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Hölder Continuity for Local Minimizers of a Nonconvex Variational Problem

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We consider integral functionals of the Calculus of Variations where the energy density is a continuous function with p-growth, $p > 1$, uniformly convex at infinity with respect to the gradient variable. We prove that local minimizers are α-Hölder continuous for all $\alpha < 1$.

Keywords: Local minimizer, regularity, nonconvex functional

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1. Introduction

Consider the integral functional of the Calculus of Variations

$$\mathcal{F}(u;\Omega) := \int_\Omega f(x,u(x),Du(x))\,dx\,, \tag{1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ and $f = f(x,u,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0,+\infty)$ is a Carathéodory function, i.e. f is measurable in x and continuous in (u,ξ), satisfying the p-growth condition $(p > 1)$

$$|\xi|^p \le f(x,u,\xi) \le L(1+|\xi|^p).$$

A function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a *local minimizer* of $\mathcal{F}$ in Ω if

$$\mathcal{F}\left(u;\operatorname{spt}(v-u)\right) \le \mathcal{F}\left(v;\operatorname{spt}(v-u)\right),$$

for every $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ such that $\operatorname{spt}(v-u) \subset\subset \Omega$.

Well known results due to Giaquinta and Giusti [13, 15] ensure that local minimizers of $\mathcal{F}$ are locally α-Hölder continuous for *some* $\alpha < 1$. According to Meyers' example in [19], when f is not continuous in $\Omega \times \mathbb{R} \times \mathbb{R}^N$, the α-Hölder continuity for *all* $\alpha < 1$ cannot be achieved, even if f is twice differentiable and uniformly convex with respect

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to ξ. However, if f is of class C^2 in ξ, uniformly elliptic in this variable and for every $x, y \in \Omega$, $u, v \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$

$$|f(x,u,\xi) - f(y,v,\xi)| \leq \omega(|x-y| + |u-v|)(1+|\xi|^p)\,, \tag{2}$$

where $\omega(t)$ is a modulus of continuity that grows as t^δ, $\delta > 0$, then Du is Hölder continuous (see Giaquinta-Giusti [14], Giaquinta-Modica [16], Manfredi [18]).

Recent results show that failing the differentiability of the integrand, the Lipschitz continuity of local minimizers, or at least the Hölder continuity for every exponent, still holds under suitable convexity assumptions.

Fonseca and Fusco in [9] study the regularity of local minimizers of functionals with a nondifferentiable integrand $f = f(\xi)$ independent of (x, u). They prove that if f has p-growth and satisfies a uniform convexity condition, i.e. there exists a constant $\nu > 0$ such that for all ξ, η in $\mathbb{R}^N$

$$f\left(\frac{\xi+\eta}{2}\right) \leq \frac{1}{2}f(\xi) + \frac{1}{2}f(\eta) - \nu(1+|\xi|^2+|\eta|^2)^{\frac{p-2}{2}}|\xi-\eta|^2\,, \tag{3}$$

then local minimizers are locally Lipschitz continuous.

Notice that when f is of class C^2, then (3) is equivalent to the usual uniform ellipticity condition

$$\langle D^2 f(\xi)\lambda, \lambda\rangle \geq \nu_0(1+|\xi|^2)^{\frac{p-2}{2}}|\lambda|^2\,,$$

for some $\nu_0 > 0$.

The general case $f = f(x, u, \xi)$ has been studied by Cupini, Fusco and Petti in [2]. They prove that if (2) and (3) hold, then local minimizers are locally α-Hölder continuous for all $\alpha < 1$. We remark that even in the simple case $f = f(x, \xi)$, $N = p = 2$, the Lipschitz continuity cannot be achieved (see Example 3.2 in [2]). Related results can be found in [4, 7, 11].

In a recent paper Fonseca, Fusco and Marcellini [10] establish existence and regularity of minimizers of energy integrals with $f = f(x, \xi)$ subject to Dirichlet boundary conditions, where the uniform convexity property of f is satisfied *at infinity*, i.e. (3) holds for all $x \in \Omega$ and every ξ, η endpoints of a segment contained in the complement of a ball $B_R(0)$ in $\mathbb{R}^N$. In particular, they prove the Lipschitz continuity of local minimizers when f is a continuous function with p-growth, uniformly convex at infinity, such that for $|\xi| > R$ the vector field $x \mapsto D_\xi f(x, \xi)$ is weakly differentiable with $|D_{\xi x} f(x, \xi)| \leq L(1 + |\xi|^{p-1})$. We explicitly notice that in the particular case $f = f(\xi)$ the differentiability assumption can be removed. In this case the p-growth and the uniform convexity at infinity are sufficient conditions to the Lipschitz continuity of local minimizes (see Theorem 2.2 below).

In this paper we consider the general case of nonconvex integrands $f = f(x, u, \xi)$ with p-growth, both uniformly convex and continuous at infinity and we prove that local minimizers of $\mathcal{F}$ are α-Hölder continuous for any exponent $\alpha < 1$. More precisely, we prove the following

Theorem 1.1. *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, +\infty)$ be a Carathéodory function satisfying*

(A1) *there exist $p > 1$ and $L > 0$ such that for a.e. $x \in \Omega$, for every $u \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$*

$$0 \leq f(x,u,\xi) \leq L(1+|\xi|^p)\,;$$

(A2) *f is uniformly convex at infinity with respect to ξ, i.e. there exist $R > 0$ and $\nu > 0$ such that if the segment $[\xi, \eta]$ is contained in the complement of the ball $B_R(0)$, then for a.e. $x \in \Omega$ and for every $u \in \mathbb{R}$*

$$f\left(x, u, \frac{\xi+\eta}{2}\right) \leq \frac{1}{2} f(x,u,\xi) + \frac{1}{2} f(x,u,\eta) - \nu(1+|\xi|^2+|\eta|^2)^{\frac{p-2}{2}} |\xi-\eta|^2 ;$$

(A3) *f is continuous at infinity with respect to the pair (x,u), i.e. for every $x, y \in \Omega$, for every $u, v \in \mathbb{R}$ and $\xi \in \mathbb{R}^N \setminus B_R(0)$*

$$|f(x,u,\xi) - f(y,v,\xi)| \leq \omega(|x-y| + |u-v|)(1+|\xi|^p) ,$$

where $\omega : [0,+\infty) \to [0,+\infty)$ is a continuous, increasing, bounded function such that $\omega(0) = 0$.

If $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a local minimizer of the functional $\mathcal{F}$ defined in (1), then $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for all $\alpha < 1$.

To better understand the type of functionals we deal with, notice that (A1) and (A2) are equivalent to assume that

$$f(x,u,\xi) = c(1+|\xi|^2)^{\frac{p}{2}} + g(x,u,\xi) ,$$

with $c > 0$, g bounded from below, with p-growth from above and convex at infinity, i.e. for a.e. $x \in \Omega$, for every $u \in \mathbb{R}$ and for every $\xi_1, \xi_2 \in \mathbb{R}^N$ endpoints of a segment contained in the complement of $B_R(0)$

$$\frac{1}{2}[g(x,u,\xi_1) + g(x,u,\xi_2)] \geq g\left(x, u, \frac{\xi_1+\xi_2}{2}\right) .$$

For more details, we refer to [10, Section 2.1], [3, Section 2] and to Section 2 below.

The plan of the paper is the following.

In Section 2 we collect preliminary results concerning functions satisfying (A1) and (A2), higher integrability results for minimizers of functionals with p-growth and an iteration lemma.

In Section 3 we study the regularity of a local minimizer u of

$$\mathcal{I}(u;\Omega) =: \int_\Omega f(x, Du(x))\, dx,$$

when $f = f(x,\xi)$ is a convex function with respect to ξ and satisfies (A1)–(A3). We approximate $\mathcal{I}(u;\Omega)$ with a sequence of functionals $(\mathcal{I}_h)$ with integrands $f_h = f_h(x,\xi)$ of class C^1 with respect to ξ, satisfying the same assumptions as f. Fixed $B_r(x_0) \subset\subset \Omega$, let u_h be the minimizer of $\mathcal{I}_h(w; B_r(x_0))$ in the Dirichlet class $u + W^{1,p}_0(B_r(x_0))$, where u is the local minimizer of the functional $\mathcal{I}$. For each u_h we establish an integral estimate with constants independent of h. Moreover, we prove that u_h converges in the weak topology of $W^{1,p}$ to a function u_∞, which turns out to be a minimizer of $\mathcal{I}$ in $u + W^{1,p}_0(B_r(x_0))$. Passing to the limit in the integral estimate proved for each u_h, we infer that the estimate is satisfied by u_∞ and, from the uniform convexity at infinity, by u too. From this estimate the Hölder continuity of u for any exponent follows by a classical argument.

