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Regularity of Optimal Convex Shapes

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We consider shape optimisation problems in the class of convex sets. Assuming that the shape functional satisfies a Lipschitz like property with respect to a distance issued from the γ-convergence, we prove that the minimiser has the boundary of class C^1. In particular, we prove that large classes of functionals depending on the eigenvalues of the Dirichlet Laplacian satisfy this property. The key point of the paper is the understanding of the asymptotic behaviour of the γ-convergence near the "angular" points of the convex set.

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1. Introduction

We consider shape optimisation problems of the form

$$\min_{A \in \mathcal{K}(D)} F(A) + \alpha |A|, \tag{1}$$

$$\mathcal{K}(D) = \{A \subseteq D : A \text{ open and convex}\},$$

D is a fixed bounded open set, α is a fixed positive constant and $|A|$ denotes the measure of A. Although in general a shape optimisation problem may fail to have a solution, as soon as the class of admissible domains is restricted to open convex subsets of D, the existence of a solution comes rather easily under mild assumptions on the cost functional F. We refer the reader to [4] for a detailed survey of the existence question in shape optimisation. The purpose of this paper is to investigate the regularity of those solutions, if the cost functional depends on the domain A via the eigenvalues of the Dirichlet Laplacian on A, or the solution of the Dirichlet problem for the Laplacian. The typical examples we consider here are

$$F_1(A) = \sum_{j=1}^{k} |\lambda_j(A) - \alpha_j| \qquad \text{and} \qquad F_2(A) = \int_D |u_{A,f} - g| dx, \tag{2}$$

where $\alpha_1, ...\alpha_k \in I\!R$ and $f, g \in L^2(D)$ are fixed. By $\lambda_1(A), ..., \lambda_k(A)$ we denote the first k eigenvalues of the Dirichlet Laplacian counted with their multiplicities and by $u_{A,f}$ the solution of the Laplace equation with Dirichlet boundary conditions (see Section 2).

The regularity of the minimising domains is, in general, difficult to treat. Sometimes, a shape optimisation problem takes the form of a usual free boundary problem. We refer the

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reader to the pioneering result of Alt and Caffarelli [1] (see also [13]), where the regularity of the free boundary is proved for a large class of energy type functionals in H_0^1-spaces. See also [7], [14] for questions regarding the regularity of optimal shapes minimising energies; we also refer to [2], [20] for optimisation problems where the unknown shape is a level set. As a general remark, in all those papers the regularity is obtained for optimal shapes minimising *energy type functionals*. Shape functionals like F_1 or F_2, defined above, are implicitly excluded. The very interesting part in all these results is that no regularity of the free boundary is *a priori* assumed. In general, one gets either the regularity of the free boundary or the regularity of the state, the free boundary being a level set of the state.

We restrict ourself to the class of convex shapes. This simplifies a lot the problem since a minimal regularity of the optimum (i.e. convex) is known *a priori*. Nevertheless, the necessary optimality conditions are difficult to write for two reasons. First, one can not perform the shape derivative to get an extra condition on the free boundary if the cost functional is not enough smooth. Second, since the class of admissible domains consists *only* of convex sets, the convexity constraint may be lost upon a small variation of the boundary, therefore getting optimality conditions would require arguments like in [6].

In the first part of the paper we prove the C^1 regularity of the domains minimising shape functionals which satisfy a Lipschitz like property with respect to a distance related to the γ-convergence (see Section 2 and [5] for the definition of the γ-convergence). There are several distances which induce a metric space structure on the family of open sets, and the choice of the distance has to be done in order to achieve the following equilibrium: strong enough to obtain the abstract regularity result, but weak enough in order to apply the result in concrete examples. The key point of the proof is the understanding of the asymptotic behaviour of the γ-convergence near the "angular" points of the boundary.

In the second part of the paper, we prove that all the eigenvalues of the Dirichlet Laplacian (more precisely the functionals $A \to 1/\lambda_k(A)$) satisfy the Lipschitz like property. As a consequence, a large class of problems of type (1) have smooth solutions (all results concerning the eigenvalues are valid in any dimension of the space). For functionals depending on the state, if $f \in L^\infty(D)$, the same kind of results hold in any dimension of space. If $f \in L^2(D) \setminus L^\infty(D)$, finer estimates of the solution $u_{A,f}$ near an angular point are needed; in this case, we can prove the regularity only in dimension 2.

We refer the reader to [6] where the authors discuss variational problems on convex functions. Although the problem and the arguments are different, the regularity results are of the same type: a solution is of class C^1. In [16], the authors perform a detailed study of the solution of (1) for $F(A) = \lambda_2(A)$. Moreover, they prove that the optimal boundary is not of class C^2. For this functional, the proof of the C^1 regularity is obtained in a direct way, using its shape differentiability. In this paper we do not assume the shape differentiability of the cost functional F. In concrete examples, the shape differentiability may not hold for two reasons: either the simplicity of the eigenvalues at the optimum can not be proved (this is a necessary condition for the shape differentiability see [17, 22]), or the functional $A \to F(\lambda_1(A), ..., \lambda_k(A))$ is not itself differentiable as a real function depending on $(\lambda_1, ..\lambda_k)$ (like F_1, for example).

2. Some basic facts: γ-convergence, eigenvalues and asymptotic behaviour near angular points

The γ-convergence. Let $f \in L^2(D)$ be fixed and denote for every A open subset of D by $u_{A,f}$ the weak solution of the following equation

$$\begin{cases} -\Delta u_{A,f} &= f \text{ in } A \\ u_{A,f} &\in H_0^1(A). \end{cases} \tag{3}$$

By extension with zero on $D \setminus A$, the function $u_{A,f}$ can be seen as an element of $H_0^1(D)$.

