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On Pseudomonotone Maps T for which $-T$ is also Pseudomonotone*

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Recently pseudomonotone variational inequalities have been studied quite extensively, hereby extending the theory of pseudoconvex minimization problems. The focus of the present work are "pseudoaffine maps", i.e., pseudomonotone maps T for which $-T$ is also pseudomonotone. A particular case of such maps are the gradients of pseudolinear functions. Our main goal is to derive the general form of pseudoaffine maps which are defined on the whole space.

Keywords: Pseudoconvex function, pseudolinear function, pseudomonotone map, pseudoaffine map

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1. Introduction

In the study of generalized convexity and its applications pseudoconvex functions play an important role [2]. Pseudoconvex functions f for which $-f$ is also pseudoconvex are often called pseudolinear. This is an interesting class of functions. An application of pseudolinear programming is linear fractional programming with its various uses, e.g. [2, 6].

Differentiable pseudoconvex functions are characterized by pseudomonotone gradients in the sense of Karamardian [9]. Hence differentiable pseudolinear functions are characterized by pseudomonotone gradients for which the negative of the gradient is also pseudomonotone. Maps (not necessarily gradient maps) with this property are the focus of the present study. They will be called pseudoaffine maps for the purposes of this paper.

In a previous study [4] some properties of pseudoaffine maps (denoted as PPM-maps therein) were derived, and characterizations of the solution set of variational inequalities

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involving such maps were obtained. The work can be viewed as an extension of some recent results for pseudolinear optimization problems in [8] to pseudoaffine maps and pseudoaffine variational inequalities. For an extension in a different direction see [1].

In the present study we focus on pseudoaffine maps which are defined on the whole space. The main goal of this paper is to derive the general form of such maps.

Let $K \subseteq \mathbb{R}^n$ be convex. A differentiable function $f : K \to \mathbb{R}$ is called pseudoconvex if for all $x, y \in K$, the following implication holds [2]:

$$\langle \nabla f(x), y - x \rangle \geq 0 \Rightarrow f(y) \geq f(x).$$

Here, $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^n$. As mentioned before, the function f is pseudolinear if both f and $-f$ are pseudoconvex. First and second order characterizations of pseudolinear functions and studies of pseudolinear programs are contained in various papers; e.g., see [5, 6, 8, 10, 11, 14] and the references therein.

Pseudolinear functions are special cases of quasilinear functions. A function $f : K \to \mathbb{R}$ is called quasilinear if both f and $-f$ are quasiconvex[1]. The study of quasilinear functions originated in the sixties by the work of Martos and continued with the work of Tuy, Martínez-Legaz and others; e.g., see [2, 12, 13, 15, 16] and the references therein. When $K = \mathbb{R}^n$, the following result is known:

Theorem 1.1. *[15]: A lower semicontinuous function $f : \mathbb{R}^n \to \mathbb{R}$ is quasilinear if and only if it has the form*

$$f(x) = h(\langle u, x \rangle) \tag{1}$$

where h is a lower semicontinuous increasing function and $u \in \mathbb{R}^n$.

According to the above theorem, there are very "few" quasilinear functions which are defined on the whole space. However, even these are very useful. Indeed, according to an important result of Martínez-Legaz [12], a lower semicontinuous function $g : \mathbb{R}^n :\to \mathbb{R}$ which is bounded from below is quasiconvex if and only if it is the supremum of differentiable quasilinear functions. The following corollary is an immediate consequence of Theorem 1.1:

Corollary 1.2. *A differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is pseudolinear if and only if it can be written in the form*

$$f(x) = h(\langle u, x \rangle) \tag{2}$$

where $u \in \mathbb{R}^n$ and h is a differentiable function whose derivative is always positive or identically zero.

Proof. If f is pseudolinear, then it is quasilinear. Thus it has the form (1) where h is differentiable and increasing, i.e., with nonnegative derivative. If h' is zero at some point, then also ∇f vanishes at some point and this implies that f is constant [10], hence h' is identically zero. Otherwise, h' is always positive. The converse is obvious. □

We recall that a map $T : K \to \mathbb{R}^n$ is called pseudomonotone [9] if for all $x, y \in K$, the following implication holds:

$$\langle T(x), y - x \rangle \geq 0 \Rightarrow \langle T(y), y - x \rangle \geq 0.$$

[1]Note that in the literature these functions are sometimes called quasiaffine or quasimonotonic.

As mentioned above, a differentiable function f is pseudoconvex if and only if ∇f is a pseudomonotone map [9]. A map $T : K \to \mathbb{R}^n$ such that both T and $-T$ are pseudomonotone will be called *pseudoaffine*. We note that in [4] such maps were called PPM maps. Accordingly, T is pseudoaffine if and only if for all $x, y \in \mathbb{R}^n$ the following equivalence holds

$$\langle T(x), y-x\rangle \geq 0 \Leftrightarrow \langle T(y), y-x\rangle \geq 0$$

or equivalently,

$$\langle T(x), y-x\rangle > 0 \Leftrightarrow \langle T(y), y-x\rangle > 0.$$

Note that if T is pseudoaffine, then for all $x, y \in K$, the following equivalence holds:

$$\langle T(x), y-x\rangle = 0 \Leftrightarrow \langle T(y), y-x\rangle = 0. \tag{3}$$

Conversely, if the above equivalence holds and T is continuous, then T is pseudoaffine [4]. We shall mostly use pseudoaffinity in the form (3). An easy consequence is the following:

$$T(0) = 0 \Rightarrow \langle T(x), x\rangle = 0, \forall x \in \mathbb{R}^n. \tag{4}$$

Pseudoaffine maps were studied in [4] in relation with variational inequalities. Obviously, a differentiable function f is pseudolinear if and only if ∇f is pseudoaffine. Inspired by Corollary 1.2, it is natural to ask: what is the most general form of a pseudoaffine map T which is defined on the whole space? In the special case where $T = \nabla f$ for some pseudolinear function f, the answer is very simple: since f has the form (2), it follows that $T(x) = h'(\langle u, x\rangle)\, u$, i.e., T is always a positive multiple of a constant vector.

Another special case of a pseudoaffine map on the whole space arises when both T and $-T$ are monotone. Then the answer is given by the following proposition:

Proposition 1.3. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be such that T and $-T$ are both monotone. Then there exists a skew-symmetric linear map A and a vector $u \in \mathbb{R}^n$ such that $T(x) = Ax+u$, $x \in \mathbb{R}^n$.*

Proof. Set $u = T(0)$ and define $T' : \mathbb{R}^n \to \mathbb{R}^n$ by $T'(x) = T(x) - u$. Then T' and $-T'$ are monotone, thus

$$\forall x, y \in \mathbb{R}^n, \langle T'(y) - T'(x), y-x\rangle = 0 \tag{5}$$

$$T'(0) = 0. \tag{6}$$

From (5), (6) it follows that $\forall x \in \mathbb{R}^n$, $\langle T'(x), x\rangle = 0$. Expanding (5), we then get

$$\forall x, y \in \mathbb{R}^n, \langle T'(x), y\rangle = -\langle T'(y), x\rangle. \tag{7}$$

Thus, for any $t \in \mathbb{R}$, $x, y \in \mathbb{R}^n$, using (7) repeatedly, we obtain

$$\langle T'(tx), y\rangle = -\langle tx, T'(y)\rangle = -t\langle x, T'(y)\rangle = t\langle T'(x), y\rangle,$$

hence

$$\forall y \in \mathbb{R}^n, \langle T'(tx) - tT'(x), y\rangle = 0$$

which implies $T'(tx) = tT'(x), \forall x \in \mathbb{R}^n$. Likewise, for all $x, y, z \in \mathbb{R}^n$, relation (7) implies

$$\begin{aligned}\langle T'(x+y), z\rangle &= -\langle x+y, T'(z)\rangle = -\langle x, T'(z)\rangle - \langle y, T'(z)\rangle \\ &= \langle T'x, z\rangle + \langle T'y, z\rangle,\end{aligned}$$

hence $\langle T'(x+y) - T'(x) - T'(y), z\rangle = 0$ for all $z \in \mathbb{R}^n$. It follows that $T'(x+y) = T'(x) + T'(y)$, i.e., T' is linear. By relation (7), it is also skew-symmetric. Setting $A = T'$, we get the result. □

The aim of this paper is to study the most general form of a pseudoaffine map defined on the whole space. In fact, our main result will be the following theorem:

