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On Λ–Convexity Conditions in the Theory of Lower Semicontinuous Functionals

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Consider the functional $I_f(u) = \int_\Omega f(u(x))\,dx$, where $u = (u_1, \ldots, u_m)$. Assume additionally that each u_j is constant along W_j, some subspace of $\mathbf{R}^n$. We find the family of cones Λ in $\mathbf{R}^m$ such that every Λ–convex function f defines a functional I_f which is lower semicontinuous under the sequential weak $*$ convergence in $L^\infty(\Omega, \mathbf{R}^m)$. Then we apply our result to functionals acting on distributional kernels of differential operators. We also discuss the relations of our problem to the rank–one conjecture of Morrey.

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1. Introduction

Assume that $\Omega \subseteq \mathbf{R}^n$, and $P = (P_1, \ldots, P_N)$ is the differential operator with constant coefficients,

$$P_k u = \sum_{i=1,\ldots,n, j=1,\ldots,m} a^k_{i,j} \frac{\partial u_j}{\partial x_i}, \ k = 1, \ldots, N. \tag{1}$$

Let $\mathcal{K} = \mathcal{K}(\Omega, P) = L^\infty(\Omega, \mathbf{R}^m) \cap \mathrm{Ker}P$, where $\mathrm{Ker}P$ is the distributional kernel of P. Take the continuous function $f : \mathbf{R}^m \to \mathbf{R}$, and consider the functional

$$I_f(u) = \int_\Omega f(u(x))\,dx, \ \text{where } u \in \mathcal{K}. \tag{2}$$

The natural problem in the theory of lower semicontinuous functionals is to look for conditions on f which guarantee lower semicontinuity and continuity of the related functional I_f (we restrict ourself to its simplest form), under the sequential weak $*$ convergence in $L^\infty(\Omega, \mathbf{R}^m)$.

The following result was established by Murat and Tartar (see e.g. [2, Theorem 3.1], [14, Theorem 2.1], [18, Theorem 10.1], [24, Corollary 10]).

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Theorem 1.1. *Define*

$$\begin{aligned} V &= \{(\xi,\lambda) : \xi \in \mathbf{R}^n, \xi \neq 0, \lambda \in \mathbf{R}^m, \sum_{i,j} a^k_{i,j}\xi_i\lambda_j = 0, \text{ for } k = 0,\dots,N\}, \\ \Lambda &= \{\lambda \in \mathbf{R}^m : \text{ there exists } \xi \in \mathbf{R}^n, \xi \neq 0, \text{ such that } (\xi,\lambda) \in V\}. \end{aligned}$$

(i) *If I_f given by (2) is lower semicontinuous with respect to the sequential L^∞–weak $*$ convergence in $\mathcal{K}$ then f is Λ–convex.*

(ii) *If I_f is continuous with respect to the sequential L^∞–weak $*$ convergence in $\mathcal{K}$ then f is Λ–affine.*

By Λ–convexity (Λ–affinity) we mean that for every $A \in \mathbf{R}^m$ and $\lambda \in \Lambda$, the function $\mathbf{R} \ni t \mapsto f(A + t\lambda)$ is convex (affine).

Theorem 1.1 states that lower semicontinuity of the functional I_f implies Λ–convexity of f, where Λ is the cone related to the system (1). In general Λ–convexity of f does not imply lower semicontinuity of the related functional I_f ([2, page 26], [14], [24], see also Šverák's celebrated result [23]).

The purpose of this paper is to contribute in finding some further relations between an algebraic structure of the operator P and the geometric properties of those continuous functions $f : \mathbf{R}^m \to \mathbf{R}$, which define lower semicontinuous functionals.

We restrict ourself to a simple model, where the equation $Pu = 0$ is equivalent to the condition $\partial_v u_i = 0$, for every $v \in W_i$, where W_i are linear subspaces of $\mathbf{R}^n$, given for $i = 1,\dots,m$. This will be denoted by

$$\partial_{W_i} u_i = 0, \text{ for } i = 1,\dots,m, \text{ or } \partial_W u = 0, \tag{3}$$

for $u = (u_1,\dots,u_m)$ and $W = (W_1,\dots,W_m)$.

Although (3) looks to be rather poor at first glance, almost nothing is known about lower semicontinuity properties of related functionals even in simple cases. Let for example $u : \mathbf{R}^2 \to \mathbf{R}^3$, and $Pu = (\frac{\partial u_1}{\partial y}, \frac{\partial u_2}{\partial x}, \frac{\partial u_3}{\partial x} - \frac{\partial u_3}{\partial y})$. Then the necessary and sufficient conditions on f for the functional I_f to be lower semicontinuous are not known. The functional related to such model plays an important role in the long time standing problem of rank–one convexity in the calculus of variations (see [21], [23], and [9], where new geometric conditions for quasiconvex functionals were obtained). Some further examples of functionals related to the system (3) can be found in [14, Section 7.3], [16], [19, 20], [24, Examples 5 and 6 and Propositions 15,16 and 17] and [25], also similar operators appear in the theory of geometric optics, see [6, 7].

When the model is related to the general system (1), and the operator P satisfies the so–called constant rank condition, then conditions on f which are equivalent to the lower semicontinuity of the related functional I_f were established by Fonseca and Müller, see [4]. In general these conditions are not equivalent to the Λ–convexity conditions. This happens for example when $P = \text{curl}$ is applied to each coordinate of $u = (u_1,\dots,u_m)$, ($u_i \in \mathbf{R}_n$) in a simply connected domain, and $m \geq 3$, $n \geq 2$ ([23]). In our case, when the system is given by (3), the constant rank condition fails in general.

Our approach is the following. At first we note that when the model is defined by (3), the cone Λ defined in Theorem 1.1 is the set theoretical sum of subspaces in $\mathbf{R}^m$, being of the form $A_1 \times \cdots \times A_m$, where $A_i = \{0\}$ or $A_i = \mathbf{R}$ (Fact 2.1). We denote such cone by $\Lambda = \Lambda_{\mathcal{B}(W)}$, where $\mathcal{B}(W)$ is a subset of $\{0,1\}^m$ that describes the intersection properties of the W_i's (see (6) and (7)). Then we consider the following problem. Let $\mathcal{B}$ be an arbitrary subset of $\{0,1\}^m$, and $\Lambda = \Lambda_{\mathcal{B}}$ (see (6) for definition). We ask what intersection properties of the W_i's in (3) guarantee the implication that every Λ convex function f defines lower semicontinuous functional I_f. Obviously, if $\mathcal{B} = \mathcal{B}(W)$, that is if $\Lambda_{\mathcal{B}}$ is the usual cone Λ defined in Theorem 1.1, then this implication guarantees the equivalence in (i) in the statement of Theorem 1.1. In Theorem 3.4 we prove that if m–tuples $\{W_j\}_{j=1,\dots,m}$ satisfy the $\mathcal{B}$–chain condition (according to the author's notation) then every $\Lambda_{\mathcal{B}}$ convex function defines the lower semicontinuous functional related to the system like (3). The $\mathcal{B}$–chain condition is defined with the help of the certain finite division process in m–products of linear subspaces of $\mathbf{R}^n$, which finally leads to the model like $\partial_{\mathbf{R}^n} u_i = 0$, for $i = 1, \dots, m$ (Definition 3.1). The approach is based on the application of the theory of Young measures, and the special reduction lemma (Lemma 3.12), which allows to reduce the investigation of the model with $\Omega \subseteq \mathbf{R}^n$ to the similar model, but with the domain in some $\mathbf{R}^k$ where k is smaller than n.

This paper is the continuation of author's work [8], where the family $\mathcal{F}$ of m–tuples $\{W_j\}_{j=0,\dots,m}$ of subspaces of $\mathbf{R}^n$, for which the equivalence in (ii) in Theorem 1.1 holds was characterized completely. The author has also found the necessary condition on m–tuples $\{W_j\}_{j=0,\dots,m}$ of subspaces of $\mathbf{R}^n$, under which the equivalence in (i) holds ([8, Theorem 3.2]). This condition was not sufficient, but most of the examples known in the literature, related to the system like (3), where the equivalence in (i) in the statement of Theorem 1.1 holds do satisfy it (see e.g. [14, Section 7.3], [16], [19, 20], [24, Examples 5 and 6 and Propositions 15, 16 and 17] and [25]). On the other hand, our previous condition in the statement of Theorem 3.2 in [8] does immediately imply our $\mathcal{B}$–chain condition from this paper for $\mathcal{B} = \mathcal{B}(W)$, and it is essentially stronger (see Remark 3.15 and Example 3.16).

We have also solved the more general problem: having the family of m–tuples $\{W_j\}_{j=0,\dots,m}$, we find the family of cones Λ for which Λ–convex functions define weakly lower semicontinuous functionals (Theorem 4.7). That leads to the condition similar to the well known polyconvexity condition (see Remark 4.8).

In Chapter 4 we explain how to apply our special approach, restricted to the model given by (3), to the general model, where the operator P is given by (1), while in Chapter 5 we show some relations and applications of our results to the famous rank–one problem in the calculus of variations.

2. Notation and preliminaries

Notation.

Let $m \in \mathbf{N}$. We recall the standard order in $\{0,1\}^m$: for $I, J \in \{0,1\}^m$ we have $I > J$ if either $i_1 > j_1$ or there is $l < m$ such that $i_s = j_s$ for $s = 1, \dots, l$, and $i_{l+1} > j_{l+1}$.

