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Continuity and Maximality Properties of Pseudomonotone Operators

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Given a Banach space X, a multivalued operator $T : X \to 2^{X^*}$ is called pseudomonotone (in Karamardian's sense) if for all (x, x^*) and (y, y^*) in its graph, $\langle x^*, y - x\rangle \geq 0$ implies $\langle y^*, y - x\rangle \geq 0$. We define an equivalence relation on the set of pseudomonotone operators. Based on this relation, we define a notion of "D-maximality" and show that the Clarke subdifferential of a locally Lipschitz pseudoconvex function is D-maximal pseudomonotone. We generalize some well-known results on upper semicontinuity and generic single-valuedness of monotone operators by showing that, under suitable assumptions, a pseudomonotone operator has an equivalent operator that is upper semicontinuous, generically single-valued etc.

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1. Introduction

Pseudomonotone operators, as introduced by Karamardian [10], are defined by making use of the order relation in $\mathbb{R}$, without any reference to topological properties. This is in sharp contrast to pseudomonotonicity in Brezis' sense[1] [1]. Another feature of these operators is that they are closely related to generalized convexity, just like the relation of monotone operators to convex functions; in fact, the subdifferential of a locally Lipschitz function is pseudomonotone if and only if the function is pseudoconvex [12, 13].

Pseudomonotone operators have been the subject of intense study during the last decade. Directions of research include the finding of criteria for pseudomonotonicity of differentiable single-valued operators [2, 6] and the pseudomonotone variational inequality problem [14, 5, 7]. However, in contrast to the theory of monotone operators, which is very rich, results on the structure of pseudomonotone operators are rare. For instance, it is known that the subdifferential of a proper, lower semicontinuous convex function is not only monotone, but also maximal monotone. Under some rather weak assumptions, monotone operators are upper semicontinuous in the interior of their domain; also, they are generically single-valued. It is a widely held belief that pseudomonotone operators do not have such properties and that, in particular, maximality is not a relevant property in generalized monotonicity, and in particular for pseudomonotone operators. Let us ten-

[1] In order to underline the distinction between pseudomonotone operators in the Karamardian and in the Brezis sense, some authors use the terms *order pseudomonotone* and *topologically pseudomonotone*, respectively.

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tatively define maximal pseudomonotone operators as those pseudomonotone operators that admit no pseudomonotone extension other than themselves. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function such that $f'(x) = 0$ for $x \in [a,b]$, $f'(x) > 0$ for $x > b$ and $f'(x) < 0$ for $x < a$. This function is pseudoconvex; its subdifferential is the single-valued pseudomonotone operator $Tx = \{f'(x)\}$ and it is not maximal pseudomonotone in the above sense since the following operator is a pseudomonotone extension:

$$\widehat{T}x = \begin{cases}]-\infty, 0[\,, & x < a \\ ]-\infty, 0]\,, & x = a \\ \{0\}\,, & x \in (a,b) \\ [0, +\infty[\,, & x = b \\ ]0, +\infty[\,, & x > b. \end{cases}$$

Note that $\widehat{T}$ has no pseudomonotone extension, $\widehat{T}x$ has the same zeros as f' (i.e., $f'(x) = 0 \iff 0 \in \widehat{T}x$), and whenever x is not a zero, $\widehat{T}x$ consists of all positive multiples of $f'(x)$. Let us note also that given any pseudomonotone operator T, say single-valued, the operator $Sx = f(x)\,Tx$ where f is any positive function is also pseudomonotone. In fact, it can also be seen that the two operators have the same solutions in a variational inequality problem (see next section for details).

Based on these observations, we will define an equivalence relation on the set of pseudomonotone operators. The idea underlying the paper is that for any pseudomonotone operator T, under assumptions similar to those of the monotone case, there might exist an equivalent operator S with better properties; for instance, we will show that S may be chosen to be upper semicontinuous, or generically single-valued. We will also define "D-maximal pseudomonotonicity" by means of this equivalence, and show that the subdifferential of a locally Lipschitz, pseudoconvex function is D-maximal pseudomonotone.

We begin by fixing the notation and recalling some definitions. Let X be a Banach space and X^* its dual. We denote by $\langle x^*, x\rangle$ the duality pairing of $x \in X$ and $x^* \in X^*$. For $C \subseteq X^*$ and $x \in X$ we set $\langle C, x\rangle = \{\langle x^*, x\rangle : x^* \in C\}$ and write $\langle C, x\rangle \geq a$ if $\langle x^*, x\rangle \geq a$ for all $x^* \in C$. We denote by $\mathrm{co}\,(K)$ the convex hull of a subset K of X and by $\mathrm{core}\,(K)$ its algebraic interior; K is called radially open if $K = \mathrm{core}\,(K)$. Given $\varepsilon > 0$, $B(x,\varepsilon)$ (resp. $B[x,\varepsilon]$) is the open (resp. closed) ball of radius ε around $x \in X$. For any $K \subseteq X^*$, we set $\mathbb{R}_+K = \cup_{t\geq 0} tK$ and $\mathbb{R}_{++}K = \cup_{t>0} tK$.

