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On the Relaxation on BV of Certain non Coercive Integral Functionals

Francesco Maggi
Dipartimento di Matematica, Università di Firenze,
Viale Morgagni 67/A, 50134 Firenze, Italy
maggi@math.unifi.it

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In this paper we prove an integral representation formula for the relaxed functional of a scalar non parametric integral of the Calculus of Variations. Similar results are known to be true under the key assumption that the integrand is coercive in the gradient variable. Here we show that the same integral representation holds for a wide class of non coercive integrands, including for example the strictly convex ones.

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1. Introduction

Let Ω be an open set in $\mathbb{R}^n$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, \infty)$ be a lower semicontinuous function, with $f(x, s, \cdot)$ convex for every $(x, s) \in \Omega \times \mathbb{R}$. Let us consider the functional

$$F[f](u, \Omega') = \begin{cases} \int_{\Omega'} f(x, u(x), \nabla u(x))dx, & \text{if } u \in C^1(\Omega), \\ \infty, & \text{if } u \in BV_{\mathrm{loc}}(\Omega) \backslash C^1(\Omega), \end{cases} \tag{1}$$

where Ω' is an open subset of Ω and $u \in BV_{\mathrm{loc}}(\Omega)$. In a fundamental paper, [19], Serrin, following Lebesgue's definition of the area functional, introduces the *lower semicontinuous envelope* of F on $BV_{\mathrm{loc}}(\Omega)$ with respect to the $L^1_{\mathrm{loc}}(\Omega)$ strong convergence. This functional, which we briefly refer to as the *relaxed functional* of F, is defined as follows:

$$\mathcal{F}[f](u, \Omega') = \inf \left\{ \liminf_{h \to \infty} F[f](u_h, \Omega') : u_h \to u \text{ in } L^1_{\mathrm{loc}}(\Omega), u_h \in C^1(\Omega) \right\}. \tag{2}$$

A natural problem is to explicitly represent $\mathcal{F}$. Confining our attention to the scalar convex case considered here, we quote Goffman-Serrin [13], Giaquinta-Modica-Souček [12], Dal Maso [6], Ferro [9], Bouchitté-Dal Maso [3] and Fonseca-Leoni [11]; the literature becomes huge either if $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, or if u is vector valued or if f is not convex with respect to the gradient variable.

The candidate to give the representation formula of $\mathcal{F}$ is the following functional H introduced by Dal Maso in [6], where for any $u \in BV_{\mathrm{loc}}(\Omega)$ we denote by Du its derivative measure, by $Du = \nabla u \mathcal{L}^n + D^s u$ the Lebesgue decomposition, by $M(u)$ the concentration set of $D^s u$, by u^+ and u^- the approximate upper and lower limits of u and we put $J(u) = \{x \in \Omega : u^-(x) < u^+(x)\}$ (it results $|D^s u| \lfloor J(u) = (u^+(x) - u^-(x))\mathcal{H}^{n-1} \lfloor J(u)$

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and $D^j u := D^s u \lfloor J(u)$ is called the jump part of Du, while $D^c u := D^s u \lfloor \{M(u) \backslash J(u)\}$ is called the Cantor part of Du, see the claims in Section 3:

$$\begin{aligned} H[f](u, \Omega') = & \int_{\Omega'} f(x, u(x), \nabla u(x)) dx \\ & + \int_{[M(u)\backslash J(u)]\cap\Omega'} f^\infty \left(x, u^+(x), \frac{Du}{|Du|}(x) \right) d|Du|(x) \\ & + \int_{J(u)\cap\Omega'} \left\{ \frac{1}{u^+(x) - u^-(x)} \int_{u^-(x)}^{u^+(x)} f^\infty \left(x, s, \frac{Du}{|Du|}(x) \right) ds \right\} d|Du|(x), \end{aligned} \tag{3}$$

where as usual f^∞ is the recession function of $f(x, s, \cdot)$, defined as $f^\infty(x, s, \xi) := \lim_{\lambda\to 0^+} \lambda f(x, s, \xi/\lambda)$.

There are many results linking H and $\mathcal{F}$. For example, $H = \mathcal{F}$ either if $f = f(\xi)$ (Goffman-Serrin [13]) or under suitable assumptions on the integrand $f(x, s, \xi)$, namely, qualified continuity in the (x, s) variable (in order to have f^∞ continuous) and coercivity in the ξ variable (see Dal Maso [6]). A problem to solve is to understand if we can further weaken the hypotheses proposed by Dal Maso and still have $H = \mathcal{F}$.

In this paper we deal with the case in which coercivity assumptions like

$$f(x, s, \xi) \geq C|\xi| - \frac{1}{C}, \quad \forall (x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n, \tag{4}$$

for a suitable $C > 0$, are dropped. In this direction Fonseca-Leoni [11] have proved the following theorem:

Theorem 1.1 (Fonseca-Leoni). *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, \infty)$ be a Borel integrand satisfying the following condition: for every $(x_0, s_0) \in \Omega \times \mathbb{R}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$|x - x_0| + |s - s_0| < \delta, \xi \in \mathbb{R}^n \Rightarrow f(x_0, s_0, \xi) \leq \varepsilon + (1 + \varepsilon) f(x, s, \xi), \tag{5}$$

and $f(x_0, s_0, \cdot)$ is convex. Then for every $u \in BV_{loc}(\Omega)$,

$$H(u) \leq \mathcal{F}(u).$$

If furthermore Ω is bounded, if there exists $C > 0$ such that

$$0 \leq f(x, s, \xi) \leq C(1 + |\xi|), \quad \forall (x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n,$$

and if f^∞ is continuous, then $H(u) = \mathcal{F}(u)$, for all $u \in BV_{loc}(\Omega)$.

