Journal of Convex Analysis
Volume 10 (2003), No. 2, 531–539

# On Weak*-Extreme Points in Banach Spaces

**S. Dutta**
*Stat–Math Unit, Indian Statistical Institute,*
*203, B. T. Road, Kolkata 700 108, India*
*sudipta_ r@isical.ac.in*

**T. S. S. R. K. Rao**
*Stat–Math Unit, Indian Statistical Institute,*
*R. V. College Post, Bangalore 560059, India*
*tss@isibang.ac.in*

Received May 23, 2002
Revised manuscript received July 15, 2003

We study the extreme points of the unit ball of a Banach space that remain extreme when considered, under canonical embedding, in the unit ball of the bidual. We give an example of a strictly convex space whose unit vectors are extreme points in the unit ball of the second dual but none are extreme points in the unit ball of the fourth dual. For the space of vector- valued continuous functions on a compact set we show that any function whose values are weak*-extreme points is a weak*-extreme point . We explore the relation between weak*-extreme points and the dual notion of very smooth points. We show that if a Banach space $X$ has a very smooth point in every equivalent norm then $X^*$ has the Radon-Nikodým property.

*Keywords:* Higher Duals, $M$-ideals, extreme points

*2000 Mathematics Subject Classification:* 46B20

## 1. Introduction

For a Banach space $X$ we denote the closed unit ball of $X$ by $X_1$ and the set of extreme points of $X_1$ by $\partial_e X_1$ . Our notation and terminology is standard and can be found in [2, 3, 9]. We always consider a Banach space as canonically embedded in its bidual.

An extreme point of $X_1$ is called weak*-extreme if it continues to be an extreme point of $X_1^{**}$. It is known from the classical work of Phelps ([19]) that for a compact set $K$, any extreme point of the unit ball of $C(K)$ is weak*-extreme . Importance of these points to the geometry of a Banach space was enunciated in [23], where it was proved that a Banach space has the Radon-Nikodým property (RNP) if and only if for every equivalent norm the unit ball has a weak*-extreme point. Several stronger forms of extreme points have been well studied in the literature, for example Kunen and Rosenthal have showed in [13] that any strongly extreme point of $X_1$ is a strongly extreme point of $X_1^{**}$. Similarly it is well known that denting (weak*-denting in dual spaces) points continue to be denting (weak*-denting) points of the bidual unit ball . Thus all these points are weak*-extreme and moreover they belong to the same class of extreme points in the unit ball of any dual of even order .

In the second section of this paper we give conditions under which extreme points get preserved from a subspace and conditions when they fail to belong to the same class of

ISSN 0944-6532 / $ 2.50 

extreme points in the unit ball of bigger space. Using this, we show that weak*-extreme points can fail to be weak*-extreme in the bidual unit ball. We also give some interesting examples of preserved extremality. We make free use of techniques from $M$-structure theory to construct these examples. The book [9] is now a standard reference for results related to $M$-ideal theory.

Recall that $x \in X_1$ is a strongly extreme point if for any sequences $\{x_n\}$ and $\{y_n\}$ in $X_1$, $\frac{1}{2}(x_n + y_n) \to x$ implies $x_n \to x$ and $y_n \to x$. This is equivalent to the definition given in [13]. We will consider weak*-extreme and strongly extreme points in the spaces of operators.

In the second section of the paper we study a new class of extreme points that have been recently considered in [6] while studying very smooth points of Banach spaces. These are extreme points of $X_1^*$ that are points of continuity for $i : (X_1^*, weak^*) \to (X_1^*, weak)$. We show that when $X^*$ or $Y$ has the compact metric approximation property any $\tau \in \mathcal{L}(X,Y)_1^*$ of the above type is of the form $x \otimes y^*$ where $x \in \partial_e X_1^{**}$ and $y^* \in \partial_e Y_1^*$ are points of weak*-weak continuity for the identity map on the respective unit balls.

Analogous to the result of Rosenthal quoted above we show that if every equivalent norm on $X$ has a very smooth point then $X^*$ has the RNP.

**Acknowledgements.** The second author's research was supported by I. F. C. P. A. R project grant No: 2601-1. He thanks Professor G. Godefroy of the University of Paris VI for his hospitality during this author's visit in April 2002, when part of this work was done.