Given a locally Lipschitz function $f : X \to \mathbb{R} \cup \{+\infty\}$, the Clarke subdifferential $\partial^o f$ is defined at any $x \in \mathrm{dom}\,(f)$ by

$$\partial^o f(x) = \{x^* \in X^* : \langle x^*, d\rangle \leq f^o(x; d)\}$$

where

$$f^o(x; d) = \limsup_{t\searrow 0,\, y\to x} \frac{f(y+td) - f(y)}{t}.$$

The locally Lipschitz function f is called pseudoconvex, if for every $x \in \mathrm{dom}\,(f)$ and $x^* \in \partial^o f(x)$, the following implication holds:

$$\langle x^*, y - x\rangle \geq 0 \Rightarrow f(y) \geq f(x).$$

Given an operator $T : X \to 2^{X^*}$ we denote by $D(T)$ its domain and by $Gr(T)$ its graph. We call T upper sign-continuous at $x \in D(T)$ if for all $v \in X$, the following implication holds:

$$\forall t \in (0,1), \inf_{x_t^* \in T(x+tv)} \langle x_t^*, v \rangle \geq 0 \Rightarrow \sup_{x^* \in Tx} \langle x^*, v \rangle \geq 0.$$

This is a very weak kind of continuity. For instance, if the restriction of T on all straight lines through x is upper semicontinuous with respect to the w* topology in X^*, then T is upper sign-continuous at x. Also, any positive function $f : \mathbb{R} \to \mathbb{R}$ is upper sign-continuous everywhere. Likewise, we call T lower sign-continuous at $x \in D(T)$ if for all $v \in X$, the following implication holds:

$$\forall t \in (0,1), \inf_{x_t^* \in T(x+tv)} \langle x_t^*, v \rangle \geq 0 \Rightarrow \inf_{x^* \in Tx} \langle x^*, v \rangle \geq 0.$$

Again, if the restriction of T on all straight lines through x is lower semicontinuous with respect to the w* topology in X^*, then T is lower sign-continuous at x.

According to Karamardian [10], T is called pseudomonotone if for all (x^*, x) and (y, y^*) in $Gr(T)$ the following implication holds:

$$\langle x^*, y - x \rangle \geq 0 \Rightarrow \langle y^*, y - x \rangle \geq 0$$

or equivalently

$$\langle x^*, y - x \rangle > 0 \Rightarrow \langle y^*, y - x \rangle > 0.$$

Given an operator T, we define the set of zeros of T by

$$Z_T = \{x \in X : 0 \in Tx\}. \tag{1}$$

2. Equivalent operators and maximal pseudomonotonicity

We will introduce an equivalence relation in the class of pseudomonotone operators. Given two pseudomonotone operators T and S, we write $T \sim S$ if the following conditions hold:

(a) $D(T) = D(S)$,

(b) $Z_T = Z_S$,

(c) for all $x \in X \backslash Z_T$, $\mathbb{R}_{++}Tx = \mathbb{R}_{++}Sx$.

This equivalence is related to variational inequalities. Given a convex subset K of X, we recall that $x \in K$ is a solution of the variational inequality problem VIP(T, K) if there exists $x^* \in Tx$ such that $\langle x^*, y - x \rangle \geq 0$ for all $y \in K$. Note that in case $0 \in Tx$, x is a solution of the VIP(T, K). It is obvious that whenever $T \sim S$, x is a solution of VIP(T, K) if and only if x is a solution of VIP(S, K).

A pseudomonotone operator T will be called D-maximal pseudomonotone if there exists an equivalent pseudomonotone operator S which has no pseudomonotone extension with the same domain, apart from itself. This means that if S' is a pseudomonotone operator such that $D(S') = D(S)$ and for all $x \in X, Sx \subseteq S'x$, then $S = S'$.

