On the Equal Hull Problem for Nontrivial Semiconvex Hulls

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We define a nontrivial semiconvex hull \( qr_\alpha(K) \) of a compact set \( K \subset M^{N \times n} \) called the \( \alpha \)-rank-one convex quadratic hull and establish the equalities of semiconvex hulls with respect to \( qr_\alpha(K) \) by showing that

\[
L_c(K) = qr_\alpha(K) \quad \text{if and only if} \quad Q(K) = qr_\alpha(K), \quad 0 < \alpha < 1,
\]

where \( Q(K) \) and \( L_c(K) \) are the quasiconvex convex hull and the closed lamination convex hull of \( K \) respectively. We also show that \( qr_\alpha(K) \) is a nontrivial semiconvex hull, that is, \( qr_\alpha(K) \neq C(K) \) if \( R(K) \neq C(K) \).

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Let \( M^{N \times n} \) the linear space of real \( N \times n \) matrices with the standard Euclidean norm of \( \mathbb{R}^{Nn} \), and assume that \( N, n \geq 2 \). In the vectorial calculus of variations, we may define, for a compact set \( K \subset M^{N \times n} \) the corresponding semiconvex hulls by using cosets. Let \( R(K), Q(K), P(K), qr(K) \) and \( C(K) \) be the rank-one convex, the quasiconvex, the polyconvex, the quadratic rank-one convex and the convex hulls of \( K \) respectively. We may also define the so called closed lamination convex hull \( L_c(K) \) of \( K \) (see below for definitions). We have

\[
K \subset L_c(K) \subset R(K) \subset Q(K) \subset qr(K) \subset P(K) \subset C(K). \tag{1}
\]

A surprising connection among these semiconvex hulls is the following equalities of semiconvex hulls with respect to the trivial hull \( C(K) \):

(a) \( Q(K) = C(K) \iff R(K) = C(K) \) [15],

(b) when \( n = M = 2 \), \( P(K) = C(K) \iff R(K) = C(K) \) [15] and in fact this result holds if and only if \( \min\{N, n\} = 2 \) [4].

(c) \( qr(K) = C(K) \iff R(K) = C(K) \) [18] and \( qr(K) = P(K) \) if and only if \( \min\{N, n\} = 2 \) [12, 4].

Item (c) above unifies results in (a) and (b) and the proof is more elementary. We may view the results a) - c) as equal hull properties with respect to the trivial hull \( C(K) \).

Naturally, one would ask whether the equal hull properties holds with respect to other semiconvex hulls such as \( qr(K) \) or \( P(K) \), that is, whether for a compact set \( K \subset M^{N \times n} \), \( L_c(K) = qr(K) \) if and only if \( Q(K) = qr(K) \), or \( L_c(K) = P(K) \) if and only if \( Q(K) = P(K) \) when \( \min\{N, n\} = 2 \).
It can be shown by using an example due to Šverák [9] that the answers to both questions are negative at least for some compact sets $K$ in $M^{N \times 2}$ (see Example 4 below). The question then is whether there are nontrivial semiconvex hulls of $K$ such that the equal hull property holds. In this paper we define a family of semiconvex hulls called $\alpha$-rank-one convex quadratic hull $qr_\alpha(K)$ lying between the convex hull $C(K)$ and the quadratic rank-one convex hull $q(K)$ by using a class of rank-one convex quadratic functions which are either convex or strictly rank-one convex. We are then able to establish the equal hull property with respect to $qr_\alpha(K)$. We have

**Theorem 1.** Let $K \subset M^{N \times n}$ be compact, then for $0 < \alpha < 1$,

1. $L_c(K) = C(K)$ if and only if $qr_\alpha(K) = C(K)$.
2. $L_c(K) = qr_\alpha(K)$ if and only if $Q(K) = qr_\alpha(K)$.

**Remark 2.** Since we have $L_c(K) \subset R(K) \subset Q(K) \subset qr_\alpha(K) \subsetqr_\alpha(K) \subset C(K)$, (2) and (i) implies that $qr_\alpha(K)$ is a nontrivial semiconvex hull, we see that the equal hull property holds for a family of nontrivial semiconvex hulls.

Before we prove our main results, let us first introduce some notation and definitions.

