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AW-Convergence and Well-Posedness of Non Convex Functions*

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Let us consider the set of lower semicontinuous functions defined on a Banach space, equipped with the AW-convergence. A function is called Tikhonov well-posed provided it has a unique minimizer to which every minimizing sequence converges. We show that well-posedness of f guarantees strong convergence of approximate minimizers of τ_{aw} -approximating functions (under conditions of equiboundedness of sublevel sets), to the minimizer of f.
Moreover we show that a lower semicontinuous function f which satisfies growth conditions at ∞ is well-posed iff its lower semicontinuous convex regularization is.
Finally we investigate the link between AW-convergence of non convex integrands and that of the associated integral functionals.

Keywords: AW-convergence, well-posedness, optimization problems

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1. Introduction

In this paper we study non convex minimization problems and the AW-convergence. In particular we deal with Tikhonov well-posed functions, i.e. functions with a unique minimum point to which every minimizing sequence converges (see [6] for a complete reference). We are also interested in another concept of well-posedness: the continuous dependence of the minimizer on problem's data. The latter means that approximate minimizers of perturbed functions f_n converge to the minimizer of f, and that $\inf f = \lim(\inf f_n)$, when we consider an appropriate convergence on the set of lower semicontinuous, proper functions. It turns out that usual notions of convergence (pointwise convergence, uniform convergence) are not well suited in this setting; this explains the attention paid to the AW-convergence, which was deeply studied in the papers of Attouch and Wets [1], [2], [3]; in fact this convergence satisfies the desired stability conditions.
The two notions of well-posedness are strongly correlated: Theorem 4.1 of [5] establishes the equivalence between Tikhonov well-posedness and continuous dependence of the minimizer on problem's data for a proper, lower semicontinuous and convex function when the set of such functions is endowed with the AW-convergence.
Purpose of this paper is to understand what happens when the functions we consider are proper and lower semicontinuous, but not convex.
In Section 3 we prove that the previous equivalence is still valid if we require the sublevel

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sets of f_n at height near to $\inf f_n$ to be equibounded.
Another way to approach the well-posedness problem of a given function f, is to study the greater lower semicontinuous and convex function which minorizes f; this function, denoted by $\hat{f}$, is called the lower semicontinuous, convex regularization of f.
We establish the equivalence between well-posedness of f and of $\hat{f}$ under the assumption that $\liminf_{\|x\|\to+\infty}(f(x)/\|x\|) \in (0,+\infty]$.
Finally, in Section 4, we investigate the case of non convex integral functionals; more precisely, we prove a dominated convergence theorem for AW-convergence. A similar result is also valid for Mosco convergence in the convex case (see [8]).

2. Definitions and preliminaries

In order to introduce definitions and results, we start with some notations.
We always denote by $(X, \| \|)$ a normed linear space and by d the distance function generated by the norm. For any subset A of X,

$$d(x, A) := \inf_{y\in A} \|x - y\|$$

denotes the distance from x to A.
In any space we consider, the unit ball is $U := \{x : \|x\| \le 1\}$ and ρU denotes the ball of radius $\rho \ge 0$. For any set $A \subset X$ and $\rho \ge 0$, we set

$$A_\rho := A \cap \rho U.$$

$\Gamma_0(X)$ denotes the set of all extended real valued, proper, lower semicontinuous and convex functions on X.
We write $v(f) = \inf\{f(x) : x \in X\}$, and argmin$f$ for the possibly empty set of points $\{x \in X : f(x) = v(f)\}$. For each $\alpha \in \mathbb{R}$, we denote by $\mathrm{lev}(f;\alpha)$ the sublevel set of f at height α, that is, $\{x \in X : f(x) \le \alpha\}$.
For $A, B \subset X$, the excess of A on B is

$$e(A, B) := \sup_{x\in A} d(x, B),$$

with the convention that $e(A, B) = 0$ if $A = \emptyset$ and $e(A, B) = +\infty$ if $B = \emptyset$. Then, the ρ-Hausdorff distance between A and B is the following number:

$$\mathrm{haus}_\rho(A, B) := \max\{e(A_\rho, B), e(B_\rho, A).\}$$

Definition 2.1. For $\rho \ge 0$, the ρ-(Hausdorff)-distance between two extended real valued functions f and g defined on X is

$$\mathrm{haus}_\rho(f, g) := \mathrm{haus}_\rho(\mathrm{epi}f, \mathrm{epi}g),$$

where the unit ball of $X \times \mathbb{R}$ is the set $U := \{(x, \alpha) : \|x\| \le 1, |\alpha| \le 1\}$.

Definition 2.2. Let $f, f_n : X \to [-\infty, +\infty]$ be lower semicontinuous functions. We say that f_n AW- converge to f, and we write $f = \tau_{aw} - \lim f_n$, iff:

$$\exists\rho_0 > 0 \text{ such that } \forall\rho > \rho_0 \ \mathrm{haus}_\rho(f, f_n) \to 0 \text{ as } n \to +\infty.$$

The following theorem gives a method to compute, or at least estimate, the ρ-Hausdorff distance between two functions, see [1].

Theorem 2.3 (Kenmochi conditions). *Suppose f, g are proper extended real valued functions defined on a normed real space X, both minorized by $-\alpha_0\|\cdot\|^p - \alpha_1$ for some $\alpha_0 \geq 0$, $\alpha_1 \in \mathbb{R}$, and $p \geq 1$. Let $\rho_0 > 0$ be such that $(\mathrm{epi} f)_{\rho_0}$ and $(\mathrm{epi} g)_{\rho_0}$ are nonempty.*

1. *Then the following conditions hold: for all $\rho > \rho_0$ and $x \in \mathrm{dom} f$ such that $\|x\| \leq \rho$, $|f(x)| \leq \rho$, for every $\epsilon > 0$ there exists some $x_\epsilon \in \mathrm{dom} g$ that satisfies*

$$\|x - x_\epsilon\| \leq \mathrm{haus}_\rho(f, g) + \epsilon, \qquad g(x_\epsilon) \leq f(x) + \mathrm{haus}_\rho(f, g) + \epsilon$$

as well as a symmetric condition with the roles of f and g interchanged.