In Section 4 we prove Theorem 1.1. Firstly we consider the supplementary assumption of convexity of f with respect to ξ. We compare the local minimizer u of $\mathcal{F}$ with the minimizer of a functional $\mathcal{I}$ of the type studied in Section 3. An essential tool for this comparison is the Ekeland variational principle. The convexity assumption on f is removed via a relaxation argument.

2. Preliminary results

Let us consider the integral functional

$$\mathcal{F}(u;\Omega) := \int_\Omega f(x,u(x),Du(x))\,dx\,, \tag{4}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ and $u:\Omega\to\mathbb{R}$ is a scalar function in $W^{1,p}_{\mathrm{loc}}(\Omega)$, with $p>1$. We denote by $B_r(x_0)$ the open ball with center x_0 and radius r; we omit x_0 if no confusion may arise. In the sequel, c is a positive constant which may take different values from line to line.

We recalled the definition of *local minimizer* of $\mathcal{F}$ in the Introduction. More generally, u is a *Q-minimizer* of $\mathcal{F}$ if there exists $Q\geq 1$ such that

$$\mathcal{F}\left(u;\mathrm{spt}\,(v-u)\right) \leq Q\mathcal{F}\left(v;\mathrm{spt}\,(v-u)\right)\,,$$

for all $v\in W^{1,p}_{\mathrm{loc}}(\Omega)$ such that $\mathrm{spt}\,(v-u)\subset\subset\Omega$.

We state some consequences of assumptions (A1) and (A2), proved in [10] in the case $f=f(\xi)$. The generalization to our case is straightforward.

Theorem 2.1. *Let $f=f(x,u,\xi):\Omega\times\mathbb{R}\times\mathbb{R}^N\to[0,+\infty)$, be a Carathéodory function. Then* (A1) *and* (A2) *imply:*

(i) *there exist $c_1(\nu)$ and $c_2(p,L,R,\nu)$ such that for a.e. $x\in\Omega$ and for every $(u,\xi)\in\mathbb{R}\times\mathbb{R}^N$*

$$f(x,u,\xi)\geq c_1|\xi|^p - c_2\,;$$

(ii) *there exist $R_0>R$ and $\nu_0>0$ depending only on p,L,R and ν, such that for a.e. $x\in\Omega$ and for every $u\in\mathbb{R}$ and $\xi\in\mathbb{R}^N\setminus B_{R_0}(0)$ there exists $q_\xi(x,u)\in\mathbb{R}^N$ such that*

$$|q_\xi(x,u)|\leq c(p,L)(1+|\xi|)^{p-1},$$

and for every $\eta\in\mathbb{R}^N$

$$f(x,u,\eta)\geq f(x,u,\xi)+\langle q_\xi(x,u),\eta-\xi\rangle+\nu_0(1+|\xi|^2+|\eta|^2)^{\frac{p-2}{2}}|\xi-\eta|^2.$$

Moreover, if $\xi\mapsto f(x,u,\xi)$ is $C^1(\mathbb{R}^N\setminus B_{R_0}(0))$ then $q_\xi(x,u)=D_\xi f(x,u,\xi)$;

(iii) *if $\xi\mapsto f^{**}(x,u,\xi)$ is the convex envelope of $\xi\mapsto f(x,u,\xi)$ then $f=f^{**}$ in $\Omega\times\mathbb{R}\times(\mathbb{R}^N\setminus B_{R_0}(0))$.*

We underline that Theorem 2.7 of [10] entails the following regularity result.

Theorem 2.2. *Let $f = f(\xi) : \mathbb{R}^N \to [0, +\infty)$ be a continuous function with p-growth, satisfying the uniform convexity* (3) *at infinity. If $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a local minimizer of the functional*

$$\int_\Omega f(Du(x))\, dx\,,$$

then u is locally Lipschitz continuous in Ω. Moreover, there exists c, depending on N, p, L, R and ν, such that for every $B_r(x_0) \subset\subset \Omega$

$$\sup_{B_{r/2}(x_0)} |Du|^p \le c \left[\fint_{B_r(x_0)} (1 + |Du|^p)\, dx \right].$$

Now we state a simple algebraic result that will be useful in the sequel (see [17], Lemma 8.5 or [2], Lemma 2.2).

Lemma 2.3. *If $p > 1$, there exists $c > 0$ such that for every ξ, $\eta \in \mathbb{R}^N$*

$$(1 + |\xi|^2)^{\frac{p}{2}} \le c\,(1 + |\eta|^2)^{\frac{p}{2}} + c\,(1 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} |\xi - \eta|^2\,.$$

We will use some regularity results of Q-minimizers of integral functionals. The following result is contained in Theorem 3.1 of [15].

Lemma 2.4. *Let $B_r(x_0) \subset\subset \Omega$ and $\phi \in L^p(B_r(x_0))$. If $u \in W^{1,p}_{\mathrm{loc}}(B_r(x_0))$ is a Q-minimizer of the functional*

$$w \mapsto \int_{B_r} \left(1 + |Dw(x)|^p + |\phi(x)|^{\frac{p+1}{2}} \right) dx\,, \tag{5}$$

then there exists $\tau > 1$ such that $u \in W^{1,p\tau}_{\mathrm{loc}}(B_r)$. Moreover, there exists $c = c(N, p, Q)$ such that for every $B_\rho(x_1) \subset B_r(x_0)$

$$\left(\fint_{B_{\rho/2}(x_1)} |Du|^{p\tau}\, dx \right)^{\frac{1}{\tau}} \le c \fint_{B_\rho(x_1)} (1 + |Du|^p + |\phi|^p)\, dx\,.$$

We need also an up-to-the-boundary higher integrability result (see e.g. Theorem 6.8 of [17] and Lemma 2.7 in [2]).

Lemma 2.5. *Let $h : B_{2r}(x_0) \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function such that $|\xi|^p \le h(x, \xi) \le L(1 + |\xi|^p)$. If $u_0 \in W^{1,q}(B_{2r}(x_0))$, for a certain $q > p$, and v is a minimizer of the functional*

$$w \mapsto \int_{B_r(x_0)} h(x, Dw(x))\, dx$$

in the Dirichlet class $u_0 + W^{1,p}(B_r(x_0))$, then there exist $s \in (p, q)$ and $c > 0$, depending on N, p and L, but neither on u_0 nor r, such that $v \in W^{1,s}(B_r(x_0))$ and

$$\left[\fint_{B_r(x_0)} |Dv|^s\, dx \right]^{\frac{1}{s}} \le c \left[\fint_{B_{2r}(x_0)} (1 + |Du_0|^q)\, dx \right]^{\frac{1}{q}}.$$

Finally, we state an iteration lemma (see the proof of Proposition 3.7 in [8]).

Lemma 2.6. *Let Z, ψ, $\omega : [0,T] \to [0,+\infty)$ be bounded and increasing functions, ω continuous and $\omega(0) = 0$. Suppose that there exist α, β, γ, $c_1 > 0$ such that for every $\epsilon > 0$ there exist $c_2(\epsilon) > 0$ such that*

$$Z(t) \leq c_1 \left[\left(\frac{t}{s} \right)^{\alpha} + \omega(s) + \epsilon \right] Z(s) + c_2(\epsilon) s^{\gamma} \psi(s)$$

for every $0 < t < s \leq T$. Then for every $0 < \delta < \alpha$, there exists $T_0 < T$ such that

$$Z(t) \leq c_3 \left(\frac{t}{s} \right)^{\alpha - \delta} Z(s) + c_4 t^{\gamma} \psi(s),$$

whenever $0 < t < s \leq T_0$. Here T_0 and c_3 are positive constants depending only on α, β, δ and c_1, while c_4 depends also on γ.

