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Convex Bodies of Optimal Shape

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Given a continuous function $f\colon S^{n-1} \to \mathbb{R}$, we consider the minimization of the functional $\int_{\partial A} f(\nu_A)\, d\mathcal{H}^{n-1}$ with respect to $A \subset \mathbb{R}^n$, included in a class of convex bodies defined by surface or shape conditions. This corresponds to non-parametric formulations of older problems, including Newton's problem of the body of minimal resistance, following an approach due to G. Buttazzo and P. Guasoni [3]. We establish existence and uniqueness results and some characterizations of the minimizers.

1. Introduction

One of the oldest problem in the Calculus of Variations is to find a body of minimal resistance, under various assumptions. It was first stated by I. Newton in his *Principia*, and has been widely studied since. In this sort of problem, it is necessary to impose global constraints on the admissible set of bodies, such as convexity [2]. Usually the considered bodies are taken as epigraphs of convex functions on a given domain $\Omega \subset \mathbb{R}^2$. Then the problem is:

$$\inf_{u \in C \cap K} \int_\Omega g(\nabla u(x))\, dx \tag{1}$$

where $g(p) = g_N(p) = 1/(1 + |p|^2)$ in the original formulation, and $C = \{u : \Omega \to \mathbb{R}, u \text{ convex }\}$ and K expresses an additional condition, for instance $0 \le u \le M$, a height constraint, or $\int \sqrt{1 + |\nabla u(x)|^2} = k$, a surface area constraint.

The latter case is considered in [4], with $g(p) = j(\sqrt{1 + |p|^2})$, j convex, and $u \in BV(\Omega)$. The authors prove there that minimizers exist, that functions of the forms $c \operatorname{dist}(x, \partial\Omega)$ are minimizers, and they are the only convex ones vanishing on the boundary. They also consider a more elaborate model with frictional effects caused by the front and lateral surfaces, that is, minimize

$$\int_\Omega \frac{dx}{1 + |\nabla u(x)|^2} + c_1 \int_\Omega \sqrt{1 + |\nabla u(x)|^2}\, dx + c_2 \int_{\partial\Omega} u(x)\, d\mathcal{H}^{n-1}(x). \tag{2}$$

Those parametric formulations actually consider bodies with a fixed face $\Omega \times \{0\}$ even though there is no requirement of this sort in the original problem. In [3] the authors

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consider a generalization of this problem in a non-parametric form:

$$\inf_{A\in\mathcal{A}} \int_{\partial A} f(x,\nu_A(x))\, d\mathcal{H}^{n-1}(x)$$

where $\mathcal{A}$ is a class of n-convex bodies of $\mathbb{R}^n$ with volume or obstacle constraints. Here $\nu_A(x)$ is the unit outward normal vector at $x\in\partial A$. They prove various existence results and suggest to take Ω as an unknown in the problem.

Their problem corresponds (in a way explained hereafter) to (1) with

$$f(x,\nu) = f(\nu) = -\nu_n g(-\nu_n^{-1}\nu')$$

(where $\nu = (\nu',\nu_n)\in S^{n-1}$, $\nu' = (\nu_1,\dots,\nu_{n-1})$) and

$$A = \Big\{(x',x_n)\in\mathbb{R}^n,\quad 0\ge x_n\ge u(x')\Big\},$$

$$\nu_A(x) = \begin{cases} \dfrac{(\nabla u(x'),-1)}{\sqrt{1+|\nabla u(x')|^2}} & \text{for } x\in \operatorname{graph} u = \{(x',u(x'))\},\\ e_n = (0,\dots,0,1) & \text{for } x\in\Omega\times\{0\}.\end{cases}$$