Definition 2.1. A sequence $(A_n)_n$ of open subsets of D γ-converges to A if for every $f \in L^2(D)$ we have that $u_{A_n,f} \to u_{A,f}$ strongly in $H_0^1(D)$.

Following [4, 23], the fact that A_n γ-converges to A is equivalent to each of the following

A.1 $u_{A_n,1} \to u_{A,1}$ strongly in $H_0^1(D)$

A.2 $R_{A_n} \to R_A$ in $\mathcal{L}(L^2(D))$, where R_A is the resolvent operator for problem (3), i.e. $R_A : L^2(D) \to L^2(D)$, $R_A(f) = u_{A,f}$.

Assertion A.1 allows us to introduce the following distance which induces a metric structure in the family of open sets. The topology generated by this distance has the same convergent sequences as the γ-convergence. For simplicity, and to respect the traditional notation, we set $w_A := u_{A,1}$, the solution of equation (3) for $f \equiv 1$. For every A_1, A_2 we set

$$d_\gamma(A_1, A_2) = \int_D |w_{A_1} - w_{A_2}| dx.$$

The choice of this distance, instead for example of $|R_{A_1} - R_{A_2}|_{\mathcal{L}(L^2(D))}$, will become more clear in Section 4. The main benefit is that, when computing the distance between two arbitrary sets, we evaluate the resolvent operators on a fixed function, i.e. $f \equiv 1$. Moreover, the functions $\{w_A\}_{A \in \mathcal{K}(D)}$ are uniformly bounded in $L^\infty(D)$. We refer the reader to the recent paper of Savaré and Schimperna [21] for a detailed study of the mapping $A \mapsto u_{A,f}$ in the family of uniformly Lipschitz domains.

Eigenvalues. Assertion A.2 gives information about the behaviour of the eigenvalues of the Dirichlet Laplacian with respect to the γ-convergence. We recall the following two inequalities (see [9, Corollaries 3 and 4, pages 1089-1090]) in a more general setting. Let $T_1, T_2 : L^2(D) \to L^2(D)$ be two linear operators which are bounded and compact. Then for every $k, m, n \in \mathbb{N}$

$$|\mu_k(T_1) - \mu_k(T_2)| \le |T_1 - T_2|_{\mathcal{L}(L^2(D))} \tag{4}$$

$$\mu_{n+m-1}(T_1 T_2) \le \mu_n(T_1)\mu_m(T_2), \tag{5}$$

where $\mu_k(T)$ are the singular values of T defined by

$$\mu_k(T) = \min_{V_{k-1} \subseteq L^2(D)} \max_{\substack{\varphi \perp V_{k-1} \\ \varphi \neq 0}} \frac{|T\varphi|}{|\varphi|},$$

V_{k-1} denoting a vector space of dimension $k-1$. If T is moreover positive and self adjoint, then $\mu_k(T)$ is the k-th eigenvalue of T, the index k taking into account the multiplicities.

Observe that for the resolvent operator R_A, we have $\mu_k(R_A)^{-1} = \lambda_k(A)$, $\lambda_k(A)$ being the k-th eigenvalue of the Dirichlet Laplacian on the open set A.

For our purposes, at some point, we will apply (5) for the product between a resolvent operator and a projector. Since the projector we consider is of finite rank, it is also compact, hence (5) has full sense.

We recall the following result from [8, Example 2.1.8].

Lemma 2.2. *Let A be a bounded open set in $I\!\!R^N$. Let λ be an eigenvalue of the Dirichlet Laplacian on Ω and ϕ a normalised eigenfunction corresponding to λ. Then*

$$|\phi|_{L^\infty(A)} \leq 3\Big(\frac{8\pi}{\lambda}\Big)^{\frac{N}{4}}.$$

This result is needed in the proof of the Lipschitz character of the eigenvalues with respect to the distance d_γ.

Asymptotic behaviour near the angular points. Let $A \subseteq I\!\!R^N$ be a bounded, open and convex set. By misuse of notation, we call a point $x \in \partial A$ "angular", if the dimension of the normal cone N_x at A in the point x is at least 2. A convex set is of class C^1 if it has no angular points on the boundary. We are interested in the asymptotic behaviour of the solution $u_{A,f}$ in a neighbourhood of an angular point of the boundary. We refer the reader to the books of Grisvard [11, 12] for a detailed study of this question.

In this paper, we denote by $\epsilon : (0,1) \to I\!\!R$ a generic function such that $\lim_{\varepsilon\to 0} \epsilon(\varepsilon) = 0$.

Lemma 2.3. *We set the dimension of the space $N = 2$. Let the origin be an angular point of a convex set $A \in \mathcal{K}(D)$ and let $f \in L^2(D)$. Then $u_{A,f}(x) = |x|\epsilon(|x|)$.*

Note that the decreasing rate depends on f (see Remark 4.3 of Section 4).