3. Regularity in the case $f = f(x, Du)$

In this section we consider the functional

$$\mathcal{I}(u; \Omega) := \int_{\Omega} f(x, Du(x))\, dx\,, \tag{6}$$

where $f = f(x, \xi) : \Omega \times \mathbb{R}^N \to [0, +\infty)$ is a Carathéodory function, satisfying the assumptions (A1)–(A3), that in this case can be stated as follows:

($\mathcal{A}$1) the p-growth condition, i.e. for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$

$$0 \leq f(x, \xi) \leq L(1 + |\xi|^p);$$

($\mathcal{A}$2) the uniform convexity at infinity, i.e. there exist R and $\nu > 0$ such that for a.e. $x \in \Omega$ if $[\xi, \eta] \subset \mathbb{R}^N \setminus B_R(0)$, then

$$f\left(x, \frac{\xi + \eta}{2}\right) \leq \frac{1}{2} f(x, \xi) + \frac{1}{2} f(x, \eta) - \nu (1 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} |\xi - \eta|^2\,;$$

($\mathcal{A}$3) the uniform continuity in x at infinity, i.e. for every $x, y \in \Omega$ and for every $\xi \in \mathbb{R}^N \setminus B_R(0)$

$$|f(x, \xi) - f(y, \xi)| \leq \omega(|x - y|)(1 + |\xi|^p)\,,$$

where $\omega : [0, +\infty) \to [0, +\infty)$ is a continuous, increasing and bounded function such that $\omega(0) = 0$. Without loss of generality, we can assume ω to be concave.

Recall that by Theorem 2.1 (ii) there exists $R_0 > R$ and $\nu_0 > 0$ such that

$$f(x, \eta) \geq f(x, \xi) + \langle q_{\xi}(x), \eta - \xi \rangle + \nu_0 (1 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} |\xi - \eta|^2, \tag{7}$$

for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^N \setminus B_{R_0}(0)$ and $\eta \in \mathbb{R}^N$, where $q_{\xi}(x) \in \mathbb{R}^N$ satisfies

$$|q_{\xi}(x)| \leq c(p, L)(1 + |\xi|)^{p-1}.$$

Moreover, by Theorem 2.1 (i) there exist $c_1, c_2 > 0$ such that for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$

$$f(x,\xi) \geq c_1|\xi|^p - c_2\,. \tag{8}$$

With a view to the proof of Theorem 1.1, fixed $B_r(x_0) \subset\subset \Omega$ we consider the functional

$$\mathcal{G}_{\vartheta_0,u_0}(w;B_r) := \mathcal{I}(w;B_r) + \vartheta_0 \int_{B_r} |Dw - Du_0|^{\frac{p+1}{2}}\,dx\,, \tag{9}$$

where $\vartheta_0 \geq 0$ and $u_0 \in W^{1,p}(B_r)$. This functional will be useful to prove the regularity of local minimizers in the general case $f = f(x,u,\xi)$.

We begin by proving an integral estimate for local minimizers of $\mathcal{G}_{\vartheta_0,u_0}$ under the supplementary assumption that f is of class C^1 with respect to ξ and convex in this variable. This assumption will be removed in Proposition 3.2.

Proposition 3.1. *Consider $B_r(x_0) \subset\subset \Omega$, $\vartheta_0 \geq 0$, $u_0 \in W^{1,p}(B_r(x_0))$. Let $\mathcal{G}_{\vartheta_0,u_0}$ be as in* (9)*, with f of class C^1 with respect to ξ and convex in this variable, satisfying* ($\mathcal{A}$1)–($\mathcal{A}$3)*. Let $u \in W^{1,p}(B_r)$ be a minimizer of $\mathcal{G}_{\vartheta_0,u_0}$ in its Dirichlet class $u + W_0^{1,p}(B_r)$. Then there exists $c = c(N,p,L,\nu,R)$ such that for every $\rho < r$*

$$\begin{aligned}\int_{B_\rho} (1+|Du|^p)\,dx \leq c\left[\left(\frac{\rho}{r}\right)^N + \omega(r)\right]\int_{B_r}(1+|Du|^p)\,dx \\ + \frac{\vartheta_0^{\frac{p}{p-1}}}{[\omega(r)]^{\frac{1}{p-1}}}\int_{B_r}|Du - Du_0|^{\frac{p}{2}}\,dx + c\,r^N\,.\end{aligned} \tag{10}$$

Proof. Let v be the minimizer in $u + W_0^{1,p}(B_r(x_0))$ of the frozen functional

$$w \mapsto \int_{B_r} f(x_0, Dw(x))\,dx\,.$$

The function $\xi \mapsto f(x_0,\xi)$ satisfies the assumptions of Theorem 2.2, hence (see [9]) v is locally Lipschitz continuous in B_r and there exists a constant c depending on N, p, L, R and ν such that for all $\rho < r$

$$\int_{B_\rho} (1+|Dv|^p)\,dx \leq c\left(\frac{\rho}{r}\right)^N \int_{B_r} (1+|Dv|^p)\,dx\,.$$

(8), the minimality of v and ($\mathcal{A}$1) imply

$$\int_{B_r} (1+|Dv|^p)\,dx \leq c\int_{B_r} (1+|Du|^p)\,dx\,, \tag{11}$$

so that for any $\rho < r$

$$\int_{B_\rho} (1+|Dv|^p)\,dx \leq c\left(\frac{\rho}{r}\right)^N \int_{B_r} (1+|Du|^p)\,dx\,. \tag{12}$$

Define

$$\begin{aligned} A(Du,\rho,\lambda) &:= \{x \in B_\rho \,:\, |Du(x)| > \lambda\}\,,\\ B(Du,\rho,\lambda) &:= \{x \in B_\rho \,:\, |Du(x)| \leq \lambda\}\,.\end{aligned} \tag{13}$$

Let R_0 and ν_0, depending only on p, L, R and ν, be as in Theorem 2.1 (ii). Using Lemma 2.3 and (12)

$$\begin{aligned}
&\int_{B_\rho} |Du|^p\,dx = \int_{B(Du,\rho,R_0)} |Du|^p\,dx + \int_{A(Du,\rho,R_0)} |Du|^p\,dx \\
&\le c\int_{B_\rho} (1+|Dv|^p)\,dx + c\int_{A(Du,\rho,R_0)} (1+|Du|^2+|Dv|^2)^{\frac{p-2}{2}}|Du-Dv|^2\,dx \qquad (14)\\
&\le c\left(\frac{\rho}{r}\right)^N\int_{B_r} (1+|Du|^p)\,dx + c\int_{A(Du,\rho,R_0)} (1+|Du|^2+|Dv|^2)^{\frac{p-2}{2}}|Du-Dv|^2\,dx.
\end{aligned}$$

Let us estimate the last integral using the Euler equation for $\mathcal{G}_{\vartheta_0,u_0}$. From Theorem 2.1 (ii), the convexity of f and the minimality of u we have

$$\begin{aligned}
&\int_{A(Du,\rho,R_0)} (1+|Du|^2+|Dv|^2)^{\frac{p-2}{2}}|Du-Dv|^2\,dx \\
&\le \frac{1}{\nu_0}\int_{A(Du,\rho,R_0)} [f(x,Dv)-f(x,Du)-\langle D_\xi f(x,Du), Dv-Du\rangle]\,dx \\
&\le \frac{1}{\nu_0}\int_{B_r} [f(x,Dv)-f(x,Du)-\langle D_\xi f(x,Du), Dv-Du\rangle]\,dx \\
&= \frac{1}{\nu_0}\int_{B_r} \Big[f(x,Dv)-f(x,Du)+\vartheta_0\langle D_\xi\Big(|Du-Du_0|^{\frac{p+1}{2}}\Big), Dv-Du\rangle\Big]\,dx \\
&\qquad -\frac{1}{\nu_0}\int_{B_r} \langle D_\xi f(x,Du)+\vartheta_0 D_\xi\Big(|Du-Du_0|^{\frac{p+1}{2}}\Big), Dv-Du\rangle\,dx \\
&= \frac{1}{\nu_0}\int_{B_r} \Big[f(x,Dv)-f(x,Du)+\vartheta_0\langle D_\xi\Big(|Du-Du_0|^{\frac{p+1}{2}}\Big), Dv-Du\rangle\Big]\,dx\,,
\end{aligned}$$

where ξ is a dummy variable for the gradient. Thus, adding and subtracting $\int_{B_r} f(x_0,Dv)dx$ and $\int_{B_r} f(x_0,Du)dx$, and using the minimality of v it follows

$$\begin{aligned}
&\int_{A(Du,\rho,R_0)} (1+|Du|^2+|Dv|^2)^{\frac{p-2}{2}}|Du-Dv|^2\,dx \\
&\le \frac{1}{\nu_0}\int_{B_r} [f(x,Dv)-f(x_0,Dv)]\,dx + \frac{1}{\nu_0}\int_{B_r} [f(x_0,Du)-f(x,Du)]\,dx \\
&\qquad + \frac{\vartheta_0}{\nu_0}\int_{B_r} \langle D_\xi\Big(|Du-Du_0|^{\frac{p+1}{2}}\Big), Dv-Du\rangle\,dx\,.
\end{aligned}$$