Roughly speaking, in this theorem the coercivity (4) is replaced by hypothesis (5). The main case in which hypothesis (5) is really weaker than coercivity is the case of product type integrands, $f(x, s, \xi) = a(x, s) g(\xi)$. However there is a wide class of integrands (including for example the ones that are just strictly convex in the gradient variable), of interest in applications as well as from a purely mathematical point of view, for which the validity of (5) is heavily related to coercivity (we give an example below).

In what follows we introduce these integrands, we explain the motivations for their study and we give a general relaxation theorem.

Definition 1.2. A function $\Psi : \mathbb{R}^n \to [0, \infty)$ is said to be *non constant on straight lines*, briefly NCL, if its restriction to any straight line is a non constant function, i.e., if for all $\xi_0 \in \mathbb{R}^n$ and $\nu \in S^{n-1}$ the map

$$\rho \in \mathbb{R} \mapsto \Psi(\xi_0 + \rho\nu) \quad \text{is not constant.} \tag{6}$$

In the case of a convex Ψ to check the NCL property it suffices to verify that (6) holds for a fixed point ξ_0 (see Proposition 2 in [14]): for example it is enough to see that for all $\nu \in S^{n-1}$ the map

$$\rho \in \mathbb{R} \mapsto \Psi(\rho\nu) \quad \text{is not constant.} \tag{7}$$

Every strictly convex function is NCL. A model case of NCL function is

$$\Psi(\xi) = \sum_{i=1}^{n} (\xi_i)^+, \tag{8}$$

where $(a)^+ := \max\{0, a\}$. If we let Ψ "turn around" the origin by introducing a continuous perturbation we can construct a family of functions $f = f(t, \xi) : \Sigma \times \mathbb{R}^n \to [0, \infty)$ such that $f(t, \cdot)$ is convex and NCL for every fixed t, and (5) is not satisfied. It suffices to take a continuous $T : \Sigma \to \{S \in \mathbb{R}^{n\times n} : \det S = 1, S^T S = I\}$ and define $f(t, \xi) = \Psi(T(t)\xi)$. When $n \geq 2$ and T is not constant in a neighborhood of a fixed $t_0 \in \Sigma$, it is not hard to verify that Fonseca-Leoni condition (5) never holds. We should note that for one dimensional problems, $n = 1$, it can be proved that, if f is lower semicontinuous and $f(t, \cdot)$ is convex and NCL, then f satisfies (5).

NCL functions were first introduced with the name of demicoercive functions by Anzellotti-Buttazzo-Dal Maso [1] as the mathematical model of the integrands appearing in the study of equilibrium problems with unilateral constraints on the stress. The term "demicoercive" follows from the fact that Ψ is NCL if and only if there exist a vector $\nu \in S^{n-1}$, and constants $a > 0$, $b, c \geq 0$ such that

$$a|\xi| - b \leq \Psi(\xi) + c\langle \nu, \xi\rangle, \quad \forall \xi \in \mathbb{R}^n.$$

In the recent paper [14] this property formulated as in Definition 1.2, has been proved to be the underlying hypothesis to the geometric conditions in Serrin's lower semicontinuity Theorem.

The following theorem is the main result of this paper.

Theorem 1.3. *Let $f \in L^\infty_{loc}(\Omega \times \mathbb{R} \times \mathbb{R}^n; [0, \infty))$ be lower semicontinuous and such that for every $(x, s) \in \Omega \times \mathbb{R}$, $f(x, s, \cdot)$ is convex and NCL. Then, for every $u \in BV_{loc}(\Omega)$,*

$$H(u) \leq \mathcal{F}(u).$$

The proof of this theorem (see Section 3) is based upon four main tools: the blow up method by Fonseca-Müller [10]; the approximation theorem for convex NCL functions by means of maximal cones given in [14] and discussed in Section 2 (see Theorem 2.1 below); a weak coercivity property of $H[\Psi]$ proved by Anzellotti-Buttazzo-Dal Maso [1] in the case when Ψ is NCL; and, finally, a lower semicontinuity theorem for the functional H due to Dal Maso [6]. Once proved Theorem 1.3, applying Theorem 1.3 in Fonseca-Leoni [11], we find immediately the following corollary.

Corollary 1.4. *Let f be as in Theorem 1.3. If furthermore Ω is bounded, if there exists a constant $C > 0$ such that*

$$0 \leq f(x, s, \xi) \leq C(1 + |\xi|), \quad \forall (x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n,$$

and if f^∞ is continuous, then $H(u) = \mathcal{F}(u)$ for all $u \in BV_{loc}(\Omega)$.

Then the joint application of Theorem 1.3 and Corollary 1.4 allows us to prove a statement analogous to the one of Theorem 1.1, where (5) has been replaced by the NCL hypothesis. In the end we would like to remark that, if we drop the NCL assumption, in general we cannot even expect that the Lebesgue part of the relaxed functional is represented by $\int_\Omega f(x, u, \nabla u)dx$ (see the introduction of the paper [14] or Example 4.1 in Dal Maso [6]).

2. Approximation of NCL functions

In this section we state an approximation result for convex NCL functions by means of certain maximal cones, proved in [14]. These cones satisfy some useful properties, that in general cannot be expected to hold for the supporting hyperplanes.