We set for $I \in \{0,1\}^m$
$D(I) = \{r \in \{1, \dots, m\} : I \text{ has } 1 \text{ on } r-\text{th place}\}$, $D^*(I) = \{1, \dots, m\} \setminus D(I)$. If $D(I) = \{i\}$ we write $I = \delta_i$. Given $I \in \{0,1\}^m$, we denote by I^* such element of $\{0,1\}^m$ that $D(I^*) = D^*(I)$.

The symbol $< \cdot, \cdot >$ will stand for the standard scalar product in $\mathbf{R}^n$ and $W^\perp$ for the space orthogonal to the subspace $W \subseteq \mathbf{R}^n$ with respect to the standard scalar product. The standard basis in $\mathbf{R}^n$ will be denoted by $\{e_1, \ldots, e_n\}$.

We denote the m–product of the sum of Grassmannians in $\mathbf{R}^n$ by

$$\widetilde{\mathcal{W}}(n,m) = \{W = (W_1, \ldots, W_m) : W_i \text{ are linear subspaces of } \mathbf{R}^n\}, \tag{4}$$

and its special subset by

$$\mathcal{W}(n,m) = \{W = (W_1, \ldots, W_m) \in \widetilde{\mathcal{W}}(n,m) : +_{i\in\{1,\ldots,m\}} W_i = \mathbf{R}^n\}, \tag{5}$$

where $+_{i\in\{1,\ldots,m\}} W_i = \{v \in \mathbf{R}^n : v = \sum_{i=1}^m v_i, v_i \in W_i\}$ is the Minkowski's sum of of the W_i's.

If $W \in \widetilde{\mathcal{W}}(n,m)$ and $W = (W_1, \ldots, W_m)$, we set $W^\perp = (W_1^\perp, \ldots, W_m^\perp)$.

If $I \in \{0,1\}^m$, we denote $W^I = \cap_{i\in D(I)} W_i$ if $D(I) \neq \emptyset$, $\;W^0 = \mathbf{R}^n$ (to abbreviate we write simply 0 instead of $(0, \ldots, 0)$). For example when $m = 3$ we have $W^{(1,1,0)} = W_1 \cap W_2$.

By $\mathbf{R}^{\times I}$ we denote $\mathbf{R}^{\times i_1} \times \cdots \times \mathbf{R}^{\times i_m}$, where $\mathbf{R}^{\times 0} = \{0\}$, $\mathbf{R}^{\times 1} = \mathbf{R}$.

For example $\mathbf{R}^{\times(1,0,1)} = \mathbf{R} \times \{0\} \times \mathbf{R}$. If $\mathcal{B} \subseteq \{0,1\}^m$, and $W \in \widetilde{\mathcal{W}}(n,m)$, we write

$$\Lambda_{\mathcal{B}} = \cup_{I\in\mathcal{B}} \mathbf{R}^{\times I}, \text{ and } E_{\mathcal{B},W} = +_{I\in\mathcal{B}} W^{I^*}. \tag{6}$$

Given $m \in \mathbf{N}$, $W = (W_1, \ldots, W_m) \in \widetilde{\mathcal{W}}(n,m)$, we introduce special subsets of $\{0,1\}^m$,

$$\begin{aligned} \mathcal{A}(W) &= \{I \in \{0,1\}^m \; : (W^\perp)^I \neq \{0\}\}, \\ \mathcal{B}(W) &= \{I \in \mathcal{A}(W) : \text{ if } J \in \{0,1\}^m, D(J) \supseteq D(I), \text{and } J \neq I \\ &\quad\; \text{then } (W^\perp)^J = \{0\}\}. \end{aligned} \tag{7}$$

One can easily verify that the following fact holds (see also [8, Theorem 3.1]).

Fact 2.1. *Consider the system* (3) *with* $W \in \widetilde{\mathcal{W}}(n,m)$. *Then the manifold* V *and the characteristic cone* Λ *associated to* (3) *are given by* $V = \bigcup_{I\in\mathcal{A}(W)} (W^\perp)^I \times \mathbf{R}^{\times I}$, $\Lambda = \bigcup_{I\in\mathcal{B}(W)} \mathbf{R}^{\times I}$.

The space of those $u \in L^\infty(\Omega, \mathbf{R}^m)$, which satisfy (3) in the sense of distributions, equipped with the topology of weak $*$ convergence in $L^\infty(\Omega, \mathbf{R}^m)$, will be denoted by $\mathcal{K}(\Omega, W)$, where $W = (W_1, \ldots, W_m)$.

If $Y \subseteq \mathbf{R}^n$ is a linear subspace, then by $\pi_Y : \mathbf{R}^n \to Y$ we denote the orthogonal projection on Y with respect to the standard scalar product. The same symbol π_Y will stand for the operator $\pi_Y : L^\infty(\Omega) \to L^\infty(\Omega)$, induced by π_Y, namely $(\pi_Y u)(x) = u(\pi_Y x)$. Note that $\partial_{Y^\perp}(\pi_Y u) = 0$. Consequently, if $Z \in \widetilde{\mathcal{W}}(n,m)$, by $\pi_Z : L^\infty(\Omega, \mathbf{R}^m) \to \mathcal{K}(\Omega, Z^\perp)$, we denote the operator, whose i–th coordinate is defined by $(\pi_Z u)_i = \pi_{Z_i} u_i$.

If $W = (W_1, \ldots, W_m) \in \widetilde{\mathcal{W}}(n,m)$, and $E \subseteq \mathbf{R}^n$ is the subspace, we define $\overrightarrow{E} = (E, \ldots, E)$, and $W + \overrightarrow{E} = (W_1 + E, \ldots, W_m + E)$, while $W_E = ((W_1)_E, \ldots, (W_m)_E)$, where $Y_E = \pi_E(Y)$.

The symbol Q (with possibly some index) will be reserved for cubes, for example $Q(r) = [-r/2, r/2]^k \subseteq \mathbf{R}^k$, while $Q(x,r) = \{x\} + Q(r)$.

Let $Q = Q(r) \subseteq \mathbf{R}^k$ be the k–dimensional cube, $i : \mathbf{R}^k \to \mathbf{R}^n$ be the linear isometric embedding, $E = i(\mathbf{R}^k)$, $\tilde{Q} = i(Q)$, $\Omega \subseteq \mathbf{R}^n$ be a bounded domain, and $\Omega(E,r) = \{z \in \Omega : z + \tilde{Q} \subseteq \Omega\}$. We define an averaging operator $\mathrm{M}_{\tilde{Q}} : L^\infty(\Omega, \mathbf{R}^m) \to L^\infty(\Omega(E,r), \mathbf{R}^m)$, $\mathrm{M}_{\tilde{Q}} u(z) = ⨍_{\{z\}+\tilde{Q}} u(\tau)\, d\tau$. Here by $⨍_A f\, dx$ we mean an average of f with respect to the k-dimensional Hausdorff measure on the k–dimensional manifold A: $|A|^{-1} \int_A f\, dx$. This operator can be computed directly in the following way. Let $e_1, \dots, e_k$ be the standard basis in $\mathbf{R}^k$, $w_j = i(e_j)$, where $j = 1, \dots, m$, and $w_{k+1}, \dots, w_n$ be the completion of $w_1, \dots, w_k$ to an orthonormal basis in $\mathbf{R}^n$ (with respect to the standard scalar product). Let $(x,y) \in \mathbf{R}^k \times \mathbf{R}^{n-k}$ be the parametrization of $\mathbf{R}^n$ along $w_1, \dots, w_n$. Using the identification $\tilde{w}(x,y) = w(\sum_{j=1}^k x_j w_j + \sum_{j=k+1}^n y_{j-k} w_j)$, we note that for $u \in L^\infty(\Omega, \mathbf{R}^m)$ the i–th coordinate w of $\mathcal{M}_{\tilde{Q}} u$ satisfies

$$\tilde{w}(x,y) = ⨍_{Q_x} \tilde{u}_i(\tau, y)\, d\tau. \tag{8}$$

Some properties of the operator $\mathrm{M}_{\tilde{Q}}$ will be described in Lemma 3.11.

As usual, $C(\Omega)$ denotes the space of continuous functions on Ω, $C_0(\mathbf{R}^n)$ are continuous functions on $\mathbf{R}^n$ vanishing at infinity, while $\mu(\Omega)$ denotes the space of Radon measures on Ω. If $f \in C(\Omega)$ and $\mu \in \mathcal{M}(\Omega)$, then (f, μ) will stand for $\int_\Omega f(\lambda)\mu(d\lambda)$.

We denote by $\to$, $\rightharpoonup$, $\overset{*}{\rightharpoonup}$ the strong, weak and weak $*$ convergence respectively.

Let $I : \mathcal{K}(\Omega, P) \to \mathbf{R}$, be an arbitrary functional. Recall that I is lower semicontinuous under the sequential weak $*$ convergence in $L^\infty(\Omega, \mathbf{R}^m)$ if for $u^\nu, u \in \mathcal{K}(\Omega, P)$, such that $u^\nu \overset{*}{\rightharpoonup} u$ as $\nu \to \infty$ in $L^\infty(\Omega, \mathbf{R}^m)$, we have $\liminf_{\nu\to\infty} I(u^\nu) \geq I(u)$. I is continuous under the sequential weak $*$ convergence in $L^\infty(\Omega, \mathbf{R}^m)$ if $u^\nu \overset{*}{\rightharpoonup} u$ implies $\lim_{\nu\to\infty} I(u^\nu) = I(u)$.