2. *Conversely, assuming that for all $\rho > \rho_0 > 0$ there exists $n(\rho) \geq 0$, such that for all $x \in \mathrm{dom} f$ with $\|x\| \leq \rho$, $|f(x)| \leq \rho$, there exists $\tilde{x} \in \mathrm{dom} g$ that satisfies*

$$\|x - \tilde{x}\| \leq n(\rho), \qquad g(\tilde{x}) \leq f(x) + n(\rho),$$

and the symmetric condition (interchanging the roles of f and g), then with $\rho_1 := \rho + \alpha_0 \rho^p + |\alpha_1|$,

$$\mathrm{haus}_\rho(f, g) \leq n(\rho_1).$$

The next theorem is a sequential characterization of AW-convergence and it will be useful in the following, see [10].

Theorem 2.4. *Let $f_n, f : X \to (-\infty, \infty]$, $n \in \mathbb{N}$, be proper lower semicontinuous functions. Assume $v(f) > -\infty$. Then $f = \tau_{aw} - \lim f_n$ if and only if the following two conditions hold.*

(i) *If x_n is a bounded sequence then $f_n(x_n)$ is bounded below. Moreover, if $f_n(x_n)$ is bounded, then there are sequences y_n in X and $\epsilon_n \geq 0$ such that*

$$\|x_n - y_n\| \to 0, \quad \epsilon_n \to 0 \quad \textit{and} \quad f(y_n) \leq f_n(x_n) + \epsilon_n, \ \forall n \in \mathbb{N}. \tag{1}$$

(ii) *For any bounded sequence x_n such that $f(x_n)$ is bounded above there are sequences y_n in X and $\epsilon_n \geq 0$, such that*

$$\|x_n - y_n\| \to 0, \quad \epsilon_n \to 0 \quad \textit{and} \quad f_n(y_n) \leq f(x_n) + \epsilon_n, \ \forall n \in \mathbb{N}. \tag{2}$$

Now, let us recall the definition of Tikhonov well-posedness:

Definition 2.5. We say that a function $f : X \to (-\infty, +\infty]$ is Tikhonov well-posed (shortly, well-posed) if it satisfies the following conditions:

1. there exists a unique global minimum point x_0 for f;
2. if x_n is any minimizing sequence, i.e. a sequence in X such that $f(x_n) \to f(x_0)$, then $x_n \to x_0$.

Let f_n be a sequence of functions bounded from below. We say that a sequence $x_n \in X$ is asymptotically minimizing iff $f_n(x_n) - \inf f_n \to 0$.

Let us recall some characterizations of well-posedness:

Definition 2.6. A function

$$c : D \to [0, +\infty)$$

is called a forcing function iff

$$0 \in D \subset [0, +\infty), c(0) = 0 \text{ and } a_n \in D, c(a_n) \to 0 \Rightarrow a_n \to 0.$$

The proof of the following fact is in [6].

Theorem 2.7. *Let $f : X \to (-\infty, +\infty]$ be proper, bounded from below and lower semicontinuous. f is well-posed iff there exists a forcing function c and a point x_0 such that*

$$f(x) \geq f(x_0) + c[d(x, x_0)] \quad \textit{for every } x \in X.$$

Finally we recall a characterization of well-posedness due to Furi and Vignoli (see [7]):

Theorem 2.8. *A proper lower semicontinuous function $f : X \to [-\infty, +\infty]$ is well-posed if and only if*

$$\inf\{\text{diam}[\text{lev}(f; \alpha)] : \alpha > v(f)\} = 0.$$

3. Well-posedness and AW-convergence for non convex functions

The main result of this section is a generalization of the following theorem, due to Beer and Lucchetti [5], in the case of non convex functions.

Theorem 3.1 (Beer-Lucchetti). *Let X be a Banach space and $f \in \Gamma_0(X)$ bounded from below. Then the following conditions hold:*

1. *if f is well-posed then the conditions $\epsilon_n > 0$, $\epsilon_n \to 0$, $f = \tau_{aw} - \lim f_n$, with $f_n \in \Gamma_0(X)$ and $x_n \in \text{lev}(f_n; v(f_n) + \epsilon_n)$ for each n, imply that x_n is convergent to the unique minimizer of f;*
2. *whenever the conditions $\epsilon_n > 0$, $\epsilon_n \to 0$, $f = \tau_{aw} - \lim f_n$, with $f_n \in \Gamma_0(X)$ and $x_n \in \text{lev}(f_n; v(f_n) + \epsilon_n)$ for each n imply that x_n is convergent, then f is well-posed.*

We start with the following lemma, which obtains the continuity of the value function v.

Lemma 3.2. *Let $\epsilon_n > 0$ be such that $\epsilon_n \to 0$, and let $f, f_n : X \to (-\infty, +\infty]$ be lower semicontinuous and proper. Assume $v(f) > -\infty$ and suppose there exists $\alpha > v(f)$ such that $\text{lev}(f; \alpha)$ is bounded. Moreover assume there is $m > 0$ such that $\text{lev}(f_n; v(f_n) + \epsilon_n) \subset mU$ for each n sufficiently large and $f = \tau_{aw} - \lim f_n$.*
Then $v(f_n) \to v(f)$.