## 2. Weak*-extreme points

In [15] the author considered the following kind of extreme points. Let $x_0 \in X_1$ be such that $|x^*(x_0)| = 1$ for all $x^* \in \partial_e X_1^*$. It is easy to see that for a compact set $K$ every extreme point of $C(K)_1$ and for a positive measure $\mu$ every extreme point of $L^1(\mu)_1$ is such a point. It was observed in [20] that any such point is a strongly extreme point. Our first result shows that these points behave the same way in the bidual.

**Proposition 2.1.** *Let $X$ be a Banach space and let $x_0 \in X_1$ be such that $|x^*(x_0)| = 1$ for all $x^* \in \partial_e X_1^*$. Then $|\tau(x_0)| = 1$ for all $\tau \in \partial_e X_1^{***}$.*

**Proof.** We first observe that if $Y \subset X$ is a subspace and $y \in Y$ is such that $|x^*(y)| = 1$ for all $x^* \in \partial_e X_1^*$ then $|y^*(y)| = 1$ for all $y^* \in \partial_e Y_1^*$.

Let $K = \{x^* \in X_1^* : x^*(x_0) = 1\}$ be equipped with the weak*-topology. Let $\Phi : X \to C(K)$ be the canonical embedding defined by $\Phi(x)(x^*) = x^*(x)$. Since $\partial_e X_1^* \subset \Gamma K$ we get that $\Phi$ is an isometry. Also $\Phi(x_0) = 1$. Note that $\Phi^{**} : X^{**} \to C(K)^{**}$ is an isometry and $\Phi^{**}(x_0) = 1$. Since $C(K)^{**}$ is again a space of the form $C(K')$ for a compact set $K'$ and $1 \in \partial_e C(K')$ it follows from our earlier remarks that $|\tau(1)| = 1$ for all $\tau \in \partial_e C(K')^*$. Thus by the observation made at the beginning of this proof we get that $|\tau(x_0)| = 1$ for all $\tau \in \partial_e X_1^{***}$. $\square$

We recall from [9] that $M \subset X$ is an $M$-ideal if there is a projection $P$ on $X^*$ such that $ker(P) = M^\perp$ and $\|P(x^*)\| + \|x^* - P(x^*)\| = \|x^*\|$ for all $x^* \in X^*$.

Our next result gives conditions under which extreme points gets preserved from a subspace and conditions when they fail to belong to the same class of extreme points in the unit ball of the bigger space.

**Theorem 2.2.** *Let $M \subset X$ be a proper subspace.*

($a$) *Suppose there is a projection of norm one $P$ in $X^*$ such that $ker(P) = M^\perp$ and $RangeP_1$ is weak*-dense in $X_1^*$. Then any strongly extreme point of $M_1$ is strongly extreme in $X_1$ and any weak*-extreme point of $M_1$ is an extreme point of $X_1$*

($b$) *If $M$ is an $M$-ideal in $X$ then no weak*-extreme point of $M_1$ can be weak*-extreme in $X_1$.*

**Proof.** ($a$) We recall the construction from Lemma 1 in [21]. For any $y^* \in M^*$ and any Hahn Banach extension $x^*$ of $y^*$, $y^* \to Px^*$ is a well defined linear map and since $P$ is of norm one, we see that $M^*$ is isometric to $PX^*$. Thus we may assume $M^*$ is embedded in $X^*$ and define $\Phi : X \to M^{**}$ by $\Phi(x) = x|_{M^*}$. Since Range $P_1$ is weak*-dense in $X^*$, we have $\Phi$ is an isometry whose restriction to $M$ agrees with the canonical embedding of $M$ in $M^{**}$.

If $m_0$ is a strongly extreme point of $M_1$ then it is easy to verify (see [13]) that $m_0$ is strongly extreme in $M_1^{**}$ as well and hence strongly extreme in $X_1$. Also if $m_0$ is a weak* -extreme point of $M_1$ then clearly it is an extreme point of $X_1$.

