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An Existence Theorem for the Neumann Problem Involving the p-Laplacian

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In the present paper we deal with the existence of weak solutions for the following Neumann problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \lambda(x)|u|^{p-2}u = f(x,u) + h(x) & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded open set with boundary $\partial\Omega$ of class C^1, ν denotes the outward unit normal to $\partial\Omega$, $p > N$, $f \in C^0(\Omega \times \mathbf{R}, \mathbf{R})$ is such that, for every $r > 0$

$$\sup_{|t|\le r} |f(\cdot,t)| \in L^1(\Omega),$$

$\lambda \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_\Omega \lambda > 0$ and $h \in L^1(\Omega)$. To prove the existence of solutions for the above problem, we use the variational methods. More precisely, we make use of the critical point theorem obtained by B. Ricceri as a consequence of a more general variational principle.

Keywords: Variational principle, critical point, weak solution, Neumann problem, p-Laplacian

1. Introduction

In the present paper we study the following Neumann problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \lambda(x)|u|^{p-2}u = f(x,u) + h(x) & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded open set with boundary $\partial\Omega$ of class C^1, ν denotes the outward unit normal to $\partial\Omega$, $p > N$, $f \in C^0(\Omega \times \mathbf{R}, \mathbf{R})$ is such that, for every $r > 0$

$$\sup_{|t|\le r} |f(\cdot,t)| \in L^1(\Omega), \tag{2}$$

$\lambda \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_\Omega \lambda > 0$ and $h \in L^1(\Omega)$.

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As usual, a weak solution of problem (1) is any $u \in W^{1,p}(\Omega)$ such that

$$\int_\Omega (|\nabla u(x)|^{p-2} \nabla u(x) \nabla v(x) + \lambda(x)|u(x)|^{p-2} u(x) v(x)) dx -$$

$$\int_\Omega (f(x, u(x)) + h(x)) v(x) dx = 0,$$

for all $v \in W^{1,p}(\Omega)$.

To prove the existence of solutions for the above problem, we use the variational methods. More precisely, we make use of the following critical point theorem obtained by B. Ricceri as a consequence of a more general variational principle:

Theorem 1.1 ([5], Theorem 2.5). *Let X be a reflexive real Banach space, and let $\Phi, \Psi : X \to \mathbf{R}$ be two sequentially weakly lower semicontinuous and Gâteaux differentiable functionals.*

Assume also that Ψ is strongly continuous and satisfies

$$\lim_{\|x\| \to +\infty} \Psi(x) = +\infty.$$

For each $\rho > \inf_X \Psi$, put

$$\varphi(\rho) = \inf_{x \in \Psi^{-1}(]-\infty, \rho[)} \frac{\Phi(x) - \inf_{\overline{(\Psi^{-1}(]-\infty,\rho[))}_w} \Phi}{\rho - \Psi(x)},$$

where $\overline{(\Psi^{-1}(] - \infty, \rho[))}_w$ is the closure of $\Psi^{-1}(] - \infty, \rho[)$ in the weak topology.

Then for each $\rho > \inf_X \Psi$ and each $\lambda > \varphi(\rho)$, the functional $\Phi + \lambda\Psi$ has a critical point which lies in $\Psi^{-1}(] - \infty, \rho[)$.

In our settings, the functionals Φ and Ψ are defined as follows:

For every $u \in W^{1,p}(\Omega)$, we put

$$\|u\|_\lambda = \left(\int_\Omega \lambda(x)|u|^p dx + \int_\Omega |\nabla u|^p dx \right)^{\frac{1}{p}},$$

$$\Psi(u) = \frac{1}{p}\|u\|_\lambda^p - \int_\Omega h(x) u(x) dx$$

and

$$\Phi(u) = -\int_\Omega \left(\int_0^{u(x)} f(x, t) dt \right) dx.$$

A way of applying Theorem 1.1, to obtain existence results for problem (1), is to find conditions on the given functionals, which guarantee that

$$\inf_{u \in \overline{\Psi^{-1}(]-\infty, r[)}_w} \Phi(u) = \Phi(u_0) \tag{3}$$

holds, for some positive real number r and $u_0 \in \Psi^{-1}(]-\infty, r[)$. This method has been already used in [1] and (for multiplicity results) in [6], where problem (1) was studied with the right-hand side of the type $\alpha(x)f(t) + \beta(x)g(t)$. In that case, we showed that, under suitable hypotheses on the functions f and g, there exists a constant function u_0 satisfying (3) for some $r > 0$.

When the dependence of f on the variables x and t is arbitrary, the function u_0 may depend on x. Here we consider this more general case. Our main result is Theorem 3.1 below. To prove it, we follow the method just described. In the present setting, the proof of the existence of an u_0 as above requires rather delicate arguments (see Lemma 2.2).