Let $f : M^{N \times n} \to \mathbb{R}$ be a continuous function. The following are some conditions related to weak lower semicontinuity of the integral (c.f. [2, 7, 5, 1])

\[ I(u) = \int_\Omega f(Du(x))dx \]

(i) $f$ is rank-one convex if for each matrix $A \in M^{N \times n}$ and each rank-one matrix $B = a \otimes b \in M^{N \times n}$, the function $t \to f(A + tB)$ is convex.

(ii) $f$ is quasiconvex at $A \in M^{N \times n}$ on $\Omega$, if for any smooth function $\phi : \Omega \to \mathbb{R}^N$ compactly supported in $\Omega$,

\[ \int_\Omega f(A + D\phi(x))dx \geq \int_\Omega f(A)dx \]

holds. $f$ is quasiconvex if it is quasiconvex at every $A \in M^{N \times n}$. The class of quasiconvex functions is independent of the choice of $\Omega$.

(iii) $f$ is polyconvex if $f(A) = \text{convex function of minors of the matrix } A$.

(iv) $f$ is a rank-one convex quadratic function if $f(A) = q(A) + l(A)$ with $q(\cdot)$ a rank-one convex quadratic form and $l(\cdot)$ an affine function.

It is well-known that (iii)$\Rightarrow$(ii)$\Rightarrow$(i), while (i)$\not\Rightarrow$(ii)$\not\Rightarrow$(iii) (cf. [2, 7, 5, 13]). However, if $f$ is a quadratic function, (i) is equivalent to (ii).

Let $E \subset M^{N \times n}$ be a linear subspace without rank-one matrices, and $E^\perp$ being its orthogonal complement. Let

\[ q_E(A) = |P_{E^\perp}(A)|^2 - \lambda_E|P_E(A)|^2, \]

(3)
where $P_{E^\perp}$ and $P_E$ are orthogonal projections to $E^\perp$ and $E$ respectively, where $\lambda_E > 0$ is defined by
\[
\frac{1}{\lambda_E} = \sup \left\{ \frac{|P_E(a \otimes b)|^2}{|P_{E^\perp}(a \otimes b)|^2} : a \in \mathbb{R}^N, b \in \mathbb{R}^n \right\} < \infty.
\] (4)

Then $q_E$ is a rank-one convex quadratic form [3]. For convenience, we also define $\lambda_E = 0$ if $E$ has rank-one matrices, and
\[
q_{\mu,E} = |P_{E^\perp}(A)|^2 - \mu \lambda_E |P_E(A)|^2
\]
for some $0 \leq \mu \leq 1$. Clearly, $q_{\mu,E}$ is a rank-one convex quadratic form. Let $E$ be the set of all non-zero linear subspaces of $M^{N \times n}$ and define for a fixed $0 \leq \alpha < 1$,
\[
QR_\alpha = \{ \sigma q_{\mu,E} + l, \sigma \geq 0, 0 \leq \mu \leq \alpha, E \in E, l \text{ affine} \}.
\]

Although the definition of the family $QR_\alpha$ of rank-one convex quadratic functions looks complicated, in fact it is among the simplest collection of strictly (when $0 < \alpha < 1$) rank one convex quadratic functions that separate points.

**Definition 3.** We define the $\alpha$-quadratic rank-one convex envelope $qr_\alpha(f)$ of $f : M^{N \times n} \to \mathbb{R}$ as
\[
qr_\alpha(f)(A) = \sup \{ q(A), q \leq f, q \in QR_\alpha \},
\] (5)

In the study of material microstructure, the following concepts of semiconvex hulls for a compact set $K \subset M^{N \times n}$ are naturally introduced.

A compact set $K \subset M^{N \times n}$ is called stable under lamination (or lamination convex) [8] if $A, B \in K$ are rank-one connected, that is, rank($A - B$) = 1, then for all $\lambda \in (0, 1)$, one has $(1 - \lambda)A + \lambda B \in K$. For a given $K \subset M^{N \times n}$, the lamination convex hull $L(K)$ is defined as the smallest lamination convex set that contains $K$ [8]. We also define the closed lamination convex hull $L_c(K)$ as the smallest closed lamination convex set that contains $K$ [15].