From $(\mathcal{A}1)$ and $(\mathcal{A}3)$

$$\begin{aligned}
&\int_{B_r} [f(x,Dv)-f(x_0,Dv)]\,dx \\
&= \int_{A(Dv,r,R)} [f(x,Dv)-f(x_0,Dv)]\,dx + \int_{B(Dv,r,R)} [f(x,Dv)-f(x_0,Dv)]\,dx \\
&\le \int_{A(Dv,r,R)} \omega(|x-x_0|)\,(1+|Dv|^p)\,dx + L\int_{B(Dv,r,R)} (1+|Dv|^p)\,dx \\
&\le \omega(r)\int_{B_r} (1+|Dv|^p)\,dx + c(N,p,L,R)\,r^N.
\end{aligned}$$

Analogously,

$$\int_{B_r} [f(x_0, Du) - f(x, Du)]\, dx \le \omega(r) \int_{B_r} (1 + |Du|^p)\, dx + c(N, p, L, R)\, r^N.$$

Therefore from (11) and the minimality of v there exists c, depending only on N, p, L, R and ν, such that

$$\begin{aligned}
&\int_{A(Du,\rho,R_0)} (1 + |Du|^2 + |Dv|^2)^{\frac{p-2}{2}} |Du - Dv|^2\, dx \\
&\le c\,\omega(r) \int_{B_r} (1 + |Du|^p) dx + c\,\vartheta_0 \int_{B_r} |Du - Du_0|^{\frac{p-1}{2}} |Du - Dv|\, dx + c\, r^N .
\end{aligned}$$

This inequality, together with (14) and Young inequality, implies that for every $\rho < r$ there exists c, depending on N, p, L, R and ν, such that

$$\begin{aligned}
\int_{B_\rho} |Du|^p\, dx &\le c \left[\left(\frac{\rho}{r}\right)^N + \omega(r) \right] \int_{B_r} (1 + |Du|^p)\, dx \\
&+ c\,\omega(r) \int_{B_r} |Du - Dv|^p\, dx + c \frac{\vartheta_0^{\frac{p}{p-1}}}{[\omega(r)]^{\frac{1}{p-1}}} \int_{B_r} |Du - Du_0|^{\frac{p}{2}}\, dx + c\, r^N
\end{aligned}$$

and from (11) the thesis follows. □

We remark that the freezing techinque emploied in the proof of the above proposition has been first used in [14, 18] and applied in the setting of non standard growth conditions, up to a certain extent, in [1].

An approximation argument allows us to remove the differentiability assumption on f.

Proposition 3.2. *Consider $B_r(x_0) \subset\subset \Omega$, $\vartheta_0 \ge 0$, $u_0 \in W^{1,p}(B_r(x_0))$. Let $\mathcal{G}_{\vartheta_0,u_0}$ be as in (9), with f convex with respect to ξ, satisfying ($\mathcal{A}1$)–($\mathcal{A}3$). Let $u \in W^{1,p}(B_r)$ be a minimizer of $\mathcal{G}_{\vartheta_0,u_0}$ in its Dirichlet class $u + W_0^{1,p}(B_r)$. Then for every $\rho < r$*

$$\begin{aligned}
\int_{B_\rho} |Du|^p dx &\le c \left[\left(\frac{\rho}{r}\right)^N + \omega(r) \right] \int_{B_r} (1 + |Du|^p + |Du_0|^p)\, dx \\
&\quad + c \frac{\vartheta_0^{\frac{2p}{p-1}}}{[\omega(r)]^{\frac{p+1}{p-1}}}\, r^N + c\, r^N,
\end{aligned}$$

with c depending on N, p, L, R and ν.

Proof. Let $\sigma \in C_c^\infty(B_1(0), [0, +\infty))$ be a radially symmetric mollifier such that $\int_{B_1(0)} \sigma(z)\, dz = 1$, and define

$$f_h(x, \xi) := \int_{B_1(0)} \sigma(z) f\left(x, \xi + \frac{1}{h} z\right)\, dz\,.$$

$(f_h)_{h\in\mathbb{N}}$ satisfies the following assumptions:

($\mathcal{H}$1) there exists $\overline{L}$, depending on L and p, but not on h, such that for every h, for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$

$$0 \leq f_h(x,\xi) \leq \overline{L}(1+|\xi|^p)\,;$$

($\mathcal{H}$2) there exists $\nu_1 > 0$, depending on p and ν, but not on h, such that if $[\xi,\eta] \subset \mathbb{R}^N \setminus B_{R+1}(0)$, then for a.e. $x \in \Omega$

$$f_h\left(x, \frac{\xi+\eta}{2}\right) \leq \frac{1}{2} f_h(x,\xi) + \frac{1}{2} f_h(x,\eta) - \nu_1 (1+|\xi|^2+|\eta|^2)^{\frac{p-2}{2}} |\xi-\eta|^2\,;$$

($\mathcal{H}$3) there exists $c = c(p)$ such that for every x, $y \in \Omega$ and for every ξ with $|\xi| \geq R+1$

$$|f_h(x,\xi) - f_h(y,\xi)| \leq c\,\omega(|x-y|)(1+|\xi|^p)\,.$$

Moreover, from (8) there exist $\overline{c}_1, \overline{c}_2 > 0$, independent of h, such that

$$f_h(x,\xi) \geq \overline{c}_1 |\xi|^p - \overline{c}_2, \quad \forall (x,\xi) \in \Omega \times \mathbb{R}^N\,. \tag{15}$$

Let u_h be a minimizer in $u + W_0^{1,p}(B_r)$ of

$$\mathcal{G}_h(w;B_r) := \int_{B_r} f_h(x,Dw)\,dx + \vartheta_0 \int_{B_r} |Dw - Du_0|^{\frac{p+1}{2}}\,dx\,.$$

Using (15), the minimality of u_h, ($\mathcal{H}$1) and Young inequality, it is not difficult to prove that there exists $c = c(N,p,L,R,\nu,\vartheta_0)$, such that

$$\int_{B_r} |Du_h|^p\,dx \leq c \int_{B_r} (1+|Dw|^p + |Du_0|^{\frac{p+1}{2}})\,dx\,,$$

for every $w \in u + W_0^{1,p}(B_r)$. In particular

$$\int_{B_r} |Du_h|^p\,dx \leq c \int_{B_r} (1+|Du|^p + |Du_0|^p)\,dx \tag{16}$$

and u_h is a Q-minimizer of

$$w \mapsto \int_{B_r} \left(1+|Dw|^p + |Du_0|^{\frac{p+1}{2}}\right)\,dx\,.$$

Therefore, from Lemma 2.4 there exist $\tau > 1$ and $c > 0$ such that $u_h \in W^{1,p\tau}_{\mathrm{loc}}(B_r)$ for every h and for every $B_\rho(x_1) \subset B_r$

$$\left(\fint_{B_{\rho/2}(x_1)} |Du_h|^{p\tau}\,dx\right)^{\frac{1}{\tau}} \leq c \fint_{B_\rho(x_1)} (1+|Du_h|^p + |Du_0|^p)\,dx\,.$$

This inequality, together with (16), implies that for every $\rho < r$ there exists $c = c(\rho, r)$ such that

$$\left(\int_{B_\rho} |Du_h|^{p\tau} dx\right)^{\frac{1}{\tau}} \leq c \int_{B_r} (1+|Du|^p + |Du_0|^p)\,dx\,. \tag{17}$$

Moreover, (16) implies that the sequence (u_h) is bounded in $W^{1,p}(B_r)$. Up to a subsequence, we may assume that there exists $u_\infty \in u + W_0^{1,p}(B_r)$ such that $u_h \rightharpoonup u_\infty$ in the weak topology of $W^{1,p}(B_r)$.

Let us prove that u_∞ is a minimizer of $\mathcal{G}_{\vartheta_0,u_0}$ defined in (9). We shall use the notations in (13).