Let us now explore the equivalence relation. Given a pseudomonotone operator T, we can construct an operator $\widehat{T}$ which is the maximum of its equivalent class with respect to graph inclusion. To do so, let us first remark that if we combine pseudomonotonicity with the relation $\langle 0, y-x\rangle \geq 0$, we deduce that the following implication holds:

$$x \in Z_T \Rightarrow \forall y \in D(T), \langle Ty, y-x\rangle \geq 0. \tag{2}$$

For any $x \in Z_T$, set

$$\begin{aligned} L_{T,x} &= \{y \in X : \exists y^* \in Ty, \langle y^*, y-x\rangle = 0\} \\ &= \{y \in X : \exists y^* \in Ty, \langle y^*, y-x\rangle \leq 0\}. \end{aligned}$$

The second equality is a consequence of (2). Let

$$N_{L_{T,x}} = \{x^* \in X^* : \langle x^*, y-x\rangle \leq 0, \forall y \in L_{T,x}\}$$

be the normal cone of $L_{T,x}$ at x. Define the operator $\widehat{T}$ by

$$\widehat{T}x = \begin{cases} N_{L_{T,x}}, & \text{if } x \in Z_T \\ \mathbb{R}_{++}Tx, & \text{if } x \in D(T)\backslash Z_T \\ \varnothing, & \text{if } x \notin D(T). \end{cases}$$

Proposition 2.1. *Let $T : X \to 2^{X^*}$ be pseudomonotone. Then:*

(i) $\widehat{T}$ *is pseudomonotone.*

(ii) $\widehat{T} \sim T$.

(iii) If $S \sim T$ then $Gr(S) \subseteq Gr(\widehat{T})$.

(iv) $S \sim T$ if and only if $\widehat{S} = \widehat{T}$.

Proof. (i) Obviously, $D(\widehat{T}) = D(T)$. Thus, we have to show that if $x, y \in D(T)$, $x^* \in \widehat{T}x$ and $\langle x^*, y-x\rangle \geq 0$, then $\left\langle \widehat{T}y, y-x\right\rangle \geq 0$.

There exists $x_1^* \in Tx$ such that $\langle x_1^*, y-x\rangle \geq 0$ (if $x \in Z_T$ we may take $x_1^* = 0$; if $x \notin Z_T$ we may take a positive multiple of x^*). Thus, by pseudomonotonicity of T, $\langle Ty, y-x\rangle \geq 0$.

We consider two cases. If $y \notin Z_T$ this obviously implies that $\left\langle \widehat{T}y, y-x\right\rangle \geq 0$ and we are done. If $y \in Z_T$ then from $\langle x_1^*, y-x\rangle \geq 0$ we deduce that $x \in L_{T,y}$. It follows that for all $y^* \in \widehat{T}y = N_{L_{T,y}}(y)$, $\langle y^*, x-y\rangle \leq 0$ thus we have again $\left\langle \widehat{T}y, y-x\right\rangle \geq 0$.

(ii) This is an obvious consequence of the definition of $\widehat{T}$.

(iii) If $S \sim T$ then from the definitions we deduce that $D(S) = D(T) = D(\widehat{T})$. For all $x \in D(T)\backslash Z_S$, $Sx \subseteq \mathbb{R}_{++}Sx = \mathbb{R}_{++}Tx = \widehat{T}x$. Now let $x \in Z_S$. We intend to prove that $L_{S,x} \subseteq L_{T,x}$. Choose any $y \in L_{S,x}$. If $y \in Z_S$ then $y \in Z_T$ thus $0 \in Ty$ and obviously $y \in L_{T,x}$. If $y \notin Z_S$ then $\mathbb{R}_{++}Sy = \mathbb{R}_{++}Ty$. Since $y \in L_{S,x}$, there exists $y^* \in Sy$ such that $\langle y^*, y-x\rangle = 0$. There exists also some $y_1^* \in Ty$ which is a positive multiple of y^*; this implies again that $y \in L_{T,x}$. It follows that in all cases, $L_{S,x} \subseteq L_{T,x}$. By symmetry, $L_{S,x} = L_{T,x}$. Thus, $\widehat{S}x = \widehat{T}x$ and $Sx \subseteq \widehat{T}x$.

(iv) If $S \sim T$ then $\widehat{S} = \widehat{T}$ since by (iii) they are both maxima of the corresponding equivalence class. Conversely, if $\widehat{S} = \widehat{T}$ then $S \sim \widehat{S} = \widehat{T} \sim T$. $\square$

As a consequence of Proposition 2.1, an operator T is D-maximal pseudomonotone if and only if $\widehat{T}$ has no pseudomonotone extension with the same domain, apart from itself. The following lemma gives a practical way to show that an operator is $D-$maximal pseudomonotone.