($b$) Let $P$ be the $L$-projection in $X^*$ with $kerP = M^\perp$. Since $Range(P^*) = M^{\perp\perp}$, if $m_0 \in M_1$ is a weak*-extreme point we have $P^*(m_0) = m_0$ is an extreme point of the unit ball of $M^{\perp\perp}$, as the latter space is isometric to $M^{**}$. Take any unit vector $\tau$ in $(M^\perp)^* = ker(P^*)$ (which is non-trivial since $M$ is proper subspace). Since $P^*$ is a $M$-projection, $\|m_0 \pm \tau\| = 1$ as $m_0$ and $\tau$ are in disjoint $M$-summands. Thus $m_0$ is not an extreme point of $X_1^{**}$. □

**Remark 2.3.** (a) If $X$ is not a reflexive space and is an $M$-ideal in its bidual then $X \subset X^{**}$ satisfies both ($a$) and ($b$) of the above hypothesis. We also note that if $X, Y$ are Banach spaces such that $\mathcal{K}(X, Y)$ is an $M$-ideal in $\mathcal{L}(X, Y)$ then again the above hypothesis is satisfied. See Chapter III of [9] for several examples of spaces that are $M$-ideals in their biduals and Chapter VI of [9] for examples of spaces $X$ and $Y$ for which $\mathcal{K}(X, Y)$ is an $M$-ideal in $\mathcal{L}(X, Y)$.

(b) For a compact set $K$ let $WC(K, X)$ denote the space of $X$-valued functions on $K$ that are continuous when $X$ has the weak topology, equipped with the supremum norm. It follows from Example 2 in [21] that $C(K, X) \subset WC(K, X)$ satisfies the hypothesis ($a$) above. Thus any weak*-extreme point of $C(K, X)_1$ is an extreme point of $WC(K, X)_1$. Any strongly extreme point of $C(K, X)_1$ is strongly extreme in $WC(K, X)_1$. This observation gives a simpler proof of Corollary 9 from [11]. Also as remarked before Corollary 9 in [11] it is not known if every extreme point of $C(K, X)_1$ is always extreme in $WC(K, X)_1$.

We now illustrate the strength of our Theorem with some specific examples.

It follows from the remarks on page 78 of [9] that a proper $M$ ideal cannot have a strongly extreme point in its unit ball. In view of our results so far it would be interesting to see examples of $M$-ideals that have weak*- extreme points. The following example is also interesting from another point of view. It was shown in [8] that for any Hilbert space every extreme point of $\mathcal{L}(H)_1$ is a strongly extreme point. For $p \neq 2$ we exhibit extreme points in $\mathcal{L}(\ell^p)_1$ that are not even weak*- extreme.

**Example 2.4.** Consider for $1 < p < \infty$, the space $\mathcal{K}(\ell^p)$. It is a proper $M$-ideal of its bidual, $\mathcal{L}(\ell^p)$.

When $p \neq 2$, it was shown in [10] that there exists $T \in \mathcal{K}(\ell^p)$ such that there is no $B \in \mathcal{L}(\ell^p)$ such that $\|T \pm B\| = 1$. Thus $T$ is an extreme point of $\mathcal{L}(\ell^p)$ and hence is weak*-extreme in $\mathcal{K}(\ell^p)$. But from Theorem 2.2 it follows that $T$ can not be weak*-extreme point in $\mathcal{L}(\ell^p)$.

The following example exhibits a situation where all the unit vectors are weak*-extreme points of the unit ball and hence are extreme points of the unit ball of the second dual but none is an extreme point of the unit ball of the fourth dual. Compare this with the example in [18] where the author exhibits a strictly convex space $X$ where none of the unit vectors of $X$ are extreme points of the unit ball of the second dual.

**Example 2.5.** Let $A$ be the disc algebra on the unit circle $\Gamma$. It is well known that $(C(\Gamma)/A)^* = H_0^1$ is a smooth space (see [9], page 167) and $C(\Gamma)/A$ is an $M$-ideal in its bidual. Thus every unit vector is a weak*-extreme point and as it is an $M$-embedded space which is not reflexive, none of them are extreme points in the unit ball of the fourth dual.

In sharp contrast to this we next give an example of Banach space $X$ having weak*-extreme points that remain weak*-extreme in the unit balls of all the duals of even order of $X$ and are not strongly extreme points.

Recall that (see [9]) a Banach space $X$ is said to be $L$-embedded if with canonical embedding of $X$ in $X^{**}$, we have $X^{**} = X \oplus_1 N$ ($\ell_1$ direct sum). Note that in this case any extreme point of $X_1$ is a weak*-extreme point.