Theorem 2.1. *Let Σ be an open set in $\mathbb{R}^d$ and $f : \Sigma \times \mathbb{R}^n \to [0, \infty)$ be a lower semicontinuous function with $f(t, \cdot)$ convex and NCL for every $t \in \Sigma$. Then there exists $(\xi_k)_{k \in \mathbb{N}} \subset \mathbb{R}^n$ such that, if we define*

$$f_k(t, \xi) = -1 + \inf \left\{ \lambda + \lambda f\left(t, \xi_k + \frac{\xi - \xi_k}{\lambda}\right) : \lambda > 0 \right\}, \tag{9}$$

then it results:

(i) *f_k is lower semicontinuous and, for every fixed $t \in \Sigma$, $f_k(t, \cdot)$ is convex, NCL and results to be the greatest function less than or equal to f such that $\xi \mapsto (1 + f_k(t, \xi + \xi_k))$ is positively homogeneous of degree one;*
(ii) *$f(t, \xi) = \sup_{k \in \mathbb{N}} f_k(t, \xi)$ for all $(t, \xi) \in \Sigma \times \mathbb{R}^n$.*

The following lemma summarizes some properties of the functions f_k in Theorem 2.1 that we shall need in the proof of Theorem 1.3.

Lemma 2.2. *Let Σ, f and f_k be as in Theorem 2.1 and suppose f to be locally bounded. Then for every $\Sigma' \subset\subset \Sigma$ there exist constants $C_k \geq 0$ such that, for every $(t, \xi) \in \Sigma' \times \mathbb{R}^n$, $\lambda > 0$, we have*

$$-1 \leq f_k(t, \xi) \leq C_k(1 + |\xi|), \tag{10}$$

$$\lambda f_k\left(t, \frac{\xi}{\lambda}\right) \geq f_k^\infty(t, \xi) - C_k \lambda. \tag{11}$$

Furthermore, for every fixed $t_0 \in \Sigma$ and $\varepsilon, \sigma > 0$, there exist $\delta > 0$ and a convex, NCL function $\Psi_k : \mathbb{R}^n \to [-1, \infty)$ such that

$$\Psi_k(\xi) \leq f_k(t, \xi), \tag{12}$$

$$f_k(t_0, \xi) + \sigma|\xi| \leq (1 + \varepsilon)[f_k(t, \xi) + \sigma|\xi|] + \varepsilon, \tag{13}$$

for every $(t, \xi) \in B_\delta(t_0) \times \mathbb{R}^n$.

Proof. By the local boundedness of f and by (9) it follows immediately that f_k is locally bounded on $\Sigma \times \mathbb{R}^n$. By (9) we can directly verify the following formula,

$$\lambda f_k\left(t, \frac{\xi}{\lambda}\right) = 1 - \lambda + f_k(t, \xi + (1-\lambda)\xi_k), \quad \forall (t,\xi) \in \Sigma \times \mathbb{R}^n, \forall \lambda > 0, \tag{14}$$

from which we see that f_k is transformed into its recession function f_k^∞ by means of a translation of its epigraph in $\mathbb{R}^{n+1}$ along the segment joining $(\xi_k, -1)$ and the origin. Hence $f_k(t,\cdot)$ is lipschitzian on $\mathbb{R}^n$ with the same Lipschitz constant $L_k(t)$ of $f_k^\infty(t,\cdot)$, i.e.,

$$L_k(t) = \operatorname{lip}(f_k^\infty(t,\cdot)) = \operatorname{lip}(f_k^\infty(t,\cdot)|B_1(0)) \leq \max_{B_2(0)} f_k^\infty(t,\cdot) - \min_{B_1(0)} f_k^\infty(t,\cdot).$$

Since $f_k^\infty \geq 0$ and f_k^∞ is locally bounded (by (14) and by the local boundedness of f_k) we have $L_k(t) \leq L_k < \infty$ for every $t \in \Sigma'$.

By these considerations we can see that (10) follows immediately while (11) is true since, by (14),

$$\begin{aligned}\left|\lambda f_k\left(t, \frac{\xi}{\lambda}\right) - f_k^\infty(t,\xi)\right| &= |f_k(t, \xi + (1-\lambda)\xi_k) - f_k(t, \xi + \xi_k) - \lambda| \\ &\leq \lambda(1 + L_k(t)|\xi_k|) \leq C_k \lambda, \quad \forall (t,\xi) \in \Sigma' \times \mathbb{R}^n.\end{aligned}$$

We prove (13). We define $\tau = \varepsilon\sigma / \min\{|\xi_k|, 1\}$. By lower semicontinuity of f_k^∞ there exists $\delta > 0$ such that

$$f_k^\infty(t_0, \nu) \leq f_k^\infty(t, \nu) + \tau, \quad \forall (t,\nu) \in B_\delta(t_0) \times S^{n-1},$$

so that, by positive homogeneity of degree one we have

$$f_k^\infty(t_0, \xi) \leq f_k^\infty(t, \xi) + \tau|\xi|, \quad \forall (t,\xi) \in B_\delta(t_0) \times \mathbb{R}^n.$$

Since $f_k^\infty(t,\xi) = 1 + f_k(t, \xi_k + \xi)$ this implies

$$f_k(t_0, \xi) \leq f_k(t, \xi) + \tau|\xi - \xi_k|, \quad \forall (t,\xi) \in B_\delta(t_0) \times \mathbb{R}^n.$$

By definition of τ it is $\tau \max\{|\xi_k|, 1\}/\sigma < \varepsilon$, and then, adding to both sides the term $\sigma|\xi|$, we find

$$[f_k(t_0,\xi) + \sigma|\xi|] \leq (1+\varepsilon)[f_k(t,\xi) + \sigma|\xi|] + \varepsilon, \quad \forall (t,\xi) \in B_\delta(t_0) \times \mathbb{R}^n,$$

and the proof is completed.