Proof. Using the maximum principle, it is enough to prove Lemma 2.3 for a bounded sector, with a vertex at the origin and an angle ω, $\omega \in (0,\pi)$. Let us denote $\hat\omega$ this sector. Using the result of Kondratiev [19] we have

$$\int_A |D^2 u_{A,f}|^2 + \frac{|\nabla u_{A,f}|^2}{|x|^2} + \frac{u_{A,f}^2}{|x|^4} dx \leq C_1 \int_A |f|^2 dx,$$

where C_1 is a constant depending only on A. On the other hand, the Sobolev embedding theorem applied in $K_{0,1,2} \cap \hat\omega$ gives that the injection $H^2(K_{0,1,2} \cap \hat\omega) \hookrightarrow L^\infty(K_{0,1,2} \cap \hat\omega)$ is continuous. Here $K_{0,a,b}$ denotes the open ring $K_{0,a,b} = B_{0,b} \setminus \overline{B}_{0,a}$. Consequently, for every $v \in H^2(K_{0,1,2} \cap \hat\omega)$

$$|v|_\infty^2 \leq C_2 \int_{K_{0,1,2}\cap\hat\omega} |D^2 v|^2 + |\nabla v|^2 + v^2 dx. \tag{6}$$

Let us denote, for every $r > 0$, $u_r(x) = v(\frac{x}{r})$. Performing the change of variable in (6) we have

$$|u_r|^2_{L^\infty(K_{0,r,2r}\cap\hat\omega)} \leq C_2 r^2 \int_{K_{0,r,2r}\cap\hat\omega} |D^2 u_r|^2 + \frac{|\nabla u_r|^2}{|r|^2} + \frac{u_r^2}{|r|^4} dx.$$

Taking $u_r = u_{A,f}|_{K_{0,r,2r}\cap\hat{\omega}}$ and using the fact that

$$\int_{K_{0,r,2r}\cap\hat{\omega}} |D^2 u_{A,f}|^2 + \frac{|\nabla u_{A,f}|^2}{|x|^2} + \frac{u_{A,f}^2}{|x|^4} dx \to 0, \text{ as } r \to 0,$$

we conclude the proof. □

In the particular case when $f \in L^\infty(A)$, the decreasing rate is stronger. Following [11, 12], one can prove that $|u_{A,f}(x)| \le c|x|^{1+\delta}$ for some $\delta > 0$. Nevertheless, for our purposes, Lemma 2.3 is enough.

In an arbitrary dimension of space $N \ge 2$ we are able to prove a similar estimate only for $f \in L^\infty(A)$. Let A be convex, such that the origin is an angular point. Suppose moreover that the dimension of $N_x \cap \{x_3 = .. = x_N = 0\}$ is two and that (up to a change of the system of coordinates)

$$A \cap \{x_3 = .. = x_N = 0\} \subseteq \{(r\cos\theta, r\sin\theta) : r > 0, \theta \in (0,\omega)\},$$

where $\omega < \pi$ is fixed. We define the following set

$$C = \{(r\cos\theta, r\sin\theta, x_3, .., x_N) : \theta \in (0,\omega), r \in (0,d), x_i \in (-d,d), i = 3,..,N\},$$

where d is the diameter of A.

Lemma 2.4. *Let $f \in L^\infty(A)$. Under the previous hypotheses on A, we have*

$$|u_{A,f}(x)| \le (x_1^2 + x_2^2)^{\frac{1}{2}} \epsilon((x_1^2 + x_2^2)^{\frac{1}{2}}).$$

Proof. From the maximum principle, it is enough to consider only the case $f \equiv 1$ and the convex C defined above. We observe that the function $C \ni \mathrm{x} \mapsto u(x_1, x_2, .., x_N) = w_S(x_1, x_2)$ satisfies

$$-\Delta u = 1, \text{ in } C \text{ and } u \ge 0 \text{ on } \partial C.$$

Here $S = \{(r\cos\theta, r\sin\theta) : \theta \in (0,\omega), r \in (0,d)\} \subseteq I\!R^2$ and $w_S := u_{S,1} \in H_0^1(S)$. Consequently, from the maximum principle we get that

$$w_A(x) \le u(x) = w(x_1, x_2).$$

We conclude using Lemma 2.3. □

Existence of optimal shapes. We briefly recall that the existence of a solution for problem (1) can be proved using the direct method of the calculus of variations. The following result is a direct consequence of the closure of $\mathcal{K}(D)$ for the H^c-convergence and of the stability property (in the sense of Keldysh-Hedberg [15]) of convex sets. See [4] for details.

Proposition 2.5. *The family $\mathcal{K}(D)$ is γ-compact.*

One can check that the Lebesgue measure is, in general, γ-lower semi-continuous but continuous in $\mathcal{K}(D)$. As a consequence, as soon as the functional F is γ-l.s.c., problem (1) has at least one solution. This follows using the direct method of the calculus of variations.

If $D = I\!R^N$, one has to use a concentration-compactness argument like in [3] or [18, Kawohl, Chapter 1] in order to discuss existence of solutions for (1).

3. Regularity of optimal convex shapes

3.1. Functionals depending on the eigenvalues of the Laplacian

Definition 3.1. Let

$$F : \mathcal{K}(D) \to \overline{I\!R}.$$

We say that F is γ-Lip, if for every $A \in \mathcal{K}(D)$ there exists a constant $L = L(A)$ such that for every $B \in \mathcal{K}(D)$, $B \subseteq A$

$$|F(A) - F(B)| \leq L d_\gamma(A, B). \tag{7}$$

Notice that this is a sort of local-Lipschitz property only for sub domains.

The main result of the paper is formulated as follows.