Let us recall the fundamental theorem of Young (see [1]).

Theorem 2.2. *Let $\Omega \subset \mathbf{R}^n$ be a bounded measurable set. Assume that $u^j : \Omega \to \mathbf{R}^m$, $j = 1, 2, \dots,$ is a sequence of measurable functions satisfying the tightness condition*

$$\sup_j |\{x \in \Omega : |u^j(x)| \geq k\}| \overset{k\to\infty}{\to} 0.$$

Then there exists a subsequence $\{u^k\}$ and a family $\{\nu_x\}_{x\in\Omega}$ of probability measures, $\nu_x \in \mathcal{M}(\mathbf{R}^m)$, such that

1. *for every $f \in C_0(\mathbf{R}^m)$ the function $x \mapsto (f, \nu_x)$ is measurable,*
2. *if $K \subseteq \mathbf{R}^n$ is a closed set, and $u^j(x) \in K$ for every j and almost every x, then* $\operatorname{supp} \nu_x \subseteq K$ *for almost every x,*
3. *if $A \subseteq \Omega$ is measurable, $f : \Omega \times \mathbf{R}^m \to \mathbf{R}$ is a Carathéodory function and the sequence $\{f(x, u^k(x))\}$ is sequentially weakly relatively compact in $L^1(A)$, then $\{f(x, u^k(x))\}$ converges weakly in $L^1(A)$ to $\bar{f}$ given by*

$$\bar{f}(x) = \int_{\mathbf{R}^m} f(x, \lambda)\nu_x(d\lambda).$$

We will say that the sequence $\{u^j\}_{j\in\mathbf{N}}$ generates the Young measure $\{\nu_x\}_{x\in\Omega}$ if $\{\nu_x\}_{x\in\Omega}$ satisfies 1. and $f(u^j) \stackrel{*}{\rightharpoonup} \bar{f} = (f, \nu_x)$ in $L^\infty(\Omega)$, for every $f \in C_0(\mathbf{R}^m)$.

3. The main results

We introduce the following definitions.

Definition 3.1. Let $W \in \widetilde{\mathcal{W}}(n,m)$, and let $\mathcal{B} \subseteq \{0,1\}^m$ be a set of indices. Introduce the following sequence of objects: $W^0 = W$, $E^i = E_{\mathcal{B},W^i}$, where $E_{\mathcal{B},W}$ is given by (6), $W^{i+1} = W^i + \vec{E}^i$. We say that W satisfies the chain condition with respect to $\mathcal{B}$ (the $\mathcal{B}$–chain condition) if there exists $k \in \mathbf{N}$ such that $W^k = (\mathbf{R}^n, \ldots, \mathbf{R}^n)$.

Remark 3.2. Note that the sequence $\{W^i\}_{i\in\mathbf{N}}$ defined above is always increasing in the sense that $W_s^i \subseteq W_s^{i+1}$ for every $i \in \mathbf{N}$ and $s \in \{1, \ldots, m\}$, and it always stabilizes on some $\overline{W} \in \widetilde{\mathcal{W}}(n,m)$.

Definition 3.3. We say that $W \in \widetilde{\mathcal{W}}(n,m)$ satisfies the chain condition if it satisfies the chain condition with respect to $\mathcal{B} = \mathcal{B}(W)$, defined by (7).

Our main results are the following.

Theorem 3.4. *Let $\Omega \subseteq \mathbf{R}^n$ be a bounded domain, $W \in \widetilde{\mathcal{W}}(n,m)$, $\mathcal{B} \subseteq \{0,1\}^m$, $\Lambda_{\mathcal{B}} = \cup_{I\in\mathcal{B}}\mathbf{R}^{\times I}$. Assume that W satisfies the chain condition with respect to $\mathcal{B}$. Then for every continuous function $f : \mathbf{R}^m \to \mathbf{R}$ which is $\Lambda_{\mathcal{B}}$–convex the functional I_f is sequentially lower semicontinuous under the L^∞ weak $*$ convergence in $\mathcal{K}(\Omega, W)$.*

The direct consequence of Theorem 3.4, Theorem 1.1 and Fact 2.1 is the following.

Corollary 3.5. *Let $\Omega \subseteq \mathbf{R}^n$ be the bounded domain, $W \in \widetilde{\mathcal{W}}(n,m)$ satisfies the chain condition, $\Lambda = \Lambda_{\mathcal{B}(W)} = \cup_{I\in\mathcal{B}(W)}\mathbf{R}^{\times I}$. Then for every continuous function $f : \mathbf{R}^m \to \mathbf{R}$ the functional I_f is sequentially lower semicontinuous under the L^∞ weak $*$ convergence on $\mathcal{K}(\Omega, W)$ if and only if f is Λ convex.*

Another corollary is of purely algebraic nature.

Corollary 3.6. *If $W \in \widetilde{\mathcal{W}}(n,m)$ satisfies the chain condition with respect to $\mathcal{B}$ then $\mathcal{B}(W) \subseteq \mathcal{B}$ and $\Lambda_{\mathcal{B}(W)} \subseteq \Lambda_{\mathcal{B}}$.*

Proof. This follows from the fact that according to Theorems 3.4 and 1.1 every $\Lambda_{\mathcal{B}}$–affine function is $\Lambda_{\mathcal{B}(W)}$–affine. □

The proof of Theorem 3.4 will be preceded by the following sequence of lemmas. The first one is a slight modification of Lemma 3.1 in [8]. We include its proof for the convenience of the reader.

Lemma 3.7. *Let $W = (W_1, \ldots, W_m) \in \widetilde{\mathcal{W}}(n,m)$, $\mathcal{B} \subseteq \{0,1\}^m$ be the certain set of indices, $\Lambda_{\mathcal{B}} = \cup_{I\in\mathcal{B}}\mathbf{R}^{\times I}$, $E = E_{\mathcal{B},W} = +_{I\in\mathcal{B}}W^{I^*} \subseteq \mathbf{R}^n$. Assume that* $\dim E = k > 0$. *Let $Q_x \subseteq \{x\} + E$ be the k–dimensional parallelepiped, whose every side is parallel to some W^{I^*}, where $I \in B$. Then for every $\Lambda_{\mathcal{B}}$–convex continuous function f, and every*

$u \in \mathcal{K}(\mathbf{R}^n, W)$,

$$\fint_{Q_x} f(u(y))dy \geq f(\fint_{Q_x} u(y)dy). \tag{9}$$

Proof. Let $w_1, \ldots, w_k$ be the basis in E such that for each l we can find $I_l \in \mathcal{B}$ with the property that $w_l \in W^{I_l^*}$. Choose $x \in \mathbf{R}^n$ and the parallelepiped $Q_x = \sum_{i=1}^k t_i w_i + x$, with $t_i \in (-1/2, 1/2)$, for $i = 1, \ldots, k$. Since $w_1 \in W_i$ for all $i \in D^*(I_1)$, and each u_i is constant along the W_i's, we see that the image of the mapping

$$\mathbf{R} \ni t_1 \mapsto \phi_1(t_1, \ldots, t_k) = u\left(\sum_{i=1}^k t_i w_i + x\right) \in \mathbf{R}^m$$

is a subset of $A + R^{\times I_1}$, where $A = u\left(\sum_{i=2}^k t_i w_i + x\right)$. By the assumption f is convex in the direction of $\mathbf{R}^{\times I_1}$, hence

$$\int_{-1/2}^{1/2} f(\phi_1(t_1, \ldots, t_k))dt_1 \geq f\left(\int_{-1/2}^{1/2} \phi_1(t_1, \ldots, t_k)dt_1\right).$$

Proceeding in the same way with variables t_i, for $i = 2, \ldots, k$, and vector–valued functions,

$$\mathbf{R} \ni t_i \mapsto \phi_i(t_i, \ldots, t_k) = \int_{-1/2}^{1/2} \cdots \int_{-1/2}^{1/2} u\left(\sum_{i=1}^k t_i w_i + x\right) dt_1 \ldots dt_{i-1}$$

we obtain

$$\begin{aligned}\int_{Q_x} f(u(x))dx &= \int_{-1/2}^{1/2} \cdots \int_{-1/2}^{1/2} f\left(u\left(\sum_{i=1}^k t_i w_i + x\right)\right) dt_1 \ldots dt_k \\ &\geq f\left(\int_{-1/2}^{1/2} \cdots \int_{-1/2}^{1/2} u\left(\sum_{i=1}^k t_i w_i + x\right) dt_1 \ldots dt_k\right) = f\left(\int_{Q_x} u(y)dy\right). \quad \Box\end{aligned}$$

Remark 3.8. It follows from the proof that if x varies along $E_{\mathcal{B},W}$ then $u(x)$ can vary along $\Lambda_{\mathcal{B}}$ only.

Remark 3.9. We do not assume that the set $\mathcal{B}$ is related to the structure of W.

The next lemma may be clear to the specialists, but we include its proof for the sake of completeness.