We prove (12). In order to simplify the notations we fix k and consider the function $g(t,\xi) := 1 + f_k(t, \xi + \xi_k)$. Then g is lower semicontinuous, with $g(t,\cdot)$ convex, NCL and positively homogeneous of degree one. We fix $t_0 \in \Sigma$ and we prove that there exists $\delta > 0$ and $\Psi : \mathbb{R}^n \to [0, \infty)$ convex and NCL such that

$$\Psi(\xi) \leq g(t,\xi), \quad \forall (t,\xi) \in B_\delta(t_0) \times \mathbb{R}^n. \tag{15}$$

To this end let us denote by $G(t) := \{p = (\xi, \alpha) \in \mathbb{R}^{n+1} : g(t,\xi) \leq \alpha\}$ the epigraph of $g(t,\cdot)$. Then $G(t)$ is always a closed, convex cone in $\mathbb{R}^{n+1}$: in particular, since $g(t,\cdot)$ is

NCL, it does not contain any straight line. For this reason we can find a half-space which intersects $G(t_0)$ only in the origin: then there exists (see the proof of Theorem 2.4 in [1]) $\varepsilon > 0$ such that the closed convex cone

$$G_\varepsilon(t_0) = \{p : \operatorname{dist}(p, G(t_0)) \leq |p|\varepsilon\},$$

does not contain any straight line. We claim there exists $\delta > 0$ such that

$$\bigcup_{|t-t_0|<\delta} G(t) \subset G_\varepsilon(t_0), \tag{16}$$

and the thesis follows defining Ψ as the convex function whose epigraph $\operatorname{epi}\Psi$ is given by

$$\operatorname{epi}\Psi := \overline{co}\left(\bigcup_{|t-t_0|<\delta} G(t)\right);$$

indeed $\operatorname{epi}\Psi$ is a closed convex set, it does not contain any straight line by (16) so that Ψ is NCL, and contains $G(t)$ for every $t \in B_\delta(t_0)$, so that (15) follows. In order to prove (16) we argue by contradiction: suppose there exists a sequence $t_h \to t_0$, and points $p_h = (\xi_h, \alpha_h) \in G(t_h)$ such that

$$\operatorname{dist}(p_h, G(t_0)) > \varepsilon|p_h|. \tag{17}$$

It must be $p_h \neq 0$ for every h, because $G(t_0)$ contains the origin. Since $G(t_h)$ is a cone it results that $\pi_h = p_h/|p_h| \in G(t_h)$. By compactness we can also assume $\pi_h \to \pi_0 = (\eta, \beta)$. Then by the lower semicontinuity and positive homogeneity of degree one of g we have

$$g(t_0, \eta) \leq \liminf_{h\to\infty} g\left(t_h, \frac{\xi_h}{|p_h|}\right) \leq \lim_{h\to\infty} \frac{\alpha_h}{|p_h|} = \beta,$$

i.e., $\pi_0 \in G(t_0)$. From (17) we can see $\operatorname{dist}(\pi_h, G(t_0)) > \varepsilon$ so that it would be $\operatorname{dist}(\pi_0, G(t_0)) > \varepsilon$, a contradiction. □

A weak coercivity property. The following is Theorem 2.7 in Anzellotti-Buttazzo-Dal Maso [1]. It states the coercivity of $H[\Psi]$ on every class of BV functions with fixed boundary data, provided Ψ is NCL.

Theorem 2.3. *Let Ψ be a lower semicontinuous, NCL, proper convex function on $\mathbb{R}^n$ and let Ω be bounded and with Lipschitz boundary. Then there exist $\alpha > 0$, $\beta, \gamma \geq 0$ such that, for every $u \in BV(\Omega)$,*

$$\alpha|Du|(\Omega) - \beta\int_{\partial\Omega} |u| d\mathcal{H}^{n-1} - \gamma\mathcal{L}^n(\Omega) \leq H[\Psi](u, \Omega).$$

A lower semicontinuity theorem. The following theorem is a lower semicontinuity result for the functional H (Theorem 3.1 in Dal Maso [6]), based upon Reshetnyak lower semicontinuity Theorem (see Theorem 2 in [17]) and on the interpretation of H as a functional defined on the subgraph of BV functions (see Lemma 2.2 in [6]). We state it in a simplified version. The reader can find related results in De Cicco [7], [8] and Braides-De Cicco [4].

Theorem 2.4. *Let $f \in L^\infty_{loc}(\Omega \times \mathbb{R} \times \mathbb{R}^n; [0,\infty))$ be lower semicontinuous, with $f(x,s,\cdot)$ convex and with*

$$a|\xi| - b \le f(x,s,\xi), \quad \forall (x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n,$$

for suitable constants $a > 0, b \ge 0$. Then the functional H is sequentially lower semicontinuous with respect to the convergence $L^1_{loc}(\Omega)$ on $BV_{loc}(\Omega)$.