Theorem 3.2. *Let $F : \mathcal{K}(D) \to I\!R$ be a γ-Lip functional. Then all solutions of problem* (1) *are of class C^1.*

Proof. Let A be a solution $A \neq \emptyset$. Suppose for contradiction that A is not of class C^1. This implies that A has an angular point x_0. Up to a translation, we can assume that x_0 is the origin of the system of coordinates. Suppose moreover that the dimension of $N_x \cap \{x_3 = .. = x_N = 0\}$ is two and that

$$A \cap \{x_3 = .. = x_N = 0\} \subseteq \{(r\cos\theta, r\sin\theta) : r > 0, \theta \in (0, \omega)\},$$

where $\omega < \pi$ is fixed.

Let $A_\varepsilon = conv(A \setminus C_\varepsilon)$ the convex hull of the set $A \setminus \overline{C}_\varepsilon$, where C_ε denotes the cylinder $\{(x_1, .., x_N) : x_1^2 + x_2^2 \leq \varepsilon\}$. For every $k = 3, .., N$ let us denote

$$g_k(\varepsilon) = \mathcal{H}^1(P_k(A \cap C_\varepsilon)) > 0,$$

where $\mathcal{H}^1$ denotes the one dimensional Hausdorff measure and P_k the orthogonal projection on the k-th axis. From the convexity of A, there exists a constant C (depending only on the geometry of A near x_0) such that

$$|A| - |A_\varepsilon| \geq C\varepsilon^2 g_3(\varepsilon) \cdot .. \cdot g_N(\varepsilon).$$

Within this notation we implicitly cover the case $N = 2$, as soon as we think to ε^2 in the previous formula, as being equivalent to $g_1(\varepsilon)g_2(\varepsilon)$. The value of the constant C could be evaluated more precisely by computing the measure of the greatest polyhedron with vertex at the origin and which is contained in $A \setminus A_\varepsilon$.

The main idea is to prove that, as soon as F is γ-Lip, the asymptotic behaviour of $|F(A) - F(A_\varepsilon)|$ when $\varepsilon \to 0$ is of the form

$$|F(A) - F(A_\varepsilon)| = \varepsilon^2 g_3(\varepsilon) \cdot .. \cdot g_N(\varepsilon)\epsilon(\varepsilon), \tag{8}$$

where, as usual, $\epsilon(\varepsilon) \to 0$ for $\varepsilon \to 0$. If we achieve this goal, then we can prove that A can not be the optimal set, in contradiction with our initial assumption. Indeed, we would have that

$$F(A_\varepsilon) + \alpha|A_\varepsilon| \leq F(A) + \alpha|A| + \varepsilon^2 g_3(\varepsilon) \cdot .. \cdot g_N(\varepsilon)(\epsilon(\varepsilon) - \alpha C).$$

Since for ε small $\epsilon(\varepsilon) - \alpha C$ is negative, we deduce that A can not be optimal.

To prove (8), we relay on the γ-Lip property of F which is assumed by hypothesis, and prove that

$$d_\gamma(A, A_\varepsilon) = \varepsilon^2 g_3(\varepsilon) \cdot ... \cdot g_N(\varepsilon)\epsilon(\varepsilon). \tag{9}$$

We have

$$d_\gamma(A, A_\varepsilon) = \int_A |w_A - w_{A_\varepsilon}| dx = \int_A w_A - w_{A_\varepsilon} dx,$$

since from the maximum principle and the fact that $A_\varepsilon \subseteq A$, we have $w_A \geq w_{A_\varepsilon}$ a.e. Since w_A and w_{A_ε} solve equation (3) for $f \equiv 1$ on A and A_ε respectively, we write

$$\begin{aligned} \int_A w_A - w_{A_\varepsilon} dx = \\ &= 2[\int_A \frac{1}{2}|\nabla w_{A_\varepsilon}|^2 - w_{A_\varepsilon} dx - \int_A \frac{1}{2}|\nabla w_A|^2 - w_A dx] \\ &= 2\Big[\min_{u \in H_0^1(A_\varepsilon)} \frac{1}{2}\int_A |\nabla u|^2 - u dx - \int_A \frac{1}{2}|\nabla w_A|^2 - w_A dx\Big]. \end{aligned} \tag{10}$$

Let us denote $\varphi \in C_c^\infty(B_{0,2}, [0,1])$ a function such that $\varphi(x) = 1$ on $B_{0,1}$, and φ_ε the mollifier defined by

$$\varphi_\varepsilon(x) = \varphi\Big(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}, \frac{x_3}{g_3(\varepsilon)}, .., \frac{x_N}{g_N(\varepsilon)}\Big).$$

The function $w_A(1 - \varphi_\varepsilon)$ belongs to $H_0^1(A_\varepsilon)$ hence can be taken as test function in (10).

Consequently we have

$$d_\gamma(A, A_\varepsilon) \leq 2[\int_A \frac{1}{2}|\nabla w_A(1 - \varphi_\varepsilon)|^2 - w_A(1 - \varphi_\varepsilon) dx - \int_A \frac{1}{2}|\nabla w_A|^2 - w_A dx].$$

Using the fact that w_A solves (3), we write $\int_A w_A \varphi_\varepsilon dx = \int_A \nabla w_A \nabla (w_A \varphi_\varepsilon) dx$. Easy computation gives

$$d_\gamma(A, A_\varepsilon) \leq \int_A |\nabla (w_A \varphi_\varepsilon)|^2 dx \leq 2\int_A w_A^2 |\nabla \varphi_\varepsilon|^2 dx + 2\int_A \varphi_\varepsilon^2 |\nabla w_A|^2 dx. \tag{11}$$