In a recent paper [8], S. Villegas studied problem (1) using a different approach. We also intend to make a comparison between our conditions and the Villegas's ones when $p = 2$ and $N = 1$. To this aim, we quote here the Villegas's main result:

Theorem 1.2 ([8], Theorem 1). *Let us suppose that the functions $h \in L^2(]0,1[)$ and $l \in C^0([0,1] \times \mathbf{R}, \mathbf{R})$ satisfy the following conditions:*

There exist $\epsilon > 0$, $s' < 0$ and $s_0 > 0$ such that

$$\frac{l\,(x,s)}{s} \leq \frac{\pi^2}{4} - \epsilon \quad \text{for each } s \leq s',\ x \in [0,1]. \qquad (f_1)$$

$$\begin{cases} l\,(x,s) + \epsilon \leq -\int_0^1 h(x)\,dx & \text{for each } s \leq s',\ x \in [0,1]. \\ -\int_0^1 h(x)\,dx \leq l\,(x,s) - \epsilon & \text{for each } s \geq s_0,\ x \in [0,1]. \end{cases} \qquad (f_2)$$

Then the Neumann problem

$$\begin{cases} -u'' = l\,(x,u) + h(x) & \text{in }]0,1[\\ u'(0) = u'(1) = 0 \end{cases}$$

has a weak solution in $W^{1,2}(]0,1[)$.

At first, we note that, in our settings, the function l is defined as follows

$$l(x,t) = f(x,t) - \lambda(x)t.$$

Our hypotheses are completely different with respect to those of Theorem 1.2. In particular, we do not require any condition about the behaviour of $l(x,t)$ for $t \to \infty$. Our assumptions are only local conditions, in a neighborhood of zero, for the function $f(x,\cdot)$ uniformly with respect to x. The next two propositions (proved in the third section) are typical examples of application of Theorem 3.1, which cannot be obtained either by Theorem 1.2 or by Theorem 2.1 of [1].

Proposition 1.3. *Let $f \in C^1([0,1] \times \mathbf{R})$, assume that there exist $a, \eta \in \mathbf{R}$, with $a > 2\eta > 0$, satisfying the following conditions:*

(a) *For every $x \in [0,1]$, one has*

$$f(x,-\eta) > 0,\ f(x,\eta) < 0.$$

(b) *One has*

$$\sup_{(x,t)\in[0,1]\times[-a,a]} f_t(x,t) < 0,$$

$$\sup_{(x,t)\in[0,1]\times[-\eta,\eta]}\left|\frac{f_x(x,t)}{f_t(x,t)}\right|\le\eta.$$

Then, for every $h\in L^1(]0,1[)$ *with* $\int_0^1|h(x)|dx<\frac{a^2-4\eta^2}{4(\eta+a)}$*, the following Neumann problem*

$$\begin{cases} -u''+u=f(x,u)+h(x) & \text{in }]0,1[\\ u'(0)=u'(1)=0 \end{cases}$$

has at least a weak solution $\overline{u}\in W^{1,2}(]0,1[)$ *such that* $\|\overline{u}\|<\frac{a}{\sqrt{2}}$.

Proposition 1.4. *For any* $h\in L^1(]0,1[)$ *such that*

$$\int_0^1|h(x)|dx\le\frac{1}{9},$$

the following Neumann problem

$$\begin{cases} -u''+2u=-\mathrm{e}^{3u-xu^3}+h(x) & \text{in }]0,1[\\ u'(0)=u'(1)=0, \end{cases}$$

has at least a weak solution $\overline{u}\in W^{1,2}(]0,1[)$ *such that* $\|\overline{u}\|<\frac{\sqrt{2}}{2}$.

Note that for a function $u\in W^{1,p}(\Omega)$, the symbol $\|u\|$ means $\|u\|_\lambda$ with $\lambda(x)=1$.

The paper is divided in three sections. The second section contains some preliminary lemmas to our result that is stated and proved in the third section. The appendix deals with the estimate of the constant $c(\lambda)$ defined in section two.

2. Basic definitions and preliminary results

We observe that $\|\cdot\|_\lambda$ is a norm on $W^{1,p}(\Omega)$ equivalent to the usual ones, we consider $W^{1,p}(\Omega)$ endowed with the norm $\|\cdot\|_\lambda$. Since $W^{1,p}(\Omega)$ is compactly embedded into $C^0(\overline{\Omega})$, the functional Ψ is well-defined, sequentially weakly lower semicontinuous, Gâteaux differentiable and such that $\lim_{\|u\|_\lambda\to+\infty}\Psi(u)=+\infty$.