Lemma 3.10. *Let $m, n \in \mathbf{N}$, $W \in \widetilde{\mathcal{W}}(n, m)$, and $f : \mathbf{R}^m \to \mathbf{R}$ be continuous. The following statements are equivalent.*

(i) *There exists a cube $Q \subseteq \mathbf{R}^n$ such that the functional I_f is lower semicontinuous in $\mathcal{K}(Q, W)$, under the sequential weak $*$ convergence in $L^\infty(Q, \mathbf{R}^m)$.*

(ii) *For an arbitrary cube $Q \subseteq \mathbf{R}^n$ the functional I_f is lower semicontinuous in $\mathcal{K}(Q, W)$, under the sequential weak $*$ convergence in $L^\infty(Q, \mathbf{R}^m)$.*

(iii) *For an arbitrary bounded domain $\Omega \subseteq \mathbf{R}^n$ the functional I_f is lower semicontinuous in $\mathcal{K}(\Omega, W)$, under the sequential weak $*$ convergence in $L^\infty(\Omega, \mathbf{R}^m)$.*

(iv) *If $\Omega \subseteq \mathbf{R}^n$ is an arbitrary bounded domain, $\{u^\nu\}_{\nu\in\mathbf{N}} \subseteq \mathcal{K}(\Omega, W)$ is an arbitrary bounded sequence, which generates the Young measure $\{\nu_x\}_{x\in\Omega}$, then for almost every $x \in \Omega$, we have*

$$(f, \nu_x) \geq f((\lambda, \nu_x)), \tag{10}$$

where $(\lambda, \nu_x) = \int_{\mathbf{R}^m} \lambda\nu_x(d\lambda)$.

Proof. The implication (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) is obvious, (i) $\Rightarrow$ (ii) follows from the fact that the space $\mathcal{K}(Q, W)$ is invariant under translations $u(x) \mapsto u(A + x)$, and dilations $u(x) \mapsto u(tx)$, where $A \in \mathbf{R}^n$ and $t \in \mathbf{R}$. For the implication (ii) $\Rightarrow$ (iv) we use the Young measure theory in a standard manner. Let $\{u^\nu\}_{\nu\in\mathbf{N}} \subseteq \mathcal{K}(\Omega, W)$ be the sequence generating the Young measure $\{\nu_x\}_{x\in\Omega}$. Then $f(u^\nu(x)) \overset{*}{\rightharpoonup} \overline{f}(x) = (f, \nu_x)$ in $L^\infty(\Omega)$, and $u^\nu(x) \overset{*}{\rightharpoonup} u(x) = (\lambda, \nu_x)$ in $L^\infty(\Omega, \mathbf{R}^m)$. Take $x_0 \in \Omega$ and $r > 0$ such that $Q(x_0, r) \subseteq \Omega$. Then by (ii) we get

$$\fint_{Q(x_0,r)} (f, \nu_x)\, dx = \lim_{\nu\to\infty} \fint_{Q(x_0,r)} f(u^\nu(x))\, dx \geq \fint_{Q(x_0,r)} f(u(x))\, dx.$$

Let $r \to 0$. Applying a Lebesgue's Differentiation Theorem, we verify that if x_0 is the Lebesque's point for $\overline{f}(x)$ and for $f(u(x))$ then (10) holds at $x = x_0$. To see that the implication (iv) $\Rightarrow$ (iii) holds true it suffices to integrate (10) over Ω and apply the theorem of Young. $\square$

Now we are going to derive some properties of the averaging operator $\mathrm{M}_{\widetilde{Q}}$ defined by (8). We have the following.

Lemma 3.11. *Assume that $W \in \widetilde{\mathcal{W}}(n, m)$, $\mathcal{B} \subseteq \{0, 1\}^m$ is a certain set of indices, $E = E_{\mathcal{B},W}$ is given by (6), $\dim E = k > 0$, $\widetilde{Q} \subseteq E$ is the k–dimensional cube, the range of $Q = Q(r) \subseteq \mathbf{R}^k$ under the linear isometry $i : \mathbf{R}^k \to \mathbf{R}^n$, and $\mathrm{M}_{\widetilde{Q}}$ is the averaging operator defined by (8). Let $W^1 = W_{E^\perp}$, $W^2 \in \widetilde{\mathcal{W}}(n, m)$ be such that $W_i^1 + W_i^2 = E^\perp$, and $W_i^1 \perp W_i^2$, for every $i = 1, \dots, m$. Define $Y = W^2 + \vec{E}$, where $\vec{E} = (E, \dots, E)$, and choose $s \in \{1, \dots, m\}$. Then the following conditions hold.*

(i) *There exists a linear operator $D_s : \mathbf{R}^n \to E$ such that $D_s = D_s\pi_{W_s^1}$, and for every $u \in \mathcal{K}(\Omega, W)$ and $z \in \mathbf{R}^n$, we have $u_s(z) = u_s(-D_s z + \pi_{Y_s} z)$. In particular*

$$(\mathrm{M}_{\widetilde{Q}}u)_s(z) = (\mathrm{M}_{\widetilde{Q}}u)_s(-D_s z + \pi_{Y_s} z). \tag{11}$$

(ii) *If $z_0 \in \mathbf{R}^n$ is such that $\|\pi_{W_s^1} z - \pi_{W_s^1} z_0\| \leq \|D_s\|^{-1} r$, while $\pi_{Y_s} z = \pi_{Y_s} z_0$, then we have*

$$|(\mathrm{M}_{\widetilde{Q}}u)_s(z) - (\mathrm{M}_{\widetilde{Q}}u)_s(z_0)| \leq C_s \cdot r^{-1} \|\pi_{W_s^1} z - \pi_{W_s^1} z_0\| \cdot \|u_s\|_{L^\infty(\Omega)}, \tag{12}$$

where $C_s = 2\sqrt{k}\|D_s\|$.

Proof. (i) Let $A_s : W_s^1 \to W_s$ be the linear embedding such that $\pi_{E^\perp} A_s = \mathrm{id}_{W_s^1}$. Define $D_s = A_s\pi_{W_s^1} - \pi_{W_s^1} = \pi_E A_s \pi_{W_s^1}$. Since $\pi_{W_s^1} + \pi_{W_s^2} = \pi_{E^\perp}$, we have $\pi_{E^\perp} = A_s\pi_{W_s^1} - D_s + \pi_{W_s^2}$, and $\mathrm{id} = (\pi_E + \pi_{W_s^2}) + A_s\pi_{W_s^1} - D_s = \pi_{Y_s} + A_s\pi_{W_s^1} - D_s$. Note that for every $z \in \mathbf{R}^n$, we have $A_s\pi_{W_s^1}z \in W_s$. In particular, if $u \in \mathcal{K}(\Omega, W)$, we have $u_s(z) = u_s(\pi_{Y_s}z - D_sz)$. This implies (i).

(ii) Let $e_1, \ldots, e_k$ be the standard basis in $\mathbf{R}^k$, and $w_j = i(e_j)$, where $j = 1, \ldots, k$. Let $w_{k+1}, \ldots, w_n$ be the completion of the $\{w_j\}_{j=1,\ldots,k}$ to the orthonormal basis in $\mathbf{R}^n$ such that $w_{k+1}, \ldots, w_{k+l}$ are parallel to W_s^1, while $w_{k+l+1}, \ldots, w_n$ are parallel to W_s^2 (where $l = \dim W_s^1$). Choose coordinates $(x, y^1, y^2) \in \mathbf{R}^k \times \mathbf{R}^l \times \mathbf{R}^{n-k-l}$ along $\{w_j\}_{j=1,\ldots,k}$, $\{w_j\}_{j=k+1,\ldots,k+l}$, and $\{w_j\}_{j=k+l+1,\ldots,n}$ respectively. Let $z = \sum_{j=1}^k x_jw_j + \sum_{j=1}^l y_j^1 w_{k+j} + \sum_{j=1}^{n-(k+l)} y_j^2 w_{k+l+j}$. Then by (i) the s–th coordinate of $v = \mathrm{M}_{\widetilde{Q}}u$ satisfies

$$v_s(z) = \tilde{v}_s(x, y^1, y^2) = \fint_{\{x-\tilde{D}_sy^1\}+Q(r)} \tilde{u}_s(\tau, 0, y^2)\, d\tau,$$

where $\tilde{D}_sy^1$ is the parametrization of $D_sz = D_s(\sum_{j=1}^l y_j^1 w_{k+j}) \in E$ in the basis $e_1, \ldots, e_k$. Hence, if $z_0 = \sum_{j=1}^k x_jw_j + \sum_{j=1}^l \overline{y}_j^1 w_{k+j} + \sum_{j=1}^{n-(k+l)} y_j^2 w_{k+l+j}$, we have

$$v_s(z_0) = \fint_{\{x-\tilde{D}_s\overline{y}^1\}+Q(r)} \tilde{u}_s(\tau, 0, y^2)\, d\tau,$$

where $\overline{y}^1 = (\overline{y}_1^1, \ldots, \overline{y}_l^1)$. In particular,

$$|v_s(z) - v_s(z_0)| \le r^{-k} \int_{Q^1\Delta Q^2} |\tilde{u}_s(\tau, 0, y^2)|\, d\tau \le \|u_s\|_{L^\infty(\Omega)} r^{-k} |Q^1\Delta Q^2|,$$

where $Q^1 = \{x - \tilde{D}_sy^1\} + Q(r)$, $Q^2 = \{x - \tilde{D}_s\overline{y}^1\} + Q(r)$, and $Q^1\Delta Q^2 := (Q^1\backslash Q^2) \cup (Q^2\backslash Q^1)$ is the symmetric difference of Q^1 and Q^2. Now it suffices to note that if $\|\tilde{D}_sy^1 - \tilde{D}_s\overline{y}^1\| = r_1 \le r$, we have $|Q^1\Delta Q^2| = 2|Q^1 \setminus Q^2| \le 2\sqrt{k}r_1r^{k-1}$. $\square$.