**Example 2.6.** Let $X$ be any infinite dimensional reflexive separable Banach space. It follows from [7] that there is a renorming on $X$ in which there are at most countably many strongly extreme points in the unit ball. Since the unit ball of an infinite dimensional reflexive space has uncountably many extreme points in its unit ball we fix a $x_0 \in \partial_e X_1$ such that it is not strongly extreme. Let $Y$ be an $\ell^1$ direct sum of countably infinitely many copies of $X$. It follows from Proposition IV.1.5 in [9] that $Y$ is a non-reflexive $L$-embedded space. Let $x_0' \in \partial_e Y_1$ have $x_0$ in the first coordinate and zeros elsewhere.

Rao has proved in [22] that any $L$-embedded space $X$ is under the appropriate canonical embedding, an $L$-ideal in all the duals of even order of $X$.

Thus $x_0'$ is a weak*-extreme point of the unit ball of any dual of even order of $Y$ and is clearly not strongly extreme .

We next study the extremal structure of the unit ball of the space of operators with special emphasis on the subspace of compact operators.

Before proving our next result we recall an equivalent formulation of a weak*-extreme point from [7]. $x \in X_1$ is a weak*-extreme point if and only if for sequences $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ in $X_1$, $\frac{x_n+y_n}{2} \to x$ implies $x_n - y_n \to 0$ in the weak topology.

**Proposition 2.7.** *Let $T \in \mathcal{K}(X,Y)_1$ be such that $T^*(y^*)$ is weak*-extreme in $X_1^*$ for all $y^* \in \partial_e Y_1^*$. Then $T$ is a weak*-extreme point.*

**Proof.** Let $\{S_n\}_{n\geq 1}$ and $\{R_n\}_{n\geq 1}$ be sequences in $\mathcal{K}(X,Y)_1$ such that $\frac{S_n+R_n}{2} \to T$. Since for each $y^* \in \partial_e Y_1^*$ the corresponding sequence of adjoint operators evaluated at $y^*$ converges to $T^*(y^*)$, by our hypothesis we get that $S_n^*(y^*) - R_n^*(y^*) \to 0$ in the weak topology. In particular for any $\tau \in \partial_e X_1^{**}$, $\tau((S_n^* - R_n^*)(y^*)) \to 0$. By a result of Ruess and Stegall [24], we know any extreme point of $\mathcal{K}(X,Y)_1^*$ is of the form $\tau \otimes y^*$ ($\tau \otimes y^*(T) = \tau(T^*(y^*))$). Now as remarked in Section 3 in [24], applying the Rainwater theorem we get that $S_n - R_n \to 0$ weakly. Therefore $T$ is weak*-extreme. □

**Remark 2.8.** Similar arguments can be used to show that $f \in C(K,X)_1$ is weak*-extreme if $f(k)$ is weak*-extreme for all $k \in K$. The converse of this statement is also true, see the added note at the end of the paper.

In the following Proposition we extend Theorem 2 in [4] to the space of compact operators.

**Proposition 2.9.** *Let $X$ be any Banach space and let $Y$ be such that $\partial_e Y_1^*$ is weak* closed. Let $T \in \mathcal{K}(X,Y)$ and $T^*(y^*)$ be strongly extreme in $X_1^*$, for all $y^* \in \partial_e Y_1^*$. Then $T$ is a strongly extreme point.*

*If $T \in \mathcal{L}(X,Y)$ satisfies the same hypothesis and if further every extreme point of $\partial_e X_1^*$ is a weak*-denting point then $T$ is a strongly extreme point.*

**Proof.** If $T$ is not strongly extreme, there exists an $\epsilon > 0$ and a sequence $\{T_n\}_{n\geq 1}$ such that $\|T_n\| > \epsilon$ and $\|T \pm T_n\| \leq 1 + \frac{1}{n}$ (see Theorem 2 in [4]). For each $n$ choose unit vectors $x_n$ with $\|T_n(x_n)\| > \epsilon$. Let $y_n^* \in \partial_e Y^*$ and $y_n^*(T_n(x_n)) = T_n^*(y_n^*)(x_n) > \epsilon$. Now using the hypothesis we get a weak* accumulation point $y^* \in \partial_e Y_1^*$ and a subnet of $\{y_n^*\}_{n\geq 1}$ that weak* converges to $y^*$. Since $T$ is a compact operator $T^*$ maps it to a norm convergent net. As $T^*(y^*)$ is a strongly extreme point we get the required contradiction as in the proof of Theorem 2 in [4].