Lemma 2.2. *Let T be a pseudomonotone operator. Suppose that for any $(x, x^*) \in (D(T) \setminus Z_T) \times X^*$ such that $\{(x, x^*)\} \cup Gr(T)$ is the graph of a pseudomonotone operator, one has $x^* \in \mathbb{R}_{++}Tx$. Then T is D-maximal pseudomonotone. The converse is also true, provided that $D(T)$ is convex.*

Proof. Since $D(T) = D(\widehat{T})$, in order to show that T is $D-$maximal pseudomonotone, we have to show that for any $(x, x^*) \in D(T) \times X^*$ such that $\{(x, x^*)\} \cup Gr(\widehat{T})$ is the graph of a pseudomonotone operator T_1, one has $x^* \in \widehat{T}x$. We consider two cases. If $x \in Z_T$ then for all $y \in L_{T,x}$ there exists $y^* \in Ty$ such that $\langle y^*, x - y\rangle \geq 0$. By pseudomonotonicity of T_1, $\langle x^*, x - y\rangle \geq 0$. Hence $x^* \in N_{L_{T,x}} = \widehat{T}x$. Now suppose that $x \notin Z_T$. By the assumption, we have again $x^* \in \mathbb{R}_{++}Tx = \widehat{T}x$. Thus, $\widehat{T}$ is D-maximal pseudomonotone.

To show the converse, suppose that T is maximal pseudomonotone with convex domain, and that for some $(x, x^*) \in (D(T) \setminus Z_T) \times X^*$, the operator S with graph $\{(x, x^*)\} \cup Gr(T)$ is pseudomonotone. We will show that the operator S' with graph $\{(x, x^*)\} \cup Gr(\widehat{T})$ is pseudomonotone. We have to show that if $(y, y^*) \in Gr(\widehat{T})$ then the following implications hold:

$$\langle x^*, y - x\rangle \geq 0 \Rightarrow \langle y^*, y - x\rangle \geq 0 \tag{3}$$

$$\langle y^*, x - y\rangle \geq 0 \Rightarrow \langle x^*, x - y\rangle \geq 0. \tag{4}$$

If $\langle x^*, y - x\rangle \geq 0$ then by pseudomonotonicity of S, $\langle Ty, y - x\rangle \geq 0$. If $y \notin Z_T$ then $y^* \in \widehat{T}y = \mathbb{R}_{++}Ty$, hence $\langle y^*, y - x\rangle \geq 0$. If $y \in Z_T$, set $z = (x + y)/2$ and choose some $z^* \in Tz$. Since S is pseudomonotone, $\langle x^*, y - x\rangle \geq 0$ implies $\langle z^*, z - x\rangle \geq 0$, hence $\langle z^*, y - z\rangle \geq 0$ and $z \in L_{T,y}$. Using $y^* \in \widehat{T}y = N_{L_{T,y}}$, we infer that $\langle y^*, y - z\rangle \geq 0$; hence we have again $\langle y^*, y - x\rangle \geq 0$ and (3) holds.

Suppose that $\langle y^*, x - y\rangle \geq 0$. If $y \notin Z_T$ then $y^* = \lambda y_1^*$ for some $\lambda > 0$ and $y_1^* \in Ty$; by pseudomonotonicity of S we infer that $\langle x^*, x - y\rangle \geq 0$. If $y \in Z_T$ then by (2) we have again $\langle x^*, x - y\rangle \geq 0$. Hence (4) holds.

Consequently, S' is pseudomonotone. By assumption, $\widehat{T}$ has no pseudomonotone extension apart from itself; hence $S' = \widehat{T}$ thus $x^* \in \widehat{T}x = \mathbb{R}_{++}Tx$. $\square$

The proof of the preceding lemma reveals that the converse is also true if, instead of convexity of $D(T)$, we assume the following: for every $x \in Z_T$ and $y \in D(T)$, there is $\varepsilon > 0$ such that $[x, x + \varepsilon(y - x)] \subseteq D(T)$. Such an assumption can be further relaxed, but not completely ommited, as shown by the following example: $X = \mathbb{R}$, $D(T) = \{0, 1\}$, $T(0) = \{-1\}$, $T(1) = \{0\}$. Then $\widehat{T}$ is given by $\widehat{T}(0) =]-\infty, 0[$, $\widehat{T}(1) = \mathbb{R}$ and admits no pseudomonotone extension with the same domain, thus T is D-maximal pseudomonotone. Also, $G(T) \cup \{(0,0)\}$ is the graph of a pseudomonotone operator, but $0 \notin \widehat{T}(0)$.

By means of Zorn's Lemma, it is easy to show that every pseudomonotone operator T has a D-maximal pseudomonotone extension. Of course, this extension needs not be equivalent to T. If it is, then T itself is D-maximal pseudomonotone.