We study each term of the last relation separately. Following Lemma 2.4, we have that

$$|w_A(x)| = \varepsilon \epsilon(\varepsilon), \tag{12}$$

where $\varepsilon = (x_1^2 + x_2^2)^{1/2}$. Consequently we have

$$\begin{aligned} \int_A w_A^2 |\nabla \varphi_\varepsilon|^2 dx &= \int_{A \cap K_{0,\varepsilon,2\varepsilon}} w_A^2 |\nabla \varphi_\varepsilon|^2 dx \\ &\leq \varepsilon^2 \epsilon(\varepsilon) \int_{A \cap K_{0,\varepsilon,2\varepsilon}} |\nabla \varphi_\varepsilon|^2 dx \end{aligned}$$

$$\leq \varepsilon^2 \epsilon(\varepsilon) \int_{B_{0,2}} \varepsilon^2 g_3(\varepsilon)..g_N(\varepsilon) \Big[\frac{1}{\varepsilon^2}\Big(\frac{\partial \varphi}{\partial x_1}\Big)^2 + \frac{1}{\varepsilon^2}\Big(\frac{\partial \varphi}{\partial x_2}\Big)^2 + \frac{1}{g_3^2(\varepsilon)}\Big(\frac{\partial \varphi}{\partial x_3}\Big)^2 + .. + \frac{1}{g_N^2(\varepsilon)}\Big(\frac{\partial \varphi}{\partial x_N}\Big)^2\Big] dx$$

$$= \varepsilon^2 g_3(\varepsilon)..g_N(\varepsilon)\epsilon(\varepsilon).$$

Indeed, the last equality holds because all the derivatives of φ are bounded (from the choice of φ) and for every k, the function $\frac{\varepsilon}{g_k(\varepsilon)}$ is bounded from the convexity of A. For the second term in (11) we have

$$\begin{aligned}\int_A \varphi_\varepsilon^2 |\nabla w_A|^2 dx &= \int_{B_{0,2\varepsilon}} \varphi_\varepsilon^2 |\nabla w_A|^2 dx \\ &= -\int_{B_{0,2\varepsilon}} \operatorname{div}(\varphi_\varepsilon^2 \nabla w_A) w_A dx \\ &= -\int_{B_{0,2\varepsilon}} \nabla \varphi_\varepsilon^2 \nabla w_A w_A dx - \int_{B_{0,2\varepsilon}} \varphi_\varepsilon^2 \Delta w_A w_A dx \\ &= -\int_{B_{0,2\varepsilon}} \nabla \varphi_\varepsilon^2 \nabla w_A w_A dx + \int_{B_{0,2\varepsilon}} \varphi_\varepsilon^2 w_A dx.\end{aligned}$$

We have

$$I = \int_{B_{0,2\varepsilon}} \nabla \varphi_\varepsilon^2 \nabla w_A w_A dx = -\int_{B_{0,2\varepsilon}} \Delta \varphi_\varepsilon^2 w_A^2 dx - I,$$

hence

$$\int_A \varphi_\varepsilon^2 |\nabla w_A|^2 dx = \int_{B_{0,2\varepsilon}} \varphi_\varepsilon^2 w_A dx + \frac{1}{2}\int_{B_{0,2\varepsilon}} \Delta \varphi_\varepsilon^2 w_A^2 dx.$$

Using again the estimation of Lemma 2.4 for w_A and using the same scheme we get

$$\int_A \varphi_\varepsilon^2 |\nabla w_A|^2 dx = \varepsilon^2 g_3(\varepsilon)..g_N(\varepsilon)\epsilon(\varepsilon),$$

hence relation (9) holds. □

A weaker version of the γ-Lip property is still enough to get the conclusion of Theorem 3.2. Let $A \in \mathcal{K}(D)$ and $x \in \partial A$ an angular point; in the sequel, the set A_ε is constructed as in the proof of Theorem 3.2.

Corollary 3.3. *Let $F : \mathcal{K}(D) \to I\!R$ be a functional such that for every $A \in \mathcal{K}(D)$ and for every angular point $x \in \partial A$ there exists a constant $L = L(A, x)$ and $\delta > 0$ such that $\forall \varepsilon \in (0, \delta)$ we have*

$$|F(A) - F(A_\varepsilon)| \le L d_\gamma(A, A_\varepsilon). \tag{13}$$

Then all solutions of problem (1) are of class C^1.

Note that if F is γ-Lip, and $A \in \mathcal{K}(D)$ is such that $F(A) \neq 0$, then $\frac{1}{F}$ satisfies (13) i.e.

$$\text{for } \varepsilon \in (0, \delta') \text{ we have } \left|\frac{1}{F(A)} - \frac{1}{F(A_\varepsilon)}\right| \le L'(A) d_\gamma(A, A_\varepsilon), \tag{14}$$

with the constant $L'(A) = |F(A)|(|F(A)| + 1)$. Here δ' is chosen such that $|F(A_\varepsilon)| \le |F(A)| + 1$.

This property is intended to be applied for the eigenvalues of the Dirichlet Laplacian. We will prove that $A \mapsto \frac{1}{\lambda_k(A)}$ is γ-Lip, hence $A \mapsto \lambda_k(A)$ satisfies (14). The mapping

$A \mapsto \lambda_k(A)$ is not itself γ-Lip, since one can take in definition (7) a sequence of sets such that $d_\gamma(B_\varepsilon, \emptyset) \to 0$, therefore $\lambda_k(B_\varepsilon) \to +\infty$.