Proof. By definition of $v(f)$, there exists $\epsilon \in (0, 1]$ such that $v(f) + \epsilon \leq \alpha$ and therefore there is ρ such that $\text{lev}(f; v(f) + \epsilon) \subset \rho U$.
Let $x \in \text{lev}(f; v(f) + \epsilon)$.
Then the pair $(x, f(x)) \in \text{epi} f \cap rU$, with $r \geq \max\{\rho, |v(f)| + 1\}$. From Theorem 2.4 we get the existence of $y_n \in X$, $\delta_n \geq 0$, such that $\|x - y_n\| \to 0$, $\delta_n \to 0$ and $f_n(y_n) \leq f(x) + \delta_n$.
Hence we have:

$$v(f_n) \leq f_n(y_n) \leq f(x) + \delta_n \leq v(f) + \epsilon + \delta_n. \tag{3}$$

Passing to the upper limit we have:

$$\limsup v(f_n) \leq v(f). \tag{4}$$

Now consider $x_n \in \operatorname{lev}(f_n; v(f_n) + \epsilon_n)$. By hypothesis x_n is bounded, so, from Theorem 2.4 we get that $f_n(x_n)$ is bounded below; moreover, from $f_n(x_n) \le v(f_n) + \epsilon_n$ and (3) we obtain that $f_n(x_n)$ is also bounded above and therefore, by Theorem 2.4, we have that there exist $y_n \in X$, $\delta_n \ge 0$, such that $\|y_n - x_n\| \to 0$, $\delta_n \to 0$ and $f(y_n) \le f_n(x_n) + \delta_n$. Then we have:

$$v(f) \le f(y_n) \le f_n(x_n) + \delta_n \le v(f_n) + \epsilon_n + \delta_n.$$

Passing to the lower limit we get:

$$v(f) \le \liminf v(f_n).$$

Hence, recalling (4), we have $v(f_n) \to v(f)$. □

Remark 3.3. Notice that we haven't used the hypothesis of equiboundedness of sublevel sets to prove upper semicontinuity of the value function v, which therefore is valid more generally for proper and lower semicontinuous functions only.

We shall use the following assumption repeatedly on a sequence ϵ_n and on a sequence $f_n : X \to (-\infty, +\infty]$:

Assumption 3.4.

1. $\epsilon_n > 0$ and $\epsilon_n \to 0$;
2. f_n are proper, lower semicontinuous and there exists $m > 0$ such that $\operatorname{lev}(f_n; v(f_n) + \epsilon_n) \subset mU$ for each n sufficiently large.

Now we are able to prove the main result of this section:

Theorem 3.5. *Let X be a Banach space and $f : X \to (-\infty, +\infty]$ be bounded from below, proper and lower semicontinuous.*

1. *Suppose that for every sequences ϵ_n, f_n satisfying Assumption 3.4 with $f := \tau_{aw} - \lim f_n$, every asymptotically minimizing sequence corresponding to f_n is convergent. Then f is well-posed.*
2. *Suppose f is well-posed. Then for every sequences ϵ_n and f_n satisfying Assumption 3.4 and $f := \tau_{aw} - \lim f_n$, whenever x_n is asymptotically minimizing, then x_n is convergent to the unique minimizer of f.*

Proof. 1. Let us define $f_n = f$ for each $n \in \mathbb{N}$. Consider $\epsilon_n > 0$, $\epsilon_n \to 0$ and two sequences x_n and y_n belonging to $\operatorname{lev}(f; v(f) + \epsilon_n)$ for each $n \in \mathbb{N}$. Suppose $\lim y_n = y$ and $\lim x_n = x$. Now consider the sequence:

$$z_n := \begin{cases} x_n & \text{if } n \text{ is even,} \\ y_n & \text{if } n \text{ is odd.} \end{cases}$$

The sequence z_n is minimizing, thus convergent and thus $x = y$.
All the sequences which satisfy the hypothesis are therefore convergent to the same point x_0.
Now, let x_n be a minimizing sequence and $x_{n_k} := y_k$ any subsequence. By the definition of minimizing sequence there exists a subsequence $y_{k_h} := z_h$ such that $z_h \in \operatorname{lev}(f; v(f) + \epsilon_h)$, $\epsilon_h \to 0$, which is convergent to x_0. Since the subsequence was arbitrary, x_n is itself

convergent. On the other hand, from the definition of minimizing sequence and of lower semicontinuity of f, we get:

$$\liminf\,(v(f)+\epsilon_h) \geq \liminf f(z_h) \geq f(x_0) \geq v(f),$$

and so $x_0 \in \operatorname{argmin} f$.
2. Let x be the minimum point of f. By well-posedness and Lemma 3.2, $v(f_n) \to f(x)$. Pick $x_n \in \operatorname{lev}(f_n; v(f_n)+\epsilon_n)$ and fix $\epsilon > 0$: we will find N_ϵ such that for each $n > N_\epsilon$, $\|x_n - x\| < \epsilon$.
Since f is well-posed, from the Furi-Vignoli characterization, Theorem 2.8, we get the existence of $\delta > 0$ such that

$$\operatorname{diam}[\operatorname{lev}(f; v(f)+\delta)] < \epsilon/2. \tag{5}$$

Moreover there exists $N \in \mathbb{N}$ such that, when $n > N$ we have:

- $\epsilon_n < \delta/3$,
- $v(f_n) < v(f) + \delta/3$.