When $T$ is not a compact operator we only get weak* convergence in the last part of the above proof. But if $T^*(y^*)$ is a weak* denting point then any net in the unit ball weak* converging to $T^*(y^*)$, converges in the norm. Thus we again get norm convergence to complete the proof. □

Our next Corollary should be compared with Theorem 2 in [8] where the authors proved that if $F$ is a uniformly rotund space, any isometry in $\mathcal{L}(E,F)$ is a strongly extreme point. We recall that $T$ is a coisometry if $T^*$ is an isometry, also any uniformly rotund space is reflexive.

**Corollary 2.10.** *Let $E^*$ be a reflexive and locally uniformly rotund space and let $F$ be such that the weak* closure of $\partial_e F_1^*$ consists of unit vectors. Let $T \in \mathcal{L}(E,F)$ be a coisometry. Then $T$ is a strongly extreme point.*

**Proof.** Let $f^* \in (\partial_e F_1^*)^{-w^*}$. Since $T^*$ is an isometry, $T^*(f^*)$ is in particular a point of mid-point locally uniform rotundity and hence a strongly extreme point. Also any net from $T^*(F_1^*)$ that converges weakly to $T^*(f^*)$ converges in the norm by local uniform rotundity. Thus the conclusion follows from the proof of the above Proposition. □

We next give a partial converse to Proposition 2.9 when $Y$ is an $L^1$- predual space, that is, $Y^*$ is isometric to $L^1(\mu)$.

**Proposition 2.11.** *Let $Y$ be an $L^1$-predual space and let $T \in \mathcal{K}(X,Y)_1$ be a strongly extreme point. Then for any $y^* \in \partial_e Y_1^*$, $T^*(y^*)$ is strongly extreme in $X_1^*$.*

**Proof.** Since $Y$ is an $L^1$-predual, by a result of Kakutani (see [14], Chapter 6) $Y^{**}$ is canonically isometric to $C(K)$ space and in the canonical identification $\partial_e Y_1^* \subset K$ (extreme points correspond to indicator functions of normalized $\mu$-atoms). Also since $Y$ has the metric approximation property (see [14] page 212), $\mathcal{K}(X,Y)$ can be identified with the injective tensor product $X^* \otimes_\epsilon Y$. We now recall the well-known canonical embedding $X^* \otimes_\epsilon Y \subset X^* \otimes_\epsilon Y^{**} \subset (X^* \otimes_\epsilon Y)^{**}$ (see [5]). Applying the result of Kunen and Rosenthal [13] once again,we get that $T$ is strongly extreme in $(X^* \otimes_\epsilon Y)_1^{**}$. In particular $T$ is a strongly extreme in $(X^* \otimes_\epsilon Y^{**})_1$. We now note that this latter space is identified with $C(K, X^*)$. Thus applying Theorem 2 from [4], as $\partial_e Y_1^* \subset K$ we conclude that $T^*(y^*)$ is strongly extreme in of $X_1^*$. $\square$

## 3. Relationship with Very smooth points

In this section we study the relationship between weak*-extreme points of $X_1^*$ and very smooth points of $X$. A unit vector $x \in X$ is said to be a very smooth point if $x$, under the canonical embedding, is a smooth point of $X^{**}$ (i.e, there is a unique norming functional for $x$ in $X^{***}$). As noted in [6] if $x^* \in X_1^*$ attains its norm at such an $x$ then $x^*$ is an extreme point of $X_1^*$ and also a point of weak*-weak continuity for the identity map on $X_1^*$.

We first note that if an extreme point $x_0^* \in \partial_e X_1^*$ is also a point of weak*-weak continuity for the identity map on $X_1^*$, then it is an weak*-extreme point. For if $x_0^* = \frac{1}{2}\{\Lambda_1 + \Lambda_2\}$ for $\Lambda_i \in X_1^{***}$ then $x_0^* = \Lambda_1/X = \Lambda_2/X$. Since $X_1^*$ is weak* dense in $X_1^{***}$ let $\{x_\alpha^*\} \subset X_1^*$ be a net converging in the weak* topology of $X^{***}$ to $\Lambda_1$. Thus $x_\alpha^* \to \Lambda_1/X$ in the weak* topology of $X^*$. Therefore by the continuity assumption this net also converges in the weak topology. Thus $x_0^* = \Lambda_1$. Hence $x_0^* \in \partial_e X^{***}$.