Theorem 1.4. *A map $T : \mathbb{R}^n \to \mathbb{R}^n$ is pseudoaffine if and only if there exist a skew-symmetric linear map A, a vector u and a positive function $g : \mathbb{R}^n \to \mathbb{R}$ such that*

$$\forall x \in \mathbb{R}^n, T(x) = g(x)(Ax + u). \tag{8}$$

One part of the theorem is trivial. Indeed, if g, A, u are as above, then for any $x, y \in \mathbb{R}^n$, we have $\langle Ay, y-x\rangle = \langle Ax, y-x\rangle$ since A is skew-symmetric. Thus we have the equivalences:

$$\begin{gathered}\langle T(x), y-x\rangle \geq 0 \Leftrightarrow \langle Ax+u, y-x\rangle \geq 0 \Leftrightarrow \\ \langle Ax, y-x\rangle + \langle u, y-x\rangle \geq 0 \Leftrightarrow \langle Ay, y-x\rangle + \langle u, y-x\rangle \geq 0 \Leftrightarrow \\ \langle T(y), y-x\rangle \geq 0.\end{gathered}$$

The converse will be studied in the following sections. In Section 2 we show that it is sufficient to prove Theorem 1.4 for continuous pseudoaffine maps. In Section 3 we study some elementary properties of pseudoaffine maps. In Section 4 we prove the theorem for $n = 2$. In Section 5 we derive some properties of pseudoaffine maps with regard to straight lines. In Section 6 we establish the theorem for pseudoaffine maps that vanish in at least one point. In Section 7 we prove the theorem for $n = 3$ without any assumption on the number of zeros of the map. In the final section we establish the theorem for the remaining case where the map never vanishes and n is arbitrary.

We introduce some terminology and notation. Given $x \in \mathbb{R}^n, S \subseteq \mathbb{R}^n$ we denote by $x + S$ the set $\{x + y : y \in S\}$. We set $\mathbb{R}^{++} = (0, +\infty)$ and denote by $\mathbb{R}^{++}S$ the set $\{tx : t \in \mathbb{R}^{++}, x \in S\}$.

For $x_1, ...x_k \in \mathbb{R}^n$, we denote by $sp(x_1, ...x_k)$ the subspace generated by $\{x_1, ...x_k\}$. More generally, if $S \subseteq \mathbb{R}^n$, then $sp(S)$ denotes the subspace generated by S. Two vectors $x, y \in \mathbb{R}^n$ will be said to have the same direction if there exists $\alpha > 0$ such that $x = \alpha y$. Given $x, y \in \mathbb{R}^n$, $x \neq y$, we denote by $l(x, y)$ the straight line generated by x and y, that is

$$l(x, y) = \{z \in \mathbb{R}^n : z = tx + (1-t)y, t \in \mathbb{R}\}.$$

A map T with values in $\mathbb{R}^n$ will be said to have a constant direction on a set K if it is identically zero on K, or else it is everywhere nonzero on K and there exists $e \in \mathbb{R}^n$ such that $T(x) = \|T(x)\| e$ for all $x \in K$.

Note that for $n = 1$ it is obvious that Theorem 1.4 is true. Indeed, a pseudoaffine map in $\mathbb{R}$ is a function which is always positive or always negative or identically zero, thus it is sufficient to take in (8) $A = 0$ and $u = 1$ or $u = -1$ or $u = 0$. Unless otherwise stated, the propositions of this paper are obviously true for $n = 1$. For this reason, we assume in all proofs that $n \geq 2$.

2. Reduction to the continuous case

The aim of this section is to show that, without loss of generality, we may suppose that T is continuous in order to prove Theorem 1.4.

The following proposition was proved in [4, Theorem 2.2] with the additional assumption that T is continuous:

Proposition 2.1. *Let $K \subseteq \mathbb{R}^n$ be open and convex and let $T : K \to \mathbb{R}^n$ be pseudoaffine . If $z_1, z_2 \in K$ are such that $T(z_1) = T(z_2) = 0$, then $T(z) = 0$ for any $z \in l(z_1, z_2) \cap K$.*

Proof. For each $v \in \mathbb{R}^n$, let $t \in \mathbb{R}$ be such that $z + tv \in K$. Since T is pseudoaffine,

$$\langle T(z_1), z + tv - z_1 \rangle = 0 \Rightarrow \langle T(z + tv), z + tv - z_1 \rangle = 0. \tag{9}$$

Since $z \in l(z_1, z_2)$, there exists $\lambda \in \mathbb{R}$ such that $z = \lambda z_1 + (1 - \lambda) z_2$. Hence, (9) gives

$$\langle T(z + tv), tv + (1 - \lambda)(z_2 - z_1) \rangle = 0. \tag{10}$$

Likewise, using $T(z_2) = 0$ we find $\langle T(z + tv), z + tv - z_2 \rangle = 0$, hence

$$\langle T(z + tv), tv + \lambda(z_1 - z_2) \rangle = 0. \tag{11}$$

From (10) and (11) we easily deduce that $\langle T(z + tv), tv \rangle = 0$. It follows that $\langle T(z + tv), z - (z + tv) \rangle = 0$, hence, using again that T is pseudoaffine, we get $\langle T(z), tv \rangle = 0$. Therefore $\langle T(z), v \rangle = 0$ for all $v \in \mathbb{R}^n$, hence $T(z) = 0$. □

By induction this result implies the following:

Proposition 2.2. *Let K, T be as in Proposition 2.1. If $z_1, z_2, \ldots z_m \in K$ are such that $T(z_1) = T(z_2) = \ldots = T(z_m) = 0$, then T vanishes on $M \cap K$ where M is the affine subspace generated by $z_1, z_2, \ldots z_m$.*

It follows from the preceding proposition that the set of zeros $V = M \cap K$ is closed as a subset of K, thus the set

$$W := K \backslash V = \{x \in K : T(x) \neq 0\}$$

is open.

Lemma 2.3. *Let K, T be as in Proposition 2.1. Then the map $S : W \to \mathbb{R}^n$ defined by $S(x) = T(x) / \|T(x)\|$ is continuous.*

Proof. Let $x \in W$ and suppose to the contrary that there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in W such that $x_n \to x$ but $S(x_n) \nrightarrow S(x)$. Since $\|S(x_n)\| = 1$ and the unit sphere is compact in $\mathbb{R}^n$, we may suppose that $S(x_n) \to A$ where $A \neq S(x)$ selecting a subsequence if necessary. Note that since $\|A\| = \|S(x)\| = 1$, the vectors $S(x)$ and A do not have the same direction. Thus we can choose $v \in \mathbb{R}^n$ such that

$$\langle S(x), v \rangle < 0 < \langle A, v \rangle .$$

We can also choose v small enough so that $w := x + v \in K$. Since S is also pseudoaffine and $\langle S(x), w - x\rangle < 0$, we deduce

$$\langle S(w), w - x\rangle < 0. \tag{12}$$

On the other hand, $0 < \langle A, v\rangle = \langle A, w - x\rangle$ implies that there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$ we have $\langle S(x_n), w - x_n\rangle > 0$. Using again the fact that S is pseudoaffine, we deduce $\langle S(w), w - x_n\rangle > 0$. Taking the limit we obtain $\langle S(w), w - x\rangle \geq 0$ which contradicts (12). □

Theorem 2.4. *Let $K \subseteq \mathbb{R}^n$ be open and convex and let $T : K \to \mathbb{R}^n$ be pseudoaffine. Then there exists a positive function $g : K \to \mathbb{R}$ and a continuous pseudoaffine map T' such that $T(x) = g(x)T'(x), x \in K$.*

Proof. If the set of zeros V is empty, the theorem follows from Lemma 2.3. Otherwise, let $d(x, V)$ be the distance of $x \in K$ from V. The function $d(\cdot, V)$ is continuous, and if $x \to x_0 \in V$, $d(x, V) \to 0$. Thus the map $T' : K \to \mathbb{R}^n$ defined by

$$T'(x) = \begin{cases} d(x, V) S(x), & x \in W \\ 0, & x \in V \end{cases}$$

is continuous. Obviously, $T(x) = g(x) T'(x)$ where $g : K \to \mathbb{R}$ is a positive function defined by

$$g(x) = \begin{cases} \frac{\|T(x)\|}{d(x,V)}, & x \in W \\ 1, & x \in V \end{cases}$$

Since $T'(x) = \frac{1}{g(x)} T(x), x \in K$, T' pseudoaffine. □

It follows from Theorem 2.4 that in order to prove Theorem 1.4, we may suppose without loss of generality that T is continuous.

3. Elementary properties of pseudoaffine maps defined on the whole space

Let us begin by exploring relation (8).