D-maximal pseudomonotone operators have some nice properties. The following proposition gives some useful information on the structure of the set of zeros:

Proposition 2.3. *Let T be $D-$maximal pseudomonotone. Then Z_T is weakly closed in $D(T)$. If in addition $D(T)$ is convex, then Z_T is convex.*

Proof. We first prove the following claim: if $z \in D(T)$ is such that $\left\langle \widehat{T}y, y-z \right\rangle \geq 0$ for all $y \in D(T)$, then $z \in Z_T$. Indeed, the assumption implies that for every $y^* \in \widehat{T}y$ the following implication is true:

$$\langle 0, y-z \rangle \geq 0 \Rightarrow \langle y^*, y-z \rangle \geq 0.$$

On the other hand we also have tautologically

$$\langle y^*, z-y \rangle \geq 0 \Rightarrow \langle 0, z-y \rangle \geq 0.$$

Hence, the operator with graph $Gr(\widehat{T}) \cup \{(z,0)\}$ is pseudomonotone. Since T is D-maximal pseudomonotone, this implies that $0 \in \widehat{T}z$, i.e., $z \in Z_{\widehat{T}} = Z_T$ and the claim is true.

Let (z_i) be a net in Z_T, weakly converging to $z \in D(T)$. By applying (2) to the pseudomonotone operator $\widehat{T}$, we infer that for every $y \in D(T)$ and for all i's, $\left\langle \widehat{T}y, y-z_i \right\rangle \geq 0$ holds; It follows that $\left\langle \widehat{T}y, y-z \right\rangle \geq 0$ and, according to the claim, $z \in Z_T$. Hence Z_T is weakly closed in $D(T)$. The fact that Z_T is convex whenever $D(T)$ is convex can be proved in the same way. □

Given a pseudomonotone operator T, define the operator $\operatorname{co} T$ by $\operatorname{co} Tx = \operatorname{co}(Tx)$ for all $x \in X$. Luc and Jeyakumar [9] have noted that $\operatorname{co} T$ is also pseudomonotone. If we apply this to $\widehat{T}$ we infer that $\operatorname{co} \widehat{T}$ is a pseudomonotone extension of $\widehat{T}$. Hence, if T is $D-$maximal pseudomonotone, then $\widehat{T} = \operatorname{co} \widehat{T}$. We arrive to the following conclusion:

Proposition 2.4. *If T is $D-$maximal pseudomonotone, then $\widehat{T}x$ is convex for all $x \in D(T)$.*

In contrast to what one might expect having in mind the behavior of maximal monotone operators, $\widehat{T}x$ is not necessarily closed, since it may be a cone from which 0 has been extracted. In fact, even $\widehat{T}x \cup \{0\}$ is not necessarily closed. As an example, consider the operator $T : \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ with domain $D(T) = \mathbb{R} \times \{0\}$ defined by $Tx = \mathbb{R}_{++} \times \mathbb{R}$ for every $x \in D(T)$. It is easy to see that T admits no pseudomonotone extension with the same domain, hence it is D-maximal pseudomonotone and $T = \widehat{T}$. It is obvious that $Tx \cup \{0\}$ is not closed. This is mainly due to the "thinness" of $D(T)$; as we will see later, under suitable assumptions this situation does not occur.

3. D-maximal pseudomonotonicity and continuity.

The first proposition gives a criterion for D-maximal pseudomonotonicity.

Proposition 3.1. *Let T be pseudomonotone and upper sign-continuous, $D(T)$ be radially open and Tx be w*-compact and convex for all $x \in D(T)$. Then T is $D-$maximal pseudomonotone.*

Proof. If T is not $D-$maximal pseudomonotone, then by Lemma 2.2 there exists $(x, x^*) \in (D(T) \backslash Z) \times X^*$ such that the operator T_1 with graph $GrT \cup (x, x^*)$ is pseudomonotone and $x^* \notin \mathbb{R}_{++}Tx$. Hence $Tx \cap \mathbb{R}_+ x^* = \varnothing$. By the separation Theorem, there exists $b \in X$ such that $\langle \mathbb{R}_+ x^*, b\rangle > \langle Tx, b\rangle$. Since $\mathbb{R}_+ x^*$ is a cone, this implies that

$$\langle x^*, b\rangle \geq 0 > \langle Tx, b\rangle . \tag{5}$$

For $t > 0$ sufficiently small, $\langle x^*, x + tb - x\rangle \geq 0$ and $x + tb \in D(T)$; using that T_1 is pseudomonotone, we infer that $\langle T(x + tb), b\rangle \geq 0$. By upper sign-continuity, $\sup_{y^* \in Tx} \langle y^*, b\rangle \geq 0$. However, by the w*-compactness of Tx, (5) implies that $\sup_{y^* \in Tx} \langle y^*, b\rangle < 0$, a contradiction. Hence T is $D-$maximal pseudomonotone. ☐

Corollary 3.2. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a pseudoconvex, locally Lipschitz function. Then $\partial^o f$ is a $D-$maximal pseudomonotone operator.*