3. Proof of the relaxation theorem

This section is devoted to the proof of Theorem 1.3. Let us fix a sequence $u_h \in C^1(\Omega)$ such that $u_h \to u$ in $L^1_{\text{loc}}(\Omega)$ for $u \in BV_{\text{loc}}(\Omega)$. We want to prove that

$$H(u,\Omega) \le \liminf_{h\to\infty} F(u_h,\Omega). \tag{18}$$

Apart from the trivial case when the minimum limit on the right hand side is equal to infinity, up to extracting a subsequence we can assume that

$$\liminf_{h\to\infty} F(u_h,\Omega) = \lim_{h\to\infty} F(u_h,\Omega) < \infty.$$

In particular the sequence $\{f(x,u_h,\nabla u_h)\}_{h\in\mathbb{N}}$ is bounded in $L^1(\Omega)$, so that by the Banach-Alaouglu-Bourbaki compactness criterion, up to a further extraction, we can suppose $f(x,u_h,\nabla u_h) \rightharpoonup^* \mu$ for some Radon measure μ in Ω. The idea behind the blow up method is to look at the densities of μ with respect to $\mathcal{L}^n$, $|D^c u|$ and $\mathcal{H}^{n-1}\lfloor J(u)$ respectively, and to prove that they are pointwise greater than or equal to the corresponding densities of the functional H. This allows to reduce the original problem (18) to proving the following three claims (see for example the first part in the proof of Theorem 1.1 in [11]; we omit the details for the sake of brevity and since this argument has been already used in many papers). We note that this reduction of the problem works under minimal assumption on the integrand f (it only needs f to be Borel non negative), so that the real difficulty is just moved to the proof of the claims. We put $Q := \{y \in \mathbb{R}^n : -1/2 \le y_i \le 1/2\}$.

Claim 1 (Lebesgue Part). Let us consider $(x_0,s_0,\xi_0) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, $\varepsilon_h \to 0^+$ and $u_h \in W^{1,1}(Q)$ such that $u_h \to v$ in $L^1(Q)$, where $v(y) := \langle \xi_0, y\rangle$ for every $y \in Q$; we have to prove that

$$\liminf_{h\to\infty} \int_Q f(x_0 + \varepsilon_h y, s_0 + \varepsilon_h u_h(y), \nabla u_h(y))dy \ge f(x_0,s_0,\xi_0).$$

Claim 2 (Cantor Part). Let us consider $(x_0,s_0) \in \Omega \times \mathbb{R}$, $s_h \to s_0$, $\varepsilon_h \to 0^+, \lambda_h \to 0^+$ with $t_h = \lambda_h/\varepsilon_h \to \infty$; $\alpha : (-1/2,1/2) \to \mathbb{R}$ a non decreasing function with $\alpha(1/2^-) - \alpha(-1/2^+) = 1$, $\int_{-1/2}^{1/2} \alpha(t)dt = 0$; $u_h \in W^{1,1}(Q)$ such that $u_h \to v$ in $L^1(Q)$, where $v(y) = \alpha(y_n)$ for every $y \in Q$; we have to prove that

$$\liminf_{h\to\infty} \int_Q \frac{1}{t_h} f\left(x_0 + \varepsilon_h y, s_h + \lambda_h u_h(y), t_h \nabla u_h(y)\right) dy \ge f^\infty(x_0,s_0,e_n).$$

Claim 3 (Jump Part). Let us consider $(x_0,a,b) \in \Omega \times \mathbb{R} \times \mathbb{R}$, $\varepsilon_h \to 0^+$; $u_h \in W^{1,1}(Q)$ such that $u_h \to v$ in $L^1(Q)$, where

$$v(y) := \begin{cases} b, & \text{if } y_n > 0, \\ a, & \text{if } y_n \le 0; \end{cases}$$

we have to prove that

$$\liminf_{h\to\infty}\int_Q \varepsilon_h f\left(x_0+\varepsilon_h y, u_h(y), \frac{\nabla u_h(y)}{\varepsilon_h}\right)dy \geq \int_a^b f^\infty(x_0,s,e_n)ds.$$

Before proving the claims we note that in each case we can suppose $a \leq v(y) \leq b$ for a.e. $y \in Q$, for $a,b\in\mathbb{R}, a\leq b$. In the following we shall consider a fixed $\tau > 0$, define $a' = a-\tau, b' = b+\tau$,

$$E_h = \{y \in Q : a' \leq u_h \leq b'\},$$
$$E_h^+ = \{y\in Q : u_h > b'\},\ E_h^- = \{y \in Q : a' < u_h\},$$

and consider the sequence

$$v_h(y) = \begin{cases} u_h(y), & y \in E_h,\\ b', & y\in E_h^+,\\ a', & y \in E_h^-. \end{cases} \tag{19}$$

Then $v_h \in W^{1,1}(Q)\cap L^\infty(Q)$ and

$$\mathcal{L}^n(Q\backslash E_h) \leq \mathcal{L}^n(\{y\in Q : |u_h(y)-v(y)|>\tau\}) \leq \frac{1}{\tau}\|u_h - v\|_{L^1(Q)} \to 0.$$

In particular it is always $v_h \to v$ in $L^1(Q)$. Let us also put $\Sigma = \Omega\times\mathbb{R}$ and denote with f_k the functions given by Theorem 2.1.

