

In recent few years, many authors have been interested to the construction of geometric, arithmetic and harmonic operator and functional means, because of their many interesting properties and applications.
Ando (Ref [3]) has constructed the geometric mean $A(g)B$ of two positive operators $A$ and $B$: $A\,(g)\,B = \max\{X \ : \ \begin{pmatrix} A & X \\ X & B \end{pmatrix} \geq 0\}$.
This operator appears, for example, in the time invariant linear optimal regulator system: Indeed, it is well known that, (Ref [1]), the particular algebraic Riccati equation, $XAX = B$ has one and only one symmetric positive solution given by $X = A^{-1}(g)B$.
Very recently, Atteia and Raïssouli (Ref [1]), have constructed the geometric mean of two convex functionals from a sequence of iterations descended from the sum and the infimal convolution. This approach permitted the authors of [1] to deduce another definition of the geometric operator mean which, of course, coincides with the given above one. Moreover, a physical illustration of the geometric mean as an equivalent resistor of an electrical circuit with matrices elements is introduced in [1].

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In another part; Fujii (Ref [9]) has defined the arithmetico-geometric and arithmetico-harmonic operator means from analogies with positive numbers: Let $A_1 = A,\ B_1 = B$, $A_{n+1} = A_n(g)B_n$ (*resp* $A_{n+1} = A_n(h)B_n$) and $B_{n+1} = \frac{1}{2}(A_n + B_n)$ for each $n \geq 1$. Then, the arithmetico-geometric mean $A(ag)B$ (resp the arithmetico-harmonic mean $A(ah)B$) of $A$ and $B$ is the same limit of $A_n$ and $B_n$ in the strong convergence. J. Fujii and M. Fujii (Ref [8]) have shown that the operator means mentioned above are reduced to the numerical means and $A(ah)B = A(g)B$ was proved by Ando.

To extend the above notions from operators to convex functionals, Raïssouli and Chergui (Ref [7] and [13]), have defined the arithmetico- geometric, arithmetico-harmonic and geometrico-harmonic means in convex analysis. Also, they have proved that the arithmetico-harmonic mean of two convex functionals coincides with their convex geometric mean constructed in [1]. In particular; in the quadratical case they have obtained again the previous operator means and their properties. Thus the theory of functional means contains that of means for positive operators.

The purpose of this article is to show that stability of the above functional means will be preserved under the necessary property of monotony. This study allows us to extend some results given by J. Fujii and M. Fujii (Ref [8]) in the positive operators case.

This paper is divided into four parts:

Firstly, we begin by presenting some definitions and results of functional means recently introduced by many authors.

In Section 3, we study the stability of the geometric functional mean when the seqences of functionals are decreasing. We deduce the stability to the increasing case by using the self-duality of the geometric operation mean.

Section 4 is devoted to the particular case of positively homogenuous functionals in finite dimension. The hypothesis of monotony is not necessary to prove the stability, but only in this case, since we prove that even if the functionals are defined in a finite dimension space and are not positively homogenuous, or the condition of monotony is not satisfied, the result cannot be obtained.

In Section 5, we apply the last properties to deduce that harmonic, arithmetic, arithmetico-geometric, aritmetico-harmonic and geometrico-harmonic functional means are still stable.

## 2. Preliminary

Let $X$ be a normed space (reflexive Banach when it is necessary), $X^*$ its topological dual, and $< .,. >$ the duality bracket between $X$ and $X^*$ .

If we denote by $\overline{IR}^X$ the space of all functions defined from $X$ into $\overline{IR} = IR \cup \{+\infty, -\infty\}$, we can extend the structure of $IR$ on $\overline{IR}$ by setting

$$\forall\ x \in \overline{IR}, \quad -\infty \leq x \leq +\infty, \quad (+\infty) + x = +\infty, \quad 0 \cdot (+\infty) = +\infty,$$

the space $\overline{IR}^X$ is equipped with the partial ordering relation defined by

$$\forall\ f, g \in \overline{IR}^X, \quad f \leq g \iff \forall x \in X \quad f(x) \leq g(x).$$

Given $f : X \to \widetilde{IR} = IR \cup \{+\infty\}$ a function, we denote by $f^*$ the Fenchel-conjugate of $f$ defined by

$$f^*(x^*) = \sup_{x \in X}\{< x, x^* > -f(x)\}, \text{ for all } x^* \in X^*$$

and by dom $f$, the effective domain of $f$, i.e.

$$\text{dom } f = \{x \in X : \quad f(x) < +\infty\}.$$

We denote by $\Gamma_0(X)$ the cone of closed (i.e. lower semi-continuous: l.s.c) convex functionals from $X$ into $I\!R \bigcup \{+\infty\}$ not identically equal to $+\infty$.
$f^{**} = (f^*)^*$ denotes the biconjugate of $f$, which coincides with the closure of $f$, i.e. the lower semi continuous hull of $f$. Below, we write $\text{cl}\,(f)$ instead of $f^{**}$.