In a first step we intend to consider shape functionals like in relation (2) or, more general,

$$F(A) = F(\lambda_1(A), ..., \lambda_k(A)). \tag{15}$$

Functionals depending on the state are discussed in the next paragraph.

First, we prove that for every $k \in \mathbb{N}$, the mappings $A \mapsto \frac{1}{\lambda_k(A)}$ are γ-Lip.

Theorem 3.4. *Let $A \subseteq D$ be a fixed open set. For every $k \in \mathbb{N}$, there exists a constant $c_k(A)$ depending only on A such that for every $j \leq k$ and for every open set $B \subseteq A$ we have*

$$\Big|\frac{1}{\lambda_j(A)} - \frac{1}{\lambda_j(B)}\Big| \leq c_k(A) d_\gamma(A, B). \tag{16}$$

Proof. Observe that $\mu_k(R_A) = \frac{1}{\lambda_k(A)}$, hence one could try to apply inequality (4) in order to prove (16). Unfortunately, we are not able to prove that $|R_A - R_B|_{\mathcal{L}(L^2(D))} \leq c d_\gamma(A, B)$. This would give straight forwardly a uniform constant c in (16) which does not depend on k. One can only prove that (see [3]) that $|R_A - R_B|_{\mathcal{L}(L^2(D))} \leq c d_\gamma(A, B)^\alpha$, for some $\alpha < 1$.

Let us fix $k \in \mathbb{N}$. We consider $V_k \subseteq L^2(D)$ the linear space generated by the first k eigenfunctions of the Dirichlet Laplacian on A. The space V_k is, a priori, a finite dimensional subspace of $H_0^1(A)$, but extending all the functions with zero on $D \setminus A$, we can see it as a subspace of $L^2(D)$. We denote

$$T_k^A = P_k \circ R_A \circ P_k,$$

$$T_k^B = P_k \circ R_B \circ P_k,$$

where $P_k : L^2(D) \to V_k$ is the orthogonal projector on V_k. Then we have the following.

Lemma 3.5. *For every $j = 1, .., k$ the following hold*

$$\mu_j(T_k^A) = \frac{1}{\lambda_j(A)} \tag{17}$$

$$\mu_j(T_k^B) \leq \frac{1}{\lambda_j(B)} \tag{18}$$

Proof of Lemma 3.5. Inequality (18) is a direct consequence of (5). Indeed, for every $j = 1, .., k$

$$\mu_j(T_k^B) = \mu_j(P_k \circ R_B \circ P_k) \leq \mu_1(P_k)\mu_j(R_B \circ P_k) \leq \mu_1(P_k)^2 \mu_j(R_B).$$

Following Section 2 we have $\mu_1(P_k) = |P_k|_{\mathcal{L}(L^2(D))} = 1$ hence (18) comes trivially.

For proving (17) we notice in the same way that

$$\mu_j(T_k^A) \leq \mu_j(R_A). \tag{19}$$

Since T_k^A is positive compact and self-adjoint, it has a spectrum consisting only of eigenvalues. Moreover, for every $j = 1, .., k$ if u_j is the j-th eigenfunction of R_B we have that

$$T_k^A u_j = P_k \circ R_A \circ T_k^A u_j = \mu_j(A) u_j, \tag{20}$$

since $P_k u_j = u_j$. Combining (19) and (20) we get (17). □

Proof of Theorem 3.4, continuation. Using Lemma 3.5 we have for every $j = 1, .., k$

$$\begin{aligned} 0 \le \frac{1}{\lambda_j(A)} - \frac{1}{\lambda_j(B)} &\le \mu_j(T_k^A) - \mu_j(T_k^B) \\ &\le |T_k^A - T_k^B|_{\mathcal{L}(L^2(D))} = |P_k \circ R_A \circ P_k - P_k \circ R_B \circ P_k|_{\mathcal{L}(L^2(D))}. \end{aligned}$$

But

$$\begin{aligned} |P_k \circ R_A \circ P_k - P_k \circ R_B \circ P_k|_{\mathcal{L}(L^2(D))} &= \\ &= \sup_{|u|_{L^2(D)} \le 1} \langle (P_k \circ R_A \circ P_k - P_k \circ R_B \circ P_k) u, u \rangle_{L^2(D) \times L^2(D)} \\ &= \sup_{|u|_{L^2(D)} \le 1} \langle (R_A - R_B) P_k u, P_k u \rangle_{L^2(D) \times L^2(D)}. \end{aligned}$$

Let us notice that from Lemma 2.2 $Range(P_k) \subseteq L^\infty(D)$ and moreover

$$P_k : L^2(D) \to L^\infty(D)$$

is bounded. Indeed, let $u \in L^2(D), |u|_{L^2(D)} \le 1$ and $P_k u = \alpha_1 u_1 + ... + \alpha_k u_k$. Here, the eigenfunctions $u_1, .., u_k$ of the Dirichlet Laplacian on A are supposed to be L^2-normalised. Since $|P_k u|_{L^2(D)} \le 1$ we get $\sum_{j=1}^k \alpha_j^2 \le 1$, hence $\alpha_j \le 1$ for every $j = 1, .., k$. From Lemma 2.2 we have that

$$|u_j|_{L^\infty(D)} \le C \lambda_j(A)^{-N/4},$$

where the constant C depends only on the dimension of the space. Finally, we observe that

$$|P_k u|_{L^\infty(D)} \le C \sum_{j=1}^k \alpha_j \lambda_j(A)^{-N/4} := C_k(A).$$

We have

$$\begin{aligned} &\langle (R_A - R_B) P_k u, P_k u \rangle_{L^2(D) \times L^2(D)} \\ &\quad \le \int_A |u_{A,P_k u} - u_{B,P_k u}| |P_k u| dx \le C_k(A) \int_A |u_{A,P_k u} - u_{B,P_k u}| dx \\ &\quad \le 2C_k(A) \int_A u_{A,|P_k u|} - u_{B,|P_k u|} dx \le 2C_k^2 \int_A w_A - w_B dx = 2C_k^2 d_\gamma(A, B). \end{aligned}$$

The last inequality is a consequence of the weak maximum principle. □

We are now able to formulate the following regularity result for shape functionals depending on eigenvalues.