Since in these cases f does not depend on x, we study in this paper the autonomous case with prescribed surface area:

$$\inf_{A\in\mathcal{A}_i} \int_{\partial A} f(\nu_A(x))\, d\mathcal{H}^{n_A-1}(x) \tag{3}$$

where $i=1$ or $i=2$ and

$$\mathcal{A}_1 := \{A \text{ convex} \subset\mathbb{R}^n,\ \mathcal{H}^{n_A-1}(\partial A) = 1\}$$

$$\mathcal{A}_2 := \Big\{A\in\mathcal{A}_1,\ A\subset\{x_n\le 0\},\\ \mathcal{H}^{n_A-1}(\Pi_{[x_n=0]}A) = \alpha = 1-\mathcal{H}^{n_A-1}(\partial A\cap\{x_n<0\})\Big\}$$

Here $\Pi_{[x_n=0]}$ designates the projection on the hyperplane $\{x_n=0\}$. Also for any convex set $A\subset\mathbb{R}^n$, n_A is the dimension of the affine space spanned by A.

Note that $A\in\mathcal{A}_2$ implies that there exists a convex set $\Omega_A\subset\mathbb{R}^{n-1}\equiv\{x_n=0\}$, with $\mathcal{L}^{n_A-1}(\Omega_A)=\alpha$ (the (n_A-1)-dimensional Lebesgue measure), and a convex function $u_A:\Omega_A\to\mathbb{R}_-$ such that $A=\{x=(x',x_n)\in\overline{\Omega}_A\times\mathbb{R}\,;\,0\ge x_n\ge u_A(x')\}$; moreover the condition $\alpha = 1-\mathcal{H}^{n_A-1}(\partial A\cap\{x_n<0\})$ implies:

$$\int_{\Omega_A}\sqrt{1+|\nabla u_A|^2}\, d\mathcal{L}^{n_A-1}(x')\le 1-\alpha.$$

Note also that we must have $\alpha\in(0,1/2)$, otherwise $\mathcal{A}_2$ is empty. In this formulation, only the area of Ω_A is given, not its shape.

Note that in the definitions of $\mathcal{A}_1$, the convexity constraint is a normalization of the considered sets: see Remark 2.1 below.

Concerning problem (3), our main concerns are existence and uniqueness of a solution; existence of solution of full dimension ($n_A = n$); existence of symetrical solutions for special values of f.

Our main result reads as follows:

Theorem 1.1. *For any $f \in C^0(S^{n-1})$, both problems* (3) *($i = 1, 2$) have a solution. Moreover, there exists a G_δ dense subset $X \subset C^0(S^{n-1})$ such that this solution is unique (up to translations) and is a n-simplex.*

We will actually prove a more detailed result: see Theorems 2.2 and 2.3 for detailed statements.

2. Equivalent formulation

We recall that any convex compact set A can be associated with a measure $\mu_A \in \mathcal{M}_+(S^{n-1})$, such that

$$\forall \varphi \in C^0(S^{n-1}), \quad \int_{S^{n-1}} \varphi(y)\, d\mu_A(y) = \int_{\partial A} \varphi(\nu_A(x))\; d\mathcal{H}^{n_A-1}(x). \tag{4}$$

In particular we have $\int_{S^{n-1}} d\mu_A(y) = \mathcal{H}^{n_A-1}(\partial A)$ and

$$\int_{S^{n-1}} y\; d\mu_A(y) = 0. \tag{5}$$

For example, if A is a polyhedron, then $\mu_A = \sum \alpha_i \delta_{\nu_i}$, where ν_i is the unit normal vector to the face i, and α_i its (relative) surface area.

Conversely, given any measure $\mu \in \mathcal{M}_+(S^{n-1})$, $\mu \neq 0$, satisfying (5), there exists a unique convex compact set A (up to translations), with dimension $n_A = \dim \operatorname{supp} \mu \in \{1, \ldots, n\}$, such that $\mu = \mu_A$. This follows from Alexandrov's Theorem [1, Theorem 19.2].