In [6] the authors gave an example of a Banach space $X$ and an $x^* \in \partial_e X_1^*$ that is a point of weak*-weak continuity of the identity map on $X_1^*$ but is not a point of weak*-weak continuity for the identity map on $X_1^{***}$. Thus these extreme points do not belong to the precise class in the bidual.

A natural question to consider is when do extreme points belong to a better class of extreme points as they pass through the higher ordered duals. Our next example illustrates such a phenomenon.

**Example 3.1.** Let $K$ be a compact set and $k_0 \in K$ be an accumulation point. Since $\chi_{\{k_0\}} \in C(K)^{**}$ it is easy to see that $\delta(k_0) \in \partial_e C(K)_1^*$ is not a point of weak*-weak continuity for the identity map on $C(K)_1^*$. However since $\delta(k_0)$ it is a denting point, it is a weak*-denting point of $C(K)_1^{***}$ and hence is a point of weak*-weak (in fact weak*-norm) continuity for the identity map on $C(K)_1^{***}$.

**Remark 3.2.** If $\Lambda \in \partial_e X_1^{***}$ is a point of weak*- weak continuity for the identity map on $X_1^{***}$ then again by the denseness of $X_1^*$ in $X_1^{***}$ we have that $\Lambda = x^* \in \partial_e X_1^*$. The above example shows that $x^*$ in general need not be a point of weak*- weak continuity of the identity map on $X_1^*$. By taking $X = c_0$, since every extreme point of $X_1^*$ is a weak*-denting point, it is easy to see that $X_1^*$ and $X_1^{***}$ have the same extreme points that are points of weak*-weak continuity for the identity map on the respective unit balls. However since $X^{(4)}$ can be identified with a $C(K)$ space we get from the above example that $\partial_e X_1^{(5)}$ has points of weak*-weak continuity that are no longer points of weak*- weak continuity for the identity map on $X_1^{***}$.

In the following theorem we describe extreme points of $\mathcal{L}(X,Y)_1^*$ that are points of weak*-weak continuity for the identity map on $\mathcal{L}(X,Y)_1^*$ under some additional hypothesis on $X$ or $Y$ involving the compact metric approximation property (see [16], page 94). Recall that for any $x^{**} \in X^{**}$ and $y^* \in Y^*$ by $x^{**} \otimes y^*$ we denote the functional defined on $\mathcal{L}(X,Y)$ by $(x^{**} \otimes y^*)(T) = x^{**}(T^*(y^*))$.

**Theorem 3.3.** *Suppose $X^*$ or $Y$ has the compact metric approximation property. Let $\tau \in \partial_e \mathcal{L}(X,Y)_1^*$ be a point of weak*-weak continuity for the identity map on $\mathcal{L}(X,Y)_1^*$. Then $\tau = x \otimes y^* \in \partial_e \mathcal{K}(X,Y)_1^*$ where $x \in \partial_e X^{**}$ and $y^* \in \partial_e Y_1^*$ are points of weak*-weak continuity in the respective unit balls.*

*If further $x$ or $y^*$ is a weak*-denting point then $x \otimes y^*$ is a point of weak*-weak continuity of the identity map on $\mathcal{K}(X,Y)_1^*$.*

**Proof.** Since $X^*$ or $Y$ has the compact metric approximation property it can be deduced from the results of [12] that there is a projection $P : \mathcal{L}(X,Y)^* \to \mathcal{L}(X,Y)^*$ of norm one such that $ker(P) = \mathcal{K}(X,Y)^\perp$ and $P(x \otimes y^*) = x \otimes y^*$ for all $x \in X$ and $y^* \in Y^*$. Thus $\mathcal{K}(X,Y) \subset \mathcal{L}(X,Y)$ satisfies condition $(a)$ of Theorem 2.2. Since $\{x \otimes y^* : x \in X_1 \text{ and } y^* \in Y_1^*\}$ is weak* dense in $\mathcal{L}(X,Y)_1^*$, let $x_\alpha \otimes y_\alpha^* \to \tau$ in the weak* topology. Since $\tau$ is a point of continuity we have that this net also converges in the weak topology. $P$ being identity on the net we have $P(\tau) = \tau$. As $\mathcal{K}(X,Y)^*$ is isometric to the range of $P$ we get that $\tau \in \partial_e \mathcal{K}(X,Y)_1^*$. Any extreme point of the latter set is of the form (see [24]) $x^{**} \otimes y^*$ for some $x^{**} \in \partial_e X_1^{**}$ and $y^* \in \partial_e Y_1^*$, we have $\tau = x^{**} \otimes y^*$. Again since $\tau$ is a point of continuity, using the remark preceding this theorem it is easy to see that $x^{**} = x$ and $y^*$ are points of weak*-weak continuity of the identity map on the respective unit balls.