Proposition 3.1. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be pseudoaffine. If T has the form (8) where $g : \mathbb{R}^n \to \mathbb{R}$ is a positive function, $u \in \mathbb{R}^n$ and A a linear map, then A is skew-symmetric. If in addition T is continuous, then g is continuous on the set $\{x \in \mathbb{R} : T(x) \neq 0\}$.*

Proof. Set $T'(x) = T(x) / g(x)$. Then T' is pseudoaffine. We want to prove that for all $x \in \mathbb{R}^n$, $\langle Ax, x\rangle = 0$. Suppose that for some $x \in \mathbb{R}^n$, $\langle Ax, x\rangle > 0$. Then for any $y \in \mathbb{R}^n$, the expression $\langle T'(tx), y - tx\rangle = \langle tAx + u, y - tx\rangle = -t^2 \langle Ax, x\rangle + t[\langle Ax, y\rangle - \langle u, x\rangle] + \langle u, y\rangle$ will be negative for all t sufficiently great. Since T' is pseudoaffine, $\langle T'(y), y - tx\rangle = \langle T'(y), y\rangle - t\langle T'(y), x\rangle$ will also be negative for t great, thus $\langle Ay + u, x\rangle$ is nonnegative for all y. For $y = tx$, $t \in \mathbb{R}$, it follows that $\langle Atx + u, x\rangle \geq 0$. Letting $t \to -\infty$ and taking into account $\langle Ax, x\rangle > 0$ we get a contradiction. Analogously, $\langle Ax, x\rangle < 0$ leads to a contradiction. Thus, A is skew-symmetric.

If T is continuous, then at any $x \in \mathbb{R}^n$ such that $T(x) \neq 0$ it follows that $g(x) = \|T(x)\| / \|Ax + u\|$ which is continuous at x. □

We remark that when T is continuous, it does not follow that g is continuous on $\mathbb{R}^n$. As an example, define T on $\mathbb{R}^2$ by (8) where $u = 0, A(x_1, x_2) = (x_2, -x_1)$ and $g(x_1, x_2) = 1/\sqrt{\|(x_1, x_2)\|}$ whenever $(x_1, x_2) \neq (0, 0)$ and $g(0, 0) = 1$. Then it is easily seen that T is continuous while g is discontinuous at 0.

The following easy result will be very useful in the sequel:

Proposition 3.2. *Let V be a subspace in $\mathbb{R}^n$, P_V the orthogonal projection on V, $x_0 \in \mathbb{R}^n$, $M = V + x_0$ an affine subspace and $T : \mathbb{R}^n \to \mathbb{R}^n$ a pseudoaffine map. Then:*

a) The translation $T_1(x) = T(x - x_0)$ is a pseudoaffine map on $\mathbb{R}^n$.
b) The orthogonal projection $P_V T$ is a pseudoaffine map on M.

Proof. Part (a) is obvious. To show part (b), it is sufficient to note that $\langle P_V T(x), y - x\rangle = \langle T(x), P_V(y - x)\rangle = \langle T(x), y - x\rangle$ for any $x, y \in M$. □

By Proposition 2.2, the set of zeros of a pseudoaffine map defined on the whole space, is either empty or an affine subspace. If T vanishes on a hyperplane M, then it vanishes on the whole space. To see this, we first include in an easy lemma an argument which will be often used in the sequel:

Lemma 3.3. *Let M be a subspace in $\mathbb{R}^n$ and $z, y \in \mathbb{R}^n$ be such that for all $x \in M$, $\langle z, y - x\rangle = 0$. Then $z \in M^\perp \cap y^\perp$.*

Proof. Taking $x = 0$ in the given relation, we deduce that $\langle z, y\rangle = 0$. Going back, we see that $\langle z, x\rangle = 0$ for all $x \in M$, i.e., $z \in M^\perp \cap y^\perp$. □

Proposition 3.4. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a pseudoaffine map. If T vanishes on a hyperplane M, then it vanishes on $\mathbb{R}^n$.*

Proof. By considering a translation of T, if necessary, we may suppose without loss of generality that $0 \in M$.

For any $z \notin M$ and any $x \in M$, $\langle T(x), z - x\rangle = 0$, hence by pseudoaffinity in the form (3) we deduce that $\langle T(z), z - x\rangle = 0$. By Lemma 3.3, $T(z) \perp M$ and $T(z) \perp z$. Since M is a hyperplane, it follows that $T(z) = 0$. □

The above result can be further generalized:

Proposition 3.5. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a pseudoaffine map. If T has constant direction on a hyperplane M, then it has constant direction on $\mathbb{R}^n$.*

Proof. By Theorem 2.4, without loss of generality we may suppose that T is continuous. If T vanishes on M, then the result follows from Proposition 3.4. Otherwise, T has the form $T(x) = \|T(x)\| e$ with $\|e\| = 1$. Note that T has no zeros on M by our definition of constant direction at the end of the Introduction. Without loss of generality, we may consider that M is a subspace.

Let $z \notin M$. We consider two cases:

If e is orthogonal to M, then for all $x \in M$, $\langle T(x), z - x\rangle = \|T(x)\| \langle e, z\rangle$, hence the quantity $\langle T(x), z - x\rangle$ has constant sign. By pseudoaffinity, also the quantity $\langle T(z), z - x\rangle$ has constant sign. In particular, for every $x \in M$ and every $t \in \mathbb{R}$ the quantity

$\langle T(z), z - tx \rangle = \langle T(z), z \rangle - t \langle T(z), x \rangle$ has constant sign which is possible only if $\langle T(z), x \rangle = 0$. Hence $T(z)$ is also orthogonal to M; it follows that $T(z)$ and e are linearly dependent.

If e is not orthogonal to M, consider the subspace $V = \{y \in \mathbb{R}^n : \langle e, y \rangle = 0\}$ and the hyperplane $M_1 = z + V$. Since V is orthogonal to e, M_1 intersects M at an affine subspace $M \cap M_1$ with dimension $n-2$. Let P be the orthogonal projection to V. The map PT is pseudoaffine on M_1 and vanishes on $M \cap M_1$ since $\forall x \in M$, $T(x) = \|T(x)\| \, e \perp V$. From Proposition 3.4 it follows that PT vanishes on M_1. Hence $T(z)$ and e are linearly dependent.

It follows that in all cases, for every $z \in \mathbb{R}^n$, $T(z)$ and e are linearly dependent. Note that $T(z)$ is never zero. Indeed, T is never zero on M by assumption. If $T(z) = 0$ for some $z \notin M$, then for all $x \in M$, $\langle T(z), z - x \rangle = 0$ would imply by pseudoaffinity that $\langle T(x), z - x \rangle = 0$, yielding $\langle e, z - x \rangle = 0$ for all $x \in M$. By Lemma 3.3, this would imply that e is orthogonal both to z and M, which is impossible since $e \neq 0$. Since $T(z), z \in \mathbb{R}^n$ is always a multiple of e and is never zero, continuity of T implies that $T(z)$ has always the same direction. $\square$

4. The case $n = 2$

We now prove that Theorem 1.4 is true when $n = 2$. As we noted before, we may restrict ourselves to continuous maps:

Theorem 4.1. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous pseudoaffine map. Then there exists a vector $u \in \mathbb{R}^2$, a skew-symmetric linear map A and a positive function $g : \mathbb{R}^2 \to \mathbb{R}$, such that $T(x) = g(x)(Ax + u)$.*

Proof. We consider three cases:

Case 1: $T(x) \neq 0, \forall x \in \mathbb{R}^2$. Let $l = \{y \in \mathbb{R}^2 : \langle T(0), y \rangle = 0\}$. Then by pseudoaffinity, for all $y \in l$, $\langle T(y), y \rangle = 0$, hence $T(y)$ is orthogonal to l. Since the space is two-dimensional and T is never zero, T has constant direction on l. By Proposition 3.5, T has constant direction on $\mathbb{R}^2$. It follows that T has the form (8) with $A = 0, u = T(0) / \|T(0)\|$ and $g(x) = \|T(x)\|$.

Case 2: There exists exactly one point x_0 such that $T(x_0) = 0$. Set $T'(x) = T(x + x_0)$, then T' is pseudoaffine and $T'(0) = 0$. By (4), for any $x \in \mathbb{R}^2$, $\langle T'(x), x \rangle = 0$. Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be the map

$$A(a,b) = (b, -a) \quad \forall (a,b) \in \mathbb{R}^2.$$

Then $(Ax, x) = 0, \forall x \in \mathbb{R}^2$. For any $x \in \mathbb{R}^2 \setminus \{0\}$, both Ax and $T'(x)$ are orthogonal to x, hence they are linearly dependent. It follows that there exists $g_1(x) \neq 0$ such that $T'(x) = g_1(x) Ax$. Then $g_1(x) = \frac{\langle T'(x), Ax \rangle}{\|Ax\|^2}$ is continuous and does not vanish on $\mathbb{R}^2 \setminus \{0\}$. Hence, g_1 has a constant sign. By changing the sign of A if necessary, we may suppose that g_1 is positive. Finally, we have

$$T(x) = T'(x - x_0) = g_1(x - x_0)(Ax - Ax_0).$$

If we set $g(x) = g_1(x - x_0)$ for $x \neq x_0$, $g(x_0) = 1$ and $u = -Ax_0$, we get the desired result.