Because of the coercivity of Ψ, for $r>0$ the number

$$k(r,h)=\inf\{\sigma>0:\Psi^{-1}(]-\infty,r[)\subseteq\{u\in W^{1,p}(\Omega):\|u\|_\lambda\le\sigma\}\},$$

is finite. Then we put

$$c(\lambda)=\sup_{u\in W^{1,p}(\Omega)\setminus\{0\}}\frac{\sup_{x\in\Omega}|u(x)|}{\|u\|_\lambda},$$

that is finite since $W^{1,p}(\Omega)$ is compactly embedded into $C^0(\overline{\Omega})$. In the sequel, $m(\Omega)$ denotes the Lebesgue-measure of Ω and we define, for every $(x,t)\in\Omega\times\mathbf{R}$ and $v\in\mathbf{R}^N$ with $|v|=1$,

$$\Delta_t f(x,t;k)=\frac{f(x,t+k)-f(x,t)}{k}$$

and

$$\Delta_v f(x,t;h)=\frac{f(x+hv,t)-f(x,t)}{h}.$$

The following Lemma follows from Theorem 1.1 of [7].

Lemma 2.1. *Let $I \subseteq \mathbf{R}$ be an open interval, $f : \Omega \times I \to \mathbf{R}$ be a continuous function such that, for each $x \in \Omega$ there exists a unique $u(x) \in I$ satisfying*

$$f(x, u(x)) = 0,$$

and $f(x, \cdot)$ changes sign in the interval I. Then the function $u : \Omega \to I$ is continuous.

Lemma 2.2. *Let I and $f : \Omega \times I \to \mathbf{R}$ be as in Lemma 2.1. We assume that there exists $L > 0$ such that, one has*

$$\liminf_{k\to 0} |\Delta_t f(x,t;k)| > \frac{1}{L} \limsup_{(h,k)\to(0,0)} |\Delta_v f(x,t+k;h)|, \tag{4}$$

for each $(x,t) \in \Omega \times I$ and $v \in \mathbf{R}^N$ with $|v| = 1$.

Then the function $u : \Omega \to I$, satisfying $f(x, u(x)) = 0$ for every $x \in \Omega$, belongs to $W^{1,p}(\Omega)$ and $\|\nabla u\|_{L^p(\Omega)} \leq L\,(m(\Omega))^{\frac{1}{p}}$.

Proof. Condition (4) guarantees that for any $(x,t) \in \Omega \times I$ and $v \in \mathbf{R}^N$ with $|v| = 1$,

$$\limsup_{(h,k)\to(0,0)} |\Delta_v f(x,t+k;h)| < +\infty, \tag{5}$$

and

$$\liminf_{k\to 0} |\Delta_t f(x,t;k)| > 0.$$

Fix $(x,t) \in \Omega \times I$ and $v \in \mathbf{R}^N$ with $|v| = 1$, if

$$\liminf_{k\to 0} |\Delta_t f(x,t;k)| = +\infty,$$

then, taking into account (5), it turns out that

$$\limsup_{(h,k)\to(0,0)} \left| \frac{\Delta_v f(x,t+k;h)}{\Delta_t f(x,t;k)} \right| = 0.$$

In the other case we have,

$$\begin{aligned}
& \limsup_{(h,k)\to(0,0)} \left| \frac{\Delta_v f(x,t+k;h)}{\Delta_t f(x,t;k)} \right| \\
\leq\ & \limsup_{(h,k)\to(0,0)} |\Delta_v f(x,t+k;h)| \limsup_{k\to 0} \frac{1}{|\Delta_t f(x,t;k)|} \\
=\ & \frac{\limsup_{(h,k)\to(0,0)} |\Delta_v f(x,t+k;h)|}{\liminf_{k\to 0} |\Delta_t f(x,t;k)|} < L.
\end{aligned}$$

Therefore, for any $(x,t) \in \Omega \times I$ and $v \in \mathbf{R}^N$ with $|v| = 1$, one has

$$a(x,t;v) = \limsup_{(h,k)\to(0,0)} \left| \frac{\Delta_v f(x,t+k;h)}{\Delta_t f(x,t;k)} \right| < L.$$

Now we prove that, for any $x \in \Omega$ and $v \in \mathbf{R}^N$ with $|v| = 1$, there exists $h(x, v) > 0$ such that for every $h \in [-h(x, v), h(x, v)]$ one has

$$|u(x + hv) - u(x)| \leq L|h|. \tag{6}$$

Fix x and v as above, we put

$$A = \{h \in \mathbf{R} : x + hv \in \Omega \text{ and } u(x + hv) = u(x)\}.$$

If A includes a neighborhood of zero then (6) obviously follows. Then we suppose that A does not include a neighborhood of zero. By Lemma 2.1, u is continuous in Ω and so

$$\lim_{h \to 0, h \in \mathbf{R} \setminus A} u(x + hv) = u(x).$$

Whence

$$\begin{aligned}
&\limsup_{h \to 0, h \in \mathbf{R} \setminus A} \left| \frac{u(x + hv) - u(x)}{f(x, u(x + hv)) - f(x, u(x))} \right| \\
&\qquad \times \left| \frac{f(x + hv, u(x + hv)) - f(x, u(x + hv))}{h} \right| \\
&\leq \limsup_{(h,k) \to (0,0)} \left| \frac{k}{f(x, u(x) + k) - f(x, u(x))} \right| \\
&\qquad \times \left| \frac{f(x + hv, u(x) + k) - f(x, u(x) + k)}{h} \right| = a(x, u(x); v).
\end{aligned}$$