We are going now to apply the following lemma, which is the key point in the proof of Theorem 3.4.

Lemma 3.12 (The Reduction Lemma). *Let $W \in \widetilde{\mathcal{W}}(n, m)$, $\mathcal{B} \subseteq \{0, 1\}^m$ be a set of indices, $E = E_{\mathcal{B},W}$, and $\Lambda = \Lambda_{\mathcal{B}}$ be given by (6), $0 < \dim E = k < n$, Asume that $\Omega \subseteq \mathbf{R}^n$ is a bounded domain, and $f : \mathbf{R}^m \to \mathbf{R}$ is continuous and $\Lambda_{\mathcal{B}}$–convex. Then the following statements are equivalent.*

(i) *The functional I_f is lower semicontinuous under the sequential L^∞–weak $*$ convergence in $\mathcal{K}(\Omega, W)$,*

(ii) *The functional I_f is lower semicontinuous under the sequential L^∞–weak $*$ convergence in $\mathcal{K}(\Omega, W + \vec{E})$.*

Remark 3.13. Note that the space $\mathcal{K}(\Omega, W + \vec{E})$ is isomorphic to the space $\mathcal{K}(\Omega_2, V)$, where $\Omega_2 \subseteq \mathbf{R}^{n-k}$, and $V \in \widetilde{\mathcal{W}}(n-k, m)$ are such that if $C : E^\perp \to \mathbf{R}^{n-k}$ is an arbitrary linear isomorphism, then $V_s = C((W_s)_{E^\perp})$, and $\Omega_2 = C(\pi_{E^\perp}(\Omega))$.

Proof. It suffices to prove the lemma under the assumption $E = \mathbf{R}^k \times \{0\}$, and $\Omega = \Omega_1 \times \Omega_2$, where $\Omega_1 \subseteq \mathbf{R}^k$, and $\Omega_2 \subseteq \mathbf{R}^{n-k}$ are bounded domains (we use Lemma 3.10).

Let W^1, W^2 and Y be as in the previous lemma, and choose coordinates $(x,y) \in \mathbf{R}^n$ in the way that $x \in \mathbf{R}^k$ and $y \in \mathbf{R}^{n-k}$. We proceed to the proof of the lemma.

(i) $\Rightarrow$ (ii): This implication is obvious since $\mathcal{K}(\Omega, W + \vec{E}) \subseteq \mathcal{K}(\Omega, W)$.
(ii) $\Rightarrow$ (i): Let $v \in \mathcal{K}(\Omega, W)$ be such that $|v_s(x)| \leq R$ for every $s \in \{1,\dots,m\}$, and for almost every $x \in \Omega$. Let $Q_1(r) = [-r/2, r/2]^k$, $\tilde{Q}_1 = Q_1 \times \{0\} \subseteq \mathbf{R}^n$, and to abbreviate let us denote $Q_1 = Q_1(x_0, r)$, where $x_0 \in \mathbf{R}^k$ is fixed. According to Lemma 3.7, for almost every $y \in \Omega_2$ we have

$$\fint_{Q_1} f(v(\tau,y))d\tau \geq f(\fint_{Q_1} v(\tau,y)d\tau). \tag{13}$$

Thus

$$\fint_{Q_1} f(v(\tau,y))d\tau \quad \geq \quad f((\mathrm{M}_{\tilde{Q}_1}v)(x_0,y)) = f(w(y)) + C(y,v), \tag{14}$$

where $C(y,v) = f((\mathrm{M}_{\tilde{Q}_1}v)(x_0,y)) - f((\pi_Y \mathrm{M}_{\tilde{Q}_1}v)(x_0,y))$, and $w(y) = (\pi_Y(\mathrm{M}_{\tilde{Q}_1}v))(x_0,y)$.
According to Lemma 3.11, we have

$$|(M_{\tilde{Q}_1}v_s - \pi_Y(M_{\tilde{Q}_1}v_s))(x_0,y)| \leq C_s R r^{-1} r_{1,s} = C(s,R,r) r_{1,s},$$

where C_s is as in (12), while $r_{1,s} = \|\pi_{W_s^1}(x_0,y)\| \leq \|y\|$. Hence

$$\begin{aligned} |C(y,v)| \leq \sup\{&|f(A+\tau) - f(A)| : |A_s| \leq R, |\tau_s| \leq C(s,R,r)\|y\|, \\ &\text{for every } s \in \{1,\dots,m\}\}, \end{aligned} \tag{15}$$

provided that $\|y\| \leq (\min_s \|D_s\|^{-1})r$.

Let $u^\nu \overset{*}{\rightharpoonup} u$ in $\mathcal{K}(\Omega, W)$, and assume that $|u_s^\nu(x)| \leq R$ for every $s \in \{1,\dots,m\}$ and almost every $x \in \Omega$. Applying (14) and (15) to $v(x,y) = u^\nu(x, y_0 + y)$, and assuming that $\|y\| \leq r_1$, where $r_1 \leq (\min_s \|D_s\|^{-1})r$, we get

$$\fint_{Q_1} f(u^\nu(\tau, y_0 + y))d\tau \geq f(w^\nu(y)) + C^\nu(y), \tag{16}$$

where

$$\begin{aligned} |C^\nu(y)| \quad &\leq \quad \sup\{|f(A+\tau) - f(A)| : |A_s| \leq R, |\tau_s| \leq C(s,R,r)r_1, \\ &\qquad \text{for every } s \in \{1,\dots,m\}\}, \\ w_s^\nu(y) \quad &= \quad \fint_{Q_1(x_0,r)} u_s^\nu((\tau,y_0) + \pi_{W_s^2}(0,y))d\tau. \end{aligned} \tag{17}$$

Note that if x_0, y_0 and r are fixed, the function
$(x,y) \mapsto w^\nu(y)$ belongs to $\mathcal{K}(\Omega, W + \vec{E})$, and $w^\nu \overset{*}{\rightharpoonup} w$, as $\nu \to \infty$, where

$$w_s(y) = \fint_{Q_1(x_0,r)} u_s((\tau,y_0) + \pi_{W_s^2}(0,y))d\tau \in \mathcal{K}(\Omega, W + \vec{E}). \tag{18}$$

After integrating (16) with respect to $y \in Q_2 = Q_2(r_1) \subseteq \mathbf{R}^{n-k}$, we get

$$\fint_{Q_2} \fint_{Q_1} f(u^\nu(\tau, y_0 + y))d\tau dy \geq \fint_{Q_2} f(w^\nu(y))dy + \fint_{Q_2} C^\nu(y)\, dy.$$

Hence, by the assumption (ii) and (17) we obtain

$$\begin{aligned}\liminf_{\nu\to\infty} \fint_{Q_2}\fint_{Q_1} f(u^\nu(\tau, y_0+y))d\tau\, dy &\geq \liminf_{\nu\to\infty}\fint_{Q_2} f(w^\nu(y))dy - B(r_1)\\ &\geq \fint_{Q_2} f(w(y))dy - B(r_1),\end{aligned}$$

where $B(r_1) = o(r_1)$. According to the Young's Theorem this implies that (passing to a subsequence) if $\{\nu_{(x,y)}\}_{(x,y)\in\Omega}$ is the Young measure generated by $\{u^\nu\}$, we have

$$\fint_{Q_2(r_1)}\fint_{Q_1(x_0,r)} (f, \nu_{(\tau,y_0+y)})d\tau\, dy \geq \fint_{Q_2(r_1)} f(w(y))dy - B(r_1), \tag{19}$$

where according to (18) $w_s(y) = \fint_{Q_1(x_0,r)} (\lambda_s, \nu_{((\tau,y_0)+\pi_{W_s^2}(0,y))})d\tau$, and λ_s is the s–th coordinate of $\lambda\in\mathbf{R}^m$.

Now it is an easy exercise to apply the Lebesque's Differentiation Theorem and prove that if x_0 and r are fixed, while $r_1\to 0$, then there exists a measurable set $\Omega_2(x_0,r)$ such that $\Omega_2(x_0,r)\subseteq\Omega_2$, $|\Omega_2\setminus\Omega_2(x_0,r)| = 0$, and for all $y_0\in\Omega_2(x_0,r)$ the left hand side of (19) tends to $\fint_{Q_1}(f,\nu_{(\tau,y_0)})d\tau$, while the right hand side of (19) tends to $f(\fint_{Q_1}(\lambda,\nu_{(\tau,y_0)})d\tau)$. That gives

$$\fint_{Q_1(x_0,r)} (f, \nu_{(\tau,y_0)})d\tau \geq f(\fint_{Q_1(x_0,r)} (\lambda, \nu_{(\tau,y_0)}d\tau)), \tag{20}$$

for all $x_0\in\Omega_1$, and all $y_0\in\Omega_2(x_0,r)$. Now we would like to let $r\to 0$ and apply the Lebesgue's Differentiation Theorem again, but we cannot do it directly, because the set of those y_0 for which (20) holds depends on x_0, while the set of Lebesgue's points for $x\mapsto(f,\nu_{(x,y_0)})$ and $x\mapsto(\lambda,\nu_{(x,y_0)})$ depends on y_0. Therefore we apply the following argument. Let $r_j=\frac{1}{j}$, $j\in\mathbf{N}$, and define the following subsets in Ω_2, being of the full measure in Ω_2: $\Omega_2(x) = \cap_j\Omega_2(x,r_j)$. Let $F(x,y) = (f,\nu_{(x,y)})$, and $G(x,y) = (\lambda,\nu_{(x,y)})$. Since $F, G\in L^\infty(\Omega_1\times\Omega_2)$, the following set is well defined and has the full measure in Ω_2:

$$\tilde\Omega_2 = \{y\in\Omega_2 : x\mapsto F(x,y) \text{ and } x\mapsto G(x,y)\in L^\infty(\Omega_1)\}.$$