Case 3: There are at least two points $x_1 \neq x_2$ such that $T(x_1) = T(x_2) = 0$. Then Propositions 2.1 and 3.4 imply that $T(x) = 0$ for any $x \in \mathbb{R}^2$, i.e., has the form (8) with $g = 1, A = 0, u = 0$. □

5. Line properties of pseudoaffine maps

We now investigate some properties of pseudoaffine maps along straight lines.

Proposition 5.1. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous pseudoaffine map. Suppose that $T = 0$ on a straight line l. Then T has constant direction on any straight line parallel to l. This direction is orthogonal to l.*

Proof. We may suppose that $0 \in l$, i.e., $l = \{w \in \mathbb{R}^n : w = te, t \in \mathbb{R}\}$ with $e \neq 0$. Consider any straight line $l' = \{y \in \mathbb{R}^n : y = x_0 + te, t \in \mathbb{R}\}$ parallel to l. If $T(y) = 0$ for some $y \in l'$, then by Proposition 2.2 the map T vanishes on the affine subspace generated by l and y, so T is identically zero on l'.

Now suppose that T is never zero on l'. For all $w \in l$ and $x \in \mathbb{R}^n$, $\langle T(w), x - w\rangle = 0$ holds, and by pseudoaffinity this implies $\langle T(x), x - w\rangle = 0$. By Lemma 3.3, $T(x) \perp l$ and $T(x) \perp x$ thus

$$\langle T(x), w\rangle = \langle T(x), x\rangle = 0 \text{ for all } x \in \mathbb{R}^n, w \in l. \tag{13}$$

Now consider the hyperplane $V = \{z \in \mathbb{R}^n : \langle T(x_0), z\rangle = 0\}$. For any $z \in V$, $\langle T(x_0), z + x_0 - x_0\rangle = 0$ implies by pseudoaffinity $\langle T(z + x_0), z\rangle = 0$. Combining with relation (13) we infer that for all $y = x_0 + w, w \in l$, $\langle T(z + x_0), y - (z + x_0)\rangle = 0$ and by pseudoaffinity $\langle T(y), y - (z + x_0)\rangle = 0$. Using again (13) we deduce $\langle T(y), z\rangle = 0$.

It follows that $T(y) \in V^\perp$, i.e., $T(y)$ and $T(x_0)$ are linearly dependent. Since there is no zero of T on the line l', by continuity $T(x_0)$ and $T(y)$ have the same direction. □

Proposition 5.2. *Let $x_1 \neq x_2$ be two points in $\mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous pseudoaffine map such that $T(x_1) = aT(x_2), a \in \mathbb{R}$. Let $l = l(x_1, x_2)$. Then for any $x \in l$ there exists $\lambda(x) \in \mathbb{R}$ such that $T(x) = \lambda(x)T(x_2)$. In particular, if T never vanishes on l, then $\lambda(x) > 0$.*

Proof. We may suppose that $0 \in l$. If $T(x_2) = 0$, we have $T(x_1) = 0$, and thus T is zero on l by Proposition 2.1. If $T(x_2) \neq 0$, we distinguish two cases:

Case 1. $\langle T(x_2), x_2 - x_1\rangle = 0$. Let $V = \{y \in \mathbb{R}^n : \langle T(x_2), y\rangle = 0\}$ and P_V the orthogonal projection on V. Then $P_V T$ is a pseudoaffine map on $M = V + x_1$. Since $x_1, x_2 \in M$ and $P_V T(x_1) = P_V T(x_2) = 0$, by Proposition 2.1, $P_V T$ vanishes on l and thus $T(x)$ and $T(x_2)$ are linearly dependent for each $x \in l$.

Case 2. $\langle T(x_2), x_2 - x_1\rangle \neq 0$. From (3) we obtain easily that $a \neq 0$. We suppose to the contrary that there exists $x \in l$ such that $T(x)$ and $T(x_2)$ are linearly independent. Then $x \neq x_1, x \neq x_2$, and the equations

$$\langle T(x_2), z - x_2\rangle = 0 \tag{14}$$

and

$$\langle T(x), z - x\rangle = 0 \tag{15}$$

define two nonparallel hyperplanes through x_2 and x respectively. Let z satisfy both (14) and (15). From pseudoaffinity we infer

$$\begin{aligned} \langle T(z), z - x_2\rangle &= 0 \\ \langle T(z), z - x\rangle &= 0. \end{aligned} \tag{16}$$

Thus $\langle T(z), x_2 - x\rangle = 0$ and consequently $\langle T(z), x_2 - x_1\rangle = 0$. Using (16) we infer $\langle T(z), x_1 - z\rangle = 0$. Again by pseudoaffinity, $\langle T(x_1), x_1 - z\rangle = 0$. Using our assumption we get $\langle T(x_2), x_1 - z\rangle = 0$ and, taking into account (14), we obtain $\langle T(x_2), x_2 - x_1\rangle = 0$, a contradiction.

Thus, in all cases there exists $\lambda(x) \in \mathbb{R}$ such that $T(x) = \lambda(x)T(x_2), x \in l$. The function λ is obviously continuous; hence, if it does not vanish, it has a constant sign. □

If l is the straight line joining two points x_1, x_2 and $T(x_1)$ and $T(x_2)$ are linearly dependent, then there are three possibilities for the various images $T(x), x \in l$. If T has at least two zeros on the line, we know that T is identically zero on l (Proposition 2.1). If T has no zeros, then all $T(x), x \in l$ belong to an open half straight line beginning at the origin (Proposition 5.2). If T has exactly one zero, then again Proposition 5.2 guarantees that all $T(x), x \in l$ belong to a straight line through the origin, but some have opposite directions according to the following proposition:

Proposition 5.3. *Let x_1, x_2, l, T be as in Proposition 5.2. If T has exactly one zero on l, then there exist $z_1, z_2 \in l$ such that $T(z_1) = \alpha T(z_2) \neq 0$, where $\alpha < 0$.*

Proof. Suppose that T has exactly one zero at x_0 on l. By making a translation, if necessary, we may suppose that $x_0 = 0$. Obviously $T(x_2) \neq 0$ since otherwise T would be identically zero on l. Let $V = sp(x_2, T(x_2))$ and P be the orthogonal projection on V. Then PT is pseudoaffine and for any $x \in l, T(x) = \lambda(x) T(x_2) \in V$. Thus, $PT = T$ on l. By Theorem 4.1, there exist a linear map A on V, a vector $u \in V$ and a positive function $g : V \to \mathbb{R}$ such that $PT(x) = g(x)(Ax + u)$ on V. Our assumption $T(0) = 0$ implies that $u = 0$. Since $A(-x_2) = -Ax_2$, it follows that $T(x_2)$ and $T(-x_2)$ have opposite directions. □

We now study the case in which $T(x_1)$ and $T(x_2)$ are linearly independent:

Proposition 5.4. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and pseudoaffine and $x_1, x_2 \in \mathbb{R}^n$ be such that $T(x_1), T(x_2)$ are linearly independent. Let $l = l(x_1, x_2)$. Then for any $z \in l$, $T(z) \in sp(T(x_1), T(x_2))$.*

Proof. Since $T(x_1), T(x_2)$ are linearly independent, the set

$$M = \{z \in \mathbb{R}^n : \langle T(x_1), z - x_1\rangle = 0, \langle T(x_2), z - x_2\rangle = 0\}$$

is an intersection of two nonparallel hyperplanes and, as such, is an $n-2$ dimensional affine subspace. By pseudoaffinity, $\forall z \in M, \langle T(z), z - x_1\rangle = 0$ and $\langle T(z), z - x_2\rangle = 0$.