Since $a(x, u(x); v) < L$, if we put $\eta = L - a(x, u(x); v)$ there exists $h(x, v) > 0$ such that for every $h \in [-h(x, v), h(x, v)] \setminus A$ one has

$$\begin{aligned}
\left| \frac{u(x + hv) - u(x)}{h} \right| &\leq \limsup_{h \to 0, h \in \mathbf{R} \setminus A} \left| \frac{u(x + hv) - u(x)}{h} \right| + \eta \\
&= \limsup_{h \to 0, h \in \mathbf{R} \setminus A} \left| \frac{u(x + hv) - u(x)}{f(x, u(x + hv)) - f(x, u(x))} \right| \\
&\quad \times \left| \frac{f(x, u(x + hv)) - f(x, u(x))}{h} \right| + \eta \\
&= \limsup_{h \to 0, h \in \mathbf{R} \setminus A} \left| \frac{u(x + hv) - u(x)}{f(x, u(x + hv)) - f(x, u(x))} \right| \\
&\quad \times \left| \frac{f(x + hv, u(x + hv)) - f(x, u(x + hv))}{h} \right| + \eta \\
&\leq a(x, u(x); v) + \eta = L.
\end{aligned}$$

Consequently, for any $h \in [-h(x, v), h(x, v)]$ one has

$$|u(x + hv) - u(x)| \leq L|h|,$$

that is (6).

Let $x, y \in \Omega$ be such that $x \neq y$ and the segment $[x, y] \subseteq \Omega$. We put

$$v = \frac{y - x}{|y - x|},$$

and

$$B = \{t \in [0,1] : |u(x + t|y - x|v) - u(x)| \leq Lt|y - x|\}.$$

The set B is not empty because $0 \in B$ and it is closed by virtue of continuity of u, then B is compact. Let $\overline{t} = \max B$, we show that $\overline{t} = 1$. If it was $\overline{t} < 1$, by (6), we could find $\delta > 0$ such that, for any $t \in [\overline{t} - \delta, \overline{t} + \delta] \subseteq [0,1]$ one has

$$|u(x + t|y - x|v) - u(x + \overline{t}|y - x|v)| \leq L|t - \overline{t}||y - x|,$$

in particular, the previous inequality holds for $t = \overline{t} + \delta$. Then, we have

$$\begin{aligned} & |u(x + (\overline{t} + \delta)|y - x|v) - u(x)| \\ \leq \; & |u(x + (\overline{t} + \delta)|y - x|v) - u(x + \overline{t}|y - x|v)| + |u(x + \overline{t}|y - x|v) - u(x)| \\ \leq \; & L(\overline{t} + \delta)|y - x|, \end{aligned}$$

whence the absurd $\overline{t} + \delta \in B$. Consequently it turns out that, for every $x, y \in \Omega$ with $[x, y] \subseteq \Omega$, one has $|u(y) - u(x)| \leq L|y - x|$. The thesis follows by Lemma 7.24 of [2]. □

3. Main results

Theorem 3.1. *Let $r, s \in \mathbf{R}_+$ be such that*

$$L = \frac{p\left(r - s\int_\Omega |h(x)|dx\right) - s^p \int_\Omega \lambda(x)dx}{m(\Omega)} > 0.$$

Moreover, assume that the function f satisfies the following conditions:

(i) For every $x \in \Omega$, one has

$$\begin{cases} f(x, \cdot) = 0 \text{ has a unique solution in }]-c(\lambda)k(r,h), c(\lambda)k(r,h)[\\ f(x, -s) > 0, \;\; f(x, s) < 0. \end{cases}$$

(ii) For every $(x, t) \in \Omega \times]-s, s[$ and $v \in \mathbf{R}^N$ with $|v| = 1$, one has

$$\liminf_{k \to 0} |\Delta_t f(x, t; k)| > L^{-\frac{1}{p}} \limsup_{(h,k)\to(0,0)} |\Delta_v f(x, t + k; h)|.$$

Then problem (1) has at least a weak solution $\overline{u} \in W^{1,p}(\Omega)$ such that $\Psi(\overline{u}) < r$ and hence $\|\overline{u}\|_\lambda < k(r, h)$.