Since $f_n(x_n)$ is bounded, by Theorem 2.4, there exist y_n in X, $\gamma_n \geq 0$, such that $\|y_n - x_n\| \to 0$, $\gamma_n \to 0$ and $f(y_n) \leq f_n(x_n) + \gamma_n$. Choosing $N_\epsilon > N$ such that for each $n > N_\epsilon$ we have $\|x_n - y_n\| < \epsilon/2$ and $\gamma_n < \delta/3$, we get:

$$f(y_n) \leq f_n(x_n) + \gamma_n \leq v(f_n) + \gamma_n + \epsilon_n \leq v(f) + \delta.$$

Therefore, by (5) we have:

$$\|x_n - x\| \leq \|x_n - y_n\| + \|y_n - x\| < \epsilon.$$

Thus the sequence x_n goes to x and the proof is complete. □

The hypothesis of equiboundedness of sublevel sets cannot be omitted, as the following example shows:

Example 3.6. Let $f, f_n : \mathbb{R} \to \mathbb{R}$ be such that $f(x) = |x|$ and:

$$f_n(x) = \begin{cases} |x| & \text{if } |x| \leq n, \\ -|x| + 2n & \text{if } n < |x| \leq 2n, \\ |x| + 2n & \text{if } |x| > 2n. \end{cases}$$

Clearly f is well-posed and 0 is the unique minimizer, $f = \tau_{aw} - \lim f_n$ since f_n converges uniformly to f on bounded sets (for a proof of this fact, see e.g. [4]), but taking $x_n = 2n$ we have $f_n(x_n) = \min f_n = 0$, however $x_n \not\to 0$.

Notice that the condition on the equiboundedness of sublevel sets is satisfied when the approximating functions are convex: this follows from Lemma 3.1 and Theorem 3.6 of [5]. The same condition is also satisfied when the functions f_n are equicoercive:

Proposition 3.7. *Let $f, f_n : X \to (-\infty, +\infty]$ with $\limsup v(f_n) < +\infty$. Moreover suppose there exists $\varphi : X \to (-\infty, +\infty]$ such that:*

$$f_n(x) \geq \varphi(x) \text{ for each } n \quad \text{and} \quad \lim_{\|x\| \to +\infty} \varphi(x) = +\infty$$

and let $f = \tau_{aw} - \lim f_n$, $\epsilon_n > 0$, $\epsilon_n \to 0$.
Then there exists $m > 0$ such that $\operatorname{lev}(f_n; v(f_n) + \epsilon_n) \subset mU$ if n is sufficiently large.

Proof. By Remark 3.3 we get $v(f_n) < v(f) + 1$ if n is sufficiently large; moreover we can assume without loss of generality $\epsilon_n \leq 1$ for each n. Taking $x_n \in \operatorname{lev}(f_n; v(f_n) + \epsilon_n)$, with n sufficiently large, we have:

$$\varphi(x_n) \leq f_n(x_n) \leq v(f_n) + 1 \leq v(f) + 2. \tag{6}$$

Since $\varphi(x) \to +\infty$ if $\|x\| \to +\infty$, there exists $R > 0$ such that

$$\varphi(x) > v(f) + 2 \quad \text{if} \quad \|x\| > R.$$

Thus, from (6), we deduce $\|x_n\| \leq R$, i. e. equiboundedness of sublevel sets. □

Let us change point of view, and find the relations between well-posedness of a given function f and well-posedness of its convex envelope $\hat{f}$.

Definition 3.8. Let $f : X \to (-\infty, +\infty]$ be proper, bounded from below and lower semicontinuous. We define the convex, lower semicontinuous regularization of f to be the function $\hat{f}$ such that

$$\operatorname{epi}\hat{f} = \operatorname{cl}\operatorname{co}(\operatorname{epi} f).$$

The definition is consistent, in the sense that the convex hull of an epigraph is still an epigraph. Moreover $\hat{f}$ is the largest convex and lower semicontinuous function minorizing f.

Remark 3.9. Let $f : X \to (-\infty, +\infty]$ be as in the definition above and suppose that $\min f = 0$. This means that $\operatorname{epi} f \subseteq X \times [0, +\infty]$ which is a closed and convex subset of $X \times (-\infty, +\infty]$ so, by the definition of $\hat{f}$, we have $\operatorname{epi}\hat{f} \subseteq X \times [0, +\infty]$.

The following theorem establishes the relation between well-posedness of a given function f and well-posedness of its convex, lower semicontinuous regularization:

Theorem 3.10. *Let $f : X \to (-\infty, +\infty]$ be proper and lower semicontinuous. Suppose $\min f = f(0) = 0$. The following properties are equivalent:*

(i) f *is well-posed and* $\liminf_{\|x\| \to +\infty} \frac{f(x)}{\|x\|} \in (0, +\infty]$;

(ii) $\hat{f}$ *is well-posed.*

Proof. Suppose condition (i) is satisfied. Then there exists $M > 0$ such that $\liminf_{\|x\| \to +\infty} \frac{f(x)}{\|x\|} \geq 4M$. By Remark 3.9 we have that $\min f \leq \inf \hat{f}$. On the other hand $\hat{f}(x) \leq f(x)$ for every $x \in X$, so $\inf \hat{f} = f(0) = \hat{f}(0)$. We observe that, since f is well-posed, by Theorem 2.7, there exists a forcing function c such that $f(x) \geq c(\|x\|)$. Without loss of generality we can assume that $\liminf_{t \to +\infty} c(t)/t \geq 2M$. In fact, if $\liminf_{t \to +\infty} c(t)/t = 0$, we can replace c by the function c_1 defined in this way:

$$c_1(t) = \begin{cases} c(t) & \text{if } t \leq r \\ 2Mt & \text{if } t > r, \end{cases}$$

with r such that $f(x) \geq 2M\|x\|$ when $\|x\| > r$. It's enough to prove that $\hat{c}$ is forcing. In fact we have:

$$\hat{f}(x) \geq \hat{c}(\|x\|),$$

which means $\hat{f}$ is well-posed in the case that $\hat{c}$ is forcing. First observe that $\hat{c}(x) \geq 0$ for every $x \in X$ and $\hat{c}(0) = c(0) = 0$ from the previous argument.
By contradiction, let us suppose that there exist $t_n \in [0, +\infty)$ such that

$$\hat{c}(t_n) \to 0, \quad \text{but } t_n \not\to 0.$$

Since $t_n \not\to 0$, we can assume (maybe considering a subsequence) that there exists $a > 0$ such that $t_n \geq 2a$ for each n. Since a forcing function is well-posed , thanks to the Furi-Vignoli characterization of well-posedness, Theorem 2.8, we can find $\delta > 0$ such that $\operatorname{lev}(c; \delta) \subset [0, a)$.
Moreover, since $\liminf c(t)/t = 2M$, there exists $R > 0$ such that $c(t) \geq Mt$ for all $t > R$.
We define $g : [0, +\infty) \to [0, +\infty)$ to be the following function:

$$g(t) = \begin{cases} 0 & \text{if } t \leq a \\ \delta & \text{if } a < t \leq K \\ Mt + \delta - MK & \text{if } t > K. \end{cases}$$