This allows us to express problem (3) as a **linear programming problem** on a subclass of $\mathcal{M}_+(S^{n-1})$:

$$\inf_{\mu \in \mathcal{M}_i} \int_{S^{n-1}} f(y)\, d\mu(y) \tag{6}$$

where $i = 1$ or $i = 2$ and

$$\mathcal{M}_1 := \left\{ \mu \in \mathcal{M}_+(S^{n-1});\quad \int_{S^{n-1}} d\mu(y) = 1, \quad \int_{S^{n-1}} y\; d\mu(y) = 0 \right\} \tag{7}$$

$$\mathcal{M}_2 := \left\{ \mu \in \mathcal{M}_1;\quad \mu\,\big|_{S_+^{n-1}} = \alpha\, \delta_{e_n} \right\}. \tag{8}$$

Here $S_+^{n-1} = S^{n-1} \cap \{x_n > 0\}$, δ_{e_n} is the Dirac mass at $e_n := (0, \ldots, 0, 1)$.

Remark 2.1. From Alexandrov's Theorem we could drop the convexity constraint in the definition of $\mathcal{A}_i$. Indeed, to any sufficiently regular compact set B corresponds a convex set A such that $\mu_A = \mu_B$; this implies that the infimum of (6) is the same for the wider class with no convexity constraint.

Theorem 2.2. *A convex compact set $A \in \mathcal{A}_i$ ($i = 1$ or $i = 2$) is a solution of* (3) *if and only if the corresponding μ_A is a solution of* (6).

For any $f \in C^0(S^{n-1})$, both problems have a solution.

Theorem 2.3. *There exists a G_δ dense subset $X \subset C^0(S^{n-1})$ such that $f \in X$ implies problem* (3) *has a unique solution, up to translations, and this solution is a n-simplex.*

Proof of Theorem 2.2. In the case $i = 1$, the Theorem follows directly from (4) and Alexandrov's Theorem recalled above.

If $i = 2$, we first observe that $A \in \mathcal{A}_2$ implies that the projection of A on $P_n := \{x_n = 0\}$ is equal to $A \cap P_n$. For otherwise, we would have $\mathcal{H}^{n_A-1}(A \cap P_n) < \mathcal{H}^{n_A-1}(\Pi A) = \alpha$ (where $\Pi = \Pi_{[x_n=0]}$); this contradicts the other condition on the surface area of A and $A \cap \{x_n < 0\}$.

Now note that if $x = (x', x_n) \in \partial A \cap \{x_n < 0\}$ then since A is convex and $\Pi(x) = (0, x_n) \in A$ we have $\langle \nu_A(x), \Pi(x) - x\rangle \leq 0$, so that

$$\langle \nu_A(x), e_n\rangle \leq 0. \tag{9}$$

Let $\varphi \in C^0(S^{n-1})$ such that φ has support in S^{n-1}_+; using (9), we have:

$$\int_{S^{n-1}} \varphi(y) d\mu_A(y) = \int_{\Pi(A)} \varphi(e_n) d\mathcal{H}^{n-1}(x) = \langle \alpha\delta_{e_n}, \varphi\rangle$$

so that $\mu_A \in \mathcal{M}_2$.

Conversely assume that $\mu \in \mathcal{M}_2$ and let us prove that there exists $A \in \mathcal{A}_2$ such that $\mu = \mu_A$. Let $A \in \mathcal{A}_1$ be such that $\mu_A = \mu$. Up to a translation of direction e_n, we may assume $0 \in \partial A$ and $A \subset \{x_n \leq 0\}$. Let us prove that (up to $\mathcal{H}^{n_A-1}$ negligible set of ∂A) one has $\{\nu_A = e_n\} = \partial A \cap \{x_n = 0\}$. First, the inclusion $\{\nu_A = e_n\} \supset \partial A \cap \{x_n = 0\}$ is straightforward. Assume now that $x \in \partial A$ and $\nu_A(x) = e_n$, by convexity we get $A \subset \{y \in \mathbb{R}^n : y_n \leq x_n\}$. Since $0 \in A$ we get $x_n = 0$, hence the desired result. Now, using $\mu_A \in \mathcal{M}_2$ we get:

$$\begin{aligned} &\mathcal{H}^{n_A-1}(\partial A \cap \{x_n = 0\}) = \mu_A(\{e_n\}) = \alpha, \\ &\mathcal{H}^{n_A-1}(\partial A \cap \{x_n < 0\}) = 1 - \alpha. \end{aligned} \tag{10}$$