Let $f \in \Gamma_0(X)$ and $\alpha > 0$ be a real, we define the functions $\alpha.f$ and $f.\alpha$ by

$$\forall\, x \in X, \quad (\alpha.f)\,(x) = \alpha \cdot f\,(x) \quad \text{and} \quad (f.\alpha)\,(x) = \alpha f\left(\frac{x}{\alpha}\right).$$

It is not hard to prove that (cf. Ref. [10]):

$$(\alpha.f)^* = f^*.\alpha \quad \text{and} \quad (f.\alpha)^* = \alpha.f^*.$$

Let $f, g : \ X \rightarrow \widetilde{I\!R}$, the infimal convolution of $f$ and $g$, denoted by $f \Box g$, is defined by

$$f \Box g(x) = \inf\{f(y) + g(x-y), \ \ y \in X\} \ \text{ for all } x \in X.$$

It is well known that $(f \Box g)^* = f^* + g^*$, (cf. Ref. [10]).

Given a nonempty subset $A$ of $X$, $\Psi_A$ denotes the indicator functional of $A$ and $\Psi_A^*$ the support function of $A$. The closure of $A$, i.e. the smallest closed set containing $A$, will be denoted by $\text{adh}\, A$, and its relative interior by $\text{ri}\, A$ (cf. Ref. [14]).
The following definition is an extension of the classical arithmetic and harmonic means of operators:

**Definition 2.1.** Let $f, g \in \Gamma_0(X)$ such that dom $f \cap$ dom $g \neq \emptyset$, we define:

**i)** The arithmetic functional mean of $f$ and $g$ by

$$f(a)g = \frac{1}{2}f + \frac{1}{2}g$$

**ii)** The harmonic functional mean of $f$ and $g$ by

$$f(h)g = \left(\frac{1}{2}f^* + \frac{1}{2}g^*\right)^* = \text{cl}(f \Box g).\frac{1}{2}.$$

**Proposition 2.2.** *Let* $f, g \in \Gamma_0(X)$ *then* $\text{dom }(f(a)g) = \text{dom } f \cap \text{dom } g$ *and* $\text{dom }\ (f(h)g) = \text{dom cl}(f \Box g).\frac{1}{2} = \text{adh}\left(\frac{1}{2}\text{dom } f + \frac{1}{2}\text{dom } g\right)$.

**Proof**. Immediate.

Suppose that $f$ and $g$ belong to $\Gamma_0(X)$, and define the following algorithm, (Ref. [1]):

$$\begin{cases} f\gamma_{n+1}g &= \frac{1}{2}\left\{(f\gamma_n g) + (f\gamma_n^* g)\right\} \\ f\gamma_n^* g &= (f^*\gamma_n g^*)^* \quad \text{for all } n \in I\!N \\ f\gamma_0 g &= \frac{1}{2}(f+g) \\ f\gamma_0^* g &= \left(\frac{1}{2}f^* + \frac{1}{2}g^*\right)^*. \end{cases}$$

**Proposition 2.3.** (Ref. [1]).

**i)** $\forall n \in IN, \quad (f\gamma_n g) \in \Gamma_0(X)$ *and* $(f\gamma_n^* g) \in \Gamma_0(X)$.

**ii)** $\forall n \in IN, \quad f\gamma_n^* g \le f\gamma_n g \,,\; f\gamma_{n+1} g \le f\gamma_n g$ *and* $f\gamma_{n+1}^* g \ge f\gamma_n^* g$.

**iii)** $\forall n \in IN, \; 0 \le f\gamma_{n+1} g - f\gamma_{n+1}^* g \le \frac{1}{2}\left(f\gamma_n g - f\gamma_n^* g\right)$.

**Theorem 2.4.** (Ref. [1]). *Let* $f,\ g \in \Gamma_0(X)$. *Assume that* $\mathrm{dom}\, f\,(a)\,g = \mathrm{dom}\, f\,(h)\,g$. *Then the sequences* $(f\gamma_n g)_{n\in IN}$ *and* $(f\gamma_n^* g)_{n\in IN}$ *converge pointwise in* $\overline{IR}$ *to the same limit* $f\,\tau\, g \in \Gamma_0(X)$, *furthermore:* $\mathrm{dom}\ (f\,\tau\, g) = \mathrm{dom}\ f\,(a)\,g = \mathrm{dom}\ f \cap \mathrm{dom}\ g$, $f\,\tau\, g = g\,\tau\, f$ *and* $(f\,\tau\, g)^* = f^*\tau g^*$.

**Remark 2.5.** The pointwise convergence of the sequence $(f\gamma_n\, g)_n$ is proved in [1]. The coincident limit of $(f\gamma_n\, g)_n$ and $(f\gamma_n^* g)_n$ follows by using Proposition 2.3, iii).

**Corollary 2.6.** *Assume that* $X$ *is an Hilbert space. Let* $A$ *and* $B$ *be two symmetric positive operators from* $X$ *into* $X$. *Put that*
$f_A(x) = \frac{1}{2} < Ax, x >$ *and* $f_B(x) = \frac{1}{2} < Bx, x >$ *for all* $x \in X$. *Then we have*

$$(f_A \tau f_B)(x) = f_{A(g)B}(x) \quad \text{for all } x \in X$$

*where* $A(g)B$ *is the geometric operator mean of* $A$ *and* $B$, *(cf.* [1], [3]).