Proof of Claim 1 and Claim 2. Let us fix $k\in\mathbb{N}$, $\varepsilon,\sigma > 0$, define $t_0 := (x_0,s_0)$ and apply Lemma 2.2 to f_k to find $\delta = \delta(k) > 0$ and a convex, NCL function $\Psi_k : \mathbb{R}^n \to [-1,\infty)$ such that (12) and (13) hold. Then we put $\Sigma' = B_{\delta/2}(t_0)$ and again by Lemma 2.2 we find a constant C_k such that (10) and (11) hold. Summarizing,

$$\begin{cases} f(t,\xi) = \sup_{k\in\mathbb{N}} f_k(t,\xi), & \forall (t,\xi)\in\Sigma\times\mathbb{R}^n,\\ -1 \leq f_k(t,\xi) \leq C_k(1+|\xi|), & \forall(t,\xi)\in\Sigma'\times\mathbb{R}^n,\\ \lambda f_k\left(t,\frac{\xi}{\lambda}\right) \geq f_k^\infty(t,\xi) - C_k\lambda, & \forall(t,\xi)\in\Sigma'\times\mathbb{R}^n,\\ \Psi_k(\xi)\leq f_k(t,\xi), & \forall(t,\xi)\in\Sigma'\times\mathbb{R}^n,\\ f_k(t_0,\xi)+\sigma|\xi| \leq (1+\varepsilon)[f_k(t,\xi)+\sigma|\xi|]+\varepsilon, & \forall(t,\xi)\in\Sigma'\times\mathbb{R}^n. \end{cases}$$

We prove Claim 1. If we take h sufficiently large, by definition of v_h and by the growth condition on f_k we have,

$$\begin{aligned} &\int_Q f(x_0+\varepsilon_h y, s_0+\varepsilon_h u_h, \nabla u_h)dy \\ &\qquad \geq \int_{E_h} f(x_0+\varepsilon_h y, s_0+\varepsilon_h v_h,\nabla v_h)dy \\ &\qquad \geq \int_{E_h} f_k(x_0+\varepsilon_h y, s_0+\varepsilon_h v_h,\nabla v_h)dy \\ &\qquad \geq \int_Q f_k(x_0+\varepsilon_h y, s_0+\varepsilon_h v_h, \nabla v_h)dy - C_k\mathcal{L}^n(Q\backslash E_h). \end{aligned} \tag{20}$$

Let us remark that, by Theorem 2.3,

$$\begin{aligned}\int_Q f_k(x_0+\varepsilon_h y, s_0+\varepsilon_h v_h, \nabla v_h)dy &\geq \int_Q \Psi_k(\nabla v_h)dy \\ &\geq \alpha_k \int_Q |\nabla v_h|dy - \beta_k \int_{\partial Q} |v_h| d\mathcal{H}^{n-1} - \gamma_k,\end{aligned}$$

where the left hand side is bounded on h and where v_h are uniformly bounded on ∂Q by $\max\{|a'|, |b'|\}$. This means that

$$\sup_{h\in\mathbb{N}} \|\nabla v_h\|_{L^1(Q;\mathbb{R}^n)} \leq C < \infty. \tag{21}$$

Let us define $g_k(x,s,\xi) = g_k(\xi) := f_k(x_0,s_0,\xi) + \sigma|\xi|$. By (20), (21) and (13) we have

$$\begin{aligned}&\int_Q f(x_0+\varepsilon_h y, s_0+\varepsilon_h u_h, \nabla u_h)dy \\ &\quad\geq \int_Q f_k(x_0+\varepsilon_h y, s_0+\varepsilon_h v_h, \nabla v_h) + \sigma|\nabla v_h|dy - C_k\mathcal{L}^n(Q\backslash E_h) - \sigma C \\ &\quad\geq \frac{1}{1+\varepsilon}\int_Q f_k(x_0,s_0,\nabla v_h) + \sigma|\nabla v_h|dy - \varepsilon - C_k\mathcal{L}^n(Q\backslash E_h) - \sigma C \\ &\quad= \frac{1}{1+\varepsilon} H[g_k](v_h,Q) - \varepsilon - C_k\mathcal{L}^n(Q\backslash E_h) - \sigma C.\end{aligned}$$

By Theorem 2.4, passing to the limit as $h\to\infty$, we find

$$\begin{aligned}\liminf_{h\to\infty}\int_Q f(x_0+\varepsilon_h y, s_0+\varepsilon_h u_h, \nabla u_h)dy &\geq \frac{1}{1+\varepsilon}H[g_k](v,Q) - \varepsilon - \sigma C \\ &= \frac{1}{1+\varepsilon} g_k(x_0,s_0,\xi_0) - \varepsilon - \sigma C.\end{aligned}$$

As $\varepsilon,\sigma\to 0$ the last term tends to $f_k(x_0,s_0,\xi_0)$. Since this is true for all $k\in\mathbb{N}$ we have achieved the proof of Claim 1.

To prove Claim 2 we note again that, for h sufficiently large,

$$\begin{aligned}&\int_Q \frac{1}{t_h} f(x_0+\varepsilon_h y, s_h+\lambda_h u_h, t_h\nabla u_h)dy \\ &\quad\geq \int_{E_h} \frac{1}{t_h} f_k(x_0+\varepsilon_h y, s_h+\lambda_h v_h, t_h\nabla v_h)dy \\ &\quad\geq \int_Q \frac{1}{t_h} f_k(x_0+\varepsilon_h y, s_0+\lambda_h v_h, t_h\nabla v_h)dy - \frac{C_k}{t_h}\mathcal{L}^n(Q\backslash E_h) \\ &\quad\geq \int_Q f_k^\infty(x_0+\varepsilon_h y, s_0+\lambda_h v_h, \nabla v_h)dy - \frac{C_k}{t_h} - \frac{C_k}{t_h}\mathcal{L}^n(Q\backslash E_h),\end{aligned} \tag{22}$$

where in the last inequality we have used the property (11). Since $f_k \geq \Psi_k$ it results $f_k^\infty \geq \Psi_k^\infty$, and hence applying again Theorem 2.3 we find

$$\sup_{h\in\mathbb{N}} \|\nabla v_h\|_{L^1(Q;\mathbb{R}^n)} \leq C < \infty. \tag{23}$$