To show that $A \in \mathcal{A}_2$ it remains to prove that $\Pi(A) = \partial A \cap \{x_n = 0\}$ and obviously it is enough to prove that $\Pi(A) \subset A$. With (10) we obtain that the set

$$B := \big\{x \in \partial A \cap \{x_n \leq 0\} : \nu_A(x) \text{ exists and } \langle \nu_A(x), e_n\rangle \leq 0\big\}$$

is of full $\mathcal{H}^{n_A-1}$ measure in $\partial A \cap \{x_n \leq 0\}$.

Since A is a convex compact set, we have:

$$A = \{x \in \mathbb{R}^n\colon\ x_n \leq 0, \text{ and } \ \langle \nu_A(z), x - z\rangle \leq 0, \text{ for all } z \in B\}$$

Now let $x = (x', x_n) \in A$ with $x_n < 0$, $y := \Pi x = (x', 0) \in \Pi(A)$. For any $z \in B$, we have

$$\langle \nu_A(z), y - z\rangle = \langle \nu_A(z), x - z\rangle - x_n \langle \nu_A(z), e_n\rangle \leq 0$$

so that $y \in A$. This proves that $A \in \mathcal{A}_2$.

Existence of solutions for (3) immediately follows since (6) consists in minimizing a continuous linear functional over a convex weakly $*$ compact subset of $\mathcal{M}(S^{n-1})$. $\square$

3. An example of uniqueness

Proposition 3.1. *Assume that f achieves a strict minimum at exactly $n+1$ points $x^0, \dots, x^n \in S^{n-1}$, such that $\forall \xi \in S^{n-1}$, $x^0 \cdot \xi, \dots, x^n \cdot \xi$ are not all nonnegative. Then* (3) *has a unique minimizer, which is a n-simplex.*

We will prove in the following that this simple case is actually the generic one for uniqueness, up to the addition of an affine function to f.

The assumption on $x^0, \dots, x^n$ means that they do not belong to the same closed half-hypersphere. This condition is equivalent to the fact that 0 is an interior point of $\mathrm{Conv}(x^0, \dots, x^n)$, from Hahn-Banach Theorem.

Proof. Since 0 is an interior point of $\mathrm{Conv}(x^0, \dots, x^n)$, the barycentric coordinates $\alpha_0, \dots, \alpha_n$ of 0 in $(x^0, \dots, x^n)$ are positive, and are the only solutions of

$$\begin{cases} \displaystyle\sum_{k=0}^{n} \alpha_k x_i^k = 0, & i = 1, \dots, n \\ \displaystyle\sum_{k=0}^{n} \alpha_k = 1, & \\ \alpha_k \geq 0, & k = 0, \dots, n. \end{cases}$$

Hence $\mu := \sum_{k=0}^{n} \alpha_k \delta_{x^k}$ belongs to $\mathcal{M}_1$, and is clearly the unique minimizer of the problem (6). The corresponding convex set is a n-simplex whose unit normal vectors are the x^k, with faces surface areas equal to α_k. $\square$

4. Optimality conditions and consequences

Lemma 4.1. *Let $\overline{\mu} \in \mathcal{M}_1$. Then $\overline{\mu}$ is a minimizer of* (6) *with $i = 1$ if and only if there exists $(\lambda_0, \dots, \lambda_n) \in \mathbb{R}^{n+1}$ such that*

$$f(y) = \lambda_0 + \sum_{i=1}^{n} \lambda_i y_i \qquad \overline{\mu} \text{ a.e. in } S^{n-1}, \tag{11}$$

$$f(y) \geq \lambda_0 + \sum_{i=1}^{n} \lambda_i y_i \qquad \forall y \in S^{n-1}. \tag{12}$$