**Definition 2.7.** (Ref. [1]). $f\,\tau\, g$ is called the convex geometric functional mean of $f$ and $g$. In particular, if $g = f_\sigma = \frac{1}{2}\|.\|^2$, then $f\,\tau\, f_\sigma$, denoted by $f^{\left[\frac{1}{2}\right]}$, is called the convex square root functional of $f$.

Now, for $f,\, g \in \Gamma_0(X)$, we consider the sequences $(f_n)_{n\in IN}$ and $(g_n)_{n\in IN}$ defined as follows:

$$\begin{cases} f_{n+1} &= \frac{1}{2}\left(f_n + g_n\right) \\ g_{n+1} &= \left(\frac{1}{2}\, f_n^* + \frac{1}{2}\, g_n^*\right)^* \end{cases} \quad \text{where} \quad \begin{cases} f_0 = f \\ g_0 = g. \end{cases}$$

Let us notice that $f_n \in \Gamma_0(X)$ and $g_n \in \Gamma_0(X)$ for all $n \ge 0$.

**Theorem 2.8.** *and* **Definition 2.9** (Ref. [7], [13]). *Let* $f$ *and* $g$ *in* $\Gamma_0(X)$ *satisfying the assumption of Theorem 2.4. Then* $(f_n)_{n\in IN}$ *and* $(g_n)_{n\in IN}$ *both converge pointwise to the same convex functional* $f(a.h)g$, *and called the arithmetico-harmonic functional mean of* $f$ *and* $g$. *Furthermore, one has* $f(a.h)g = f\tau g$.

**Corollary 2.10.** *With the same hypotheses and notations as Corollary 2.6, we have*

$$f_A(a.h)f_B = f_{A(a.h)B} = f_{A(g)B}$$

*where* $A(a.h)B$ *is the arithmetico-harmonic operator mean of* $A$ *and* $B$ *(cf* [3], [8]) *and thus* $A(a.h)B = A(g)B$.

**Theorem 2.11.** *and* **Definition 2.12** (Ref. [7], [13]). *Let* $f\,,\, g \in \Gamma_0(X)$ *as in Theorem 2.4.*

- *The arithmetico-geometric functional mean of* $f$ *and* $g$, *denoted by* $f(a.g)g$, *is defined* *as the pointwise limit of the algorithm:*

$$\begin{cases} f_{n+1} &= \frac{1}{2}(f_n + g_n) \\ g_{n+1} &= f_n \ \tau \ g_n \end{cases} \quad \textit{with} \quad \begin{cases} f_0 = f \\ g_0 = g. \end{cases}$$

• *The geometrico-harmonic functional mean of* $f$ *and* $g$*, denoted* $f(g.h)g$*, is the pointwise limit of the following algorithm*

$$\begin{cases} f_{n+1} &= f_n \ \tau \ g_n \\ g_{n+1} &= \left(\frac{1}{2} \ f_n^* + \frac{1}{2} \ g_n^*\right)^* \end{cases} \quad \textit{where} \quad \begin{cases} f_0 = f \\ g_0 = g. \end{cases}$$

*Moreover there hold*

$$f(h)g \leq f(g.h)g \leq f\tau g = f(a.h)g \leq f(a.g)g \leq f(a)g.$$

**Corollary 2.13.** *With the above notations, we have*

$$f_A(a.g)f_B = f_{A(a.g)B} \ \textit{and} \ f_A(g.h)f_B = f_{A(g.h)B}$$

*where* $A(a.g)B$ *(resp.* $A(g.h)B$*) is the arithmetico-geometric (resp. the geometrico-harmonic) operator mean constructed in* [9].

Below, we write $\lim\limits_{n\to+\infty} f_n = f$ when $(f_n)_n$ converges pointwise to $f$.

**Definition 2.14.** (Ref [2]). Let $X$ be a reflexive Banach space. We say that the sequence $(f_n)_n$ defined in $\overline{IR}^X$ is Mosco convergent to the functional $f$ if:

**i)** For all $x \in X$, for all sequence $(x_n)_n$converging in the weak topology to $x$ one has $f(x) \leq \liminf\limits_{n\to+\infty} \ f_n(x_n)$.

**ii)** For all $x \in X$, there exits a sequence $(x_n)_n$ converging in the strong topology to $x$ satisfying that, $f(x) \geq \limsup\limits_{n\to+\infty} \ f_n(x_n)$.

In this case, we write $f = \ \text{M-}\lim\limits_{n\to+\infty} \ f_n$

**Theorem 2.15.** (Ref. [2]). *Let* $(f_n)_{n\in IN}$ *be a sequence of closed convex proper function from* $X$ *a reflexive Banach space into* $]-\infty,+\infty]$*. Then the following properties hold:*

**i)** *If the sequence* $(f_n)_{n\in IN}$ *increases, then it Mosco-converges to* $f = \sup\limits_{n\in IN} f_n$.

**ii)** *If the sequence* $(f_n)_{n\in IN}$ *decreases, then it Mosco-converges to* $f = \text{cl}\left(\inf\limits_{n\in IN} \ f_n\right)$.