Theorem 3.6. *Let $F : I\!R^k \to [0, +\infty)$ be a Lipschitz function. Every solution of the problem*

$$\min_{A \in \mathcal{K}(D)} F(\lambda_1(A), \lambda_2(A), .., \lambda_k(A)) + \alpha|A|, \tag{21}$$

is of class C^1.

Proof. It is a direct consequence of Theorem 3.4 and Corollary 3.3. □

Remark 3.7. The existence of a solution for problem (21) is a consequence of the fact that $\mathcal{K}(D)$ is γ-compact, under a *coerciveness*-like assumption on F. In order to get the existence, one has only to rule out the situation in which a minimising sequence γ-converges to the empty set; this would imply that all the eigenvalues diverge to $+\infty$.

3.2. Functionals depending on the state

We consider in the sequel shape functionals depending on the state, i.e. of the form

$$F(A) = \int_D j(x, u_{A,f})dx, \tag{22}$$

where $j : D \times I\!R \to I\!R$ is a Carathéodory function ($u_{A,f}$ is assumed extended by zero on $D \setminus A$ in (22)), i.e. $s \mapsto j(x, s)$ is continuous on $I\!R$ for almost every $x \in D$ and $x \mapsto j(x, s)$ is measurable for every $s \in I\!R$.

If $j : D \times I\!R \to I\!R$ is Lipschitz in the second variable, then for every $f \in L^\infty(D)$, the functional

$$A \to \int_D j(x, u_{A,f}(x))dx$$

is γ-Lip, thus Theorem 3.2 applies. Indeed, let us consider $A_1 \subseteq A_2$. Then

$$|\int_D j(x, u_{A_1,f}(x))dx - \int_D j(x, u_{A_2,f}(x))dx| \leq L \int_D |u_{A_1,f}(x) - u_{A_2,f}(x)|dx$$

$$\leq 2L|f|_\infty \int_D |w_{A_1}(x) - w_{A_2}(x)|dx = 2L|f|_\infty d_\gamma(A_1, A_2).$$

This argument works only for $f \in L^\infty(D)$. It is not clear whether for $f \in L^2(D)$ the previous functional is γ-Lip. Nevertheless, in dimension 2 of the space, the existence of smooth solutions for $f \in L^2(D)$ can be obtained in a direct way, using the same type of arguments as in Theorem 3.2. The important feature which allows us to repeat the arguments of Theorem 3.2 is that $f \in L^2(D)$ is *fixed* (see Remark 4.3 of the last section).

Lemma 3.8. *We fix the dimension of space $N = 2$ and consider $f \in L^2(D)$. Let $A \in \mathcal{K}(D)$ be such that the origin is an angular point of ∂A. Let A_ε be constructed like in Theorem 3.2. The following estimate holds for $\varepsilon \to 0$*

$$\int_D |u_{A,f} - u_{A_\varepsilon,f}|dx = \varepsilon^2 \epsilon(\varepsilon). \tag{23}$$

Proof. We can fix $f \geq 0$, if not we decompose $f = f^+ - f^-$. We have

$$\int_D (u_{A,f} - u_{A_\varepsilon,f})dx = \int_{A \setminus A_\varepsilon} u_{A,f} dx + \int_{A_\varepsilon} (u_{A,f} - u_{A-\varepsilon,f})dx.$$

For the first term of the right hand side, we readily get

$$0 \leq \int_{A \setminus A_\varepsilon} u_{A,f} dx \leq |A \setminus A_\varepsilon| |u_{A,f}|_{L^\infty(A \setminus A_\varepsilon)} = \varepsilon^3 \epsilon(\varepsilon).$$

For the estimate of the second term, we use the fact that $u_{A,f} - u_{A_\varepsilon,f}$ is harmonic in A_ε; there exists a constant $C > 0$ such that for every $\varepsilon \in (0,1)$

$$\int_{A_\varepsilon} (u_{A,f} - u_{A_\varepsilon,f})dx \leq C \int_{\partial A_\varepsilon} (u_{A,f} - u_{A_\varepsilon,f})d\sigma = \varepsilon^2 \epsilon(\varepsilon).$$

For the last inequality we used the fact that on $\partial A_\varepsilon \cap \partial A$ the function $u_{A,f} - u_{A_\varepsilon,f}$ vanishes. On $\partial A_\varepsilon \setminus \partial A$ the function $u_{A_\varepsilon,f}$ vanishes, while $u_{A,f}$ satisfies the estimate of Lemma 2.3. Since there exists a positive constant C' such that $\mathcal{H}^1(\partial A_\varepsilon \setminus \partial A) \leq C'\varepsilon$ the previous inequality follows. □

We give now the following.