It follows that $\forall t \in \mathbb{R}, \langle T(z), z-(tx_1+(1-t)x_2)\rangle = 0$ and again by pseudoaffinity $\langle T(tx_1+(1-t)x_2), z-(tx_1+(1-t)x_2)\rangle = 0, \forall z \in M$. Thus,

$$\forall w \in l, \forall z \in M, \langle T(w), z-w\rangle = 0. \tag{17}$$

Fix $z_0 \in M$. If $N = (T(x_1))^{\perp} \cap (T(x_2))^{\perp}$, then obviously $M = z_0 + N$. From (17) we deduce that $\forall z \in N, \langle T(w), z+z_0-w\rangle = 0$. By Lemma 3.3, $T(w) \in N^{\perp} = sp(T(x_1), T(x_2))$, for all $w \in l$. $\square$

6. The case where T has at least one zero

We already know that the set of zeros of T is either empty or an affine subspace. We shall begin by showing that Theorem 1.4 is true under the additional assumption that T has exactly one zero. First we recall two basic results from algebraic topology and projective geometry:

Theorem 6.1. ***(Schauder domain invariance**, [7, Theorem II2.8.4]) If U is open in a normed space E and $f: U \to E$ is an injective completely continuous field, then $f(U)$ is open.*

We recall that f is a completely continuous field if the map $F(x) = x - f(x)$ is continuous and it maps bounded sets to relatively compact sets.

Theorem 6.2. ***(Fundamental Theorem of Projective Geometry**, [3]) Let $F: L \to L$ be an automorphism of the lattice of subspaces of $\mathbb{R}^n, n \geq 3$ (i.e., an application onto L such that $V_1 \subseteq V_2$ is equivalent to $F(V_1) \subseteq F(V_2)$). Then there exists a linear map A on $\mathbb{R}^n$ such that $F(V) = A(V), \forall V \in L$.*

Actually the above is a simplified form of the fundamental Theorem of Projective Geometry.

Proposition 6.3. *Suppose that T is continuous and pseudoaffine and has a zero only at 0. Then there exists a positive function g and a linear map A such that $T(x) = g(x)Ax$ for all $x \in \mathbb{R}^n$.*

Proof. Since we know that the Theorem is true for $n = 2$, we may suppose that $n \geq 3$.

For any straight line l through 0 and any nonzero $y \in l$, $T(y)$ is also nonzero by assumption. According to Proposition 5.2, for all $x \in l, T(x) \in sp(T(y))$. Thus the image of a straight line though 0 is contained in a straight line through 0. Hence, if D is the set of all lines through 0, T defines a map $F_1: D \to D$ such that for all $x \neq 0$, $T(x) \in F_1(sp(x))$. This map is 1-1. To prove this, first we show:

$$\forall x, y \in \mathbb{R}^n, T(sp(x,y)) \subseteq sp(T(x), T(y)). \tag{18}$$

Indeed, each $z \in sp(x,y)$ can be written as $z = ax + by$ with $a, b \in \mathbb{R}$. By Propositions 5.2 and 5.4, $T(z) \in sp(T(ax), T(by))$. But we know that $T(ax) \in sp(T(x))$ and $T(ay) \in sp(T(y))$. Hence, $T(z) \in sp(T(x), T(y))$ and (18) follows.

Now suppose F_1 is not 1-1. Then there exist x, y linearly independent such that $T(x), T(y)$ are linearly dependent. From (18) it follows that $T(sp(x,y)) \subseteq sp(T(x))$. By our assumption on zeros, $T(sp(x,y) \setminus \{0\})$ does not contain 0. In addition, by Proposition 5.3, it contains two points z and z' such that $T(z), T(z')$ have opposite directions. Thus, $T(sp(x,y) \setminus \{0\})$ cannot be connected, while $sp(x,y) \setminus \{0\}$ is connected. This is not possible, since T is continuous. Hence F_1 is 1-1.

Next we show that F_1 is onto. The set D is a differentiable manifold; actually it is the real projective $n-1$ manifold $\mathbb{R}P^{n-1}$. Since D is compact and connected and F_1 is continuous, the image $F_1(D)$ is also compact and connected. Thus, if F_1 is not onto, then there exists $l \in D$ such that $F_1(l)$ is a boundary point of the range of F_1. By considering local coordinate systems (U, ϕ) and (V, φ) around l and $F_1(l)$ we get the coordinate representation $\widetilde{F_1} = \varphi F_1 \phi^{-1}$ of F_1 from the open bounded set $\phi(U)$ to the open bounded set $\varphi(V)$ in $\mathbb{R}^{n-1}$. Note that the image $\widetilde{F_1}(U)$ is not open in $\mathbb{R}^n$ since $\varphi(F_1(l))$ is a boundary point of $\widetilde{F_1}(U)$. However, $\widetilde{F_1}$ is obviously a completely continuous field, and this contradicts Theorem 6.1.

Let L be the lattice of subspaces of $\mathbb{R}^n$. We define a map $F : L \to L$ by $F(V) = sp(T(V)), V \in L$. Note that from the definition of F and relation (18) it follows that

$$\forall x_1, x_2, \ldots x_k \in \mathbb{R}^n, F(sp(x_1, x_2, \ldots x_k)) = sp(T(x_1), T(x_2), \ldots T(x_k)) \tag{19}$$

We show that this map is a lattice automorphism.

First we show that it preserves the dimension of subspaces. Indeed, for any $V \in L$, let k_1, k_2 be the dimensions of $V, F(V)$. From (19) it follows that if $x_1, x_2, \ldots x_{k_1}$ span V, then $T(x_1), T(x_2), \ldots T(x_{k_1})$ span $F(V)$. Hence, $k_2 \leq k_1$. Let D_1, D_2 be the sets of lines through the origin in $V, F(V)$ respectively. Then D_1, D_2 are differentiable manifolds with dimensions $k_1 - 1, k_2 - 1$ respectively. The map $F_1 : D_1 \to D_2$ is continuous and 1-1. It is well known that this implies $k_2 - 1 \geq k_1 - 1$. Hence, V and $F(V)$ have the same dimension.

We now show that F is onto. If $U \in L$, let $\{y_1, y_2, \ldots y_k\}$ be a basis of U. Since F_1 is onto, there exist $x_1, x_2, \ldots x_k \in \mathbb{R}^n$ such that $F(sp(x_i)) = sp(y_i), i = 1, 2, \ldots k$. Then (19) implies

$$F(sp(x_1, x_2, \ldots x_k)) = sp(T(x_1), T(x_2), \ldots T(x_k)) = U.$$

Let us show that F preserves the lattice-theoretic union, i.e., the sum of subspaces. For $V_1, V_2 \in L$, we may find a basis of $V_1 + V_2$ in the form $K_1 \cup K_2 \cup K_3$ such that $K_1 \cup K_3$ is a basis of V_1, $K_2 \cup K_3$ is a basis of V_2 and K_3 is a basis of $V_1 \cap V_2$ (K_3 is empty if $V_1 \cap V_2 = \{0\}$). Then $F(V_1 \cup V_2) = sp(T(K_1), T(K_2), T(K_3)) = sp(T(K_1), T(K_3)) + sp(T(K_2), T(K_3)) = F(V_1) + F(V_2)$.

It is obvious that for $V_1, V_2 \in L$, $V_1 \subseteq V_2$ implies $F(V_1) \subseteq F(V_2)$. We show that the converse is true. If $F(V_1) \subseteq F(V_2)$, then $F(V_1 + V_2) = F(V_1) + F(V_2) = F(V_2)$. It follows that $V_2, F(V_2), F(V_1 + V_2), V_1 + V_2$ all have the same dimension. Hence, $V_2 = V_1 + V_2$ which implies $V_1 \subseteq V_2$.

By definition, this last property together with the fact that F is onto, imply that F is a lattice automorphism. By Theorem 6.2, there exists a linear map A such that for all $V \in L$, $F(V) = A(V)$. In particular, for any $x \neq 0$ it follows that $T(x) \in sp(T(x)) = F(sp(x)) = A(sp(x))$. Thus, there exists a number $g(x) \neq 0$ such that $T(x) = g(x) Ax$. The function g is obviously continuous on $\mathbb{R}^n \setminus \{0\}$, thus it has a constant sign. By changing the sign, if necessary, we may suppose that g is positive. We set $g(0) = 1$ and get the desired result. □

Theorem 6.4. *Suppose that T has at least one zero. Then there exist a positive function g, a skew-symmetric map A and $u \in \mathbb{R}^n$ such that $T(x) = g(x)(Ax + u), x \in \mathbb{R}^n$.*

Proof. By Proposition 3.1 it is sufficient to show that T has the form (8) with g positive and A linear. Pick x_0 such that $T(x_0) = 0$ and set $T_1(x) = T(x + x_0), x \in \mathbb{R}^n$. Then T_1 is pseudoaffine, and by Proposition 2.2 its set of zeros is a subspace V. We proceed by induction on the dimension of V to show that T_1 has the form (8) with $u = 0$. By Proposition 6.3, this is true if the dimension of V is 0. We suppose that it is true if the dimension is $k - 1$ and proceed to show that it is also true if the dimension is k. Pick $e \in V \setminus \{0\}$.