## 3. Stability of the geometric operation mean

We will begin by the following proposition which simplify the assumption of convergence of the geometric functionnal mean.

**Proposition 3.1.** *Let* $X$ *be a Banach space and* $f,\ g \in \Gamma_0(X)$*, then the following equivalence holds*

$$\text{dom} \ f(a)\,g = \text{dom} \ f(h)\,g \iff \text{dom} \ f = \text{dom} \ g. \tag{1}$$

**Proof.** In Theorem 2.4, the hypothesis dom $(f(a)g) =$ dom $(f(h)g)$ which allows us the convergence of the algorithm to a proper closed convex functional, is equivalent to the following:

$$\text{dom } f \cap \text{dom } g = \text{dom } \left(\text{cl}(f \Box g)\right).\frac{1}{2} = \text{adh}(\frac{1}{2}\text{dom } f + \frac{1}{2}\text{dom } g) \tag{2}$$

Remark that dom $f =$ dom $g$ implies immediately dom $f\,(a)\,g =$ dom $f\,(h)\,g$. Now, we shall prove that relation (2) implies that dom $f =$ dom $g$.

Indeed, (2) gives successively

$$\frac{1}{2}\text{dom}\, f + \frac{1}{2}\text{dom}\, g \subset \text{dom}\, g, \tag{3}$$

and

$$\frac{1}{2}\text{dom}\, f + \frac{1}{2}\text{dom}\, g \subset \text{dom}\, f, \tag{4}$$

Let $x \in \text{dom}\, f$. The inclusion (3) yields:

$$\forall y \in \text{dom } g \quad \exists y' \in \text{dom } g \quad \text{such that} \quad \frac{1}{2}x + \frac{1}{2}y = y'.$$

If $y' \in$ dom $g$ such that $y = y'$ then $x = y \in$ dom $g$ and dom $f$ will be included in dom $g$.

Suppose now that for all $y \in \text{dom}\, g$ there exists $y' \in$ dom $g$ which is different from $y$ such that $\frac{1}{2}x + \frac{1}{2}y = y'$. So, dom $g$ is infinite and we can construct a sequence $(y_n)_{n \in IN} \in$ dom $g$ as the following:

$$\begin{cases} \frac{1}{2}x & + & \frac{1}{2}y_0 & = & y_1, & y_1 \neq y_0 \\ \frac{1}{2}x & + & \frac{1}{2}y_1 & = & y_2, & y_2 \neq y_1 \\ \vdots & & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{2}x & + & \frac{1}{2}y_{n-1} & = & y_n, & y_n \neq y_{n-1}. \end{cases}$$

Multiplying each line $i$ by $\frac{1}{2^{n-i}}$ and adding them, we find $\frac{1}{2}x \times \left(\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}\right) + \frac{1}{2^n}y_0 = y_n$. Letting $n \to +\infty$, we obtain $y_n \to x$. Since $y_n = \frac{1}{2}x + \frac{1}{2}y_{n-1} \in \frac{1}{2}\text{dom } f + \frac{1}{2}\text{dom}\, g$, we deduce $x \in \text{adh}\left(\frac{1}{2}\text{dom}\, f + \frac{1}{2}\text{dom}\, g\right)$ which, combined with (2), implies that $x \in \text{dom}\, f \cap \text{dom } g$, i.e $x \in \text{dom}\, g$.

Using the same way and relation (4) we can prove that dom $g \subset$ dom $f$, and the result follows.

Below, we assume that $\text{dom}\, f = \text{dom}\, g$.

**Lemma 3.2.** *Let $X$ be a Banach space and $f,\ g \in \Gamma_0(X)$. Assume that $(f_n)_{n\in IN}$ and $(g_n)_{n\in IN}$ be two decreasing sequences in $\Gamma_0(X)$ pointwise convergent to $f$ and $g$ respectively. If* $\operatorname{dom} f = \operatorname{dom} g$ *then there exists $N \in IN$ such that $f_n\tau g_n$ exists and*

$$\lim_{p\to+\infty} f_n\gamma_p g_n \ =\ f_n\tau g_n \ \text{for all}\ \ n \geq N.$$

**Proof.** By Proposition 3.1, we obtain the result if we verify that there exists $N \in IN$ such that $\operatorname{dom} f_n = \operatorname{dom} g_n$ for all $n \geq N$.

Indeed, $(f_n)_{n\in IN}$ and $(g_n)_{n\in IN}$ are two decreasing sequences, so $\operatorname{dom} f_n \subset \operatorname{dom} f$ and $\operatorname{dom} g_n \subset \operatorname{dom} g$. Knowing that $(f_n)_n$ and $(g_n)_n$ are pointwise convergent to $f$ and $g$ respectively, then there exists $N \in IN$ such that for all $n \geq N$ $\operatorname{dom} f \subset \operatorname{dom} f_n$ and $\operatorname{dom} g \subset \operatorname{dom} g_n$, and the result follows immediately.