Proof. Since f is locally Lipschitz, it follows that $\mathrm{dom}(f)$ is open. Also, it is known that $D(\partial^o f) = \mathrm{dom}(f)$, $\partial^o f(x)$ is w*-compact and convex for all $x \in D(\partial^o f)$, and $\partial^o f$ is upper semicontinuous in the strong-to-w* topology [3]. In particular, $\partial^o f$ is upper sign-continuous. In addition, $\partial^o f$ is pseudomonotone [12, 13]. Hence, by Proposition 3.1, $\partial^o f$ is $D-$maximal. ☐

Remark. Let us call T "maximal pseudomonotone" if $\widehat{T}$ has no pseudomonotone extension. In contrast to what holds for convex functions, $\partial^o f$ is not maximal pseudomonotone in general. For instance, if f is any locally Lipschitz pseudoconvex function with domain $]0, 1[$, then the operator T defined by

$$Tx = \begin{cases} \{-1\}, & x \leq 0 \\ \widehat{\partial^o f}(x), & x \in]0, 1[\\ \{1\}, & x \geq 1 \end{cases}$$

is a pseudomonotone extension of $\widehat{\partial^o f}$, thus $\partial^o f$ is not maximal pseudomonotone. This is the reason why we focus our study on $D-$maximal pseudomonotone operators, rather than maximal pseudomonotone ones.

We will prove that pseudomonotone operators have a property which reminds of the so-called M-property of operators [8]. We first need two lemmas that establish a small but necessary refinement of the argument used in Proposition 3.1.

Lemma 3.3. *Let $x^* \in X^* \backslash \{0\}$ and $C \subseteq X^*$ be nonempty, w*-compact and convex. If $C \cap \mathbb{R}_+ x^* = \varnothing$, then there exists $b \in X$ such that $\langle x^*, b\rangle > 0 > \langle C, b\rangle$.*

Proof. Set $K = \mathbb{R}_+ C$. Then K is a closed convex cone and $x^* \notin K$. Let $d > 0$ be the distance of x from K. Then $D := \mathbb{R}_+ B[x^*, d/2]$ is a closed convex cone and it can be easily seen that $D \cap C = \varnothing$. By the separation Theorem, there exists $b \in X$ such that $\langle D, b\rangle > \langle C, b\rangle$. Since D is a cone, it follows that $\langle D, b\rangle \geq 0 > \langle C, b\rangle$. Finally, using $x^* \in intD$ we infer that $\langle x^*, b\rangle > 0 > \langle C, b\rangle$. □

Lemma 3.4. *Assumptions as in Proposition 3.1. Let* $(x, x^*) \in (D(T) \setminus Z) \times (X^* \setminus \{0\})$ *be such that the following implication holds for all* $(y, y^*) \in Gr(T)$:

$$\langle x^*, y - x\rangle > 0 \Rightarrow \langle y^*, y - x\rangle \geq 0. \qquad (6)$$

Then $x^* \in \widehat{T}x$.

Proof. Suppose that $x^* \notin \widehat{T}x$. Since $0 \notin Tx$, this implies that $Tx \cap \mathbb{R}_+ x^* = \varnothing$. Using the previous lemma we deduce that there exists $b \in X$ such that

$$\langle x^*, b\rangle > 0 > \langle Tx, b\rangle .$$

For all $t > 0$ sufficiently small, $x + tb \in D(T)$. Since $\langle x^*, x + tb - x\rangle > 0$, from (6) we infer that $\langle T(x + tb), b\rangle \geq 0$. We conclude as in the proof of Proposition 3.1. □

Proposition 3.5. *Assumptions as in Proposition 3.1. Let* $(x_i, x_i^*) \in GrT$ *and* $a_i > 0$, $i \in I$ *be nets such that* $w^*\text{-}\lim a_i x_i^* = x^* \neq 0$, $w\text{-}\lim x_i = x \in D(T) \setminus Z$, *and* $\limsup \langle a_i x_i^*, x_i\rangle \leq \langle x^*, x\rangle$. *Then* $x^* \in \widehat{T}x$.

Proof. Let $y \in D(T)$ be such that $\langle x^*, y - x\rangle > 0$. Using the assumptions we infer that $\lim \langle a_i x_i^*, y\rangle > \limsup \langle a_i x_i^*, x_i\rangle$, hence for sufficiently large i, $a_i \langle x_i^*, y - x_i\rangle > 0$. Since T is pseudomonotone, $\langle y^*, y - x_i\rangle > 0$ for all $y^* \in Ty$. Thus, $\langle y^*, y - x\rangle \geq 0$. The result follows from Lemma 3.4. □

If T satisfies the assumptions of Proposition 3.1 then we know that it is D-maximal pseudomonotone; hence, according to Corollary 2.4, $\widehat{T}$ has convex values. Actually, $\widehat{T}x \cup \{0\}$ is a closed convex cone:

Corollary 3.6. *Assumptions as in Proposition 3.1. Then* $\widehat{T}x \cup \{0\}$ *is* w^**-closed for all* $x \in D(T)$.