We show first that the subspace $\{e\}^{\perp}$ is invariant under T_1. Indeed, if $x \in \{e\}^{\perp}$, then $\langle T_1(e), x - e\rangle = 0$, hence $\langle T_1(x), x - e\rangle = 0$. By Lemma 3.3, $T_1(x) \in \{e\}^{\perp}$. The set of zeros of the restriction of T_1 on $\{e\}^{\perp}$ has dimension $k - 1$. Since the theorem is true in this case, there exists a linear map A on $\{e\}^{\perp}$ and a nonnegative function g on $\{e\}^{\perp}$ such that

$$T_1(x) = g(x)\, Ax, x \in \{e\}^{\perp}. \tag{20}$$

Let P be the orthogonal projection on $\{e\}^{\perp}$. For any $x \in \mathbb{R}^n$, x and Px are on a straight line parallel to e, and by Proposition 5.1 we know that $T_1(x)$ and $T_1 P(x)$ have the same direction, i.e., there exists a positive function $g_1 : \mathbb{R}^n \to \mathbb{R}$ such that $T_1(x) = g_1(x)\, T_1 P(x), x \in \mathbb{R}^n$. Combining with (20) we get

$$\forall x \in \mathbb{R}^n, T_1(x) = g_1(x)\, T_1 P(x) = g_1(x)\, g(Px)\, APx.$$

It follows that

$$\forall x \in \mathbb{R}^n, T(x) = g_1(x - x_0)\, g(P(x - x_0))\,(APx - APx_0).$$

Hence T has the form (8). □

7. The case $n = 3$

In this section we prove Theorem 1.4 for $n = 3$. Since we know that the theorem is true if T has at least one zero, we may suppose that T has no zeros.

Proposition 7.1. *Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a continuous pseudoaffine map that has no zeros and let T have constant direction on a straight line $l = \{y \in \mathbb{R}^n : y = te, t \in \mathbb{R}\}$. Let $x' \neq 0$ be such that $\langle T(0), x'\rangle = 0$, $\langle e, x'\rangle = 0$. Then T has constant direction on $l' = x' + l$.*

Proof. We distinguish two cases:

Case 1. $\langle T(0), e\rangle = 0$. Then for any $x' + se \in l'$ and any $te \in l$, $\langle T(0), x' + se - te\rangle = 0$. Since $T(te)$ and $T(0)$ are linearly dependent, we deduce that $\langle T(te), x' + se - te\rangle = 0$. Pseudoaffinity implies that $\langle T(x' + se), x' + se - te\rangle = 0$, for each $t \in \mathbb{R}$. Hence, $\langle T(x' + se), x'\rangle = 0$ and $\langle T(x' + se), e\rangle = 0$. Since the space is 3-dimensional, $T(0)$ and $T(x' + se)$ are linearly dependent, for each $s \in \mathbb{R}$. Since T has no zeros, it has constant direction on l'.

Case 2. $\langle T(0), e\rangle \neq 0$. We suppose to the contrary that $T(x')$ and $T(x'+se)$ are linearly independent for some $s \in \mathbb{R}\setminus\{0\}$. Thus we can find $z \in \mathbb{R}^3$ such that $\langle T(x'), z-x'\rangle = 0$ and $\langle T(x'+se), z-(x'+se)\rangle = 0$. By pseudoaffinity, we obtain $\langle T(z), z-x'\rangle = 0$ and $\langle T(z), z-(x'+se)\rangle = 0$. Thus $\langle T(z), e\rangle = 0$. It follows that $z \notin l$ since $z \in l$ would imply by assumption that $T(z)$ and $T(0)$ have the same direction while $\langle T(0), e\rangle \neq 0$. We consider now a plane V containing l and z and let P_V the orthogonal projection on V. The map $P_V T$ is pseudoaffine with constant direction on l and thus by Proposition 3.5 with constant direction on V. But $\langle P_V T(z), e\rangle = \langle T(z), e\rangle = 0$ and $\langle P_V T(0), e\rangle = \langle T(0), e\rangle \neq 0$, a contradiction. Hence $T(x')$ and $T(x'+se)$ are linearly dependent for all $s \in \mathbb{R}$, and as before this implies that T has a constant direction on l'. □

Proposition 7.2. *Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a continuous pseudoaffine map with no zeros. There exists a straight line $l = \{y \in \mathbb{R}^n : y = x_0 + te, t \in \mathbb{R}\}$ such that $T(y) = \|T(y)\| e$ for all $y \in l$.*

Proof. By Borsuk's antipodal Theorem ([7]) there exists $w \in \mathbb{R}^n$ with $\|w\| = 1$ such that $T(-w) = \alpha T(w)$, where $\alpha > 0$. Hence Proposition 5.2 implies that there exists $v \neq 0$ such that $T(y) = \|T(y)\| v$ for all y on the line $l' = \{tw, t \in \mathbb{R}\}$.

Consider the subspace $M_1 = \{x \in \mathbb{R}^3 : \langle x, w\rangle = 0\}$ and the orthogonal projection P_1 on M_1. If for some $x_1 \in M_1$ we have $P_1 T(x_1) = 0$, then $T(x_1) = \beta w$ for some $\beta \in \mathbb{R}$ and $\langle T(x_1), x_1\rangle = 0$. Thus by pseudoaffinity we obtain $\langle T(0), x_1\rangle = 0$. Applying Proposition 7.1 we deduce that T has constant direction on the line $l = x_1 + l'$. It follows that for all $y \in l'$ one has $T(y) = \|T(y)\| \sigma w$ where $\sigma = 1$ or -1. Setting $e = \sigma w$ we obtain the desired result.

Now suppose that $P_1 T(x) \neq 0$ for each $x \in M_1$. Then from the proof of Theorem 4.1 (case 1) we know that the pseudoaffine map $P_1 T$ has constant direction on M_1. Consider any $y \in M_1$ with $\langle y, T(0)\rangle \neq 0$. Note that $T(0)$ is not orthogonal to M_1 since $P_1 T(0) \neq 0$ by assumption. Then there exists $\gamma > 0$ such that $P_1 T(y) = \gamma P_1 T(0)$. Consider further the two dimensional subspace M_2 containing y and w and let P_2 be the orthogonal projection on M_2. Since $l' \subset M_2$, by Proposition 3.5 the pseudoaffine map $P_2 T$ has constant direction on M_2. Thus there exists $\delta > 0$ such that $P_2 T(y) = \delta P_2 T(0)$. Let $\{e_1, e_2, e_3\}$ be an orthonormal basis with $e_1 = w, e_2 = \frac{y}{\|y\|}$ and let (a, b, c) and (a_1, b_1, c_1) be the coordinates of $T(0)$ and $T(y)$ respectively with respect to this basis. Then $(0, b_1, c_1) = \gamma(0, b, c)$, $(a_1, b_1, 0) = \delta(a, b, 0)$ and $b \neq 0$. It follows that $\delta = \gamma$ and $T(y) = \gamma T(0)$, i.e., $T(y)$ and $T(0)$ have the same direction. This holds for all $y \in M_1$ such that $\langle y, T(0)\rangle \neq 0$ and, by continuity, it holds for all $y \in M_1$. Applying again Proposition 3.5, we obtain that T has constant direction on $\mathbb{R}^3$, and thus to finish the proof, we can choose $x_0 = 0$ and $e = \frac{T(0)}{\|T(0)\|}$. □

Theorem 7.3. *Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a continuous pseudoaffine map. Then there exists a vector $u \in \mathbb{R}^3$, a skew-symmetric linear map A and a positive function $g : \mathbb{R}^3 \to \mathbb{R}$ such that $T(x) = g(x)(Ax + u)$.*