**Theorem 3.3.** *Let $X$ be a reflexive Banach space and $f,\ g,\ (f_n)_n$ and $(g_n)_n$ as in Lemma 3.2. For all $x \in \operatorname{dom} f$, there holds*

$$\operatorname{cl}\inf_n f_n\tau g_n(x) \ =\ f\tau g(x)\,. \tag{5}$$

**Proof.** Let us show first that the sequence $(f_n\gamma_p g_n)_{n\in IN}$ converges pointwise to $f\gamma_p g$ for all $p \in IN$.

It is obvious that $f_n\gamma_0 g_n(x) = \dfrac{1}{2}f_n(x) + \dfrac{1}{2}g_n(x)$ which converges pointwise to $f\gamma_0 g(x)$.

We have $f_n\gamma_1 g_n(x) = \dfrac{1}{2}\left(f_n\gamma_0 g_n + f_n\gamma_0^* g_n\right)(x)$, with $f_n\gamma_0^* g_n = \left(f_n^*\gamma_0 g_n^*\right)^*$.

The sequences $(f_n)_{n\in IN}$ and $(g_n)_{n\in IN}$ are decreasing; knowing that the pointwise and the Mosco convergences are equivalent in the monotone case (Ref [2]), we deduce that $(f_n)_{n\in IN}$ and $(g_n)_{n\in IN}$ are Mosco convergent to $f$ and $g$ respectively. We also know that the Mosco convergence makes the Fenchel transformation bicontinuous from $\Gamma_0(X)$ to $\Gamma_0(X)$ (Ref [2]); then $\operatorname*{M-lim}_{n\to+\infty} f_n^* = f^*$ and $\operatorname*{M-lim}_{n\to+\infty} g_n^* = g^*$.

It is clear that the sequences $(f_n^*)_n$ and $(g_n^*)_n$ become increasing and so $\lim_{n\to+\infty} f_n^* = f^*$ and $\lim_{n\to+\infty} g_n^* = g^*$ which yields that $\lim_{n\to+\infty} f_n^*\gamma_0 g_n^* = f^*\gamma_0 g^*$. Remarking that the sequence $(f_n^*\gamma_0 g_n^*)_n$ remains increasing and its conjugate $(f_n^*\gamma_0 g_n^*)_n^*$ is a decreasing one, we can deduce by using the same way that $\lim_{n\to+\infty}(f_n^*\gamma_0 g_n^*)^* = (f^*\gamma_0 g^*)^*$ which implies that:

$$\lim_{n\to+\infty} f_n\gamma_1 g_n \ =\ \frac{1}{2}\left(f\gamma_0 g + f\gamma_0^* g\right) = f\gamma_1 g.$$

By induction, we show without difficulties that the sequences $(f_n\gamma_p g_n)_n$ and $(f_n^*\gamma_p g_n^*)_n^*$ are decreasing. By Theorem 2.15, ii) and using the preceding properties we deduce the following equality:

$$\lim_{n\to+\infty} f_n\gamma_p g_n = \operatorname{cl}\left(\inf_n\ f_n\gamma_p g_n\right) \ =\ f\gamma_p g \qquad \text{for all}\ p \in IN. \tag{6}$$

We now study the convergence of the sequence when depending on $p$.

We remark that the sequences $\left(f\gamma_p g\right)_p$ and $\left(f_n\gamma_p g_n\right)_p$ are also decreasing when they depend on $p$. Using Lemma 3.2, $(f_n\gamma_p g_n)_p$ will converge pointwise to $f_n\,\tau\, g_n$, so we obtain by Theorem 2.15, ii) again

$$\mathrm{cl}\left(\inf_p\ f_n\gamma_p g_n\right) = f_n\tau g_n \quad\text{and}\quad \mathrm{cl}\left(\inf_p\ f\gamma_p g\right) = f\tau g. \tag{7}$$

According to relation (6), we deduce that

$$\mathrm{cl}\inf_p \mathrm{cl}\left(\inf_n\ f_n\gamma_p g_n\right) = \mathrm{cl}\inf_p f\gamma_p g = f\tau g.$$

Since $\mathrm{cl}\inf_p \mathrm{cl}\left(\inf_n\ f_n\gamma_p g_n\right) = \mathrm{cl}\inf_p \inf_n\ f_n\gamma_p g_n = \mathrm{cl}\left(\inf_n\ f_n\tau g_n\right)$, we obtain

$$\mathrm{cl}\left(\inf_n\ f_n\tau g_n\right) = f\tau g.$$

This completes the proof of Theorem.

**Corollary 3.4.** *Let $X$ be a reflexive Banach space, $f$ and $g$ belong to $\Gamma_0(X)$. Assume that $(f_n)_n$ and $(g_n)_n$ are increasing sequences in $\Gamma_0(X)$, pointwise converging to $f$ and $g$ respectively. If* $\mathrm{dom}\, f^* = \mathrm{dom}\, g^*$, *then for all $x\in\mathrm{dom}\, f$, we have:*

$$\sup_n\ f_n\tau g_n(x) = f\tau g(x). \tag{8}$$

**Proof.** The sequences $(f_n^*)_n$ and $(g_n^*)_n$ are decreasing and converge pointwise to $f^*$ and $g^*$ respectively. Using the previous results, one has $\lim_{n\to+\infty} f_n^*\tau g_n^*(x) = f^*\tau g^*(x)$ for all $x\in\mathrm{dom}\, f^*$. $(f_n^*\tau g_n^*)_n$ is a decreasing sequence and the relation $\tau$ is self-dual in the sense of Fenchel duality, then for all $x\in\mathrm{dom}\, f$, we obtain

$$\lim_{n\to+\infty}(f_n\tau g_n)(x) = \lim_{n\to+\infty}(f_n^*\tau g_n^*)^*(x) = (f^*\tau g^*)^* = f\,\tau\, g(x),$$

and Corollary 3.4 is proved.