It is also apparent that

$$f_k^\infty(t_0,\xi)+\sigma|\xi|\le(1+\varepsilon)[f_k^\infty(t,\xi)+\sigma|\xi|],\quad \forall(t,\xi)\in\Sigma'\times\mathbb{R}^n. \tag{24}$$

If we define $g_k(x,s,\xi)=g_k(\xi):=f_k^\infty(x_0,s_0,\xi)+\sigma|\xi|$, we find, using (22), (23), (24) and applying Theorem 2.4, that

$$\begin{aligned}
&\liminf_{h\to\infty}\int_Q \frac{1}{t_h} f(x_0+\varepsilon_h y, s_h+\lambda_h u_h, t_h\nabla u_h)dy\\
&\quad\ge \liminf_{h\to\infty}\int_Q f_k^\infty(x_0+\varepsilon_h y, s_0+\lambda_h v_h,\nabla v_h)+\sigma|\nabla v_h|dy-\sigma C\\
&\quad\ge \frac{1}{1+\varepsilon}\liminf_{h\to\infty} H[g_k](v_h,Q)-\sigma C\\
&\quad\ge \frac{1}{1+\varepsilon}H[g_k](v,Q)-\sigma C.
\end{aligned}$$

Since it results

$$\lim_{\sigma\to 0} H[g_k](v,Q)=f_k^\infty(x_0,s_0,e_n),$$

letting ε,σ tends to zero and then taking the supremum for $k\in\mathbb{N}$, by Lemma 3.1.3 in Buttazzo [5], we conclude the proof. □

Proof of Claim 3. Let us fix $\varepsilon,\sigma>0$ and, for every $s'\in[a',b']$. Arguing as in the proof of Lemma 2.2 we find $\delta(s')=\delta(s',k)>0$ and a convex, NCL function $\Psi_k[s']:\mathbb{R}^n\to[-1,\infty)$ such that $\Psi_k[s'](\xi)\le f_k(t,\xi)$ for every $(t,\xi)\in B_{\delta(s')}(x_0)\times B_{\delta(s')}(s')\times\mathbb{R}^n$ and

$$\begin{aligned}
f_k(x_0,s,\xi)+\sigma|\xi|\le(1+\varepsilon)[f_k(x,s,\xi)+\sigma|\xi|]+\varepsilon,&\\
\forall(x,s,\xi)\in B_{\delta(s')}(x_0)\times B_{\delta(s')}(s')\times\mathbb{R}^n;&
\end{aligned}$$

then we apply Lemma 2.2 to $\Sigma[s']=B_{\delta(s')/2}(x_0)\times B_{\delta(s')}(s')$, to find a constant $C_k[s']$ such that (10) and (11) hold in $\Sigma[s']$.

Since $[a,b]\subset\bigcup_{s'\in[a,b]}B_{\delta(s')}(s')$ we can find $\{s_i\}_{i=1}^N\subset[a,b]$ such that

$$[a,b]\subset\bigcup_{i=1}^N B_{\delta_i/2}(s_i)\subset[a',b'],$$

where we define $\delta_i:=\delta(s_i)$. We put $\delta=\min\{\delta_i:1\le i\le N\}$, $C_k=\max\{C_k[s_i]:1\le i\le N\}$, and we consider the sets $\Sigma_i=B_\delta(x_0)\times B_{\delta_i}(s_i)$, and $\Sigma'=B_{\delta/2}(x_0)\times[a',b']$. Then the following holds true:

$$\begin{cases}
f(t,\xi)=\sup_{k\in\mathbb{N}}f_k(t,\xi), & \forall(t,\xi)\in\Sigma\times\mathbb{R}^n,\\
-1\le f_k(t,\xi)\le C_k(1+|\xi|), & \forall(t,\xi)\in\Sigma'\times\mathbb{R}^n,\\
\lambda f_k\left(t,\frac{\xi}{\lambda}\right)\ge f_k^\infty(t,\xi)-C_k\lambda, & \forall(t,\xi)\in\Sigma'\times\mathbb{R}^n,\\
\Psi_k[s_i](\xi)\le f_k(t,\xi), & \forall(t,\xi)\in\Sigma_i\times\mathbb{R}^n,\\
f_k(x_0,s,\xi)+\sigma|\xi|\le(1+\varepsilon)[f_k(x,s,\xi)+\sigma|\xi|]+\varepsilon, & \forall(t,\xi)\in\Sigma'\times\mathbb{R}^n.
\end{cases}$$

We can repeat this construction for every $j \in \{1,...,k\}$, define $g_k(t,\xi) = \max\{f_j(t,\xi) : 1 \le j \le k\}$ and find:

$$\begin{cases} g_k(t,\xi) \uparrow f(t,\xi) \text{ as } k \to \infty, & \forall (t,\xi) \in \Sigma \times \mathbb{R}^n, \\ -1 \le g_k(t,\xi) \le M_k(1+|\xi|), & \forall (t,\xi) \in \Sigma' \times \mathbb{R}^n, \\ \lambda g_k\left(t, \frac{\xi}{\lambda}\right) \ge g_k^\infty(t,\xi) - M_k \lambda, & \forall (t,\xi) \in \Sigma' \times \mathbb{R}^n, \\ \Psi_k[s_i](\xi) \le g_k(t,\xi), & \forall (t,\xi) \in \Sigma_i \times \mathbb{R}^n, \\ g_k(x_0,s,\xi) + \sigma|\xi| \le (1+\varepsilon)[g_k(x,s,\xi)+\sigma|\xi|] + \varepsilon, & \forall (t,\xi) \in \Sigma' \times \mathbb{R}^n. \end{cases}$$