We also introduce for $y\in\tilde\Omega_2$,

$$\begin{aligned}\Omega_1(y) &= \{x\in\Omega_1 : x \text{ is a Lebesgue's point for } F(x,y) \text{ and for } G(x,y)\}\\ \tilde\Omega_1(y) &= \{x\in\Omega_1 : y\in\Omega_2(x)\}.\end{aligned}$$

Since

$$|\Omega_1\times\Omega_2| = \int_{\Omega_1}\int_{\Omega_2} 1dxdy = \int_{\Omega_1}(\int_{\Omega_2}\chi_{\{y\in\tilde\Omega_2\cap\Omega_2(x)\}}dy)dx = \int_{\tilde\Omega_2}(\int_{\Omega_1}\chi_{\tilde\Omega_1(y)}(x)dx)dy,$$

it follows that there exists a set $\Omega_2^*\subseteq\tilde\Omega_2$ such that $|\Omega_2\setminus\Omega_2^*| = 0$, and for every $y\in\Omega_2^*$ the set $\tilde\Omega_1(y)$ is of full k–dimensional Lebesgue's measure in Ω_1. Let

$$\tilde\Omega = \{(x,y) : y\in\Omega_2^*,\ x\in\Omega_1(y)\cap\tilde\Omega_1(y)\}.$$

Note that the set $\tilde{\Omega}$ is of full measure in Ω. Now let us apply (20) to $(x_0, y_0) \in \tilde{\Omega}$, $r = r_j = \frac{1}{j}$, and let $j \to \infty$. According to the Lebesgue's Differentiation Theorem, and to continuity of f, we verify that

$$(f, \nu_{(x_0,y_0)}) \geq f((\lambda, \nu_{(x_0,y_0)})),$$

for all $(x_0, y_0) \in \tilde{\Omega}$. Now the assertion follows from Lemma 3.10. □

Remark 3.14. It would seem at first glance that if $W \in \widetilde{\mathcal{W}}(n,m)$, and $v \in \mathcal{K}(\Omega, W)$, then the function $w(x,y) = M_{\tilde{Q}_1} v(x_0, y) \in \mathcal{K}(\Omega, W + \vec{E})$ if x_0 is fixed. It is easy to see that such property does not hold, as the function w may change in the directions of $(W)_{E^\perp}$, see (i) of Lemma 3.11. This was the reason to employ Lemma 3.11.

Proof of Theorem 3.4. According to Definition 3.1 and Lemma 3.12 if f is $\Lambda_{\mathcal{B}}$ convex then the functional I_f is sequentially weakly $*$ lower semicontinuous in $\mathcal{K}(\Omega, W^i)$ if and only if it is sequentially weakly $*$ lower semicontinuous in $\mathcal{K}(\Omega, W^{i+1})$. Hence the functional I_f is sequentially weakly $*$ lower semicontinuous in $\mathcal{K}(\Omega, W)$ if and only if it is sequentially weakly $*$ lower semicontinuous in $\mathcal{K}(\Omega, \vec{\mathbf{R}}^n)$. But the space $\mathcal{K}(\Omega, \vec{\mathbf{R}}^n)$ consists of constant functions only, where the weak convergence is the same as the strong one. Hence, since f is continuous, the last condition is always satisfied. □

Remark 3.15. If $\{W^{I^*}\}_{I \in B(W)}$ span the whole of $\mathbf{R}^n$ then obviously W does satisfy the chain condition. Such condition was introduced in [8] and defined as parallelness condition (see also Chapter 4 of [8] for equivalent characterization). The author has proved there that under such assumption Theorem 3.4 holds (Theorem 3.2 of [8]). The following example shows that the chain condition is essentially weaker than parallelness condition from the author's previous paper.

Example 3.16. Let $m = n = 3$, $W_1 = \mathrm{span}\{e_2\}$, $W_2 = \mathrm{span}\{e_2 + e_3\}$, $W_3 = \mathrm{span}\{e_1 - e_2, e_3\}$. Here $\mathcal{B}(W) = \{(1,1,0), (0,0,1)\}$, and $\Lambda = (\mathbf{R} \times \mathbf{R} \times \{0\}) \cup (\{0\} \times \{0\} \times \mathbf{R})$. Hence $W^0 = W$, $E^0 = \mathrm{span}\{e_1 - e_2, e_3\}$, $W^1 = (\mathbf{R}^3, \mathbf{R}^3, \mathrm{span}\{e_1 - e_2, e_3\})$, $E^1 = \mathbf{R}^3$, $W^2 = (\mathbf{R}^3, \mathbf{R}^3, \mathbf{R}^3)$, and W does satisfy the chain condition, while the subspaces $\{W^{I^*}\}_{I \in B(W)}$ do not span the whole of $\mathbf{R}^n$.

Some more examples will be presented in the next chapter.

4. Generalizations. A wider class of functionals

Let $W \in \widetilde{\mathcal{W}}(n,m)$, and assume that W does not satisfy the chain condition. We would like to know what is the characterization of the set of those continuous functions $f : \mathbf{R}^m \to \mathbf{R}$ which define the sequentially lower semicontinuous functionals acting on the space $\mathcal{K}(\Omega, W)$, equipped with the L^∞ weak $*$ topology. Before we formulate the general theorem, let us start with some definitions and remarks.

Definition 4.1. Let $W \in \widetilde{\mathcal{W}}(n,m)$, and $\Lambda \in \{\Lambda_{\mathcal{B}} : \mathcal{B} \in 2^{\{0,1\}^m}\}$. We say that (W, Λ) is a chain pair if $\Lambda = \Lambda_{\mathcal{B}}$ (see (6)) for such $\mathcal{B}$ that W satisfies the chain condition with respect to $\mathcal{B}$.

Remark 4.2. Suppose that $W \in \widetilde{\mathcal{W}}(n,m)$, $\Lambda_1, \Lambda_2 \in \{\Lambda_{\mathcal{B}} : \mathcal{B} \in 2^{\{0,1\}^m}\}$, and $\Lambda_1 \subseteq \Lambda_2$. Then if (W, Λ_1) is a chain pair then (W, Λ_2) is also a chain pair.

Definition 4.3. Let

$$\mathcal{D}(W) = \{\Lambda \in \{\Lambda_{\mathcal{B}} : \mathcal{B} \in 2^{\{0,1\}^m}\} : (W, \Lambda) \text{ is the chain pair}\}. \tag{21}$$

By $\mathcal{D}^0(W)$ we will denote the subset of $\mathcal{D}(W)$, consisting of all its elements which are minimal with respect to the inclusion "$\subseteq$". Note that the set $\mathcal{D}^0(W)$ is well defined and, as follows from Corollary 3.6, every element of $\mathcal{D}^0(W)$ contains $\Lambda_{\mathcal{B}(W)}$.

Remark 4.4. According to Theorem 3.4, if $W \in \widetilde{\mathcal{W}}(n,m)$, and $\mathcal{D}^0(W) = \{\Lambda_1, \ldots, \Lambda_r\}$, then every Λ_i convex function $f : \mathbf{R}^m \to \mathbf{R}$ defines the sequentially weakly $*$ lower semi-continuous functional I_f, acting on $\mathcal{K}(\Omega, W)$ equipped with the topology of $L^\infty(\Omega, \mathbf{R}^m)$.

Let me illustrate the above reasoning on the following simple example.

Example 4.5. Let $n = 2, m = 3$, $W = (\text{span}\{e_2\}, \text{span}\{e_1\}, \text{span}\{e_1 - e_2\})$. Then there are four Λ's such that (W, Λ) is a chain pair: $\Lambda_1 = (\{0\} \times \mathbf{R} \times \mathbf{R}) \cup (\mathbf{R} \times \{0\} \times \{0\})$, $\Lambda_2 = (\mathbf{R} \times \{0\} \times \mathbf{R}) \cup (\{0\} \times \mathbf{R} \times \{0\})$, $\Lambda_3 = (\mathbf{R} \times \mathbf{R} \times \{0\}) \cup (\{0\} \times \{0\} \times \mathbf{R})$, and $\Lambda_4 = \mathbf{R}^3$, while $\Lambda_{\mathcal{B}(W)} = (\mathbf{R} \times \{0\} \times \{0\}) \cup (\{0\} \times \mathbf{R} \times \{0\}) \cup (\{0\} \times \{0\} \times \mathbf{R})$. Hence $\mathcal{D}^0(W) = \{\Lambda_1, \Lambda_2, \Lambda_3\}$. According to Remark 4.4 every Λ_i convex function defines the lower semicontinuos functional on $\mathcal{K}(\Omega, W)$. This can be easily checked directly.

The next two theorems extend the result of Theorem 3.4. Their easy proofs are left to the reader.