Proof. We may suppose that T has no zeros. Let l be the straight line whose existence is asserted in Proposition 7.2. We may suppose that $x_0 = 0$. Consider an orthonormal basis $\{e_1, e_2, e_3\}$ such that $e_3 = e$ and denote by x_1, x_2, x_3 the coordinates of $x \in \mathbb{R}^3$ in this basis. Let $M_i = \{x \in \mathbb{R}^3 : \langle x, e_i \rangle = 0\}$ and P_i be the orthogonal projection on M_i. Then P_3T is pseudoaffine on M_3 and $P_3T(0) = 0$. If $P_3T = 0$ on M_3, then T has constant direction on M_3, hence by Proposition 3.5 it has constant direction on $\mathbb{R}^3$. So in this case (8) holds with $A = 0$ and $u = e_3$. Otherwise, by the proof of Theorem 4.1 (see case 2) there exists a function with constant sign $g : \mathbb{R}^2 \to \mathbb{R}$, continuous on $\mathbb{R}^2 \setminus \{0\}$ and such that

$$\forall (x_1, x_2, 0) \in M_3, P_3T(x_1, x_2, 0) = g(x_1, x_2) A(x_1, x_2, 0),$$

where

$$A(x_1, x_2, x_3) = (x_2, -x_1, 0). \tag{21}$$

Define the map $T' : \mathbb{R}^3 \to \mathbb{R}^3$ by $T'(x_1, x_2, x_3) = \frac{T(x_1,x_2,0)}{g(x_1,x_2)}$. For any $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, by Proposition 7.1, T has constant direction on the line $x + l$. Hence,

$$\begin{aligned} T(x_1, x_2, x_3) &= \frac{\|T(x_1, x_2, x_3)\|}{\|T(x_1, x_2, 0)\|} T(x_1, x_2, 0) \\ &= \frac{\|T(x_1, x_2, x_3)\|}{\|T(x_1, x_2, 0)\|} g(x_1, x_2) T'(x_1, x_2, x_3). \end{aligned} \tag{22}$$

Since T and T' always have the same or always opposite direction, T' is pseudoaffine. In addition, it is continuous on $\mathbb{R}^3 \setminus l$ and $P_3T'(x) = Ax$ for all $x \in M_3$. Hence,

$$\forall x = (x_1, x_2, 0) \in M_3, T'(x) = (x_2, -x_1, f(x_1, x_2)) \tag{23}$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous on $\mathbb{R}^2 \setminus \{0\}$.

Take any $x_2 \neq 0, x_1 \neq 0$ and consider the affine subspace $M_2 + x_2e_2$. Then T' is constant on the line $l + x_2e_2$ which belongs to M_2, hence by Proposition 3.5, P_2T' has constant direction on $M_2 + x_2e_2$. Thus, the vectors $P_2T'(x_1, x_2, 0) = (x_2, 0, f(x_1, x_2))$ and $P_2T'(0, x_2, 0) = (x_2, 0, f(0, x_2))$ have the same direction. It follows that $f(x_1, x_2) = f(0, x_2)$. Likewise, we can prove that $f(x_1, x_2) = f(x_1, 0)$. Since x_1 and x_2 are arbitrary, for any $x_1, x_2, x_1', x_2' \in \mathbb{R} \setminus \{0\}$ it follows that

$$f(x_1, x_2) = f(x_1, 0) = f(x_1, x_2') = f(0, x_2') = f(x_1', x_2').$$

Continuity implies that f is constant on $\mathbb{R}^2 \setminus \{0\}$. Thus, combining (21) with (23) we see that T' has the form $T'x = Ax + \lambda e_3$ for all $x \in M_3 \setminus \{0\}$.

Setting $g_1(x_1, x_2, x_3) = \frac{\|T(x_1,x_2,x_3)\|}{\|T(x_1,x_2,0)\|} g(x_1, x_2)$ we deduce from (22) that for all $x \in \mathbb{R}^3 \setminus l$,

$$T(x) = g_1(x)(Ax + \lambda e_3).$$

By changing the sign of A and λ, if necessary, we can take g_1 to be positive. Since T has no zeros, $Ax + \lambda e_3$ is never zero on $\mathbb{R}^3$ and its norm has a positive infimum. Thus $g_1(x) = \|T(x)\| / \|Ax + \lambda e_3\|$, $x \in \mathbb{R}^3 \setminus l$ has a positive continuous extension g on $\mathbb{R}^3$. By continuity,

$$T(x) = g(x)(Ax + \lambda e_3), x \in \mathbb{R}^3,$$

which completes the proof of the theorem. □

8. The case where T has no zeros

In this section we prove Theorem 1.4 for maps that have no zeros. This is the only case left unproved so far. First we obtain some more information on the image of a straight line through T; see also Proposition 5.4.

Proposition 8.1. *Let T be continuous and pseudoaffine. Let further $x \neq y \in \mathbb{R}^n$ and $l = l(x, y)$. If $T(x), T(y)$ are linearly independent, then $\mathbb{R}^{++}T(l)$ is an open half-space in the plane $sp(T(x), T(y))$.*

Proof. We know already that $T(l) \subseteq sp(T(x), T(y))$ (Proposition 5.4). Without loss of generality we suppose that $x = 0$. Let V be the subspace $sp(y, T(0), T(y))$ and P be the orthogonal projection on V. Then PT is pseudoaffine on V and $PT = T$ on $sp(y)$. Thus, by Theorem 7.3, PT has the form $PT(z) = g(z)(Az + u), z \in V$ with g positive and $A : V \to V$ linear. Consequently, $PT(ty) = T(ty) = g(ty)(tAy + u)$ hence $T(l) = \{g(ty)(tAy + u) : t \in \mathbb{R}\}$. Since $T(0)$ and $T(y)$ are linearly independent, u and Ay also are linearly independent. Then $\mathbb{R}^{++}T(l) = \{tsAy + su : t \in \mathbb{R}, s \in \mathbb{R}^{++}\}$ is obviously a half-space. □

We now investigate the sets where T has a constant direction. The following proposition generalizes both Propositions 2.2 and 5.2:

Proposition 8.2. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous pseudoaffine map and $z_1, z_2, ...z_m \in \mathbb{R}^n$ be such that $T(z_i) \in sp(v), i = 1, 2, \ldots m$, for some $v \in \mathbb{R}^n$. Then $T(z) \in sp(v)$ for all z on the affine subspace M generated by $z_1, z_2, ...z_m$. In particular, if T has no zeros, then T has a constant direction on M.*

Proof. We proceed by induction. If $m = 2$, the statement is true by Proposition 5.2. Suppose that it is true for $m - 1$ and let $z = \sum_{i=1}^{m} a_i z_i$ with $\sum_{i=1}^{m} a_i = 1, a_i \neq 0$ and $T(z_i) \in sp(v), i = 1, 2, \ldots m$. At least one of the a_i's is different from 1. If, say, $a_1 \neq 1$ then $z = a_1 z_1 + (1 - a_1) z_1'$ where $z_1' = \sum_{i=2}^{m} \frac{a_i}{1-a_1} z_i$ belongs to the affine subspace generated by $z_2, \ldots z_m$. By assumption, $T(z_1') \in sp(v)$. The proposition follows by applying again Proposition 5.2. □

The following proposition generalizes Proposition 7.1:

Proposition 8.3. *Let T be continuous and pseudoaffine with no zeros. If T has constant direction on a straight line l, then it has a constant direction on any straight line l' parallel to l.*

Proof. We may assume with no loss of generality that l contains the origin. Let $T(w)$ have the same direction with a vector u for all $w \in l$. Consider the two-dimensional subspace V generated by l, l' and let $x_0 \in V \backslash l$. If x is any point on V not belonging to the line $l + x_0$, then the straight line through x_0 and x intersects l at some point y. By Propositions 5.2 and 5.4, $T(x) \in sp(T(x_0), T(y)) = sp(T(x_0), u)$. By continuity, it follows that the same holds for any $x \in V$.

Note that if $T(x')$ has the same direction with u for some $x' \in V \backslash l$, then by Proposition 8.2, T has a constant direction on V. Thus, T has a constant direction on l'. Hence, we may assume that $T(x')$ has a direction different from u for all $x' \in V \backslash l'$. Now consider any straight line l_1 in V intersecting l' at z. Then $T(l_1)$ and $T(l')$ also intersect at $T(z)$, and *a fortiori* $\mathbb{R}^{++}T(l_1)$ and $\mathbb{R}^{++}T(l')$ intersect. Since l_1 intersects l, by assumption T has not a constant direction on l_1; thus, by Proposition 8.1, $\mathbb{R}^{++}T(l_1)$ is an open half-space in the two-dimensional space $sp(T(x_0), u)$. If $\mathbb{R}^{++}T(l')$ is also an open half-space, then there exists another common point of $\mathbb{R}^{++}T(l_1)$ and $\mathbb{R}^{++}T(l')$, linearly independent from $T(z)$. Thus, there exist two points $z_1 \in l_1$ and $z' \in l'$ such that $T(z)$ and $T(z')$ have the same direction. Let l_2 be the straight line joining z_1, z'. Then T has a constant direction on l_2. But l_2 intersects l. Thus T has the direction u on l_2. This contradicts our assumption on the direction of T outside l. Thus, $\mathbb{R}^{++}T(l')$ is not an open half-space. Hence by Proposition 8.1, for any $w, w' \in l'$, $T(w), T(w')$ are linearly dependent. Since by our assumption T has no zeros, T has a constant direction on l'. □

Proposition 8.4. *Let T be pseudoaffine and continuous with no zeros and let M be the set of all $x \in \mathbb{R}^n$ such that $T(x)$ has the same direction as $T(0)$. Then M is a non-trivial subspace. In addition, if we set $N = M^{\perp}$, then for any distinct $x, y \in N$, $T(x), T(y)$ are linearly independent. Finally, if x, y and 0 are all distinct, then $T(0) \notin sp(T(x), T(y))$.*

Proof. By Proposition 8.2 the set M is indeed a subspace. Arguing as in the beginning of the proof of Proposition 7.2, we deduce that M contains at least one line through the origin.