By using the coset definition, we may define semiconvex hulls $S(K)$ of $K$ as follows,
\[
S(K) = \{ X \in M^{N \times n}, f(X) \leq \sup_{Y \in K} f(Y) \text{ for every } S\text{-convex } f : M^{N \times n} \to \mathbb{R} \}.
\]

If we replace $S$-convex by rank-one convex, quasiconvex, polyconvex, quadratic rank-one convex functions respectively, we obtain the rank-one convex hull $R(K)$, the quasiconvex hull $Q(K)$, the polyconvex hull $P(K)$ [14], the quadratic rank-one convex hull $qr(K)$ [18] and the convex hull $C(K)$. Clearly, if $K$ is closed, (1) holds. If $L_c(K)$ is convex, obviously, all other ‘semiconvex’ hulls are identical to $C(K)$.

Later we will use some facts from the theory of gradient Young measures and homogeneous (gradient) Young measures supported on compact sets in $M^{N \times n}$ [10, 6]: (i) If $X_0 \in Q(K)$, there is a homogeneous gradient Young measure $\nu$ supported in $K$ such that the integral average $\bar{\nu} := \int_K \lambda d\nu = X_0$ (also see [16]). (ii) For a rank-one convex quadratic form $q$ satisfying $q(a \otimes b) \geq c|a|^2|b|^2$ for $a \in \mathbb{R}^N, b \in \mathbb{R}^n$, with $c > 0$ a constant, one has for the Young measure above, $\int_K q(\lambda) d\nu \geq q(X_0) + c \int_K |\lambda - X_0|^2 d\nu$ (also see [17]).
Before we introduce the $\alpha$-quadratic rank-one convex hull $qr_\alpha(K)$, let us examine the following counter-example of the equal hull property with respect to $qr(K)$ and $P(K)$ due to Šverák [9].

**Example 4.** Let $u : \mathbb{R}^2 \to \mathbb{R}^6$ be defined as

$$u(x_1, x_2) = (\cos x_1, \sin x_1, \cos x_2, \sin x_2, \cos(x_1 + x_2), \sin(x_1 + x_2)),$$

and notice that $u$ is a periodic smooth mapping. We define $K$ as the image of the gradient of $u$,

$$K = \{Du(x) \in M^{6 \times 2}, \, x \in \mathbb{R}^2\}.$$

It was shown by Šverák that $L_c(K) = R(K) = K$ while $P(K) = Q(K) = K \cup \{0\}$. We have $P(K) = qr(K) \neq L_c(K)$ while $qr(K) = Q(K)$. Thus the equal hull property does not hold in $M^{6 \times 2}$.

**Definition 5.** For a compact set $K \subset M^{N \times n}$, the $\alpha$-quadratic rank-one convex hull $qr_\alpha(K)$ is defined by

$$qr_\alpha(K) = \{A \in M^{N \times n}, \, q(A) \leq \sup_{B \in K} q(B), \, q \in QR_\alpha\}. \quad (6)$$

If $qr_\alpha(K) = K$, we call $K$ an $\alpha$-quadratic rank-one convex set. Clearly, $qr_\alpha(K)$ satisfies (2).

**Remark 6.** Our choice of $QR_\alpha$ of quadratic functions is because it gives us several advantages.

(i) $QR_\alpha$ separates points for any plane $E \subset M^{N \times n}$ without rank-one connection in the sense that for every $X \in E$, there is some $q \in QR_\alpha$ such that $q(Y) < 0$ for $Y \in E, \, Y \neq X$ and $q(X) = 0$. We may construct such a quadratic function as follows. Let $E_0$ be the subspace parallel to $E$ given by $E_0 = \{Y - X, \, Y \in E\}$, then we define

$$q(A) = |P_{E_0}^\perp(A - X)|^2 - \alpha \lambda_0 |P_{E_0}(A - X)|^2,$$

then $q \in QR_\alpha$ and for every $Y \in E, \, Y \neq X, \, q(Y) = -\alpha \lambda_0 |P_{E_0}(Y - X)|^2 < 0 = q(X)$.

(ii) Every ‘bounded’ set of $QR_\alpha$ is sequentially compact. To be more precisely, let

$$q_k(A) = \sigma_k \left(|P_{E_k}(A)|^2 - \mu_k \lambda_k |P_{E_k}(A)|^2\right) + B_k \cdot A + c_k$$

be a sequence in $QR_\alpha$ such that $\sigma_k, \, B_k$ and $c_k$ are bounded. Then there is a subsequence $q_{k_j}$ and some $q \in QR_\alpha$ such that for each fixed $A \in M^{N \times n}$, $q_{kj}(A) \to q(A)$.