Since $\mathcal{G}_{\vartheta_0,u_0}$ is lower semicontinuous with respect to the weak topology of $W^{1,p}$, then for every $\rho < r$ and $k \in \mathbb{N}$

$$\begin{aligned} \mathcal{G}_{\vartheta_0,u_0}(u_\infty; B_\rho) &\le \liminf_{h\to\infty} \int_{B_\rho} \left[f(x, Du_h) + \vartheta_0 |Du_h - Du_0|^{\frac{p+1}{2}} \right] dx \\ &\le \limsup_{h\to\infty} \int_{A(Du_h,\rho,k)} f(x, Du_h)\, dx \\ &+ \limsup_{h\to\infty} \left\{ \int_{B(Du_h,\rho,k)} f(x, Du_h) dx + \vartheta_0 \int_{B_\rho} |Du_h - Du_0|^{\frac{p+1}{2}} dx \right\}. \end{aligned} \tag{18}$$

Since f is convex with respect to ξ and $(\mathcal{A}1)$ holds, then for a.e. $x \in \Omega$ and for any ξ and η in $\mathbb{R}^N$,

$$|f(x,\xi) - f(x,\eta)| \le c(p, L)(1 + |\xi| + |\eta|)^{p-1} |\xi - \eta|\,;$$

therefore

$$\begin{aligned} &\lim_{h\to\infty} \int_{B(Du_h,\rho,k)} |f(x, Du_h) - f_h(x, Du_h)|\, dx \\ &\le c \lim_{h\to\infty} \frac{1}{h} \int_{B(Du_h,\rho,k)} (1 + |Du_h|)^{p-1}\, dx \\ &\le c \lim_{h\to\infty} \frac{1}{h} (1+k)^{p-1} |B(Du_h, \rho, k)| = 0\,. \end{aligned} \tag{19}$$

Thus, from (18) and (19)

$$\begin{aligned} \mathcal{G}(u_\infty; B_\rho) \le \limsup_{h\to\infty} &\int_{A(Du_h,\rho,k)} f(x, Du_h)\, dx \\ &+ \limsup_{h\to\infty} \int_{B_r} \left[f_h(x, Du_h) + \vartheta_0 |Du_h - Du_0|^{\frac{p+1}{2}} \right] dx\,. \end{aligned}$$

From $(\mathcal{A}1)$, Hölder inequality, (17) and the minimality of u_h we get that

$$\begin{aligned} \mathcal{G}_{\vartheta_0,u_0}(u_\infty; B_\rho) &\le L \limsup_{h\to\infty} \left\{ \left(\int_{B_\rho} (1 + |Du_h|^{\frac{p}{\tau}}) dx \right)^{\frac{1}{\tau}} |A(Du_h, \rho, k)|^{\frac{\tau-1}{\tau}} \right\} \\ &+ \limsup_{h\to\infty} \int_{B_r} \left[f_h(x, Du) + \vartheta_0 |Du - Du_0|^{\frac{p+1}{2}} \right] dx \\ &\le c\, k^{p(1-\tau)} \limsup_{h\to\infty} \int_{B_\rho} (1 + |Du_h|^{p\tau})\, dx + \int_{B_r} \left[f(x, Du) + \vartheta_0 |Du - Du_0|^{\frac{p+1}{2}} \right] dx\,. \end{aligned}$$

The last inequality and (17) imply

$$\mathcal{G}_{\vartheta_0,u_0}(u_\infty;B_\rho)\le c\,k^{p(1-\tau)}+\mathcal{G}_{\vartheta_0,u_0}(u;B_r)\,,$$

where $c=c(N,p,L,u,u_0,\vartheta_0,\rho,r)$. Letting k go to infinity and then ρ go to r we infer $\mathcal{G}_{\vartheta_0,u_0}(u_\infty;B_r)=\mathcal{G}_{\vartheta_0,u_0}(u;B_r)$, so that u_∞ is a minimizer of $\mathcal{G}_{\vartheta_0,u_0}(w;B_r)$.

Now we compare u_∞ with u. For every $x\in B_r$ we apply Theorem 2.1 (ii) with $\xi=\frac{1}{2}(Du(x)+Du_\infty(x))$ and $\eta=Du(x)$. Thus, from (7) we have

$$\begin{aligned}&\int_{A(Du+Du_\infty,r,2R_0)}\left[f(x,Du)-f\left(x,\frac{Du+Du_\infty}{2}\right)\right]dx\\
&\ge\int_{A(Du+Du_\infty,r,2R_0)}\left\langle q_\xi(x),\frac{Du-Du_\infty}{2}\right\rangle dx\\
&\quad+\nu\int_{A(Du+Du_\infty,r,2R_0)}\left(1+\left|\frac{Du+Du_\infty}{2}\right|^2+|Du|^2\right)^{\frac{p-2}{2}}\left|\frac{Du-Du_\infty}{2}\right|^2dx.\end{aligned}\tag{20}$$

Analogously, Theorem 2.1 (ii) with $\eta=Du_\infty(x)$ yields

$$\begin{aligned}&\int_{A(Du+Du_\infty,r,2R_0)}\left[f(x,Du_\infty)-f\left(x,\frac{Du+Du_\infty}{2}\right)\right]dx\\
&\ge\int_{A(Du+Du_\infty,r,2R_0)}\left\langle q_\xi(x),\frac{Du_\infty-Du}{2}\right\rangle dx\\
&\quad+\nu\int_{A(Du+Du_\infty,r,2R_0)}\left(1+\left|\frac{Du+Du_\infty}{2}\right|^2+|Du_\infty|^2\right)^{\frac{p-2}{2}}\left|\frac{Du-Du_\infty}{2}\right|^2dx;\end{aligned}\tag{21}$$

adding (20) and (21)

$$\begin{aligned}&\int_{A(Du+Du_\infty,r,2R_0)}[f(x,Du)+f(x,Du_\infty)]\,dx\\
&\ge 2\int_{A(Du+Du_\infty,r,2R_0)}f\left(x,\frac{Du+Du_\infty}{2}\right)dx\\
&\quad+\nu\int_{A(Du+Du_\infty,r,2R_0)}\left(1+\left|\frac{Du+Du_\infty}{2}\right|^2+|Du|^2\right)^{\frac{p-2}{2}}\left|\frac{Du-Du_\infty}{2}\right|^2dx\\
&\quad+\nu\int_{A(Du+Du_\infty,r,2R_0)}\left(1+\left|\frac{Du+Du_\infty}{2}\right|^2+|Du_\infty|^2\right)^{\frac{p-2}{2}}\left|\frac{Du-Du_\infty}{2}\right|^2dx.\end{aligned}\tag{22}$$

By convexity

$$\begin{aligned}&\int_{B(Du+Du_\infty,r,2R_0)}f\left(x,\frac{Du+Du_\infty}{2}\right)dx+\vartheta_0\int_{B_r}\left|\frac{Du+Du_\infty}{2}-Du_0\right|^{\frac{p+1}{2}}dx\\
&\le\frac{1}{2}\int_{B(Du+Du_\infty,r,2R_0)}f(x,Du)\,dx+\frac{\vartheta_0}{2}\int_{B_r}|Du-Du_0|^{\frac{p+1}{2}}dx\\
&\quad+\frac{1}{2}\int_{B(Du+Du_\infty,r,2R_0)}f(x,Du_\infty)\,dx+\frac{\vartheta_0}{2}\int_{B_r}|Du_\infty-Du_0|^{\frac{p+1}{2}}dx\,,\end{aligned}$$

therefore, from (22) we get

$$
\begin{aligned}
&\mathcal{G}_{\vartheta_0,u_0}\Big(\frac{u+u_\infty}{2};B_r\Big) \le \frac{1}{2}\mathcal{G}_{\vartheta_0,u_0}(u;B_r)+\frac{1}{2}\mathcal{G}_{\vartheta_0,u_0}(u_\infty;B_r)\\
&-\frac{\nu}{2}\int_{A(Du+Du_\infty,r,2R_0)}\bigg(1+\Big|\frac{Du+Du_\infty}{2}\Big|^2+|Du|^2\bigg)^{\frac{p-2}{2}}\Big|\frac{Du-Du_\infty}{2}\Big|^2\,dx\\
&-\frac{\nu}{2}\int_{A(Du+Du_\infty,r,2R_0)}\bigg(1+\Big|\frac{Du+Du_\infty}{2}\Big|^2+|Du_\infty|^2\bigg)^{\frac{p-2}{2}}\Big|\frac{Du-Du_\infty}{2}\Big|^2\,dx.
\end{aligned}
$$