As a consequence of Theorem 3.3 and Corollary 3.4, we find again the following result [8]:

**Corollary 3.5.** *Assume that $X$ is an Hilbert space. Let $(A_n)_n$ and $(B_n)_n$ be two monotone sequences of symmetric positive operators from $X$ into $X$. If $(A_n)_n$ and $(B_n)_n$ converge strongly to $A$ and $B$ respectively, then $(A_n(g)B_n)_n$ converges strongly to $A(g)B$.*

**Proof.** For all $x\in X$, put that

$$f_n(x) = \frac{1}{2} < A_n x, x > \quad\text{and}\quad g_n(x) = \frac{1}{2} < B_n x, x >$$

where $< .,. >$ is the inner product of $X$.
It is clear that $(A_n)_n$ (resp. $(B_n)_n$) is monotone if and only if $(f_n)_n$ (resp. $(g_n)_n$) is also monotone. Theorem 3.3 and Corollary 3.4, applied to $(f_n)_n$ and $(g_n)_n$, give the desired result.

**Proposition 3.6.** *Let $X$ be a reflexive Banach space. $f$, $g$, $(f_n)_n$ and $(g_n)_n$ for all $n \in IN$ be defined as in Theorem 3.3 or in Corollary 3.4. Then there hold:*

**i)** $\lim_{n\to+\infty} f_n^{\left[\frac{1}{2}\right]} = f^{\left[\frac{1}{2}\right]}$.

**ii)** *Let $S$ be an isomorphism defined from $X$ into $X$ then:*

$$\lim_{n\to+\infty} (f_n \tau g_n) \circ S = (f \tau g) \circ S.$$

**iii)** *Let $(\lambda_n)_n$ and $(\alpha_n)_n$ be two sequences of positive reals and having the same monotony as $(f_n)_n$ and $(g_n)_n$. If $\lim_{n\to+\infty} \lambda_n.f_n = \lambda.f$ and $\lim_{n\to+\infty} \alpha_n.f_n = \alpha.f$ then*

$$\lim_{n\to+\infty} (\lambda_n \tau \alpha_n).f_n = (\lambda \tau \alpha).f.$$

**Proof.**

**i)** Comes from the relation $\lim_{n\to+\infty} f_n \tau \frac{1}{2}\|.\|^2 = f \tau \frac{1}{2}\|.\|^2$ .

**ii)** Use the relation $(f \tau g) \circ S = (f \circ S) \tau (g \circ S)$, for the proof, the reader is referred to [1] .

**iii)** One has $(\lambda.f) \tau (\alpha.f) = (\lambda \; \tau \; \alpha).f$, from which the desired result follows.

## 4. Stability in the positively homogenuous (p.h) case

In this paragraph, we suppose that $X = IR^m$, and we study the case when the stability does not need the property of monotonicity. This case holds for example when the geometric mean coincides with the infimal convolution.

**Proposition 4.1.** *Let $f$ and $g$ be two positively homogenuous functionals $(p.h)$ in $\Gamma_0(IR^m)$. Then $f \tau g$ exists and equals to* $\mathrm{cl}(f \square g)$. *In particular, if* $\mathrm{ri}\,\mathrm{dom}\, f^* \cap \mathrm{ri}\,\mathrm{dom}\, g^* \neq \emptyset$, *then $f \tau g = f \square g$.*

**Proof**. All $(p.h)$ functional in $\Gamma_0(IR^m)$ is a support function of a nonempty, closed and convex set (Ref [14]). We can thus assume that there exist $A$ and $B$ closed convex sets such that $g(x) = \Psi_A^*(x)$ and $f(x) = \Psi_B^*(x)$, for all $x \in X$. Then $f\gamma_0 g(x) = \Psi^*_{\mathrm{adh}(\frac{1}{2}A+\frac{1}{2}B)}(x) = \Psi^*_{(\frac{1}{2}A+\frac{1}{2}B)}(x)$ and $f\gamma_0^* g(x) = \Psi^*_{A\cap B}(x)$, if we set that $A\gamma_0 B = \mathrm{adh}(\frac{1}{2}A + \frac{1}{2}B)$ and $A\gamma_0^* B = A \cap B$.