Now suppose $x$ or $y^*$ is a weak*-denting point. Since the hypothesis $(a)$ of Theorem 2.2 is satisfied we have :

$$\mathcal{K}(X,Y) \subset \mathcal{L}(X,Y) \subset \mathcal{K}(X,Y)^{**} \subset L(X,Y)^{**}$$

under the canonical embedding. As any point of weak*-weak continuity has a unique norm preserving extension to the bidual (see [9] Lemma III.2.14), we have that $\tau$ has unique norm preserving extension to $\mathcal{L}(X,Y)^{**}$. Now since we are assuming that $x$ or $y^*$ is a weak*-denting point, it follows from Theorem 3.7 in [17] that $\tau$ also has unique norm preserving extension from $\mathcal{K}(X,Y)$ to $\mathcal{L}(X,Y)$. Thus applying Lemma III. 2. 14 in [9] again, we have $\tau$ is indeed a point of weak*-weak continuity of the identity map on $\mathcal{K}(X,Y)_1^*$. $\square$

We recall from [25] that $X$ is said to be a very smooth space if all the unit vectors are very smooth points. It was shown in [25] that if $X$ is a very smooth space then $X^*$ has the RNP. Our next result is local version this result.

**Proposition 3.4.** *If every equivalent norm on $X$ has a very smooth point then $X^*$ has the RNP.*

**Proof.** Suppose $X^*$ fails the RNP. From Corollary 1.4 in [23] we get an equivalent norm on $X^*$ whose unit ball $B$ has no weak*- extreme points. Thus $B$ contains no extreme points of its weak* closure in $X^{***}$. Let $B_1 = B^-$, where the closure is taken in the weak*-topology of $X^*$. Thus there is an equivalent norm on $X$ whose dual unit ball is $B_1$. Hence from our hypothesis and the remarks made above, it follows that $B_1$ contains a weak*-extreme point say $x_0^*$, which is also a point of weak*-weak continuity for the identity map on $B_1$. Therefore $x_0^* \in B$. Clearly $x_0^*$ is a weak*- extreme point of $B$. This contradicts our choice of $B$. □

If $x^*$ is a very smooth point then since $X_1$ is (under the canonical embedding) weak*-dense in $X_1^{**}$, it follows from our remarks that $x^*$ attains its norm at a weak*-extreme point of $X_1$. Sullivan also showed in [25] that if $X^*$ is very smooth then $X$ is reflexive.

We next give an example of a non-reflexive space $X$ in which every norm attaining unit vector of $X^*$ is a very smooth point.

**Example 3.5.** Let $X$ be any infinite dimensional non-reflexive Banach space with a separable dual. It follows from Proposition 11 in [1] that there is an equivalent norm $|.|$ on $X$ with $(X, |.|)^{**}$ rotund, weak and norm sequential convergence coincide for unit vectors in the new norm. Thus if a unit vector $x^*$ of $(X, |.|)^*$ attains its norm, by rotundity on $X^{**}$ and since weak and norm sequential convergence coincide on the surface, we conclude that $x^*$ is a very smooth point (see [6]).

We conclude the paper with the following questions.

**Question 3.6.** Is there a notion of an extreme point that remains the same in all the duals of even order and coincides with the usual notion of an extreme point when the space is reflexive? In particular can one completely describe extreme points that remain extreme in the unit ball of all the duals of even order of $X$, in terms of $X$ alone?

**Question 3.7.** For every positive integer $n$, can one construct a strictly convex space $X$ all of whose unit vectors are extreme points of the unit ball of $X^{(2n)}$ but fail to be extreme in the unit ball of $X^{(2n+2)}$?