The minimality of u_∞, together with the previous inequality, yields

$$
\begin{aligned}
&\mathcal{G}_{\vartheta_0,u_0}\Big(\frac{u+u_\infty}{2};B_r\Big) \le \mathcal{G}_{\vartheta_0,u_0}(u;B_r)\\
&-\frac{\nu}{2}\int_{A(Du+Du_\infty,r,2R_0)}\bigg(1+\Big|\frac{Du+Du_\infty}{2}\Big|^2+|Du|^2\bigg)^{\frac{p-2}{2}}\Big|\frac{Du-Du_\infty}{2}\Big|^2\,dx\\
&-\frac{\nu}{2}\int_{A(Du+Du_\infty,r,2R_0)}\bigg(1+\Big|\frac{Du+Du_\infty}{2}\Big|^2+|Du_\infty|^2\bigg)^{\frac{p-2}{2}}\Big|\frac{Du-Du_\infty}{2}\Big|^2\,dx,
\end{aligned}
$$

which implies that the Lebesgue measure of the set

$$\{x\in B_r : |Du(x)+Du_\infty(x)|>2R_0\}\cap\{x\in B_r : |Du(x)-Du_\infty(x)|>0\}$$

is zero. Therefore, for every $\rho<r$

$$
\begin{aligned}
&\int_{B_\rho}|Du|^p\,dx \le c\int_{B_\rho}|Du_\infty|^p\,dx + c\int_{B_\rho}|Du+Du_\infty|^p\,dx\\
&\le c\int_{B_\rho}|Du_\infty|^p\,dx + c\,2^pR_0^p\,\rho^N + c\int_{A(Du+Du_\infty,\rho,2R_0)}|Du+Du_\infty|^p\,dx\\
&= c\int_{B_\rho}|Du_\infty|^p\,dx + c\,\rho^N + c\int_{A(Du+Du_\infty,\rho,2R_0)\cap\{|Du-Du_\infty|=0\}}|Du_\infty|^p\,dx\,,
\end{aligned}
$$

thus there exists $c=c(N,p,L,R,\nu)$ such that

$$\int_{B_\rho}|Du|^p\,dx \le c\int_{B_\rho}(1+|Du_\infty|^p)\,dx\,. \tag{23}$$

Let us estimate the right-hand side. Since f_h is of class C^1 with respect to ξ and satisfies $(\mathcal{H}1)$–$(\mathcal{H}3)$, from Proposition 3.1 estimate (10) holds with u replaced by u_h. Hence, for all $\rho<r$,

$$
\begin{aligned}
&\int_{B_\rho}|Du_\infty|^p\,dx \le \liminf_{h\to\infty}\int_{B_\rho}(1+|Du_h|^p)\,dx\\
&\le \limsup_{h\to\infty}\bigg\{c\bigg[\Big(\frac{\rho}{r}\Big)^N+\omega(r)\bigg]\int_{B_r}(1+|Du_h|^p)\,dx + c\,r^N\bigg\}\\
&+\limsup_{h\to\infty}\frac{c\,\vartheta_0^{\frac{p}{p-1}}}{[\omega(r)]^{\frac{1}{p-1}}}\int_{B_r}|Du_h-Du_0|^{\frac{p}{2}}\,dx,
\end{aligned}
$$

where c is independent of h, ρ and r. Young inequality implies

$$\begin{aligned}
&\frac{c\vartheta_0^{\frac{p}{p-1}}}{[\omega(r)]^{\frac{1}{p-1}}}\int_{B_r}|Du_h-Du_0|^{\frac{p}{2}}\,dx\\
&=\frac{c\vartheta_0^{\frac{p}{p-1}}}{[\omega(r)]^{\frac{1}{p-1}}}\int_{B_r}\left[\frac{\vartheta_0}{\omega(r)}\right]^{\frac{p}{2(p-1)}}\left[\frac{\omega(r)}{\vartheta_0}\right]^{\frac{p}{2(p-1)}}|Du_h-Du_0|^{\frac{p}{2}}\,dx\\
&\le c\,\omega(r)\int_{B_r}|Du_h-Du_0|^p\,dx+c\frac{\vartheta_0^{\frac{2p}{p-1}}}{[\omega(r)]^{\frac{p+1}{p-1}}}r^N\\
&\le c\,\omega(r)\int_{B_r}(|Du_h|^p+|Du_0|^p)\,dx+c\frac{\vartheta_0^{\frac{2p}{p-1}}}{[\omega(r)]^{\frac{p+1}{p-1}}}\,r^N\,.
\end{aligned}$$

Therefore, from (16), there exists c, depending only on N, p, L, R and ν, such that for every $\rho < r$

$$\begin{aligned}
\int_{B_\rho}|Du_\infty|^p dx \le c\left[\left(\frac{\rho}{r}\right)^N+\omega(r)\right]\int_{B_r}(1+|Du|^p+|Du_0|^p)\,dx\\
+\,c\frac{\vartheta_0^{\frac{2p}{p-1}}}{[\omega(r)]^{\frac{p+1}{p-1}}}\,r^N+c\,r^N.
\end{aligned}$$

Finally, by (23) the thesis follows. □

If $\vartheta_0 = 0$ we get a regularity result for local minimizers of the functional $\mathcal{I}$ in (6) when f is convex in ξ.

Theorem 3.3. *Let u be a local minimizer of the functional $\mathcal{I}$, whose integrand $f(x,\xi)$ is a Carathéodory function, convex with respect to the last variable and satisfies $(\mathcal{A}1)$–$(\mathcal{A}3)$. Then u is locally in $C^{0,\alpha}(\Omega)$ for all $\alpha < 1$. Moreover, for all $\alpha < 1$ there exists $c > 0$, depending on N, p, L, R, ν and α, such that for every $B_r \subset\subset \Omega$ and $\rho < r$*

$$\int_{B_\rho}|Du|^p\,dx \le c\left(\frac{\rho}{r}\right)^{N-p+p\alpha}\int_{B_r}\left(1+|Du|^p\right)\,dx\,. \tag{24}$$

Proof. When $\vartheta_0 = 0$, Proposition 3.2 implies that for every $\rho < r$

$$\int_{B_\rho}|Du|^p\,dx \le c\left[\left(\frac{\rho}{r}\right)^N+\omega(r)\right]\int_{B_r}(1+|Du|^p)\,dx+c\,r^N\,.$$

A standard iteration argument (see [12], p.170) leads to estimate (24). The Hölder continuity of u follows from a characterization of Campanato spaces (see [17], p.57). □

4. Proof of Theorem 1.1

In this section we prove Theorem 1.1. First we shall suppose that f is convex with respect to ξ. This supplementary assumption will be removed observing that a local minimizer of $\mathcal{F}$ is a local minimizer for the relaxed functional, too.

The Ekeland variational principle enables us to compare the local minimizer of $\mathcal{F}$ with a minimizer of a functional whose integrand does not explicitly depend on u.

Theorem 4.1 (Ekeland variational principle [5]). *Let (V,d) be a complete metric space and let $\mathcal{H} : V \to (-\infty, +\infty]$ be a lower semicontinuous functional with finite infimum. If $u \in V$ is such that $\mathcal{H}(u) \le \inf_V \mathcal{H} + \epsilon$ for some $\epsilon > 0$, then there exists $v_0 \in V$ such that*

(i) $d(u, v_0) \le 1$,
(ii) $\mathcal{H}(v_0) \le \mathcal{H}(u)$,
(iii) *v_0 minimizes in V the functional $\mathcal{E}(w) := \mathcal{H}(w) + \epsilon\, d(v_0, w)$.*

Proof of Theorem 1.1. Without loss of generality we assume ω in (A3) to be a concave function.

From Theorem 2.1 (i) a local minimizer of $\mathcal{F}$ is a Q-minimizer of (5) with $\phi = 0$, therefore from Lemma 2.4 there exists $q > p$ such that $|Du| \in L^q_{\mathrm{loc}}(\Omega)$. Since we aim to prove a local result, we can assume that $u \in W^{1,q}(\Omega)$ and that for any ball $B_r(x_1) \subset \Omega$

$$\left[\fint_{B_{r/2}(x_1)} |Du|^q \, dx \right]^{\frac{1}{q}} \le c \left[\fint_{B_r(x_1)} (1 + |Du|^p) \, dx \right]^{\frac{1}{p}} . \tag{25}$$

Moreover (see [15]) we can assume $u \in C^{0,\gamma}(\Omega)$ for some $0 < \gamma < 1$. We denote by $[u]_\gamma$ the Hölder constant of u in Ω. Let us fix $B_r(x_0)$ such that $B_{4r}(x_0) \subset\subset \Omega$.