Proof. If $x \in Z_T$ then it is obvious that $\widehat{T}x$ is a w*-closed convex cone. Suppose that $x \in D(T) \setminus Z_T$. Let $x_i^* \in \widehat{T}x$, $i \in I$ be a net such that $\text{w}^*\text{-}\lim x_i^* = x^* \neq 0$. Then $x_i^* = a_i y_i^*$ for some $a_i > 0$ and $y_i^* \in Tx$. If we apply Proposition 3.5 to the net (x, y_i^*) we infer that $x^* \in \widehat{T}x$. Hence $\widehat{T}x \cup \{0\}$ is w*-closed. □

It is known that a maximal monotone operator is upper semicontinuous on the interior of its domain, with respect to the w*- topology in X^*. However, it is not true that a $D-$maximal pseudomonotone operator has an equivalent operator which is even upper sign-continuous. For instance, on $\mathbb{R}^2$ take

$$T(x, y) = \begin{cases} (0, 1) & \text{if } y \geq 0 \\ (1, 1) & \text{if } y < 0. \end{cases}$$

Then T is D-maximal pseudomonotone but has no equivalent upper sign-continuous operator. Nevertheless, the following theorem shows that pseudomonotonicity "helps" continuity:

Theorem 3.7. *Let $T : \mathbb{R}^k \to 2^{\mathbb{R}^k}$ be pseudomonotone, upper sign-continuous on $D(T)$, with compact convex values. Suppose that $D(T)$ is an open convex set. Then there exists an equivalent pseudomonotone operator T_1 with compact convex values which is upper semicontinuous on $D(T)$.*

Proof. Define an operator $T_0 : D(T) \setminus Z_T \to 2^{\mathbb{R}^k}$ by

$$T_0 x = \{x^*/\|x^*\| : x^* \in Tx\}, x \in D(T) \setminus Z_T.$$

We show that T_0 is upper semicontinuous (as a map defined only on $D(T) \setminus Z_T$): If $\left(x_n, \frac{x_n^*}{\|x_n^*\|}\right) \to (x, x^*)$ with $x_n \in D(T) \setminus Z_T$, $x_n^* \in Tx_n$ and $x \in D(T) \setminus Z_T$, then $\|x^*\| = 1$ and by Proposition 3.5 applied to the original operator T, $x^* \in \widehat{T}x$. Hence, $x^* \in T_0 x$ and T_0 is closed (as a map on $D(T) \setminus Z_T$). Since obviously T_0 is locally bounded, it follows that T_0 is upper semicontinuous.

Define an operator $T_1 : \mathbb{R}^k \to 2^{\mathbb{R}^k}$ with domain $D(T_1) = D(T)$ as follows. If $Z_T \neq \varnothing$, set $\rho(x) = d(x, Z_T)$, the distance of $x \in \mathbb{R}^k$ from the set Z_T; otherwise, set $\rho(x) = 1$ for all $x \in \mathbb{R}^k$. Define

$$T_1 x = \begin{cases} \rho(x) \operatorname{co} T_0 x\ , & \text{if } x \in D(T) \setminus Z_T \\ \{0\}, & \text{if } x \in Z_T \end{cases} \tag{7}$$

(thus, $T_1 = \operatorname{co} T_0$ whenever $Z_T \neq \varnothing$). For all $x \in D(T)$ the set $T_1 x$ is compact and convex. Also, Propositions 3.1 and 2.3 entail that Z_T is convex and closed in $D(T)$; hence the function $\rho(x)$ is strictly positive on $D(T) \setminus Z_T$ and continuous on $\mathbb{R}^k$.

For every $x \in D(T) \setminus Z_T$ and $x^* \in T_1 x$, there exist $x_i^* \in Tx$ and $\lambda_i > 0$, $i = 1, 2, \ldots n$, with $\sum_i \lambda_1 = 1$, such that

$$x^* = \rho(x) \sum_i \lambda_i \frac{x_i^*}{\|x_i^*\|}.$$

If we set $\lambda_i' = \lambda_i/\|x_i^*\|, t_i = \lambda_i'/\sum_i \lambda_i'$ and $a = \rho(x) \sum_i \lambda_i'$, then we get $x^* = a \sum_i t_i x_i^* \in \mathbb{R}_{++} Tx$. Hence, $T_1 x \subseteq \mathbb{R}_{++} Tx$. Since we also have $Tx \subseteq \mathbb{R}_{++} T_1 x$, it follows that $T_1 \sim T$.