This can be easily proved because $\dim(E_k)$ is a sequence of bounded integers. Therefore, there is a subsequence such that $\dim(E_k) = m$. For each $k$ we take an orthonormal basis $\{E_k^{(1)}, \ldots, E_k^{(m)}\} \subset E_k$ and that of $E_k^\perp$. Then up to a subsequence, the two basis of $E_k$ and $E_k^\perp$ converge to basis of $E$ and $E^\perp$ respectively (this is easy to verify). Note also that $\lambda_{E_k}$ is bounded and equals zero if $E_k$ has a rank-one matrix. We may also assume that $\sigma_k \to \sigma \geq 0, \, \mu_k \to \mu \leq \alpha, \, B_k \to B, \, c_k \to c$, and $\lambda_{E_k} \to \lambda_E$ up to a subsequence. We will check the last assertion later. Now, let

$$q(A) = \sigma \left(|P_{E_k}^\perp(A)|^2 - \mu \lambda_E |P_E(A)|^2\right) + B \cdot A + c,$$
then clearly \( q_k(A) \to q(A) \) for each fixed \( A \in M^{N \times n} \) and \( q \in QR_\alpha \).

To show that \( \lambda_{E_k} \to \lambda_E \), we have, by definition, \( |P_{E_k}(a \otimes b)|^2 \geq \lambda_{E_k} |P_{E_k}(a \otimes b)|^2 \), and for some \( a_k \in \mathbb{R}^n \), \( b_k \in \mathbb{R}^N \) with \( |a_k| = |b_k| = 1 \), \( |P_{E_k}(a_k \otimes b_k)|^2 = \lambda_{E_k} |P_{E_k}(a_k \otimes b_k)|^2 \). Passing to the limit \( k \to \infty \), one obtains, \( |P_{E_k}(a \otimes b)|^2 \geq \lambda |P_{E}(a \otimes b)|^2 \), so that \( \lambda \leq \lambda_E \).

On the other hand, up to a subsequence, \( a_k \to a_0, b_k \to b_0 \) as \( k \to \infty \), we also have \( |P_{E_k}(a_0 \otimes b_0)|^2 = \lambda |P_{E}(a_0 \otimes b_0)|^2 \). Thus \( \lambda = \lambda_E \).

(iii) The semiconvex hull \( qr_\alpha(K) \) can be represented as the intersection of the convex hull \( C(K) \) and sub-level sets of a family of simple rank-one convex quadratic functions.

From the definition of \( qr_\alpha(K) \), for any \( X_0 \in qr_\alpha(K) \), \( q(X_0) \leq \max \{ q(X), X \in K \} := c_\alpha \), for \( q \in QR_\alpha \), if we let \( q_c = q - c_\alpha \), then \( q_c \in QR_\alpha \) and \( q_c(X_0) \leq 0 \). Let \( \mathcal{K}_q = \{ X \in M^{N \times n}, q_c(X) \leq 0 \} \), then

\[
qr_\alpha(K) = \bigcap_{q \in QR_\alpha} \mathcal{K}_q.
\]

Since every \( q \in QR_\alpha \) can be written as

\[
q(X) = \sigma \left( |P_{E_k}(X)|^2 - \mu \lambda_E |P_E(X)|^2 \right) + B \cdot A + c,
\]

where \( \sigma \geq 0 \). If \( \sigma = 0 \), \( q \) is affine hence \( q_c \) is affine. If \( \sigma > 0 \), we may divide \( q_c \) by \( \sigma \) and consider quadratic functions in the form

\[
q^{(*)}(X) = \left( |P_{E_k}(X)|^2 - \mu \lambda_E |P_E(X)|^2 \right) + B^{(*)} \cdot X + c^{(*)},
\]

under the assumption that \( E \) does not have rank-one matrices and \( 0 < \mu \leq \alpha \). Clearly \( q_c(X) \leq 0 \) if and only of \( q^{(*)}(X) \leq 0 \). If \( \mu = 0 \) or \( E \) has rank-one matrices, \( q^{(*)} \) is then convex. However, recall [11] that convex functions can be represented by ‘sup’ of affine functions, we define a subset \( QR_\alpha(K) \) of \( QR_\alpha \) by

\[
QR_\alpha(K) = \{ q \in QR_\alpha, \text{ } q \text{ satisfies (7), } \mu \lambda_E > 0, q(X) \leq 0, X \in K, \exists Y_q \in K, q(Y_q) = 0 \}. \tag{8}
\]