By the stated properties we can see as before that

$$\begin{aligned} &\int_Q \varepsilon_h f\left(x_0 + \varepsilon_h y, u_h(y), \frac{\nabla u_h}{\varepsilon_h}\right) dy \\ &\qquad \ge \int_Q \varepsilon_h g_k\left(x_0 + \varepsilon_h y, v_h(y), \frac{\nabla v_h}{\varepsilon_h}\right) dy - M_k \varepsilon_h \mathcal{L}^n(Q \setminus E_h) \\ &\qquad \ge \int_Q g_k^\infty(x_0 + \varepsilon_h y, v_h(y), \nabla v_h) dy - M_k \varepsilon_h - M_k \varepsilon_h \mathcal{L}^n(Q\setminus E_h). \end{aligned} \tag{25}$$

Now we prove the boundedness of $(\nabla v_h)_{h\in\mathbb{N}}$ in $L^1(Q;\mathbb{R}^n)$. For every $i = 1,...,N$, $y \in Q$, we define

$$T_i v_h(y) = \begin{cases} v_h(y), & \text{if } s_i - \delta_i \le v_h(y) \le s_i + \delta_i, \\ s_i + \delta_i, & \text{if } v_h(y) > s_i + \delta_i, \\ s_i - \delta_i, & \text{if } v_h(y) < s_i - \delta_i, \end{cases}$$

so that, by $g_k^\infty(x,s,0) = 0$, it always results

$$\begin{aligned} &\int_Q g_k^\infty(x_0 + \varepsilon_h y, v_h(y), \nabla v_h) dy \\ &\qquad \ge \int_{\{s_i - \delta_i \le v_h(y) \le s_i + \delta_i\}} g_k^\infty(x_0 + \varepsilon_h y, v_h(y), \nabla v_h) dy \\ &\qquad = \int_{\{s_i - \delta_i \le v_h(y) \le s_i + \delta_i\}} g_k^\infty(x_0 + \varepsilon_h y, T_i v_h(y), \nabla T_i v_h) dy \\ &\qquad = \int_Q g_k^\infty(x_0 + \varepsilon_h y, T_i v_h(y), \nabla T_i v_h) dy \\ &\qquad \ge \int_Q \Psi_k^\infty[s_i](\nabla T_i v_h) dy \\ &\qquad \ge \alpha_k \int_Q |\nabla T_i v_h| dy - \beta_k \int_{\partial Q} |T_i v_h| d\mathcal{H}^{n-1}(y) - \gamma_k, \end{aligned}$$

where in the last inequality we have used Theorem 2.3. Since the left hand side is bounded on h and since $|T_i v_h| \le |v_h| \le \max\{|a'|,|b'|\}$ we deduce that, for every $i = 1,...,N$,

$$\sup_{h\in\mathbb{N}} \|\nabla T_i v_h\|_{L^1(Q;\mathbb{R}^n)} \le C < \infty.$$

In particular

$$\alpha := \sup_{h\in\mathbb{N}} \|\nabla v_h\|_{L^1(Q;\mathbb{R}^n)} \le NC < \infty. \tag{26}$$

Then if we define $l_k(x,s,\xi) = l_k(s,\xi) := g_k^\infty(x_0,s,\xi) + \sigma|\xi|$, by (25), (26), Theorem 2.4 and by the properties of g_k we have

$$
\begin{aligned}
&\liminf_{h\to\infty} \int_Q \varepsilon_h f\left(x_0+\varepsilon_h y, u_h(y), \frac{\nabla u_h}{\varepsilon_h}\right) dy \\
&\quad \geq \liminf_{h\to\infty} \int_Q g_k^\infty(x_0+\varepsilon_h y, v_h(y), \nabla v_h) + \sigma|\nabla v_h| dy - \sigma\alpha \\
&\quad \geq \frac{1}{1+\varepsilon} \liminf_{h\to\infty} \int_Q g_k^\infty(x_0, v_h(y), \nabla v_h) + \sigma|\nabla v_h| dy - \sigma\alpha \\
&\quad = \frac{1}{1+\varepsilon} \liminf_{h\to\infty} H[l_k](v_h, Q) - \sigma\alpha \\
&\quad \geq \frac{1}{1+\varepsilon} H[l_k](v,Q) - \sigma\alpha.
\end{aligned}
$$

Since it results

$$\lim_{\sigma\to 0} H[l_k](v,Q) = \int_a^b g_k^\infty(x_0, s, e_n) ds,$$

when $\varepsilon, \sigma \to 0$ we finally find

$$\liminf_{h\to\infty} \int_Q \varepsilon_h f\left(x_0+\varepsilon_h y, u_h(y), \frac{\nabla u_h}{\varepsilon_h}\right) dy \geq \int_a^b g_k^\infty(x_0, s, e_n) ds.$$

It suffices to apply Beppo Levi's Theorem as $k \to \infty$ and remark that, by Lemma 3.1.3 in Buttazzo [5],

$$g_k^\infty = \left(\max_{1\leq j\leq k} f_j\right)^\infty = \max_{1\leq j\leq k} f_j^\infty \uparrow \sup_{j\in\mathbb{N}} f_j^\infty = f^\infty, \text{ as } k\to\infty,$$

in order to achieve the proof. $\square$

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