We show by induction that

$$\begin{cases} f\gamma_n g(x) = \Psi^*_{\left\{(A\gamma_{n-1}B)\gamma_0\left((A\gamma^*_{n-1}B)\right)\right\}}(x) \\ f\gamma_n^* g(x) = \Psi^*_{\left\{(A\gamma_{n-1}B)\gamma_0^*\left((A\gamma^*_{n-1}B)\right)\right\}}(x). \end{cases}$$

Where from the preceding algorithm of functions, we deduce another algorithm of sets:

$$\begin{cases} A\gamma_n B &= \left\{(A\gamma_{n-1}B)\gamma_0\left((A\gamma^*_{n-1}B)\right)\right\} \\ A\gamma_n^* B &= \left\{(A\gamma_{n-1}B)\gamma_0^*\left((A\gamma^*_{n-1}B)\right)\right\} \\ A\gamma_0 B &= \mathrm{adh}(\frac{1}{2}A + \frac{1}{2}B) \;\text{ and }\; A\gamma_0^* B = A \cap B. \end{cases}$$

$(f\gamma_n^* g)_n$ is an increasing sequence, thus $A\cap B = A\gamma_0^* B \subset A\gamma_n^* B$.

$A\gamma_1^* B = \text{adh}(\frac{1}{2}A + \frac{1}{2}B) \cap A\cap B \subset A\cap B$, we show by induction that $A\gamma_n^* B \subset A\cap B$ for all $n \in I\!N$, so

$$A\gamma_n^* B = A\cap B \quad \text{for all } \; n \in I\!N.$$

Substituting $A\gamma_n^* B$ by $A\cap B$ in the algorithm of sets we obtain:

$$A\gamma_n B \;=\; \text{adh}\left(\frac{1}{2^{n+1}}\,(A+B) + \left(1 - \frac{1}{2^{n+1}}\right) A\cap B\right)$$

The sequence $(A\gamma_n B)_{n\in IN}$ converges to $A\cap B$. Thus
$\lim_{n\to\infty} A\gamma_n B = \lim_{n\to\infty} A\gamma_n^* B = A\cap B$, this implies that $\Psi_{A\gamma_n B}$ converges pointwise to $\Psi_{A\cap B}$.
The sequences $(A\gamma_n B)_n$ and $(A\gamma_n^* B)_n$ are decreasing, so by using again the equivalence between the pointwise convergence and the Mosco convergence in this case, we deduce that
$\lim_{n\to\infty} \Psi^*_{(A\gamma_n B)}(x) = \lim_{n\to\infty} f\gamma_n g\,(x) = \lim_{n\to\infty} \Psi^*_{(A\gamma_n^* B)}(x) = \lim_{n\to\infty} f\gamma_n^* g\,(x) = \Psi^*_{A\cap B}(x)$, for all $x \in X$.

The uniqueness of the limit of those sequences permits us to deduce that $f\,\tau\, g\,(x) = \Psi^*_{A\cap B}(x) = (\Psi_A + \Psi_B)^*(x) = (f^* + g^*)^*(x) = \text{cl}\,(f\Box g)\,(x)\,.$

In particular if ri dom $f^* \cap$ ri dom $g^* \neq \emptyset$ then $f\,\tau\, g\,(x) = f\Box g\,(x)$. This concludes the proof.

**Corollary 4.2.** *Let $f$, $g$, $(f_n)_n$ and $(g_n)_n$ be convex (p.h) functionals defined from $I\!R^m$ to $I\!R$. If $(f_n)_n$ and $(g_n)_n$ are pointwise converging to $f$ and $g$ respectively then*

$$\forall x \in X \qquad \lim_{n\to+\infty} f_n \tau g_n\,(x) \;=\; f\tau g\,(x)\,.$$

**Proof.** As in the preceding proposition, we set $\forall x \in X$, $g(x) = \Psi_A^*(x)$, $f(x) = \Psi_B^*(x)$, $g_n(x) = \Psi^*_{A_n}(x)$ and $f_n(x) = \Psi^*_{B_n}(x)$.
Since $f$, $(f_n)_n$, $g$ and $(g_n)_n \in \Gamma_0(I\!R^m)$ are finite elsewhere, thus we can assume that $A$, $B$, $A_n$ and $B_n$ are compact sets for all $n \in I\!N$ (Ref [11]). So, we can deduce the following equivalence

$$\lim_{n\to+\infty} \Psi^*_{A_n}(x) \;=\; \Psi^*_A(x) \Longleftrightarrow \;\; \text{M-}\lim_{n\to+\infty} \; \Psi^*_{A_n}(x) \;=\; \Psi^*_A(x)\,. \tag{9}$$

And a similar equivalence for $B_n$ and $B$ holds. $f_n\tau g_n\,(x) = \Psi^*_{A_n\cap B_n}(x) = (\Psi_{A_n} + \Psi_{B_n})^*(x)\,.$

If $\lim_{n\to+\infty} \Psi^*_{A_n}(x) = \Psi^*_A(x)$ and $\lim_{n\to+\infty} \Psi^*_{B_n}(x) = \Psi^*_B(x)$ then
$\text{M-}\lim_{n\to+\infty} \; \Psi_{A_n}(x) = \Psi_A(x)$ and $\text{M-}\lim_{n\to+\infty} \; \Psi_{B_n}(x) = \Psi_B(x)\,.$
Using the stability theorem for the Mosco convergence (Ref [5]), one deduces that

$$\text{M-}\lim_{n\to+\infty} \; (\Psi_{A_n} + \Psi_{B_n})\,(x) \;=\; (\Psi_A + \Psi_B)\,(x)\,. \tag{10}$$

Since $A_n \cap B_n$, for all $n \in I\!N$, and $A\cap B$ are compact sets; the relation (10) will be equivalent to

$$\lim_{n\to+\infty} (\Psi_{A_n} + \Psi_{B_n})^*(x) \;=\; (\Psi_A + \Psi_B)^*(x)\,.$$

The desired result follows.