Theorem 4.6. *Let $W \in \widetilde{\mathcal{W}}(n,m)$, and $\mathcal{D}^0(W) = \{\Lambda_1, \ldots, \Lambda_r\}$. Assume that $g_1, \ldots, g_r : \mathbf{R}^m \to \mathbf{R}$, g_i are Λ_i convex and continuous for every $i \in \{1, \ldots, m\}$, and let $h : \mathbf{R}^r \to \mathbf{R}$ be convex, and increasing with respect to every variable. Then the function $f(\lambda) = h(g_1(\lambda), \ldots, g_r(\lambda))$ defines the sequentially weakly $*$ lower semicontinuous functional I_f, acting on $\mathcal{K}(\Omega, W)$ equipped with the topology of $L^\infty(\Omega, \mathbf{R}^m)$.*

Theorem 4.7. *Let $W \in \widetilde{\mathcal{W}}(n,m)$, and $\mathcal{D}^0(W) = \{\Lambda_1, \ldots, \Lambda_r\}$. Assume that $g_1, \ldots, g_r : \mathbf{R}^m \to \mathbf{R}$, g_i are Λ_i affine and continuous for every $i \in \{1, \ldots, m\}$, and let $h : \mathbf{R}^r \to \mathbf{R}$ be convex. Then the function*

$$f(\lambda) = h(g_1(\lambda), \ldots, g_r(\lambda)) \tag{22}$$

defines the sequentially weakly $*$ lower semicontinuous functional I_f, acting on $\mathcal{K}(\Omega, W)$, equipped with the topology of $L^\infty(\Omega, \mathbf{R}^m)$.

Remark 4.8. Note that the condition (22) is the analogue of the polyconvexity condition known in the calculus of variations (see e.g. [3]), while the condition (9) is the poor cousin of the A–quasiconvexity condition, see [4].

Remark 4.9. If $\Lambda_{\mathcal{B}(W)}$ spans the whole $\mathbf{R}^m$, then every Λ_i–affine function in the statement of Theorem 4.7 is a polynomial (see e. g. Theorem 4.5 of [2]). Then the continuity assumption on g_i in the statement of Theorem 4.7 can be dropped.

Let me explain my motivations to investigate functionals related to the model like (3). It is easy to see that the following theorem holds.

Theorem 4.10. *Assume that*

(i) *$\Omega \subseteq \mathbf{R}^n$ is an open bounded domain, P is the operator given by (1), $\mathcal{K} = KerP \cap L^\infty(\Omega, \mathbf{R}^m)$, V and Λ are defined by Theorem 1.1.*

(ii) *There are $n_1, m_1 \in \mathbf{N}$ such that $m_1 \leq m$, $n_1 \leq n$, the bounded domain $\Omega_1 \subseteq \mathbf{R}^{n_1}$, $W \in \widetilde{\mathcal{W}}(n_1, m_1)$, and the linear imbedding $J : \mathbf{R}^{m_1} \to \mathbf{R}^m$, which induces the linear embedding $\mathcal{K}(\Omega_1, W) \subseteq \mathcal{K}$, defined by the expression $Ju(x) = J(u(x))$, $x \in \Omega_1$.*

(iii) *$f : \mathbf{R}^m \to \mathbf{R}$ is continuous and defines the lower semicontinuous functional I_f, acting on the space $\mathcal{K}$.*

Then the function $f_J(\lambda) = f \circ J : \mathbf{R}^{m_1} \to \mathbf{R}$ defines the sequentially weakly $*$ lower semicontinuous functional on the space $\mathcal{K}(\Omega_1, W)$, equipped with the topology of $L^\infty(\Omega_1, \mathbf{R}^{m_1})$.

The above theorem can be used to test conjectures for lower semicontinuous functionals related to the general model, as well as to derive some their further properties. Let me illustrate this possibility on the following example.

Example 4.11. Let P be given by (1), $\mathcal{K} = KerP \cap L^\infty(\Omega, \mathbf{R}^m)$, Λ and V be given by Theorem 1.1. Take $r \in \mathbf{N}$, and $(\xi_1, \lambda_1), \ldots, (\xi_r, \lambda_r) \in V$. Note that for an arbitrary $v \in L^\infty(\mathbf{R})$, and an arbitrary $(\xi, \lambda) \in V$, we have $u(x) = v(< x, \xi >) \cdot \lambda_i \in \mathcal{K}$. Since $\mathcal{K}$ is linear, we verify that

$$\begin{aligned} \mathcal{K}^r &= \mathcal{K}^r_{(\xi_1,\lambda_1),\ldots,(\xi_r,\lambda_r)} \\ &= \{u \in \mathcal{K} : u = \sum_{i=1}^{r} v_i(< x, \xi_i >) \cdot \lambda_i, \text{ where } v_1, \ldots, v_r \in L^\infty(\mathbf{R})\} \end{aligned} \tag{23}$$

is the subset of $\mathcal{K}$. Moreover, if λ_i's are linearly independent, then $\mathcal{K}^3$ is the subspace of $\mathcal{K}$, isomorphic to $\mathcal{K}(\Omega_1, W)$, for a suitably choosen domain Ω_1 and $W \in \widetilde{\mathcal{W}}(n, m)$. Hence, if the functional I_f is lower semicontiunous on $\mathcal{K}$, then the functional I_f restricted to functions from $\mathcal{K}^r$ must also be lower semicontinuous.

The above approach does exist in the literature in some special cases. Let me give a few more examples.

Example 4.12. The proof of Theorem 1.1 uses spaces $K^1_{(\xi,\lambda)}$, see e. g. Theorem 3.1 of [2].

Example 4.13. The proof of the known result of Murat and Tartar about weakly continuous functionals (see e.g. Theorem 3.3 of [2]) uses spaces of the form $\mathcal{K}^N_{(\xi_1,\lambda_1),\ldots,(\xi_N,\lambda_N)}$, where $\xi_1, \ldots, \xi_N$ are linearly dependent.

Example 4.14. The space K^3 was used in Šverák's famous paper [23], in Pedregal's paper [21], and in the paper of the author [9] in the context of the application to the rank–one problem. Some more details will be explained in the next section.

5. Applications to the rank–one problem

Now we are going to discuss the relations of our results with the rank–one problem. Suppose that $m, n \in \mathbf{N}$, $\Omega \subseteq \mathbf{R}^n$ is the bounded domain, and

$$\mathcal{K} = \{\nabla v = (\nabla v_1, \ldots, \nabla v_m), v_i \in W^{1,\infty}(\Omega)\} \tag{24}$$

If Ω is simply connected, then $\mathcal{K}$ is exactly the kernel of the operator of rotation applied to every column of the vector $w = (w_1, \ldots, w_m)$, where $w_i \in \mathbf{R}_n$ for $i = 1, \ldots, m$. Take the continuous function $f : \mathbf{R}^m_n \to \mathbf{R}$ and consider the functional

$$I_f(v) = \int_\Omega f(\nabla v(x))\, dx,\ v \in W^{1,\infty}(\Omega, \mathbf{R}^m).$$

According to Theorem 1.1, the set V related to the space of gradients is equal to $V = \{(\xi, \xi \otimes b) : \xi \in \mathbf{R}^n, b \in \mathbf{R}^m\}$, and $\Lambda = \{\xi \otimes b : \xi \in \mathbf{R}^n, b \in \mathbf{R}^m\}$ is the set of rank–one matrices in $\mathbf{R}^m_n$. The space of those continuous functions $f : \mathbf{R}^m_n \to \mathbf{R}$ which define the lower semicontinuous functionals I_f has been completely characterized and every such function is called quasiconvex (see e.g. [2, 3, 11, 12]). Unfortunately the condition which defines the quasiconvex function is not geometrically transparent. As the direct consequence of Theorem 1.1 one obtains that every quasiconvex function must be convex in the direction of matrices of rank–one (rank–one convex). It has been posed by Morrey in [11] that every continuous function $f : \mathbf{R}^m_n \to \mathbf{R}$ which is rank–one convex is quasiconvex. This conjecture has been answered in negative by Šverák in 1992 ([23]) in the case $m \geq 3, n \geq 2$, and it remains open when $m = 2$ and $n \geq 2$ (we refer to the interesting paper of Iwaniec [5] where the nontrivial connections of this conjecture and some other disciplines of mathematics were shown, and for example to [10, 17, 22] and their references for more results concerning the rank–one conjecture).

Let me make some comments concerning Šverák's construction for the case $m = 3$, $n = 2$, and for the case $m = n = 2$, referring to the original paper [23] for details. At first we consider the case $n = 2, m = 3$.

Remark 5.1. Let $W \in \widetilde{\mathcal{W}}(2, 3)$ be as in Example 4.5. Then W does not satisfy the chain condition and one can show directly that in this case the set of all $\tilde{\Lambda}$ convex functions, where $\tilde{\Lambda} = \Lambda_{\mathcal{B}(W)} = (\mathbf{R} \times \{0\} \times \{0\}) \cup (\{0\} \times \mathbf{R} \times \{0\}) \cup (\{0\} \times \{0\} \times \mathbf{R})$ is essentially bigger than the set of those functions, which define the lower semicontinuous functionals acting on $\mathcal{K}(\Omega, W)$ (where $\Omega \subseteq \mathbf{R}^2$ is an arbitrary bounded domain). For example every function of the form

$$f_\epsilon(r, s, t) = -rst + \epsilon \|(r, s, \sqrt{2}t)\|^2, \tag{25}$$

is $\tilde{\Lambda}$ convex, but the functional I_{f_ϵ} is not lower semicontinuous on $\mathcal{K}(\Omega, W)$ if $\epsilon \geq 0$ is sufficiently small (for $\epsilon = 0$ this observation is due to Murat and Tartar, see e. g. page 26 of [2], [3, 13, 14, 24]).