Proof. At first, we observe that $s \leq c(\lambda)k(r, h)$, since by hypothesis

$$p\left(r - s\int_\Omega |h(x)|dx\right) - s^p \int_\Omega \lambda(x)dx > 0,$$

taking into account Proposition 4.1 and the definition of the number $k(r,h)$. Condition (i) implies the existence of the function $u_0 : \Omega \to]-s, s[$ such that, for every $x \in \Omega$ $f(x, u_0(x)) = 0$. Then condition (ii) assures, by virtue of Lemma 2.2, that $u_0 \in W^{1,p}(\Omega)$ and $\|\nabla u_0\|_{L^p(\Omega)} \le (Lm(\Omega))^{\frac{1}{p}}$. Moreover, for every $x \in \Omega$, one has

$$\int_0^{u_0(x)} f(x,t)dt = \sup_{\|\xi\| \le c(\lambda)k(r,h)} \int_0^{\xi} f(x,t)dt. \tag{7}$$

We define the functional $\Phi : W^{1,p}(\Omega) \to \mathbf{R}$, setting for every $u \in W^{1,p}(\Omega)$,

$$\Phi(u) = -\int_\Omega \left(\int_0^{u(x)} f(x,t)dt \right) dx,$$

that is well-defined, sequentially weakly continuous and Gâteaux differentiable, owing to the compact embedding of $W^{1,p}(\Omega)$ into $C^0(\overline{\Omega})$ and the assumption (2). The critical points of the functional $\Psi(\cdot) + \Phi(\cdot)$ are the weak solutions of problem (1). Our end is to apply Theorem 1.1. Then, to complete the proof, it is enough to prove that

$$\varphi(r) = \inf_{u \in \Psi^{-1}(]-\infty,r[)} \frac{\Phi(u) - \inf_{\overline{\Psi^{-1}(]-\infty,r[)}_w} \Phi}{r - \Psi(u)} = 0,$$

where $\overline{\Psi^{-1}(]-\infty,r[)}_w$ denotes the weak closure of the set $\Psi^{-1}(]-\infty,r[)$. By the definition of the number $k(r,h)$, one has

$$\varphi(r) \le \inf_{u \in \Psi^{-1}(]-\infty,r[)} \frac{\Phi(u) - \inf_{\|v\|_\lambda \le k(r,h)} \Phi(v)}{r - \Psi(u)}.$$

Furthermore, for every $v \in W^{1,p}(\Omega)$ with $\|v\|_\lambda \le k(r,h)$, we have

$$\sup_{x \in \Omega} |v(x)| \le c(\lambda)k(r,h).$$

Whence from (7) it follows that

$$\Phi(u_0) \le \Phi(v)$$

and so $\Phi(u_0) = \inf_{\|v\|_\lambda \le k(r,h)} \Phi(v)$. Then, we also have

$$\begin{aligned} \Psi(u_0) &= \frac{1}{p}\|u_0\|_\lambda^p - \int_\Omega h(x)u_0(x)dx \\ &< \frac{1}{p}\left(s^p \int_\Omega \lambda(x)dx + L\, m(\Omega) \right) + s\int_\Omega |h(x)|dx = r. \end{aligned}$$

Consequently $u_0 \in \Psi^{-1}(]-\infty,r[)$, thus it follows that

$$\inf_{u \in \Psi^{-1}(]-\infty,r[)} \frac{\Phi(u) - \inf_{\|v\|_\lambda \le k(r,h)} \Phi(v)}{r - \Psi(u)} = 0,$$

and the thesis is proved. □

Corollary 3.2. *Let $r, s, L \in \mathbf{R}_+$ be as in Theorem 3.1 and assume that the function f satisfies (ii) and the following condition:*

(i') *For each $x \in \Omega$, one has*

$$\begin{cases} f(x,\cdot) \text{ is nonincreasing in }]-c(\lambda)k(r,h), c(\lambda)k(r,h)[\\ f(x,-s) > 0, \ f(x,s) < 0. \end{cases}$$

Then problem (1) *has at least a weak solution $\overline{u} \in W^{1,p}(\Omega)$ such that $\Psi(\overline{u}) < r$, hence $\|\overline{u}\|_\lambda < k(r,h)$.*

Proof. Conditions (i') and (ii) imply that for every $x \in \Omega$ the function $f(x,\cdot)$ is strictly decreasing in $[-s,s]$. Hence there exists a unique

$$u(x) \in]-c(\lambda)k(r,h), c(\lambda)k(r,h)[$$

such that $f(x,u(x)) = 0$. Consequently the thesis follows by Theorem 3.1. □

Corollary 3.3. *Let $r, s, L \in \mathbf{R}_+$ be as in Theorem 3.1 and $f \in C^1(\overline{\Omega}\times\mathbf{R}, \mathbf{R})$. Moreover assume that f satisfies the following conditions:*