Choose K such that $K \geq \max\{\frac{\delta}{M} + a,\ R\}$. By construction, we get $g(t) \leq c(t)$ for every t. In this way $\hat{g}$ is the following function:

$$\hat{g}(t) = \begin{cases} 0 & \text{if } t \leq a \\ \frac{\delta}{K-a} t - \frac{\delta a}{K-a} & \text{if } a < t < K \\ Mt + \delta - MK & \text{if } t \geq K. \end{cases}$$

$\hat{g}$ is a convex and lower semicontinuous function which minorizes c, so we have $\hat{c} \geq \hat{g}$, and $\min c = \hat{g}(0) = 0$. The sequence t_n is therefore minimizing also for $\hat{g}$; but $t_n \geq 2a$ and $\hat{g}(t_n) \to 0$, a contradiction. We then have that $\hat{c}$ is forcing and $\hat{f}$ is well-posed.

Now suppose $\hat{f}$ is well-posed. We get $\min \hat{f} = \min f = \hat{f}(0) = f(0)$.
Let x_n be a minimizing sequence for f: then x_n is a minimizing sequence also for $\hat{f}$ therefore $x_n \to 0$. This means that f is well-posed.
For the sake of contradiction, assume $\liminf_{\|x\| \to +\infty} (f(x)/\|x\|) = 0$.
Then it's possible to find a sequence y_k in X such that $\|y_k\| > k$ and $f(y_k) < \frac{1}{k}\|y_k\|$.
Recalling the definition of epigraph $(y_k, \frac{1}{k}\|y_k\|) \in \operatorname{epi} f$.
Since $\operatorname{epi}\hat{f} = \operatorname{cl}\operatorname{co}(\operatorname{epi} f)$, the segments joining $(0, 0)$ to $(y_k, \frac{1}{k}\|y_k\|)$ are entirely contained in $\operatorname{epi}\hat{f}$.
Consider the sequence $z_k := t_k y_k$, with $t_k = \frac{1}{\|y_k\|}$.
Then $\|t_k y_k\| = |t_k|\|y_k\| = 1$, and so $z_k \not\to 0$. Moreover $(t_k y_k, (t_k/k)\|y_k\|) \in \operatorname{epi}\hat{f}$ and $\frac{t_k}{k}\|y_k\| = \frac{1}{k} \to 0$; that is z_k is minimizing for $\hat{f}$, but $z_k \not\to 0$, which is a contradiction, because $\hat{f}$ is well-posed. □

4. A dominated convergence theorem

It is useful to obtain criteria which guarantee that a sequence f_n of lower semicontinuous functions converges in the AW-sense to a certain function f. This is relevant e. g. in order to apply Theorem 3.5.
In this section we study some class of integral functionals, and we prove that under suitable hypotheses convergence of the sequence of the integrands is a sufficient condition

to assure convergence of the sequence of the integrals (both in the AW-sense).

Let $(T, \mathcal{A}, \mu)$ a measure space and let μ be a positive finite measure on $\mathcal{A}$ which is complete.

Definition 4.1. An integrand is a function $f : T \times \mathbb{R}^k \to (-\infty, +\infty]$. If for each $t \in T$, the set epi$f(t, \cdot)$ is closed and nonempty, and if the multifunction $t \mapsto$ epi$f(t, \cdot)$ is measurable, we say that f is a normal integrand.

Let f be a normal integrand and let $\mathcal{L}$ be a class of measurable functions from T to R^k; then for each $u \in \mathcal{L}$, the function $t \mapsto f(t, u(t))$ is measurable; if it is summable or it is majorized by a summable function, a natural value can be assigned to the integral

$$F(u) = \int_T f(t, u(t))\, d\mu. \tag{7}$$

Otherwise, we set $F(u) = +\infty$. In this way, F is a well-defined extended real-valued functional on the space $\mathcal{L}$; we say that F is the integral functional associated with the normal integrand f. See [9].

Let h be a function on $\mathbb{R}^k$ bounded from below and lower semicontinuous; then for each $z \in \mathbb{R}^k$

$$H_z(x) = h(x) + |x - z|$$

has at least one global minimum point on $\mathbb{R}^k$. Let us define

$$\mathrm{Prox}[z, h] = \mathrm{argmin} H_z.$$

Given f a normal integrand defined on $T \times \mathbb{R}^k$ bounded from below and a measurable function $u : T \to \mathbb{R}^k$, we can consider the multifunction P given by:

$$P(t) = \mathrm{Prox}[u(t), f(t, \cdot)].$$

Since the function $(t, x) \mapsto f(t, x) + |x - u(t)|$ is itself a normal integrand, we get that P is a measurable multifunction from Theorem 2K of [9]. Moreover, by Theorem 1C of [9], there exists a measurable selection of P, which we will denote by $t \mapsto \mathrm{prox}[u(t), f(t, \cdot)]$.