Now take $\xi_1 = (1, 0), \xi_2 = (0, 1), \xi_3 = (1, 1) \in \mathbf{R}^2$, $b_1 = (1, 0, 0)$, $b_2 = (0, 1, 0)$, $b_3 = (0, 0, 1)$, and let $\lambda_1 = \xi_1 \otimes b_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\lambda_2 = \xi_2 \otimes b_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $\lambda_3 = \xi_3 \otimes b_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, and construct the space $\mathcal{K}^3 = \mathcal{K}^3_{(\xi_1,\lambda_1),(\xi_2,\lambda_2),(\xi_3,\lambda_3)}$ by (23). Then $\mathcal{K}^3 = \begin{bmatrix} u_1(y) & 0 & u_3(x+y) \\ 0 & u_2(x) & u_3(x+y) \end{bmatrix}$, where $u = (u_1, u_2, u_3) \in \mathcal{K}(\Omega, W)$. Let $X = \{X(r, s, t) = \begin{bmatrix} r & 0 & t \\ 0 & s & t \end{bmatrix}, r, s, t \in \mathbf{R}\}$ be the space spanned by $\lambda_1, \lambda_2, \lambda_3$ in $\mathbf{R}^3_2$. Then X is linearly isomorphic to $\mathbf{R}^3$, where the isomorphism $I : \mathbf{R}^3 \to X$ is defined by $I(r, s, t) = X(r, s, t)$. This isomorphism induces the isomorphism I between the space $\mathcal{K}^3$ and $\mathcal{K}(\Omega, W)$, namely

$I(u_1(y), u_2(x), u_3(x+y)) = \begin{bmatrix} u_1(y) & 0 & u_3(x+y) \\ 0 & u_2(x) & u_3(x+y) \end{bmatrix}$. Šverák's countrexample was based on the following argument. Take function f_ϵ of the form (25) acting on $\mathbf{R}^3$, and transform it to the function $f_{\tilde{\epsilon}}$ acting on X, given by $\tilde{f}_\epsilon(X(r,s,t)) = f_\epsilon(r,s,t)$. Then obviously $\tilde{f}_\epsilon$ is not lower semicontinuous on $\mathcal{K}^3$ (f_ϵ was not lower semicontinuous on $\mathcal{K}(\Omega, W)$), but when $\epsilon > 0$ it does extend to a rank–one convex function F_ϵ defined on the whole space of matrices $\mathbf{R}^3_2$. In particular F_ϵ cannot define the lower semicontinuous functional on the space of gradients, so F_ϵ cannot be quasiconvex. This implies that in the case $m \geq 3$, $n = 2$ the set of all rank–one convex functionals is essentially biger than that of quasiconvex ones.

Let me comment now the remaining case when $n = m = 2$.

Remark 5.2. Let $m = n = 2$ now, and $f : \mathbf{R}^2_2 \to \mathbf{R}$ be continuous and rank–one convex. Take $\xi_1, \xi_2, \xi_3 \in \mathbf{R}^2$ such that ξ_1, ξ_2 are linearly independent, $b_1, b_2 \in \mathbf{R}^2$, and $(\xi_1, \lambda_1 = \xi_1 \otimes b_1), (\xi_2, \lambda_2 = \xi_2 \otimes b_2) \in V$. Then vectors λ_1, λ_2 are linearly independent, and $\mathcal{K}^2 = \mathcal{K}^2_{(\xi_1,\lambda_1),(\xi_2,\lambda_2)} = \{u_1(< x, \xi_1 >)\lambda_1 + u_2(< x, \xi_2 >)\lambda_2, u_i \in L^\infty\}$ is a subspace of $\mathcal{K}$. It can be easily computed that the functional I_f reduced to $\mathcal{K}^2$ is lower semicontiuous. This is also the simple ilustration of Theorem 3.4. Since the space $\mathcal{K}^2$ can be identified with the space $\mathcal{K}(Q, W)$, where $Q \subseteq \mathbf{R}^2$ is an arbitrary cube, and $W = (\mathrm{span}\{e_2\}, \mathrm{span}\{e_1\}) \in \mathcal{W}(2,2)$, the question about lower semicontinuity of the functional I_f reduced to the space $\mathcal{K}^2$ is the question about lower semicontinuity of the functional $I_{\tilde{f}}$ defined on the space $\mathcal{K}(Q, W)$, where $\tilde{f}(z_1, z_2) = f(z_1\lambda_1 + z_2\lambda_2)$. Note that for the problem transported to $\mathcal{K}(Q, W)$ we have $\Lambda = (\mathbf{R} \times \{0\}) \cup (\{0\} \times \mathbf{R}) = \Lambda_{\mathcal{B}(W)}$, $\mathcal{B}(W) = \{(1,0), (0,1)\}$, and $\tilde{f}$ is Λ convex. Since according to Definition 3.1 W does satisfy the chain condition, this implies that the functional $I_{\tilde{f}}$ is lower semicontinuous. Then also I_f, restricted to $\mathcal{K}^2$ is lower semicontinuous. When $\xi_1 = (1,0)$, $\xi_2 = (0,1)$, and $b_1 = (1,0), b_2 = (0,1)$ then $\mathcal{K}^2$ is exactly the space of diagonal matrices of the form $\begin{bmatrix} u_1(x_1) & 0 \\ 0 & u_2(x_2) \end{bmatrix}$. This model has been deeply analyzed by Müller in his paper [16].

Remark 5.3. Let $f : \mathbf{R}^2_2 \to \mathbf{R}$ be rank–one convex. Take $\xi_1, \xi_2, \xi_3 \in \mathbf{R}^2$ such that every pair ξ_i, ξ_j is linearly independent for $i \neq j$, $b_1, b_2, b_3 \in \mathbf{R}^2$, let $(\xi_1, \lambda_1 = \xi_1 \otimes b_1), (\xi_2, \lambda_2 = \xi_2 \otimes b_2), (\xi_3, \lambda_3 = \xi_3 \otimes b_3) \in V$, and consider $\mathcal{K}^3 = \mathcal{K}^3_{(\xi_1,\lambda_1),(\xi_2,\lambda_2),(\xi_3,\lambda_3)} =$

$$= \{u_1(< x, \xi_1 >)\lambda_1 + u_2(< x, \xi_2 >)\lambda_2 + u_3(< x, \xi_3 >)\lambda_3 : u_1, u_2, u_3 \in L^\infty(\mathbf{R})\}.$$

Then the question about lower semicontinuity of the functional I_f when reduced to $\mathcal{K}^3$ is the question about the lower semicontinuity of the corresponding functional $I_{\tilde{f}}(u)$ where $u \in \mathcal{K}(Q, W)$, $Q \subseteq \mathbf{R}^2$ is an arbitrary cube in $\mathbf{R}^2$, $W \in \widetilde{\mathcal{W}}(2,3)$, $W = (\mathrm{span}\{e_2\}, \mathrm{span}\{e_1\}, \mathrm{span}\{e_1 - e_2\})$ is as in Example 4.5, and $\tilde{f}(z_1, z_2, z_3) = f(z_1\lambda_1 + z_2\lambda_2 + z_3\lambda_3)$. Obviously such W does not satisfy the chain condition and convexity of f along vectors from the standard basis $e_1, e_2, e_3 \in \mathbf{R}^3$ is not the sufficient condition for lower semicontinuity of the functional $I_{\tilde{f}}$. On the other hand, if f is rank–one convex, then f is convex along all walls of the form $\Sigma^b = \{\xi \otimes b : \xi \in \mathbf{R}^2\}$, and $\Sigma_\xi = \{\xi \otimes b : b \in \mathbf{R}^2\}$, two–dimensional subspaces of $\mathbf{R}^2_2$. In particular, if f is rank–one convex, then $\tilde{f}$ is convex along all walls $I(\Sigma^b)$, $I(\Sigma_\xi)$, where $I : \mathbf{R}^2_2 \to \mathbf{R}^3$ is given by $I(t_1\lambda_1 + t_2\lambda_2 + t_3\lambda_3 + t_4\lambda_4) = (t_1, t_2, t_3)$, and λ_4 is the completion of $\lambda_1, \lambda_2, \lambda_3$ to the basis in $\mathbf{R}^2_2$. If we take $b_1 = b_2$, and $b_3 \neq b_2$ then $\lambda_1, \lambda_2, \lambda_3$ are still linearly independent, but f is convex along the whole wall $\{\xi \otimes b_1 : \xi \in \mathbf{R}^2\}$, which containes λ_1 and λ_2. This implies that $\tilde{f}$ is Λ convex, where $\Lambda = (\mathbf{R} \times \mathbf{R} \times \{0\}) \cup (\{0\} \times \{0\} \times \mathbf{R}) = \Lambda_{\mathcal{B}}$, with $\mathcal{B} = \{(1,1,0),(0,0,1)\}$. Since W does satisfy the chain condition with respect to $\mathcal{B}$ (see Example 4.5), we see that $I_{\tilde{f}}$ is lower semicontinuous on $\mathcal{K}(Q, W)$. Thus I_f is lower semicontinuous when reduced to $\mathcal{K}^3 = \mathcal{K}^3_{(\xi_1,\lambda_1),(\xi_2,\lambda_2),(\xi_3,\lambda_3)}$, provided that every pair ξ_i, ξ_j is linearly independent for $i \neq j$. I do not know what happens if the pair b_1, b_2 is linearly independent.

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