Lemma 4.2. *Let $\overline{\mu} \in \mathcal{M}_2$. Then $\overline{\mu}$ is a minimizer of* (6) *with $i = 2$ if and only if there exists $(\lambda_0, \dots, \lambda_{n-1}) \in \mathbb{R}^n$ such that*

$$f(y) = \lambda_0 + \sum_{i=1}^{n-1} \lambda_i y_i + \lambda_n(\alpha + (1-\alpha) y_n) \qquad \overline{\mu} \text{ a.e. in } S_-^{n-1}. \tag{13}$$

$$f(y) \geq \lambda_0 + \sum_{i=1}^{n-1} \lambda_i y_i + \lambda_n(\alpha + (1-\alpha) y_n) \qquad \forall y \in S_-^{n-1}. \tag{14}$$

Here $S^{n-1}_- = S^{n-1} \cap \{x_n \leq 0\}$. In both lemmas, note that λ_0 is the optimal value of the functional.

Proof. We prove the last lemma first.

Let $\overline{\mu}$ be a solution, and $\lambda_0 := \int f\, d\overline{\mu}$, the optimal value of the functional. Define $g := f - \lambda_0$; then $\overline{\mu}$ is a minimizer of $\mu \mapsto \int g\, d\mu$ in $\mathcal{M}_2$, with a minimal value of zero. In other terms, g is non-negative on $\mathcal{M}_2$, that is $g \in (\mathbb{R}_+\mathcal{M}_2)^+$, the positive polar set of the cone $\mathbb{R}_+\mathcal{M}_2$. Since $\mathbb{R}_+\mathcal{M}_2$ is the intersection of $\mathcal{M}_+(S^{n-1})$ and a finite number of hyperplanes of $\mathcal{M}(S^{n-1})$:

$$\mathbb{R}_+\mathcal{M}_2 = \Big\{\mu \in \mathcal{M}_+(S^{n-1});\ \int_{S^{n-1}_-} (\alpha + (1-\alpha)y_n)\, d\mu = 0,$$
$$\int_{S^{n-1}} y_i\, d\mu = 0,\ i = 1, \ldots, n-1\Big\}$$

we get, in $\mathcal{M}(S^{n-1})'$:

$$(\mathbb{R}_+\mathcal{M}_2)^+ = \operatorname{clos}\left[\mathcal{M}_+(S^{n-1})^+ + \operatorname{Span}\{y_1, \ldots, y_{n-1}, w\}\right]$$

with $w(y) := (\alpha + (1-\alpha)y_n)\ 1_{S^{n-1}_-}(y)$.

Since C^0_+ is dense in $\mathcal{M}_+(S^{n-1})^+$, there exists sequences $(\Lambda^k) \subset C^0_+$, $(a^k) \in \mathbb{R}$, $(\lambda^k_i) \subset \mathbb{R}$ $(i = 1, \ldots, n-1)$ such that

$$g = \lim_k g^k \quad \text{with} \quad g^k := \Lambda^k + a^k w + \sum_{i=1}^{n-1} \lambda^k_i y_i.$$

Since $\alpha \in (0, 1/2)$, w changes sign in S^{n-1}_-; more precisely, $w > 0$ for $y_n \in (\beta, 0]$, where $\beta := \alpha/(\alpha - 1)$, and $w < 0$ for $y_n \in [-1, \beta)$. Then since

$$\int_{y_n \in (\beta,0)} \Lambda^k + a^k \int_{y_n \in (\beta,0)} w = \int_{y_n \in (\beta,0)} g_k$$

converges to $\int_{y_n \in (\beta,0)} g$, it is bounded. Hence a_k is bounded from above (recall that $\Lambda_k \geq 0$). Similarly we obtain a lower bound for a_k by integrating with respect to $y_n \in [-1, \beta)$. Up to subsequences, we can assume that $a_k \to \overline{a}$ as $k \to \infty$.