(i") *For each $x \in \overline{\Omega}$ one has*

$$\begin{cases} \frac{\partial f}{\partial t}(x,\cdot) < 0 \text{ in }]-c(\lambda)k(r,h), c(\lambda)k(r,h)[\\ f(x,-s) > 0, \ f(x,s) < 0. \end{cases}$$

(ii") *One has*

$$\max_{(x,t)\in\overline{\Omega}\times[-s,s]} \left| \frac{\nabla_x f(x,t)}{\partial_t f(x,t)} \right| < L^{\frac{1}{p}}.$$

Then problem (1) *has at least a weak solution $\overline{u} \in W^{1,p}(\Omega)$ such that $\Psi(\overline{u}) < r$, hence $\|\overline{u}\|_\lambda < k(r,h)$.*

Proof. Condition (i") implies (i') of the previous Corollary. So it is enough to show that condition (ii") implies (ii) of Theorem 3.1. By (ii"), for every $v \in R^n$ with $|v| = 1$, one has

$$\left|\frac{\partial f}{\partial v}(x,t)\right| < L^{1/p} \left|\frac{\partial f}{\partial t}(x,t)\right|. \qquad ((x,t) \in \overline{\Omega}\times[-s,s])$$

Since $\partial f/\partial v$ is continuous, by Lagrange's Theorem, it follows that

$$\lim_{(h,k)\to(0,0)} \Delta_v f(x,t+k;h) = \frac{\partial f}{\partial v}(x,t),$$

for every $(x,t) \in \overline{\Omega}\times\mathbf{R}$. And so (ii) is satisfied. □

Remark 3.4. It is difficult, in general, to calculate exactly the constants $c(\lambda)$ and $k(r,h)$. However, to apply Theorem 3.1 and Corollary 3.1, it is enough to verify conditions (i) (resp. (i') in Corollary 3.2) and (ii), using some explicit upper bounds of those constants.

Proof of Proposition 1.3. Fix $\varepsilon > 0$ and $h \in L^1(]0,1[)$ with

$$\int_0^1 |h(x)|dx < \frac{a^2 - 4\eta^2}{4(\eta + a)}.$$

We set $s = \eta,\ r_\varepsilon = \eta^2 + \sigma\eta + \varepsilon$ where $\sigma = \int_0^1 |h(x)|dx$. It is easily seen that, under our assumptions, it results

$$c(\lambda)k(r_{\overline{\varepsilon}}, h) \leq a$$

for some $\overline{\varepsilon} > 0$, since $c(\lambda) \leq \sqrt{2}$ as it follows by Proposition 4.2 taking into account that $\lambda \equiv 1$. Then, condition (a) and the first of (b) imply that (i') of Corollary 3.2 holds. Moreover, we have

$$\sqrt{L} = \sqrt{\eta^2 + \overline{\varepsilon}} > \eta,$$

whence condition (ii) is also satisfied. Thus the conclusion follows by Corollary 3.2. □

Example 3.5. Here is an example relative to Proposition 1.3. Let us consider the following Neumann problem

$$\begin{cases} -u'' + 11u = |u|u - 5\sin(x+u) + h(x) & \text{in }]0,1[\\ u'(0) = u'(1) = 0. \end{cases}$$

with $h \in L^1(]0,1[)$. Taking $a = \frac{25}{12}$, $\eta = \frac{5}{6}$ and

$$f(x,t) = t|t| - 10t - 5\sin(x+t),$$

the hypotheses of Proposition 1.3 are satisfied. Then, for every $h \in L^1(]0,1[)$ with $\int_0^1 |h(x)|dx \leq 1$, the problem has, at least, a weak solution with norm less than $\frac{25}{12\sqrt{2}}$.

Proof of Proposition 1.4. We put, for every $x \in [0,1], t \in \mathbf{R}$

$$f(x,t) = -\mathrm{e}^{3t - xt^3} - t,$$

$$\lambda(x) = 1.$$

By Proposition 4.2, it follows that $c(\lambda) \leq \sqrt{2}$.

Let $h \in L^1(]0,1[)$ with $\int_0^1 |h(x)|dx \leq \frac{1}{9}$, we prove that both conditions (i') and (ii) of Corollary 3.2 are satisfied. Then the thesis follows by it.