Step 1. Assume that f is convex in ξ. Consider the space $V = u + W_0^{1,\frac{p+1}{2}}(B_r)$ and let v be the minimizer in V of the functional

$$\mathcal{H}(w; B_r) := \int_{B_r} h(x, Dw(x)) \, dx \, , \tag{26}$$

where

$$h(x, \xi) := f(x, u(x), \xi). \tag{27}$$

By Theorem 2.1 (i), the minimality of v and (A1)

$$\int_{B_r} |Dv|^p \, dx \le c \int_{B_r} (1 + |Du|^p) \, dx \, . \tag{28}$$

Using notations (13), the minimality of u and (A3) we have

$$\begin{aligned}
&\mathcal{H}(u; B_r) = \mathcal{F}(u; B_r) \le \mathcal{F}(v; B_r) \\
&= \mathcal{H}(v; B_r) + \int_{B(Dv,r,R)} [f(x, v, Dv) - f(x, u, Dv)] \, dx \\
&\quad + \int_{A(Dv,r,R)} [f(x, v, Dv) - f(x, u, Dv)] \, dx \\
&\le \mathcal{H}(v; B_r) + L \int_{B(Dv,r,R)} (1 + |Dv|^p) \, dx + \int_{A(Dv,r,R)} \omega(|v-u|)(1 + |Dv|^p) dx \\
&\le \inf_V \mathcal{H}(w; B_r) + \int_{B_r} \omega(|v-u|)(1 + |Dv|^p) \, dx + c\, r^N .
\end{aligned} \tag{29}$$

Let $s \in (p,q)$ be the exponent of the higher integrability of Dv which follows from Lemma 2.5. Using the Hölder inequality, (25), the boundedness and the concavity of ω, we estimate the last integral in (29) as follows:

$$\begin{aligned}
&\int_{B_r} \omega(|v-u|)(1+|Dv|^p)\,dx \\
&\le c\,|B_r| \left(\fint_{B_r} (1+|Dv|^s)\,dx \right)^{\frac{p}{s}} \left(\fint_{B_r} \omega^{\frac{s}{s-p}}(|v-u|)\,dx \right)^{1-\frac{p}{s}} \\
&\le c\,|B_r| \left(\fint_{B_{2r}} (1+|Du|^q)\,dx \right)^{\frac{p}{q}} \left(\fint_{B_r} \omega(|v-u|)\,dx \right)^{1-\frac{p}{s}} \qquad (30)\\
&\le c\,|B_r| \fint_{B_{4r}} (1+|Du|^p)\,dx \cdot \omega^{1-\frac{p}{s}}\left(\fint_{B_r} |v-u|\,dx \right) \\
&\le c \int_{B_{4r}} (1+|Du|^p)\,dx \cdot \omega^{\sigma}\left(\fint_{B_r} |v-u|\,dx \right),
\end{aligned}$$

with $\sigma = 1 - \frac{p}{s}$. Caccioppoli inequality for the minimizer u (see e.g. [15]) gives

$$\int_{B_\rho(x_1)} |Du|^p\,dx \le c \int_{B_{2\rho}(x_1)} \left(1 + \frac{|u-u_{x_1,2\rho}|^p}{\rho^p} \right) dx\,,$$

which holds for every $B_{2\rho}(x_1) \subseteq B_{2r}(x_0)$ (here $u_{x_1,\rho}$ stands for $\fint_{B_\rho(x_1)} u(x)dx$). Thus, using the Poincaré inequality and (28), we get that

$$\begin{aligned}
&\fint_{B_r} |v-u|\,dx \le \left(c\,r^p \fint_{B_r} |Dv-Du|^p\,dx \right)^{\frac{1}{p}} \le \left(c\,r^p \fint_{B_r} (1+|Du|^p)\,dx \right)^{\frac{1}{p}} \\
&\le \left[c\,r^p \fint_{B_{2r}} \left(1 + \frac{|u-u_{x_0,2r}|^p}{r^p} \right) dx \right]^{\frac{1}{p}} \le \left(c\,r^p + c\,[u]_\gamma^p\, r^{p\gamma} \right)^{\frac{1}{p}} \le \overline{c}\, r^\gamma,
\end{aligned}$$

where $\overline{c}$ depends also on $[u]_\gamma$; then

$$\omega^\sigma\left(\fint_{B_r} |v-u|dx \right) \le \omega^\sigma(\overline{c} r^\gamma)\,.$$

This estimate, together with (29), (30) and (28), leads to

$$\mathcal{H}(u;B_r) \le \inf_V \mathcal{H}(w;B_r) + c\,\omega^\sigma\left(\overline{c} r^\gamma\right) \int_{B_{4r}} (1+|Du|^p)dx + c\,r^N\,. \qquad (31)$$

Step 2. Let us define

$$H(r) := c\,\omega^\sigma(\overline{c} r^\gamma) \int_{B_{4r}} (1+|Du|^p)\,dx + c\,r^N \qquad (32)$$

and apply Theorem 4.1 with V endowed with the distance

$$d(w_1,w_2) := \left[\frac{H(r)}{r^N} \right]^{-\frac{p+1}{2p}} r^{-N} \int_{B_r} |Dw_1 - Dw_2|^{\frac{p+1}{2}}\,dx\,, \qquad (33)$$

and $\mathcal{H}$ defined in (26). Thus, by (31), there exists $v_0 \in V$ such that

$$\int_{Br} |Du - Dv_0|^{\frac{p+1}{2}} \, dx \ \le \ \left[\frac{H(r)}{r^N}\right]^{\frac{p+1}{2p}} r^N, \tag{34}$$

$$\mathcal{H}(v_0; B_r) \ \le \ \mathcal{H}(u; B_r)\,, \tag{35}$$

$$v_0 \ \text{ is a minimizer of } \ \mathcal{E}(w; B_r) \text{ in } V\,,$$

where

$$\mathcal{E}(w; B_r) := \mathcal{H}(w; B_r) + \left[\frac{H(r)}{r^N}\right]^{\frac{p-1}{2p}} \int_{B_r} |Dw - Dv_0|^{\frac{p+1}{2}} \, dx\,. \tag{36}$$

The minimality of v_0 implies that for every $\varphi \in W_0^{1,p}(B_r)$

$$\mathcal{H}(v_0; \operatorname{spt}\varphi) \le \mathcal{H}(v_0 + \varphi; \operatorname{spt}\varphi) + \left[\frac{H(r)}{r^N}\right]^{\frac{p-1}{2p}} \int_{\operatorname{spt}\varphi} |D(v_0+\varphi) - Dv_0|^{\frac{p+1}{2}} \, dx.$$

Using Young inequality and noting that $(\frac{2p}{p+1})' = \frac{2p}{p-1}$, we obtain

$$\begin{aligned}\mathcal{H}(v_0; \operatorname{spt}\varphi) &\le \mathcal{H}(v_0 + \varphi; \operatorname{spt}\varphi) + \frac{c_1}{2}\int_{\operatorname{spt}\varphi} |Dv_0|^p \, dx \\ &\qquad + c\int_{\operatorname{spt}\varphi} |Dv_0 + D\varphi|^p \, dx + c\,\frac{H(r)}{r^N}|\operatorname{spt}\varphi|\,.\end{aligned}$$

From this inequality, Theorem 2.1 (i) and (A1), it follows that

$$\int_{\operatorname{spt}\varphi} |Dv_0|^p \, dx \le c\int_{\operatorname{spt}\varphi} \left(1 + |Dv_0 + D\varphi|^p + \frac{H(r)}{r^N}\right) dx\,,$$

therefore v_0 is a Q-minimizer (with Q depending on N, p, L, R and ν) of the functional

$$w \mapsto \int_{B_r} \left(1 + |Dw|^p + \frac{H(r)}{r^N}\right) dx\,.$$

Thus, there exist $\lambda \in (p,q)$ and $c > 0$, independent of v_0, such that

$$\begin{aligned}\left(\fint_{B_{r/2}} |Dv_0|^\lambda \, dx\right)^{\frac{p}{\lambda}} &\le c\fint_{B_r} (1 + |Dv_0|^p) \, dx + c\left[1 + \frac{H(r)}{r^N}\right] \\ &\le c\fint_{B_{4r}} (1 + |Du|^p) \, dx\,,\end{aligned} \tag{37}$$

where the last inequality follows from (32), Theorem 2.1 (i), (35) and (A1). Notice that h defined in (27) satisfies assumptions ($\mathcal{A}1$)–($\mathcal{A}3$), with a slightly different modulus of continuity. Precisely, for any $0 < \alpha < 1$, h satisfies ($\mathcal{A}3$) with ω replaced by

$$\tilde{\omega}_\alpha(t) := \max\left\{[\omega^\sigma(\overline{c}t^\gamma)]^{\frac{p-1}{2p}},\, \omega(t + [u]_\gamma t^\gamma)\,,\, t^{p(1-\alpha)\frac{p-1}{p+1}}\right\}. \tag{38}$$