It is easy to verify that

\[
qr_\alpha(K) = C(K) \cap \left( \bigcap_{q \in QR_\alpha(K)} \mathcal{K}_q \right), \text{ where } K_q = \{ X \in M^{N \times n}, q(X) \leq 0 \}.
\]

We only prove Theorem 1 (ii) here and leave the proof for (i) at the end of this paper.

**Proof of Theorem 1 (ii).** We may assume that \( K = L_c(K) \). In other words, \( K \) is a closed laminated convex set. If \( qr_\alpha(K) \neq K \), we show that \( qr_\alpha(K) \neq Q(K) \). Let \( m \) be the smallest affine dimension of \( C(K) \) such that our statement fails. It is easy to show that this dimension \( m \) is greater than 1.

In fact, if the affine dimension of \( C(K) \) is 1, \( K \) is contained in a straight line. If the line contains rank-one connections, then \( L_c(K) = C(K) \) and our claim is true. If the line does not have rank-one connections, then it separates points as shown in Remark 6.(i). Thus in this case \( qr_\alpha(K) = K = L_c(K) \). We examine two cases.
Clearly, we may write $X$ while we need the following Lemma for Case (I).

**Lemma 7.** Let $E$ be a proper supporting plane [11] of $C(K)$ then $qr_α(K) ∩ E = qr_α(K ∩ E)$.

We prove Lemma 7 after we finish the proof of Theorem 1.

**Proof of Theorem 1, (ii), Case (I).** Accepting Lemma 7 for the moment. We seek to prove that $Q(K) ≠ qr_α(K)$. Lemma 7 implies that there is a supporting plane $E$ of $C(K)$ passing through $X_0$, hence $qr_α(K) ∩ E = qr(K ∩ E)$. On the other hand, we have $Q(K ∩ E) = Q(K) ∩ E$ and $K ∩ E$ is still a closed lamination convex set. Since $dim(E ∩ C(K)) < m$, we see that $Q(K ∩ E) ≠ qr_α(K ∩ E)$. Thus $Q(K) ∩ E ≠ qr_α(K) ∩ E$ so that $Q(K) ≠ qr_α(K)$. Therefore the proof of Case (I) is finished pending the proof of Lemma 7.

**Proof of Theorem 1, (ii), Case (II).** Without loss of generality, we may assume that $0 ∈ K$. Since

$$qr_α(K) = C(K) ∩ (∩_{q ∈ QR_α(K)} K_q) ⊂ \text{span}[C(K)] := E_0,$$

we see that $E_0$ must have rank-one matrices. Otherwise $qr_α(K) = Q(K) = L_c(K) = K$.

We further consider two different subcases:

**Case (IIa)** The relative boundary of $qr_α(K)$ in $E_0$ is contained in $K$ and there is a relative interior point $X_0$ of $qr_α(K)$ such that $X_0 ∉ Q(K)$.

**Case (IIb)** There is a relative boundary point $X_0$ of $qr_α(K)$ such that $X_0 ∉ Q(K)$.

**Proof of Case (IIa).** Let $A_0 ∈ E_0$ be a rank-one matrix, and let us consider the line $X_0 + tA_0$. It is easy to see that since $qr_α(K)$ is compact and $X_0$ is a relative interior point, there are $t_1 > 0$, $t_2 < 0$ such that $A_1 = X_0 + t_1A_0$, $A_2 = X_0 + t_2A_0$ are both on the boundary $∂qr_α(K) ⊂ K$. Thus $X_0 ∈ L_c(K)$, which is a contradiction.