We show that $T(x), T(y)$ are linearly independent. Indeed, if they are linearly dependent then by Proposition 5.2, T has a constant direction on the straight line $l \subseteq N$ through x, y. Then by Proposition 8.3, T has constant direction on the straight line l' through 0 which is parallel to l. Since $l' \subseteq N$ this means that M and N intersect at l', a contradiction.

Now suppose that $x, y \neq 0$ and $T(0) \in sp(T(x), T(y))$. For any $z \in sp(x, y)$, let $a, b \in \mathbb{R}$ be such that $z = ax + by$. Applying Proposition 5.4 we find $T(2ax) \in sp(T(0), T(x)) \subseteq sp(T(x), T(y))$ and likewise $T(2by) \in sp(T(x), T(y))$. Since $z = \frac{1}{2}2ax + \frac{1}{2}2by$, we also find $T(z) \in sp(T(2ax), T(2by)) \subseteq sp(T(x), T(y))$. Hence the restriction of T on $sp(x, y)$ is a continuous map of $sp(x, y)$ into the 2-dimensional subspace $sp(T(x), T(y))$. By the first part of the proof, this map sends distinct points to linearly independent points. If we consider any simple continuous closed curve C in $sp(x, y)$, then its image through T will be a simple continuous closed curve in the 2-dimensional space $sp(T(x), T(y))$. It is easy to see that at least two points of the latter curve are linearly dependent, a contradiction. □

We are now in a position to prove Theorem 1.4.

Proof of Theorem 1.4. According to Theorem 2.4 we may suppose that T is continuous. According to Theorems 4.1, 6.4, 7.3 and the remark at the end of Section 1, we may suppose that $n > 3$ and T has no zeros. Let M and N be as in Proposition 8.4.

For any straight line $l \subseteq N$ through 0 we know by Propositions 5.4 and 8.4 that $T(l)$ is included in a subspace of dimension 2. Let V_l be the subspace generated by l and $T(l)$, and let P_l be the orthogonal projection on V_l. Then V_l is two- or three-dimensional, P_lT is pseudoaffine on V_l and $P_lT = T$ on l. Thus we know that there exist a positive function $g_l : V_l \to \mathbb{R}$, a linear map $A_l : V_l \to V_l$ and a vector u_l such that $\forall z \in V_l, P_lTz = g_l(z)(A_lz + u_l)$. We can choose g_l so that $g_l(0) = 1$. In particular, on l it follows that

$$\forall x \in l, T(x) = g_l(x)(A_lx + u_l). \tag{24}$$

For $x=0$ we get $T(0)=u_l$, so u_l does not depend on l and we shall denote it by u. Now define the positive function $g : N \to \mathbb{R}$ by $g(0)=1$ and $g(x)=g_{sp(x)}(x), \forall x \in N\setminus\{0\}$. Then (24) gives

$$\forall x \in N, T(x) = g(x)\left(A_{sp(x)}x + u\right)$$

where we put $A_{sp(0)}0 = 0$. If further we define the map $T' : N \to \mathbb{R}^n$ by $T'(x) = T(x)/g(x)$, then T' is pseudoaffine. Finally, if we define the map $A : N \to \mathbb{R}^n$ by $Ax = A_{sp(x)}x$ for all $x \in N$, we deduce

$$\forall x \in N, T'(x) = Ax + u. \tag{25}$$

We want to show that the map A is linear. Note that for all $t \in \mathbb{R}, x \in N$,

$$A(tx) = A_{sp(x)}(tx) = tA_{sp(x)}x = tAx. \tag{26}$$

We now show that for any $x, y \in N, A(x+y) = Ax + Ay$. This is evident if x, y are linearly dependent, because then they belong to the same straight line through the origin. Thus we suppose that they are linearly independent. First we show that for any linearly independent $x, y \in N$ and any $t \in \mathbb{R}$, $A(tx + (1-t)y)$ belongs to the straight line joining $T(x), T(y)$.

By Proposition 5.4 and the definition of T' there exist $a, b \in \mathbb{R}$ (depending on x, y, t) such that $T'(tx + (1-t)y) = aT'(x) + bT'(y)$; thus,

$$A(tx + (1-t)y) + u = aAx + bAy + (a+b)u. \tag{27}$$

The above relation is true for all x, y, t. If we take $2x, 2y$ instead of x, y and the same t we deduce that there exist $a', b' \in \mathbb{R}$ such that

$$A(t2x + (1-t)2y) + u = a'A2x + b'A2y + (a'+b')u$$

which, in view of (26), becomes

$$2A(tx + (1-t)y) + u = 2a'Ax + 2b'Ay + (a'+b')u. \tag{28}$$

Then (27) with (28) imply that

$$(a-a')Ax + (b-b')Ay = (-a - b + \frac{a'+b'+1}{2})u. \tag{29}$$

Let us prove that $u \notin sp(Ax, Ay)$ and that Ax, Ay are linearly independent. Indeed, suppose first that $u = a''Ax + b''Ay$. Then a'', b'' cannot be both zero since $u = T(0) \neq 0$. From (25) we deduce that

$$u = a''T'x + b''T'y - (a''+b'')u. \tag{30}$$

Note that $a''+b'' \neq -1$ since otherwise (30) would imply that $T'(x)$ and $T'(y)$ are linearly dependent, and this is excluded by Proposition 8.4. Solving (30) with respect to u implies that $u \in sp(T'x, T'y)$; thus $u \in sp(Tx, Ty)$. But this contradicts Proposition 8.4. Hence

$u \notin sp(Ax, Ay)$. Writing $Ax = T'(x) - u$ and $Ay = T'(y) - u$ shows easily that Ax, Ay are linearly independent.

Then (29) implies that $a = a', b = b', a+b = 1$. From (27) it follows that $A(tx + (1-t)y) = aAx + bAy$ where $a + b = 1$.

Thus, if $w = \frac{1}{2}(x+y)$, then $\exists \alpha, \beta \in \mathbb{R}$ with $\alpha + \beta = 1$ and

$$Aw = \alpha Ax + \beta Ay. \tag{31}$$

For any $0 < t < 1$, $w = t\left(\frac{x}{2t}\right) + (1-t)\left(\frac{y}{2(1-t)}\right)$ is also in the straight line joining $\frac{x}{2t}$ and $\frac{y}{2(1-t)}$, hence

$$Aw = \alpha(t) A\left(\frac{x}{2t}\right) + \beta(t) A\left(\frac{y}{2(1-t)}\right) = \frac{\alpha(t)}{2t}Ax + \frac{\beta(t)}{2(1-t)}Ay$$

where $\alpha(t), \beta(t)$ are functions of t such that $\alpha(t) + \beta(t) = 1$. Since Ax, Ay are linearly independent, a comparison with (31) shows that $\alpha(t) = 2t\alpha, \beta(t) = 2(1-t)\beta$. Hence $2t\alpha + 2(1-t)\beta = 1, 0 < t < 1$ and this implies $\alpha = \beta = \frac{1}{2}$. Hence $A(x+y) = 2A\left(\frac{1}{2}(x+y)\right) = 2Aw = Ax + Ay$ and A is linear.

It follows from (25) that

$$\forall x \in N, T(x) = g(x)(Ax + u) \tag{32}$$

where $A : N \to \mathbb{R}^n$ is linear and $g : N \to \mathbb{R}$ is positive.

Let P be the orthogonal projection on N. Then for any $x \in \mathbb{R}^n$, the straight line l joining x and Px is parallel to a line $l' \subseteq M$. Since T has a constant direction on l', it also has a constant direction on l by Proposition 8.3. Hence $T(x)$ and $T(Px)$ have the same direction. Thus for any $x \in \mathbb{R}^n$, there exists $g_1(x) > 0$ such that $T(x) = g_1(x)T(Px)$. Using (32) we obtain

$$\forall x \in \mathbb{R}^n, T(x) = g_1(x)g(Px)(APx + u)$$

where $AP : \mathbb{R}^n \to \mathbb{R}^n$ is linear and $g_1(x)g(Px)$ is positive. Proposition 3.1 finishes the proof of the theorem. □

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