**Remark 4.3.** The stability for geometric mean is not verified in general case. If we take the example where the sequences have not the same monotony; the result is not satisfied. Indeed; let $X = IR$, $f_n(x) = \dfrac{n}{n+1}\left(\dfrac{x^2}{2} - 1\right)$ and $g_n(x) = \dfrac{1}{n}\left(\dfrac{x^2}{2} - 1\right)$; $(f_n)_n$ is an increasing sequence and $(g_n)_n$ is a decreasing one.

$f_n(x)$ converges to $f(x) = \dfrac{x^2}{2} - 1$ and $g_n(x)$ converges to $g(x) = 0$.

$f_n \,\tau\, g_n(x) = \dfrac{1}{\sqrt{n+1}}\left(\dfrac{x^2}{2} - 1\right)$ which converges to 0, but

$f \,\tau\, g(x) = -\dfrac{1}{2} f^*(0) = -\dfrac{1}{2}$ (see [1]).

## 5. Stability for the $(h)$, $(a)$, $(a.g)$, $(a.h)$ and $(g.h)$ means

**Proposition 5.1.** *Let $X$ be a reflexive Banach space. $f$, $g$, $(f_p)_p$ and $(g_p)_p$ for all $p \in IN$ be given functions satisfying the assumption of Theorem 3.3. If we denote by $\pi$ one of these functional means $(h)$, $(a)$, $(a.g)$, $(a.h)$ and $(g.h)$ then the sequence $(f_p \pi g_p)_p$ converges pointwise to $f \pi g$.*

**Proof.**

- If $\pi = (h)$ and $\pi = (a)$ the proof is immediate. If $\pi = (a.h)$, $f \tau g$ coincides with $f \pi g$.
- If $\pi = (a.g)$ we get

$$\left\{\begin{array}{lcl} f_{n+1,p} & = & \frac{1}{2}(f_{n,p} + g_{n,p}) \\ g_{n+1,p} & = & f_{n,p} \,\tau\, g_{n,p} \end{array}\right. \quad \text{where} \quad \left\{\begin{array}{l} f_{0,p} = f_p \\ g_{0,p} = g_p. \end{array}\right.$$

$$\left\{\begin{array}{lcl} f_{1,p} & = & \frac{1}{2}(f_p + g_p) \\ g_{1,p} & = & f_p \,\tau\, g_p \end{array}\right.$$

$(f_{1,p})_p$ converges pointwise to $f_1 = \dfrac{1}{2}(f + g)$, and using Theorem 3.3 we obtain $g_{1,p} = f_p \,\tau\, g_p$ which converges pointwise to $f \tau g = g_1$.

The sequences $(f_{n,p})_{p \in IN}$ and $(g_{n,p})_{p \in IN}$ are decreasing for all $n \in IN$. By induction, it follows that $\lim_{p \to \infty} f_{n,p} = f_n$ and $\lim_{p \to \infty} g_{n,p} = g_n$.

We deduce that

$$\operatorname{cl} \inf_n \inf_p g_{n,p} = \operatorname{cl} \inf_n g_n = f \pi g = \lim_{n \to \infty} f_n.$$

- If $\pi = (g.h)$ we set,

$$\left\{\begin{array}{lcl} f_{n+1,p} & = & f_{n,p} \,\tau\, g_{n,p} \\ g_{n+1,p} & = & \left(\frac{1}{2} f_{n,p}^* + \frac{1}{2} g_{n,p}^*\right)^* \end{array}\right. \quad \text{where} \quad \left\{\begin{array}{l} f_0 = f_p \\ g_0 = g_p \end{array}\right.$$

For $n = 1$ we get:

$$\begin{cases} f_{1,p} &= f_p \,\tau\, g_p \\ g_{1,p} &= \left(\dfrac{1}{2} f_p^* + \dfrac{1}{2} g_p^*\right)^* . \end{cases}$$

According to Theorem 3.3, one deduces that $(f_{1,p})_p$ converges pointwise to $f_1 = f \,\tau\, g$ and using the property of equivalence between the pointwise and Mosco convergences in the monotone case, we show that $(g_{1,p})_p$ converges pointwise to $\left(\dfrac{1}{2} f^* + \dfrac{1}{2} g^*\right)^*$.

As in the previous case, we prove by induction that the sequences $(f_{n,p})_{p\in IN}$ and $(g_{n,p})_{p\in IN}$ are decreasing for all $n \in IN$ and converge pointwise to $f_n$ and $g_n$ respectively. We deduce the desired result.

One can obtain the same result if $f$, $g$, $(f_p)_{p\in IN}$, and $(g_p)_{p\in IN}$ verify the hypothesis of Corollary 3.4.

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