Then, if $\mu \in \mathcal{N} := \{\mu \in \mathcal{M}(S^{n-1}); \int w\, d\mu = 0, \int y_i\, d\mu = 0\ \forall i < n\}$, we have

$$\int_{S^{n-1}_-} g\, d\mu = \lim \int_{S^{n-1}_-} g^k\, d\mu = \int_{S^{n-1}_-} \Lambda^k\, d\mu.$$

Hence (Λ^k) is bounded in $\mathcal{N}'$ and therefore admits a convergent subsequence in the weak-* topology, with limit $\overline{\Lambda}$. Since $\overline{\mu} \in \mathcal{N}$ and $\int g\, d\overline{\mu} = 0$, we get $\int \overline{\Lambda}\, d\overline{\mu} = 0$. Since $\overline{\Lambda}\, d\overline{\mu} \geq 0$, that yields $\overline{\Lambda} = 0$, $\overline{\mu}$ a.e.

Now g_k converges to g, a_k is bounded and $\int \Lambda^k$ is bounded; so (λ^k_i) is bounded, and we can assume, extracting subsequences, that $\lambda^k_i \to \lambda_i$, $i = 1, \ldots, n-1$. Therefore $g - \sum \lambda_i y_i$ is equal to $\overline{\Lambda} + \overline{a}w$ in $\mathcal{M}'$. This implies that $\overline{\Lambda}$ is a continuous function, and

$$g - \sum \lambda_i y_i = \overline{a}w = \overline{a}\alpha + \overline{a}(1-\alpha)y_n \qquad \overline{\mu} \text{ a.e. in } S^{n-1}_-.$$

Then (14) follows from the nonnegativity of $\overline{\Lambda}$. This ends the proof of the second lemma.

The proof of the first lemma is similar, except that $i = 1, \dots, n$, and the term λ_n plays a role similar to the other λ_i. □

Let us give a few simple consequences of these lemmas.

Proposition 4.3. *Assume that f is continuous, and not affine. Then any optimal set in problem (3) is singular, in the sense that $\nu_A(\partial A)$ has a complement with nonempty interior in S^{n-1}.*

Note that this indicates that the set of minimizers is not generic: for a smooth (C^1) n-dimensional convex set A, we have $\nu_A(\partial A) = S^{n-1}$.

From Lemma 4.1 there exists an affine function θ such that $f = \theta$ in the support of μ_A. By assumption $\{f \neq \theta\}$ has nonempty interior.

Proposition 4.4. *Let f be continuous, $i = 1$ or 2. There exists $\overline{\mu} \in \mathcal{M}_i$, solution of (6), such that the support of $\overline{\mu}$ contains the support of any other solution of (6). In particular, in Lemmas 4.1, 4.2, the coefficients $(\lambda_i)_i$ can be chosen independently from the minimizer.*

Proof. Let M be the minimal set of the functional, which is a convex closed subset of $\mathcal{M}(S^{n-1})$. Let $(\mu_k) \subset M$ be a dense sequence in M, and $\overline{\mu} := \sum_k \alpha_k \mu_k$ where $\alpha_k > 0$, $\sum \alpha_k = 1$. Then $\overline{\mu} \in M$, the support of $\overline{\mu}$ is the closure of the union of the supports of μ_k, and it contains the support of any element of M; for if not, that would contradict the density of (μ_k) in M. □

Corollary 4.5. *Let $\overline{\Sigma}$ be the support of the solution $\overline{\mu}$ defined in the last proposition. Then the set of solutions of (6) is exactly $\mathcal{M}(\overline{\Sigma}) \cap \mathcal{M}_i$. In particular, the solution of (6) is unique if and only if $\operatorname{Card} \overline{\Sigma} = 1 + \dim \operatorname{Span} \overline{\Sigma} \leq n + 1$.*

Note that $\overline{\Sigma}$ is not known explicitely (it comes from the set of solutions). Consequently, 0 lies in the relative interior of $\operatorname{co} \overline{\Sigma}$ by construction. The additional condition in the corollary means that 0 has a unique barycentric decomposition $0 = \sum_{P \in \overline{\Sigma}} \alpha_P P$, with $\alpha_P > 0$.