Choose $s = \frac{2}{5}$, it is easily seen that

$$\begin{array}{ll} f_t(x,t) < 0 & \text{for any } (x,t) \in [0,1] \times]-1,1[\\ f(x,-s) > 0,\ f(x,s) < 0 & \text{for any } x \in [0,1]. \end{array}$$

Moreover, one has

$$\sup_{(x,t)\in[0,1]\times[-s,s]} \left| \frac{f_x(x,t)}{f_t(x,t)} \right| < \frac{13}{100}.$$

Then, at this point, it is enough to prove that there exists $r > 0$ such that

$$\left[2\left(r - \frac{2}{5}\int_0^1 |h(x)|dx \right) - \frac{4}{25} \right]^{\frac{1}{2}} = \frac{13}{100}, \tag{8}$$

and

$$c(\lambda)k(r,h) \leq 1. \tag{9}$$

We set

$$r = \frac{1}{2}\left(\frac{13}{100}\right)^2 + \frac{2}{25} + \frac{2}{5}\int_0^1 |h(x)|dx.$$

The inequality (8) is obviously satisfied with this choice. We also have

$$c(\lambda)k(r,h) \leq 2\left(\int_0^1 |h(x)|dx + \sqrt{r + \left(\int_0^1 |h(x)|dx\right)^2}\right).$$

Then, taking into account the choice of r and the assumption on the function h, it is easily seen that the condition (9) is satisfied. Then the conclusion is proved. □

4. Appendix

In order to apply Theorem 3.1, it is essential to know, at least, an upper bound of $c(\lambda)$. Here we deal with this problem. But first we prove

Proposition 4.1. *We have that*

$$c(\lambda) > \frac{1}{\left(\int_\Omega \lambda(x)dx\right)^{\frac{1}{p}}}.$$

Proof. Let $x_0 \in \Omega$ and $\epsilon > 0$ be such that

$$B(x_0,\epsilon) = \{x \in \mathbf{R}^N : |x - x_0| < \epsilon\} \subseteq \Omega.$$

Define, for $k > (\operatorname{diam}(\Omega))^2$,

$$u_k(x) = k - |x - x_0|^2 \quad (x \in \Omega).$$

It results $\sup_{x\in\Omega} |u_k(x)| = k$. Moreover, we have

$$\begin{aligned}
&\int_\Omega \lambda(x)|u_k(x)|^p dx + \int_\Omega |\nabla u_k(x)|^p dx \\
= &\int_\Omega k^p \lambda(x)dx - \int_\Omega \lambda(x)\left(k^p - (k - |x-x_0|^2)^p\right)dx + 2^p \int_\Omega |x - x_0|^p dx \\
< \; &k^p \int_\Omega \lambda(x)dx - \operatorname*{ess\,inf}_\Omega \lambda \, m(\Omega \setminus B(x_0,\epsilon))\left(k^p - (k-\epsilon^2)^p\right) + 2^p(\operatorname{diam}(\Omega))^p m(\Omega).
\end{aligned}$$

Since $\lim_{k\to\infty}(k^p - (k-\epsilon^2)^p) = +\infty$, we can choose $\overline{k} > (\operatorname{diam}(\Omega))^2$ such that

$$\overline{k}^p - (\overline{k} - \epsilon^2)^p > \frac{2^p(\operatorname{diam}(\Omega))^p m(\Omega)}{\operatorname{ess\,inf}_\Omega \lambda m(\Omega \setminus B(x_0,\epsilon))}.$$

Consequently it follows that

$$\int_\Omega \lambda(x)|u_{\overline{k}}(x)|^p dx + \int_\Omega |\nabla u_{\overline{k}}(x)|^p dx < \overline{k}^p \int_\Omega \lambda(x)dx.$$

Then the conclusion follows by the definition of $c(\lambda)$. □

Now, we state and prove the propositions about the upper bound of $c(\lambda)$.

Proposition 4.2. *Let $N = 1$ and $\Omega =]a,b[$. Then*

$$c(\lambda) \leq \left[\max\left\{ \left(\frac{1}{\int_a^b \lambda(x)dx} + \frac{p-1}{\operatorname{ess\,inf}_{]a,b[} \lambda} \right), 1 \right\} \right]^{\frac{1}{p}}.$$

Proof. For every $x, y \in]a,b[$, one has

$$\begin{aligned} |u(x)|^p &= |u(y)|^p + \int_x^y p|u(t)|^{p-2}u(t)u'(t)dt \\ &\leq |u(y)|^p + \left| \int_y^x p|u(t)|^{p-1}|u'(t)|dt \right| \\ &\leq |u(y)|^p + \int_a^b p|u(t)|^{p-1}|u'(t)|dt \\ &\leq |u(y)|^p + (p-1)\int_a^b |u(t)|^p dt + \int_a^b |u'(t)|^p dt. \end{aligned}$$