Lemma 4.2. *Let $f, f_n : T \times \mathbb{R}^m \to [0, +\infty)$ be normal integrands with $\rho_0 > 0$. Fix $G \subset T$, G measurable and write for all $\rho > \rho_0$,*

$$\sup_{t \in G} \mathrm{haus}_\rho(f_n(t, \cdot), f(t, \cdot)) := h_n(\rho).$$

Consider $x : T \to \mathbb{R}^m$ such that, for some ρ,

$$\{x(t) : t \in G\} \subset \rho U \quad \textit{and} \quad \{f(t, x(t)) : t \in G\} \subset \rho U$$

and let $G_n \subset T$ be measurable and $x_n : T \to \mathbb{R}^m$ be a sequence such that:

$$\{x_n(t) : t \in G_n,\, n \in \mathbb{N}\} \subset \rho U \quad \textit{and} \quad \{f_n(t, x_n(t)) : t \in G_n, n \in \mathbb{N}\} \subset \rho U.$$

Then, for every $\epsilon > 0$, $t \in G$ and for each $n \in \mathbb{N}$, we can find $x_{n,\epsilon}(t) \in \mathbb{R}^m$ such that:

$$|x(t) - x_{n,\epsilon}(t)| \le h_n(\rho) + \epsilon$$

and

$$f_n(t, x_{n,\epsilon}(t)) \leq f(t, x(t)) + h_n(\rho) + \epsilon$$

and for every $\epsilon > 0$, $t \in G_n$, $n \in \mathbb{N}$ *we can find* $y_{n,\epsilon}(t)$ *such that:*

$$|x_n(t) - y_{n,\epsilon}(t)| \leq h_n(\rho) + \epsilon$$

and

$$f(t, y_{n,\epsilon}(t)) \leq f_n(t, x_n(t)) + h_n(\rho) + \epsilon.$$

Proof. By Theorem 2.3, for every $\epsilon > 0$, $t \in G$ and for each $n \in \mathbb{N}$, we can find $x_{n,\epsilon}(t) \in \mathbb{R}^m$ such that:

$$|x(t) - x_{n,\epsilon}(t)| \leq \text{haus}_\rho(f(t,\cdot), f_n(t,\cdot)) + \epsilon$$

and

$$f_n(t, x_{n.\epsilon}(t)) \leq f(t, x(t)) + \text{haus}_\rho(f(t,\cdot), f_n(t,\cdot)) + \epsilon.$$

In particular we obtain:

$$|x(t) - x_{n,\epsilon}(t)| \leq h_n(\rho) + \epsilon$$

and

$$f_n(t, x_{n,\epsilon}(t)) \leq f(t, x(t)) + h_n(\rho) + \epsilon.$$

Finally, by Theorem 2.3, for every $\epsilon > 0$ and $t \in G_n$ there exist $y_{n,\epsilon}(t)$ such that:

$$|x_n(t) - y_{n,\epsilon}(t)| \leq h_n(\rho) + \epsilon$$

and

$$f(t, y_{n,\epsilon}(t)) \leq f_n(t, x_n(t)) + h_n(\rho) + \epsilon.$$

□

Given $f, f_n : T \to [0, +\infty)$ normal integrands we can consider the associated integral functionals F, F_n defined by (7) on $L^p = L^p(T; \mathbb{R}^m)$, with $p \in (1, +\infty)$. In the sequel we denote by p' the conjugate exponent of p.
Using Lemma 4.2 we can prove the following Theorem:

Theorem 4.3. *Let* $f, f_n : T \times \mathbb{R}^m \to [0, +\infty)$ *be normal integrands. Suppose there exist* $\varphi : T \to [0, +\infty)$ *in* $L^{p'}$ *and* $\psi : T \to [0, +\infty)$ *in* L^1 *such that*

$$f_n(t, x) \leq \varphi(t)|x| + \psi(t) \text{ and } f(t, x) \leq \varphi(t)|x| + \psi(t)$$

for each $n \in \mathbb{N}$, $t \in T$, $x \in \mathbb{R}^m$. *Moreover assume*

$$f(t,\cdot) = \tau_{aw} - \lim f_n(t,\cdot)$$

for almost every t. *Then:*

$$F = \tau_{aw} - \lim F_n.$$

Proof. We shall prove:

(i) For all $r > 0$ and $\epsilon > 0$ there exists $\eta_\epsilon \in \mathbb{N}$ such that for all $u \in L^p$ with $\|u\|_p \leq r$ and $F(u) \leq r$, and for each $n > \eta_\epsilon$ we can find $w_n \in L^p$ which satisfy:

$$\|u - w_n\|_p \leq \epsilon \text{ and } F_n(w_n) \leq F(u) + \epsilon.$$

(ii) For all $r > 0$ and $\epsilon > 0$ there exists $\nu_\epsilon \in \mathbb{N}$ such that for all $u_n \in L^p$ with $\|u_n\| \leq r$ and $F_n(u_n) \leq r$, and for each $n > \nu_\epsilon$ we can find $y_n \in L^p$ which satisfy:

$$\|u_n - y_n\|_p \leq \epsilon \text{ and } F(y_n) \leq F_n(u_n) + \epsilon.$$

Since F, F_n are proper, lower semicontinuous and bounded from below by the same constant, we can apply part 2 of Theorem 2.3, therefore the previous conditions will imply that for every $r > 0$ and $\epsilon > 0$, there exists $N_\epsilon := \eta_\epsilon + \nu_\epsilon$ such that $\text{haus}_r(F, F_n) < \epsilon$ if $n > N_\epsilon$, that is exactly what we have to prove.
(i) Fix $r > 0$ an let $u \in L^p(T; \mathbb{R}^m)$ such that

$$\|u\|_p \leq r \text{ and also } F(u) \leq r. \tag{8}$$

Fix $\epsilon > 0$ and let $K > 0$ to be chosen later.
Define:

$$S_K := \{t \in T : |u(t)| \geq K\} \textit{ and } T_K := \{t \in T : f(t, u(t)) \geq K\}.$$