On the other hand, $\overline{\Sigma} = (f - \theta)^{-1}(0)$, where $\theta(y) = \lambda_0 + \sum_i \lambda_i y_i$, since adding a measure with support in the latter set does not change the value of the functional.

Note that, if all minimizers of (3) have empty interior in $\mathbb{R}^n$, then $\overline{\Sigma} = (f - \theta)^{-1}(0)$ is included in an hyperplane of $\mathbb{R}^n$.

We are now in position to prove our second main theorem.

Proof of Theorem 2.3. Generic uniqueness is a consequence of Mazur's Theorem (see [6], see also [5, Theorem 1.20]) which states that both (concave Lipschitz) functionals

$$f \mapsto \inf_{\mu \in \mathcal{M}_i} \langle f, \mu \rangle, \quad i = 1, 2$$

are Gâteaux-differentiable on a G_δ dense subset of $C^0(S^{n-1})$ so that the corresponding minimizing measure is unique.

Corollary 4.5 shows that, in case of uniqueness, the corresponding body is a k-simplex, with $k \leq n$. On the other hand, the case $k < n$ occurs only if $(f - \theta)^{-1}(0)$, the minimal set of $f-\theta$, lies in a hyperplane. The opposite of this property is generic. Indeed if $f \in C^0(S^1)$ has only one minimum, then there exists $r_1, r_2 \in \mathbb{Q}$ such that $\max_{[r_1,r_2]} f \leq \min_{S^1\setminus[r_1,r_2]} f$; for each r_1, r_2 the set of such f is closed and has empty interior. $\square$

5. The cylindrical case

We now specialize to the case where $i = 2$ and f, in problem (3), depends only on y_n. This is particularly interesting in view of Newton's problem of the body of minimal resistance, where $f(y) = -y_n^3$. In the following we write $f(y) = g(y_n)$.

Theorem 5.1. *Assume $f(y) = g(y_n)$ for $y \in S_-^{n-1}$, with g strictly convex on $[-1, 0]$. The problem (3) with $i = 2$ and $f(y) = g(y_n)$ admits a unique solution A among convex sets invariant by SO^{n-2}, the group of rotations of $S^{n-2} = S^{n-1} \cap \{x_n = 0\}$. Moreover, this solution is a cone, that is, up to translations, Ω_A is a disc with center 0, and $u_A(x) = k_1 - k_2 \operatorname{dist}(x, 0)$.*

As explained in the introduction, a similar result was already established in the parametric case in [4]. Note that the proof here is completely different.

Proof. Let μ be a solution. Let us define $\overline{\mu}$ as follows: for all $\varphi \in C^0(S^{n-1})$,

$$\langle \overline{\mu}, \varphi \rangle = \int_{SO^{n-2}} \langle \mu, \varphi \circ R \rangle \, dR$$

where dR is the Haar probability measure on SO^{n-2}.

Then $\overline{\mu}$ is rotationnaly invariant, and $\int g(y_n)\, d\overline{\mu} = \int g(y_n)\, d\mu$: therefore $\overline{\mu}$ is is a minimizer.