Apply Proposition 3.2 to $\mathcal{E}(w; B_r)$ defined in (36), with ϑ_0 and u_0 replaced by $\left[\frac{H(r)}{r^N}\right]^{\frac{p-1}{2p}}$ and v_0, respectively. Then for every $\rho \leq r/2$

$$\begin{aligned}
&\int_{B_\rho} |Du|^p dx \leq 2^{p-1} \int_{B_\rho} |Dv_0|^p \, dx + 2^{p-1} \int_{B_{r/2}} |Du - Dv_0|^p \, dx \\
&\leq c \left[\left(\frac{\rho}{r}\right)^N + \tilde{\omega}_\alpha(r) \right] \int_{B_r} (1 + |Dv_0|^p) \, dx \\
&\quad + c \frac{H(r)}{[\tilde{\omega}_\alpha(r)]^{\frac{p+1}{p-1}}} + c\, r^N + 2^{p-1} \int_{B_{r/2}} |Du - Dv_0|^p \, dx \\
&\leq c \left[\left(\frac{\rho}{r}\right)^N + \tilde{\omega}_\alpha(r) \right] \int_{B_r} (1 + |Du|^p) \, dx + c \frac{\omega^\sigma(\bar{c} r^\gamma)}{[\tilde{\omega}_\alpha(r)]^{\frac{p+1}{p-1}}} \int_{B_{4r}} (1 + |Du|^p) \, dx \\
&\quad + c\, r^{N-p+p\alpha} + c\, r^N + 2^{p-1} \int_{B_{r/2}} |Du - Dv_0|^p \, dx \\
&\leq c \left[\left(\frac{\rho}{r}\right)^N + \tilde{\omega}_\alpha(r) \right] \int_{B_{4r}} (1 + |Du|^p) dx + c\, r^{N-p+p\alpha} + 2^{p-1} \int_{B_{r/2}} |Du - Dv_0|^p dx.
\end{aligned} \tag{39}$$

In order to estimate the last integral, let $\vartheta \in (0,1)$ be such that $\frac{\vartheta}{\lambda} + \frac{2(1-\vartheta)}{p+1} = \frac{1}{p}$, where λ is the exponent in (37). Using (25), (37), (34) and (35), we get

$$\begin{aligned}
&\int_{B_{r/2}} |Du - Dv_0|^p dx \\
&\leq c |B_r| \left(\fint_{B_{r/2}} |Du - Dv_0|^\lambda \, dx \right)^{\vartheta \frac{p}{\lambda}} \left(\fint_{B_r} |Du - Dv_0|^{\frac{p+1}{2}} \, dx \right)^{(1-\vartheta) \frac{2p}{p+1}} \\
&\leq c\, r^N \left(\fint_{B_{4r}} (1 + |Du|^p) \, dx \right)^\vartheta \left[\frac{H(r)}{r^N} \right]^{1-\vartheta} \\
&\leq c \left[\omega^\sigma(\bar{c} r^\gamma)\right]^{1-\vartheta} \int_{B_{4r}} (1 + |Du|^p) \, dx + c\, r^{N(1-\vartheta)} \left(\int_{B_{4r}} (1 + |Du|^p) \, dx \right)^\vartheta .
\end{aligned} \tag{40}$$

Finally, collecting (39), (40) and using Young inequality, we deduce that for every $\epsilon > 0$ there exists $c(\epsilon)$ such that

$$\begin{aligned}
\int_{B_\rho} |Du|^p dx &\leq c \left[\left(\frac{\rho}{r}\right)^N + \tilde{\omega}_\alpha(r) + \epsilon \right] \int_{B_{4r}} (1 + |Du|^p) \, dx \\
&+ c [\tilde{\omega}_\alpha(r)]^{(1-\vartheta)\frac{2p}{p-1}} \int_{B_{4r}} (1 + |Du|^p) \, dx + c\, r^{N-p+p\alpha} + c(\epsilon)\, r^N,
\end{aligned}$$

which implies

$$\int_{B_\rho} |Du|^p \, dx \leq c \left[\left(\frac{\rho}{r}\right)^N + [\tilde{\omega}_\alpha(r)]^\delta + \epsilon \right] \int_{B_{4r}} (1 + |Du|^p) \, dx + c_\epsilon r^{N-p+p\alpha},$$

for a certain $\delta > 0$ independent of ρ and r. From this inequality the thesis follows by Lemma 2.6 and by a standard iteration argument (see [12], p.170).

Step 3. Now we consider a general f non convex with respect to the last variable. We observe that local minimizers of $\mathcal{F}$ are also local minimizers of the relaxed functional

$$\mathcal{F}^{**}(u; B_r(x_0)) = \inf\left\{ \liminf_{h\to+\infty} \int_{B_r(x_0)} f(x, u_h, Du_h)\, dx : u_h \rightharpoonup u \text{ in } W_0^{1,p}(B_r(x_0)) \right\}$$

which is equal to

$$\int_{B_r(x_0)} f^{**}(x, u(x), Du(x))\, dx\,,$$

where $\xi \mapsto f^{**}(x,u,\xi)$ is the bipolar of $\xi \mapsto f(x,u,\xi)$, see [6, Corollary 3.8]. The result follows immediately from the fact that by Theorem 2.1 (ii) there exists R_0 such that

$$f(x,u,\xi) = f^{**}(x,u,\xi)$$

in $\Omega \times \mathbb{R} \times (\mathbb{R}^N \setminus B_{R_0}(0))$. □

References

[1] E. Acerbi, G. Mingione: Regularity results for a class of functionals with non-standard growth, Arch. Rat. Mech. Anal. 156 (2001) 121–140.

[2] G. Cupini, N. Fusco, R. Petti: Hölder continuity of local minimizers, J. Math. Anal. Appl. 235 (1999) 578–597.

[3] G. Cupini, M. Guidorzi, E. Mascolo: Regularity of minimizers of vectorial integrals with $p-q$ growth, Nonlinear Anal., Ser. A: Theory Methods 54 (2003) 591–616.

[4] G. Cupini, R. Petti: Hölder continuity of local minimizers of vectorial integral functionals, NoDEA 10 (2003) 269–285.

[5] I. Ekeland: Nonconvex minimization problems, Bull. Am. Math. Soc. 1(3) (1979) 443–474.

[6] I. Ekeland, R. Temam: Convex Analysis and Variational Problems, North Holland, Amsterdam (1976).

[7] L. Esposito, F. Leonetti, G. Mingione: Regularity results for minimizers of irregular integrals with (p,q) growth, Forum Mathematicum 14 (2002) 245–272.

[8] V. Ferone, N. Fusco: Continuity properties of minimizers of integral functionals in a limit case, J. Math. Anal. Appl. 202 (1996) 27–52.

[9] I. Fonseca, N. Fusco: Regularity results for anisotropic image segmentation models, Ann. Sc. Norm. Sup. Pisa 24 (1997) 463–499.

[10] I. Fonseca, N. Fusco, P. Marcellini: An existence result for a nonconvex variational problem via regularity, ESAIM; Control, Optim. Calc. Var. 7 (2002) 69–95.

[11] N. Fusco, G. Mingione, C. Trombetti: Regularity of minimizers for a class of anisotropic free discontinuity problems, J. Convex Analysis 8 (2001) 349–368.

[12] M. Giaquinta: Multiple Integrals in the Calculus of Variations and Non Linear Elliptic Systems, Annals of Math. Studies 105, Princeton Univ. Press, Princeton NJ (1983).

[13] M. Giaquinta, E. Giusti: On the regularity of the minima of variational integrals, Acta Math. 148 (1982) 31–46.

[14] M. Giaquinta, E. Giusti: Differentiability of minima of non-differentiable functionals, Invent. Math. 72 (1983) 285–298.

[15] M. Giaquinta, E. Giusti: Quasi-minima, Ann. Inst. H. Poincare, Non Lineaire 1 (1984) 79–107.

[16] M. Giaquinta, G. Modica: Remarks on the regularity of the minimizers of certain degenerate functionals, Manuscripta Math. 57 (1986) 55–99.

[17] E. Giusti: Metodi Diretti nel Calcolo delle Variazioni, Unione Matematica Italiana, Bologna (1994).

[18] J. J. Manfredi: Regularity for minima of functionals with p-growth, J. Differential Equations 76 (1988) 203–212.

[19] N. G. Meyers: An L^p-estimate for the gradient of solutions of second order elliptic divergence equations, Ann. Scuola Norm. Sup. Pisa 17 (1963) 189–206.

[20] K. Uhlenbeck: Regularity for a class of nonlinear elliptic systems, Acta Math. 138 (1977) 219–240.