We have

$$(\|u\|_p)^p = \int_T |u(t)|^p \, d\mu = \int_{T \setminus S_K} |u(t)|^p \, d\mu + \int_{S_K} |u(t)|^p \, d\mu \leq r^p.$$

Hence:

$$|S_K| \leq \frac{r^p}{K^p}, \tag{9}$$

where $|\cdot|$ denotes measure. In the same way, from (8):

$$\int_T f(t, u(t)) \, d\mu = \int_{T \setminus T_K} f(t, u(t)) \, d\mu + \int_{T_K} f(t, u(t)) d\mu \leq r$$

and therefore:

$$|T_K| \leq \frac{r}{K}.$$

By assumptions $\psi \in L^1$ and so, by the absolute continuity of the integrals, for each $\delta > 0$ there exists $m_\delta > 0$ such that, given a measurable set G for which $|G| \leq m_\delta$, we have $\int_G \psi(t) \, d\mu \leq \delta$.
On the other hand $\varphi^{p'} \in L^1$; thus for all $\delta > 0$ there exists $c_\delta > 0$ such that, given a measurable set G for which $|G| \leq c_\delta$, we get $\int_G \varphi^{p'}(t) \, d\mu \leq \delta$.
Now, choose K to have

$$\frac{r^p}{K^p} + \frac{r}{K} < \min\left\{\frac{m_{(\epsilon/3)}}{2}, \frac{c_{(\epsilon/3r)^{p'}}}{2}\right\}. \tag{10}$$

By hypothesis $\text{haus}_\rho(f(t, \cdot), f_n(t, \cdot)) \to 0$ for every ρ and almost every t; by the Severini-Egoroff Theorem we have that if $\lambda > 0$ there exists a measurable set $E_\lambda \subset T$ such that $|T \setminus E_\lambda| \leq \lambda$ and $\text{haus}_\rho(f(t, \cdot), f_n(t, \cdot))$ approaches 0 uniformly on E_λ. In particular, choosing $\lambda = \min\{\frac{m_{(\epsilon/3)}}{2}, \frac{c_{(\epsilon/3r)^{p'}}}{2}\}$ we can find some measurable E_ϵ such that:

$$|T \setminus E_\epsilon| \leq \min\left\{\frac{m_{(\epsilon/3)}}{2}, \frac{c_{(\epsilon/3r)^{p'}}}{2}\right\} \tag{11}$$

and $\mathrm{haus}_\rho(f(t,\cdot), f_n(t,\cdot))$ goes to 0 uniformly on E_ϵ. Let $t \in E_\epsilon \setminus (S_K \cup T_K)$, then:

$$\{(u(t), f(t,u(t)))\} \subset KU \times [0,K].$$

From Lemma 4.2, when $t \in E_\epsilon \setminus (S_K \cup T_K)$, it's possible to find $u_{n,\epsilon}(t)$ such that

$$|u(t) - u_{n,\epsilon}(t)| < h_n(K) + \frac{1}{2}\min\left\{\frac{\epsilon}{2|T|^{1/p}}, \frac{\epsilon}{6|T|}\right\} \tag{12}$$

where $h_n(K) = \sup_{t\in G} \mathrm{haus}_K(f_n(t,\cdot), f(t,\cdot))$ with $G := E_\epsilon \setminus (S_K \cup T_K)$ and

$$f_n(t, u_{n,\epsilon}(t)) < f(t,u(t)) + h_n(K) + \frac{1}{2}\min\left\{\frac{\epsilon}{2|T|^{1/p}}, \frac{\epsilon}{6|T|}\right\}. \tag{13}$$

On the other hand, from uniform convergence there exists η_ϵ such that, given $n > \eta_\epsilon$, we get:

$$h_n(K) \le \frac{1}{2}\min\left\{\frac{\epsilon}{2|T|^{1/p}}, \frac{\epsilon}{6|T|}\right\}. \tag{14}$$

Now, using (14), and rewriting (12) and (13), we obtain:

$$|u(t) - u_{n,\epsilon}(t)| < \min\left\{\frac{\epsilon}{2|T|^{1/p}}, \frac{\epsilon}{6|T|}\right\} \tag{15}$$

and

$$f_n(t, u_{n,\epsilon}(t)) < f(t,u(t)) + \min\left\{\frac{\epsilon}{2|T|^{1/p}}, \frac{\epsilon}{6|T|}\right\}. \tag{16}$$

Consider the functions:

$$v_n(t) := \mathrm{prox}[u(t), \max\{0, f_n(t,\cdot) - f(t,u(t))\}].$$

Now fix $n > \eta_\epsilon$. For any $t \in E_\epsilon \setminus (S_K \cup T_K)$, by (15) and (16) we have:

$$\begin{aligned} 0 &\le |u(t) - v_n(t)| + \max\{0, f_n(t, v_n(t)) - f(t,u(t))\} \\ &\le |u(t) - u_{n,\epsilon}(t)| + \max\{0, f_n(t, u_{n,\epsilon}(t)) - f(t,u(t))\} \\ &< \min\left\{\frac{\epsilon}{3|T|}, \frac{\epsilon}{|T|^{1/p}}\right\} \end{aligned} \tag{17}$$

Define $C_{K,\epsilon} := (T \setminus E_\epsilon) \cup S_K \cup T_K$ and

$$w_n(t) := \begin{cases} v_n(t) & \text{if } t \in E_\epsilon \setminus (S_K \cup T_K), \\ u(t) & \text{if } t \in C_{K,\epsilon}. \end{cases}$$