From (13), there exists a constant c such that $g(y_n) - cy_n$ is an affine function of $y_1, \ldots, y_{n-1}$, $\overline{\mu}$ a.e. This implies $\lambda_i = 0$, $i = 1, \ldots, n-1$ since $\overline{\mu}$ is rotationnaly invariant. Now $g(y_n) - cy_n = \lambda_0$, $\overline{\mu}$ a.e and from (14) $g(y_n) - cy_n \geq \lambda_0$ in $[-1, 0]$. Therefore the support of $\overline{\mu}$ in S_-^{n-1} reduces to the minimal set of $g(y_n) - cy_n$, which is a sphere with dimension $n-2$ since g is strictly convex. The corresponding convex body is a cone as indicated in the statement of the Theorem. $\square$

Remark 5.2. The proof of the theorem can be easily extended to the case where f is invariant under the action of an arbitrary subgroup $G \subset O_n$. Then $\overline{\mu}$ is also invariant under the action of the group. For instance, if $f \circ \sigma = f$ for some symmetry σ, then $\overline{\mu} \circ \sigma = \overline{\mu}$ and $\sigma(A) = A$ for the corresponding body.

6. Frictional effects

In [4], the authors suggest to take into account frictional effects on the front and lateral sides of the body, minimizing the functional in (2) among all function $u \leq 0$, $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$. In a non-parametric formulation, this can be expressed as the minimizing problem:

$$\inf_{A \in \mathcal{A}_3} \int_{\partial A} f(\nu_A(x))\, d\mathcal{H}^{n_A-1}(x) + \\ + c_1 \mathcal{H}^{n_A-1}(\{x; \nu_A(x) \cdot e_n > 0\}) + c_2 \mathcal{H}^{n_A-1}(\{x; \nu_A(x) \cdot e_n = 0\}).$$

where $\mathcal{A}_3$ is an appropriate set of bodies. Here again we can normalize the minimizers by considering convex sets. Assuming that A is the intersection of the graph of a convex function u with $\{x_n \leq 0\}$ is equivalent to requiring that $\mu_A \big|_{S_+^{n-1}} = \alpha\delta_{e_n}$ for some $\alpha > 0$ as before. Then the compact set A is a minimizer if and only if μ_A minimizes

$$F_3(\mu) := \int_{S^{n-1}} f(y)\, dy + c_1\mu(\{y_n < 0\}) + c_2\mu(\{y_n = 0\}).$$

among all μ in $\mathcal{M}_3 := \left\{\mu \in \mathcal{M}_+(S^{n-1});\ \int_{S^{n-1}} y\, d\mu(y) = 0;\ \mu_A \big|_{S_+^{n-1}} = \alpha\delta_{e_n}\right\}$.

Clearly, all the results in previous sections remain true for this functional. In particular, for $f(y) = -y_n^3$, $c_1 > 0$, $c_2 > 0$, the case considered in [4], there exists a unique cylindrical minimizer which can be found as follows: write $\mu(y) = \alpha\delta_{e_n} + \eta(-y_n)$, with $\eta \in \mathcal{M}_+([0,1])$ and minimizes

$$H(\eta) := \int_0^1 t^3\, d\eta(t) + c_1\eta([0,1]) + c_2\eta(\{0\})$$

under the constraint $\int_0^1 t\, d\eta(t) = \alpha$. This obviously implies $\eta(0) = 0$ for the minimizer. Moreover there exists $\lambda \in \mathbb{R}$ such that $t^3 + c_1 = \lambda t$, η a.e. in $[0,1]$, and $t \mapsto t^3 - \lambda t + c_1$ is minimal on the support of η. Then $H(\eta) = (1-\alpha)\lambda + c_1\eta([0,1])$, which implies that λ must be the smallest number such that $t^3 - \lambda t + c_1$ has a nonnegative root. The corresponding root a satisfies $a^3 - \lambda a + c_1 = 0$ and $3a^2 - \lambda = 0$, that is $\lambda = 3a^2$ and $a = (c_1/2)^{1/3}$, if $c_1 \leq 2$. If $c_1 \geq 2$, then $a = 1$.

Hence the optimal body is flat if $c_1 \geq 2$. If $c_1 < 2$, it is a cone with slope p given by $1/\sqrt{1+p^2} = \nu_n = a = (c_1/2)^{1/3}$ *i.e.* $p = ((2/c_1)^{2/3} - 1)^{1/2}$. This result was obtained in [4] using different arguments.

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