**Proof of Case (IIb).** Since $X_0$ is a boundary point of $qr_α(K) = C(K) ∩ (∩_{q ∈ QR_α(K)} K_q)$ while $X_0 ∉ ∂C(K)$, we see that there is a sequence $(q_k)$ in $QR_α(K)$ such that $q_k(X) ≤ 0$, $-1/k ≤ q_k(X_0) ≤ 0$, $k = 1, 2, \ldots$. Clearly, we may write $q_k$ as $q_k(X) = q_k(0)(X) + B_k · (X - X_0) + c_k$, where

$$q_k(0)(X) = |P_{E_k^⊥}(X)|^2 - μ_kλ_{Ek}|P_{Ek}(X)|^2, \quad X ∈ M^{N × n}.$$

We see from $-1/k ≤ q_k(X_0) ≤ 0$ that

$$-\frac{1}{k} ≤ q_k(0)(X_0) + c_k ≤ 0.$$
Since \( q_k^{(0)}(X_0) \) is bounded, \( c_k \) is then bounded. We also claim that \( |B_k| \) is bounded. Otherwise, up to a subsequence, \( |B_k| \to \infty \). Hence up to a subsequence \( B_k/|B_k| \to B_0 \) for some \( B_0 \in M^{N \times n} \), \( |B_0|=1 \). Now we dividing \( q_k \) by \( |B_k| \) and pass to the limit \( k \to \infty \) for each fixed \( X \), we see that \( q_k(X)/|B_k| \to l(X) = B_0 \cdot (X-X_0) \). We also have \( l(X) \leq 0 \) for \( X \in K \) and \( l(X_0) = 0 \). Thus \( X_0 \in \partial C(X) \), which is a contradiction.

Now, since \( B_k \) and \( c_k \) are both bounded, we may pass to the limit \( k \to \infty \) (up to a subsequence) so that \( q_k(X) \to q(X) \) where

\[
q(X) = |P_{E^\perp}(X)|^2 - \mu \lambda_E |P_E(X)|^2 + B \cdot (X-X_0) + c,
\]

and we let

\[
q^{(0)}(X) = |P_{E^\perp}(X)|^2 - \mu \lambda_E |P_E(X)|^2.
\]

If \( E \) has rank-one matrices, hence \( \lambda_E = 0 \) or \( \mu = 0 \), \( q \) is then convex which again implies \( X_0 \in \partial C(K) \) and leads to a contradiction. Thus we may claim that \( 0 < \mu \leq \alpha \) and \( \lambda_E > 0 \).

Since \( X_0 \in Q(K) \), there is a homogeneous gradient Young measure \( \nu \) supported in \( K \) such that \( \int_K \lambda d\nu = X_0 \). Due to the fact that \( q \) is quasiconvex, we have

\[
\int_K q(\lambda) d\nu \geq q(X_0).
\]

On the other hand, \( q(X) \leq 0 \) for \( X \in K \), we have \( \int_K q(\lambda) d\nu = q(X_0) \). Thus \( \nu \) is supported on the level set \( \{ X \in K, q(X) = 0 \} \). Since \( \sigma \leq \alpha \), we see that the quadratic form \( q^{(0)} \) above satisfies

\[
q^{(0)}(a \otimes b) \geq (1-\alpha)\lambda_E |P_{E^\perp}(a \otimes b)|^2 \geq (1-\alpha)\lambda_E \frac{\lambda_E}{1+\lambda_E} |a|^2 |b|^2,
\]

so that for the homogeneous Young measure \( \nu \) above, we have

\[
0 = \int_K (q(\lambda) - q(X_0)) d\nu \geq (1-\alpha)\lambda_E \frac{\lambda_E}{1+\lambda_E} \int_K |\lambda - X_0|^2 d\nu \geq 0.
\]

Thus \( \nu = \delta_{X_0} \) is a Dirac mass, hence \( X_0 \in K \), a contradiction. The proof for Case (IIb) is complete. \( \Box \)

**Proof of Lemma 7.** Let \( E_1 \) be the plane in \( M^{N \times n} \) containing \( C(K) \) with the same dimension as \( C(K) \) \([11]\). Obviously, \( qr_{\alpha}(K \cap E) \subset qr_{\alpha}(K) \cap E \). Let \( X \in qr_{\alpha}(K) \cap E \). There is an affine function \( l \) defined on \( M^{N \times n} \) such that \( l < 0 \) on the open half plane in \( E_1 \) containing \( C(K) \setminus E \), \( l = 0 \) on \( E \) and \( l > 0 \) on the opposite half plane to \( C(K) \) in \( E_1 \). We also define

\[
E(\epsilon) = \{ A \in E_1, \text{dist}(A, E) \leq \epsilon, l(A) \leq 0 \}
\]