By construction, the functions w_n are measurable and we have

$$\|w_n - u\|_\infty < \frac{\epsilon}{|T|^{1/p}}$$

which implies $\|w_n - u\|_p < \epsilon$.
Moreover, by (9), (10) and (11), we get:

$$\begin{aligned} |C_{K,\epsilon}| &= |S_K \cup T_K \cup (T \setminus E_\epsilon)| \\ &\le \frac{r^p}{K^p} + \frac{r}{K} + |T \setminus E_\epsilon| \\ &\le \min\{m_{(\epsilon/3)}, c_{(\epsilon/3r)^{p'}}\}. \end{aligned}$$

Hence:

$$\int_{C_{K,\epsilon}} \varphi(t)^{p'}\, d\mu \leq \left(\frac{\epsilon}{3r}\right)^{p'} \text{ and } \int_{C_{K,\epsilon}} \psi(t)\, d\mu \leq \frac{\epsilon}{3}.$$

Then, using Hölder inequality and recalling (8) and (17), it follows that:

$$\begin{aligned} F_n(w_n) &= \int_T f_n(t, w_n(t))\, d\mu \\ &= \int_{T\setminus C_{K,\epsilon}} f_n(t, v_n(t))\, d\mu + \int_{C_{K,\epsilon}} f_n(t, u(t))\, d\mu \\ &\leq \int_{T\setminus C_{K,\epsilon}} (f(t, u(t)) + \frac{\epsilon}{3|T|})\, d\mu + \int_{C_{K,\epsilon}} (\varphi(t)|u(t)| + \psi(t))\, d\mu \\ &\leq \int_T f(t, u(t))\, d\mu + \frac{\epsilon}{3} + \int_{C_{K,\epsilon}} \varphi(t)|u(t)|\, d\mu + \int_{C_{K,\epsilon}} \psi(t)\, d\mu \\ &\leq F(u) + \frac{\epsilon}{3} + \left(\int_{C_{K,\epsilon}} \varphi(t)^{p'}\, d\mu\right)^{1/p'} \|u\|_p + \frac{\epsilon}{3} \\ &\leq F(u) + \epsilon. \end{aligned}$$

Therefore the condition (i) is satisfied.
(ii) Let $u_n \in L^p(T;\mathbb{R}^m)$ such that $\|u_n\|_p \leq r$ and $F_n(u_n) \leq r$. Considering $S^n_K := \{t \in T : |u_n(t)| \geq K\}$, for each n it follows that: $|S^n_K| \leq r^p/K^p$. Similarly, defining $T^n_K = \{t \in T : f_n(t, u_n(t)) \geq K\}$, for each n it happens that $|T^n_K| \leq \frac{r}{K}$.
Choose K such that

$$\frac{r^p}{K^p} + \frac{r}{K} \leq \min\left\{\frac{m_{(\epsilon/3)}}{2}, \frac{c_{(\epsilon/3r)^{p'}}}{2}\right\}$$

for each $n \in \mathbb{N}$.
Let E_ϵ as in (i).
Following the same proof as in (i), since $\mathrm{haus}_\rho(f_n(t,\cdot), f(t,\cdot)) \to 0$ uniformly with respect to t on E_ϵ for each $\rho > 0$, as a consequence of Lemma 4.2 there exists ν_ϵ such that, given $n > \nu_\epsilon$, for all $t \in E_\epsilon \setminus (S^n_K \cup T^n_K)$ we can find $z_{n,\epsilon}(t)$ such that

$$|z_{n,\epsilon}(t) - u_n(t)| < \min\left\{\frac{\epsilon}{2|T|^{1/p}}, \frac{\epsilon}{6|T|}\right\} \tag{18}$$

and

$$f(t, z_{n,\epsilon}(t)) \leq f_n(t, u_n(t)) + \min\left\{\frac{\epsilon}{2|T|^{1/p}}, \frac{\epsilon}{6|T|}\right\}. \tag{19}$$

Define $v_n(t) := \mathrm{prox}[u_n(t), \max\{0, f(t,\cdot) - f_n(t, u_n(t))\}]$.
Now fix $n > \nu_\epsilon$. For any $t \in E_\epsilon \setminus (S^n_K \cup T^n_K)$, by (18) and (19), we have:

$$\begin{aligned} 0 &\leq |u_n(t) - v_n(t)| + \max\{0, f(t, v_n(t)) - f_n(t, u_n(t))\} \\ &\leq |u_n(t) - z_{n,\epsilon}(t)| + \max\{0, f(t, z_{n,\epsilon}(t)) - f_n(t, u_n(t))\} \\ &< \min\left\{\frac{\epsilon}{3|T|}, \frac{\epsilon}{|T|^{1/p}}\right\}. \end{aligned} \tag{20}$$

Let us construct the functions

$$y_n(t) := \begin{cases} v_n(t) & \text{if } t \in E_\epsilon \setminus (S^n_K \cup T^n_K) \\ u_n(t) & \text{if } t \in (T \setminus E_\epsilon) \cup S^n_K \cup T^n_K. \end{cases}$$

For every $n > \nu_\epsilon$ we have:

$$\|u_n - y_n\|_p < \epsilon$$

and

$$\begin{aligned}
F(y_n) &= \int_T f(t, y_n(t))\, d\mu \\
&= \int_{E_\epsilon \setminus (S_K^n \cup T_K^n)} f(t, v_n(t))\, d\mu + \int_{(T\setminus E_\epsilon)\cup S_K^n \cup T_K^n} f(t, u_n(t))\, d\mu \\
&\leq \int_T (f_n(t, u_n(t)) + 2\frac{\epsilon}{6|T|})\, d\mu + \int_{(T\setminus E_\epsilon)\cup S_K^n \cup T_K^n} (\varphi(t)|u_n(t)| + \\
&\quad +\psi(t))\, d\mu \\
&\leq F_n(u_n) + \epsilon.
\end{aligned}$$

Taking $N_\epsilon = \eta_\epsilon + \nu_\epsilon$, the proof is complete. □

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