Now, multiplying by $\lambda(y)$ and integrating with respect to y over the interval $]a,b[$, we obtain

$$\begin{aligned} |u(x)|^p &\leq \frac{1}{\int_a^b \lambda(y)dy} \left[\int_a^b \lambda(y)|u(y)|^p dy + \frac{(p-1)\int_a^b \lambda(y)dy}{\operatorname{ess\,inf}_{]a,b[} \lambda} \right. \\ &\times \left. \int_a^b \lambda(y)dy \int_a^b \lambda(t)|u(t)|^p dt + \left(\int_a^b \lambda(y)dy \right) \int_a^b |u'(t)|^p dt \right] \\ &\leq \max\left\{ \left(\frac{1}{\int_a^b \lambda(x)dx} + \frac{p-1}{\operatorname{essinf}_{]a,b[} \lambda} \right), 1 \right\} \\ &\times \left(\int_a^b \lambda(y)|u(y)|^p dy + \int_a^b |u'(y)|^p dy \right), \end{aligned}$$

that implies the thesis. □

Proposition 4.3. *Let Ω be an open, bounded and convex subset of $\mathbf{R}^N$. Then the following inequality holds*

$$c(\lambda) \leq 2^{\frac{p-1}{p}} \max\left\{ \frac{1}{(\int_\Omega \lambda(x)dx)^{\frac{1}{p}}}, \frac{d}{N^{\frac{1}{p}}} \left(\frac{p-1}{p-N} m(\Omega) \right)^{\frac{p-1}{p}} \frac{\|\lambda\|_\infty}{\int_\Omega \lambda(x)dx} \right\},$$

where $d = \operatorname{diam}(\Omega)$.

Proof. Put $p' = \frac{p}{p-1}$ and fix $u \in C^\infty(\Omega) \cap W^{1,p}(\Omega)$. For every $x, y \in \Omega$ one has

$$|u(x)| \leq |u(y)| + \int_0^1 |\nabla u(tx + (1-t)y)||x-y|dt.$$

Multiplying by $\lambda(y)$ and integrating with respect to y over Ω, we obtain that

$$\begin{aligned} |u(x)| &\leq \frac{1}{\int_\Omega \lambda(x)dx}\left[\int_\Omega \lambda(y)|u(y)|dy\right. \\ &+ \left.\int_\Omega \lambda(y)\left(\int_0^1 |\nabla u(tx+(1-t)y)||x-y|dt\right)dy\right] \\ &\leq \frac{1}{\int_\Omega \lambda(x)dx}\left[\left(\int_\Omega \lambda(y)dy\right)^{\frac{1}{p'}}\left(\int_\Omega \lambda(y)|u(y)|^p dy\right)^{\frac{1}{p}}\right. \\ &+ \left.\|\lambda\|_\infty \int_\Omega\left(\int_0^1 |\nabla u(tx+(1-t)y)||x-y|dt\right)dy\right]. \end{aligned} \tag{10}$$

Now, following the same arguments as in the proof of Theorem 1 of [3] (see also [4]), one has, taking into account (10),

$$\begin{aligned} |u(x)| &\leq \frac{1}{\int_\Omega \lambda(x)dx}\left[\left(\int_\Omega \lambda(y)dy\right)^{\frac{1}{p'}}\left(\int_\Omega \lambda(y)|u(y)|^p dy\right)^{\frac{1}{p}}\right. \\ &+ \left.\frac{d}{N^{\frac{1}{p}}}\left(\frac{p-1}{p-N}m(\Omega)\right)^{\frac{p-1}{p}}\|\lambda\|_\infty\left(\int_\Omega |\nabla u|^p dy\right)^{\frac{1}{p}}\right] \\ &\leq \max\left\{\frac{1}{(\int_\Omega \lambda(x)dx)^{\frac{1}{p}}}, \frac{d}{N^{\frac{1}{p}}}\left(\frac{p-1}{p-N}m(\Omega)\right)^{\frac{p-1}{p}}\frac{\|\lambda\|_\infty}{\int_\Omega \lambda(x)dx}\right\} \\ &\times \left[\left(\int_\Omega \lambda(y)|u(y)|^p dy\right)^{\frac{1}{p}} + \left(\int_\Omega |\nabla u|^p dy\right)^{\frac{1}{p}}\right] \\ &\leq 2^{\frac{p-1}{p}}\max\left\{\frac{1}{(\int_\Omega \lambda(x)dx)^{\frac{1}{p}}}, \frac{d}{N^{\frac{1}{p}}}\left(\frac{p-1}{p-N}m(\Omega)\right)^{\frac{p-1}{p}}\frac{\|\lambda\|_\infty}{\int_\Omega \lambda(x)dx}\right\} \\ &\times \|u\|_\lambda. \end{aligned}$$

Whence, the conclusion follows by the density of $C^\infty(\Omega)\cap W^{1,p}(\Omega)$ in $W^{1,p}(\Omega)$. □

Remark 4.4. By the previous proposition it follows that, if the diameter d satisfies

$$d \leq \frac{(\operatorname{ess\,inf}_{x\in\Omega}\lambda(x))^{\frac{p-1}{p}}}{\|\lambda\|_\infty}N^{\frac{1}{p}}\left(\frac{p-N}{p-1}\right)^{\frac{p-1}{p}},$$

then the estimate has the simpler expression

$$c(\lambda) \leq \left(\frac{2^{p-1}}{\int_\Omega \lambda(x)dx}\right)^{\frac{1}{p}}.$$

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