which is a set on the same side as \( C(K) \) in \( E_1 \), where \( \text{dist}(A, E_1) \) is the Euclidean distance from \( A \) to \( E_1 \). For any fixed \( q \in QR_{\alpha} \) we consider, for every integer \( n > 0 \) the quadratic function \( q(\cdot) + nl(\cdot) \in QR_{\alpha} \). Since for any \( A \in E \), \( l(A) = 0 \), we have, for each fixed \( X \in qr_{\alpha}(K) \cap E \),

\[
q(X) = q(X) + nl(X) \leq \sup_{A \in K} [q(A) + nl(A)].
\]
Since $q + nl$ is continuous and $K$ compact, the maximum is attained at some $A_n \in K$, that is, $\sup_{A \in K} [q(A) + nl(A)] = q(A_n) + nl(A_n)$, so that $q(X) \leq q(A_n) + nl(A_n)$. Since $K$ is compact there is a subsequence $A_{n_k} \to A_0 \in K$ as $k \to \infty$. Notice that $l(A_n) \leq 0$ for all $n$. If we let $k \to \infty$ we see that $\delta_k := \text{dist}(A_{n_k}, E) \to 0$. Otherwise $q(X)$ cannot be finite. Now we have

$$q(X) \leq q(A_{n_k}) + n_k l(A_{n_k}) \leq \sup \{ q(A), A \in K \cap E(\delta_k) \}. \tag{9}$$

Again the ‘sup’ in (9) can be reached by, say $B_k \in K \cap E(\delta_k)$, and up to a subsequence, we have $B_k \to B_0 \in K \cap E$ as $k \to \infty$.

Passing to the limit $k \to \infty$ on both side of the inequality $q(X) \leq q(B_k)$ and noticing that $B_0 \in K \cap E$, we have $q(X) \leq q(B_0) \leq \sup_{A \in K \cap E} q(A)$, hence $X \in qr_\alpha(K \cap E)$, Lemma 7 is then proved by noticing also that $C(K) \cap E = C(K \cap E)$. \hfill $\square$

**Remark 8.** Our argument breaks down if we allow all rank-one convex quadratic functions to be considered. The reason is that non-convex quadratic functions are no longer strictly quasiconvex. The problem is reduced to the study of homogeneous Young measures $\nu$ supported on a compact set $K \subset \{ X \in M^{N \times n}, q(X) = 0 \}$ for some quadratic rank-one convex function $q$ with $q(0) = 0$ under the conditions that $0 \notin L_c(K)$. Example 4 shows that it is possible that the average of Young measure $\bar{\nu} = 0$ hence the equal hull property fails.

However, it is interesting to find examples of non-convex rank-one convex quadratic functions and compact sets $K$ contained in the level set $q = 0$ such that $qr(K) \neq R(K)$ while $qr(K) = Q(K)$.

**Proof of Theorem 1 (i).** For any nontrivial supporting plane of $C(K)$, we have, from Lemma 7 that $qr_\alpha(K) \cap E = qr_\alpha(K \cap E)$. Notice also that $C(K) \cap E_1 = C(K \cap E_1)$.

Suppose $L_c(K) \neq C(K)$, while $qr_\alpha(K) = C(K)$. We may assume that $K$ is a closed laminated convex set. Then among all these $K$’s there is one for which the affine dimension $\dim C(K) \geq 1$ of $C(K)$ is the smallest. For such $K$ we claim that the plane $E$ spanned by $C(K)$ does not have rank-one connections. Otherwise it is easy to see [15] that there is a supporting plane $E$ of $C(K)$ such that $E \cap K$ is not convex and is still a closed laminated convex set while

$$qr_\alpha(K \cap E) = qr_\alpha(K) \cap E = C(K) \cap E$$

is convex. This contradicts to the fact that the dimension $\dim C(K)$ is the smallest. Now since $C(K) \subset E$ and $E$ does not have rank-one connection, there is some $X \in C(K) \neq K$. If we define $q_X$ as in Remark 6(i), there is some $\delta > 0$, such that $q_X(X) = 0 > -\delta = \sup_{A \in K \subset E} q_X(A)$. Hence $X \notin qr_\alpha(K)$ and $qr_\alpha(K) \neq C(K)$, a contradiction. \hfill $\square$

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**References**