Since T_0 is upper semicontinuous, the operator $\operatorname{co} T_0 : D(T) \setminus Z_T \to 2^{\mathbb{R}^k}$ is also upper semicontinuous [8, Proposition 2.42], hence closed. Consequently, the restriction of T_1 on $D(T) \setminus Z_T$ is closed, thus it is upper semicontinuous. Finally, for any $x \in Z_T$ and any $\varepsilon > 0$, if $y \in B(x, \varepsilon)$ then $T_1 y \subseteq B(x, \varepsilon)$ i.e., T_1 is upper semicontinuous at x. Therefore, T_1 is upper semicontinuous on $D(T)$. □

If T is single-valued, then T_1 defined by (7) is also single-valued. If T is radially continuous (i.e., its restriction on straight line segments of $D(T)$ is continuous) then it is certainly upper sign-continuous. We deduce the following result:

Corollary 3.8. *Let $T : \mathbb{R}^k \to \mathbb{R}^k$ be a single-valued, pseudomonotone operator with open, convex domain. If T is radially continuous , then there exists a function $f : D(T) \to \mathbb{R}_{++}$ and a continuous operator $T_1 : D(T) \to \mathbb{R}^k$ such that $Tx = f(x) T_1 x$ for all $x \in D(T)$.*

In the study of variational inequalities with single-valued pseudomonotone operators, a set of assumptions which is frequently used to show existence of solutions for $\mathrm{VIP}(T, K)$ is the following:

1. $K \subseteq D(T)$ is weakly compact,
2. T is pseudomonotone,
3. The restriction of T on finite-dimensional subspaces [4] (or: on one-dimensional subspaces [14]) is continuous with respect to the w* topology on X^*.

Actually, when the continuity assumption is that the restriction of T on finite-dimensional subspaces is continuous, the following idea is used (see for instance [4]): for any finite-dimensional subspace N that intersects K, if E is the projection on N and E^* the adjoint operator of E, then E^*TE (i.e., the projection of T on E) is continuous; by the Hartman-Stampachia Theorem, $\mathrm{VIP}(E^*TE, N \cap K)$ has a solution. Then, a limit process ensures the existence of a solution for $\mathrm{VIP}(T, K)$. The above corollary shows that whenever $K \subseteq \operatorname{int} D(T)$, the same proof applies if we suppose that the projection of T on *one*-dimensional subspaces is continuous; for in this case, we can replace E^*TE on each finite-dimensional subspace by an equivalent continuous operator. Analogous considerations are valid for multivalued operators.

As with monotone operators, lower semicontinuity implies a kind of single-valuedness for pseudomonotone operators:

Proposition 3.9. *Let T be a pseudomonotone operator, which is lower sign-continuous at $x \in \operatorname{core} D(T)$. Then $Tx = \{0\}$ or there exists $x^* \in X^*$ such that $Tx \subseteq \mathbb{R}_{++}x^*$.*

Proof. Suppose that the conclusion is not true; then there exist $x^* \in Tx$, $y^* \in Tx \setminus \{0\}$ such that there exists no $\lambda > 0$ with $x^* = \lambda y^*$. Hence there exists $z \in X$ such that $\langle x^*, z\rangle \geq 0 > \langle y^*, z\rangle$. In particular, $\inf_{u^* \in Tx} \langle u^*, z\rangle < 0$. Since T is lower sign-continuous and $x \in \operatorname{core} D(T)$, we can find $t > 0$ and $w^* \in T(x + tz)$ such that $\langle w^*, z\rangle < 0$. This implies that $\langle w^*, x - (x + tz)\rangle > 0$. Using pseudomonotonicity we infer that $\langle x^*, x - (x + tz)\rangle > 0$, i.e., $\langle x^*, z\rangle < 0$, a contradiction. $\square$

We remark that the same proof applies to show that an analogous result holds for quasimonotone operators.

By applying the same proof as for monotone operators [11], we obtain:

Corollary 3.10. *Let X be separable. If $T : X \to 2^{X^*}$ is pseudomonotone, upper semicontinuous on $D(T)$ with w*-compact values and $\operatorname{int} D(T) \neq \varnothing$, then the set $C = \{x \in X : Tx$ is not contained in $\{0\}$ or $\mathbb{R}_{++}x^*$ for some $x^*\}$ is of the first category.*

Proof. The operator $T : D(T) \to 2^{X^*}$ is nonempty valued, upper semicontinuous with w*-compact values. Also, since X is separable, there exists a metrizable topology τ on X^* such that τ is weaker than w*. Hence, according to [11, Corollary 1.4], the set $\{x \in X : T$ is not lower semicontinuous at $x\}$ is of the first category. Applying Proposition 3.9 we get the desired result. $\square$

An immediate consequence of the corollary is that, under its assumptions, there exists an equivalent operator T_1 which is generically single-valued. For instance, this conclusion is true under the assumptions of Theorem 3.7.

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