Note added on 14-07-2003 : In a recent work K. Jarosz and the second author (Weak*-extreme points of injective tensor product spaces, Contemporary Mathematics Vol 238, Amer. Math. Soc., 2003) have showed that any weak*-extreme point of $C(K, X)_1$ takes weak*-extremal values.

## References


[1] P. Bandyopadhyay, Da Huang, Bor-Luh Lin, S. L. Troyanski: Some generalizations of local uniform rotundity, J. Math. Anal. Appl 252 (2000) 906–916.

[2] R. D. Bourgin: Geometric Aspects of Convex Sets with the Radon-Nikodým Property, Lecture Notes in Mathematics 993, Springer, Berlin (1983).

[3] R. Deville, G. Godefroy, V. Zizler: Smoothness and Renormings in Banach Spaces, Pitman Monographs and Surveys 64, London (1993).

[4] P. N. Dowling, Z. Hu, M. A. Smith: Extremal structure of the unit ball of $C(K, X)$, Contemp. Math. 144 (1993) 81–85.

[5] G. Emmanuele: Remarks on weak compactness of operators defined on injective tensor products, Proc. Amer. Math. Soc. 116 (1992) 473–476.

[6] G. Godefroy, V. Indumathi: Norm-weak upper semicontinuity of the duality and preduality mappings, Set-Valued Anal. 10 (2002) 317–330.

[7] B. V. Godun, Bor-Luh Lin, S. L. Troyanski: On the strongly extreme points of convex bodies in separable Banach spaces, Proc. Amer. Math. Soc. 114 (1992) 673–675.

[8] R. Grząślewicz, P. Scherwentke: On strongly extreme and denting points in $\mathcal{L}(H)$, Math. Japonica 41 (1995) 283–284.

[9] P. Harmand, D. Werner, W. Werner: $M$-Ideals in Banach Spaces and Banach Algebras, Lecture Notes in Mathematics 1547, Springer, Berlin (1993).

[10] J. Hennefeld: Compact extremal operators, Illinois J. Math. 21 (1977) 61–65.

[11] Z. Hu, M. A. Smith: On the extremal structure of the unit balls of Banach spaces of weakly continuous functions and their duals, Trans. Amer. Math. Soc. 349 (1997) 1901–1918.

[12] J. Johnson: Remarks on Banach spaces of compact operators, J. Funct. Anal. 32 (1979) 304–311.

[13] K. Kunen, H. P. Rosenthal: Martingale proofs of some geometrical results in Banach space theory, Pacific J. Math. 100 (1982) 153–175.

[14] H. E. Lacey: The Isometric Theory of Classical Banach Spaces, Grundlehren der mathematischen Wissenschaften 208, Springer, Berlin (1974).

[15] J. Lindenstrauss: Extension of Compact Operators, Mem. Amer. Math. Soc. 48 (1964).

[16] J. Lindenstrauss, L. Tzafriri: Classical Banach Spaces I, Springer, Berlin (1973).

[17] Å. Lima, E. Oja, T. S. S. R. K. Rao, D. Werner: Geometry of operator spaces, Michigan Math. J. 41 (1994) 473–490.

[18] P. Morris: Disappearance of extreme points, Proc. Amer. Math. Soc. 88 (1983) 244–246.

[19] R. R. Phelps: Extreme points of polar convex sets, Proc. Amer. Math. Soc. 12 (1961) 291–296.

[20] T. S. S. R. K. Rao: Denting and strongly extreme points in the unit ball of spaces of operators, Proc. Indian Acad. Sci. (Math. Sci.) 109 (1999) 75–85.

[21] T. S. S. R. K. Rao: On ideals in Banach spaces, Rocky Mountain J. Math. 31 (2001) 595–609.

[22] T. S. S. R. K. Rao: Geometry of higher duals of a Banach space, Illinois J. Math. 45 (2001) 1389–92.

[23] H. Rosenthal: On the Non-Norm Attaining Functionals and the Equivalence of the Weak*-**KMP** with the **RNP**, Longhorn Notes (1985-86).

[24] W. Ruess, C. Stegall: Extreme points in duals of operator spaces, Math. Ann. 261 (1982) 535–546.

[25] F. Sullivan: Geometric properties determined by the higher duals of Banach spaces, Illinois J. Math. 21 (1977) 351–331.