Theorem 3.9. *Let $j : D \times I\!R \to I\!R$ be a Carathéodory function which is uniformly Lipschitz in the second variable, and $f \in L^2(D)$ and $\alpha > 0$ be fixed. Every solution of the shape optimisation problem*

$$\min_{A \in \mathcal{K}(D)} \int_D j(x, u_{A,f}(x))dx + \alpha|A|, \tag{24}$$

is of class C^1.

Proof. The proof follows, step by step, as in Theorem 3.2. In order to estimate the asymptotic behaviour of

$$\int_D j(x, u_{A,f}(x))dx - \int_D j(x, u_{A_\varepsilon,f}(x))dx,$$

we use Lemma 3.8. □

For a function $f \in L^2(D)$, the previous theorem does not apply to the typical case when the energy is to be minimised, i.e. for the functional

$$F(A) = -\frac{1}{2} \int_A f u_{A,f} dx,$$

since $j(x,u) = f(x)u$ is not uniformly Lipschitz in u. To cover this case, we give the following lemma. We still fix the dimension of the space $N = 2$.

Lemma 3.10. *Let $f \in L^2(D)$ be fixed. Let $A \in \mathcal{K}(D)$ be such that the origin is an angular point of ∂A. Let A_ε be constructed like in the proof of Theorem 3.2. The following estimate holds for $\varepsilon \to 0$*

$$\int_D |f(u_{A,f} - u_{A_\varepsilon,f})| dx = \varepsilon^2 \epsilon(\varepsilon). \tag{25}$$

Proof. The proof of (25) follows the same lines as the proof of Theorem 3.2, only relation (12) is to be replaced by the estimate of Lemma 2.3. Since

$$\Big| \int_D f(u_{A,f} - u_{A_\varepsilon,f}) dx \Big| \leq \int_D |f|(u_{A,|f|} - u_{A_\varepsilon,|f|}) dx$$

it is enough to consider only $f \geq 0$. So let us fix $f \geq 0$. Then

$$\begin{aligned} 0 &\leq \int_D f(u_{A,f} - u_{A_\varepsilon,f}) dx \\ &\leq 2\Big[\int_A \frac{1}{2} |\nabla u_{A_\varepsilon,f}|^2 - f u_{A_\varepsilon,f} dx - \int_A \frac{1}{2} |\nabla u_{A,f}|^2 - f u_{A,f} dx \Big] \\ &\leq 2 \int_{B_{0,2\varepsilon}} u_{A,f}^2 |\nabla \varphi_\varepsilon|^2 dx + 2 \int_{B_{0,2\varepsilon}} \varphi_\varepsilon^2 f u_{A,f} dx + \int_{B_{0,2\varepsilon}} \Delta \varphi_\varepsilon^2 u_{A,f}^2 dx. \end{aligned} \tag{26}$$

The previous inequality was obtained as in Theorem 3.2 taking as test function on A_ε the function $u_{A,f}(1 - \varphi_\varepsilon)$, where φ_ε is the usual mollifier. We clearly get

$$\int_{B_{0,2\varepsilon}} u_{A,f}^2 |\nabla \varphi_\varepsilon|^2 dx \leq |u_{A,f}|^2_{L^\infty(B_{0,2\varepsilon})} \int_{B_{0,2\varepsilon}} |\nabla \varphi_\varepsilon|^2 dx = \varepsilon^2 \epsilon(\varepsilon),$$

$$\int_{B_{0,2\varepsilon}} \varphi_\varepsilon^2 f u_{A,f} dx \leq |u_{A,f}|_{L^\infty(B_{0,2\varepsilon})} \int_{B_{0,2\varepsilon}} f dx \leq |u_{A,f}|_{L^\infty(B_{0,2\varepsilon})} |f|_{L^2(B_{0,2\varepsilon})} |B_{0,2\varepsilon}|^{\frac{1}{2}} = \varepsilon^2 \epsilon(\varepsilon),$$

$$\Big| \int_{B_{0,2\varepsilon}} \Delta \varphi_\varepsilon^2 u_{A,f}^2 dx \Big| \leq |u_{A,f}|^2_{L^\infty(B_{0,2\varepsilon})} \int_{B_{0,2\varepsilon}} |\Delta \varphi_\varepsilon^2| dx = \varepsilon^2 \epsilon(\varepsilon).$$

All these relations are a consequence of Lemma 2.3. □

Lemma 3.10 is useful in proving regularity for shape functionals of energy type.

Theorem 3.11. *Let $N = 2$ and $j : D \times I\!R \to I\!R$ be a Carathéodory function which is uniformly Lipschitz in the second variable. Let $f \in L^2(D)$ and $\alpha > 0$ be fixed. Every solution of the shape optimisation problem*

$$\min_{A \in \mathcal{K}(D)} \int_D f(x) \cdot j(x, u_{A,f}(x)) dx + \alpha |A|, \tag{27}$$

is of class C^1.

Proof. The proof is a consequence of Lemma 3.10 and follows the steps as Theorem 3.2. □

4. Further remarks

Remark 4.1. The choice of a bounded design region plays a role only in the existence question. The proof of the regularity for unbounded design regions is the same, since an optimal set, if it exists, it is bounded (since there does not exist unbounded open convex sets with finite measure). If $D = I\!R^N$ for example, the existence of minimisers has to be treated in a more careful way; in particular one should handle situations in which a minimising sequence becomes longer and thiner. In this case, all the eigenvalues diverge to $+\infty$, thus a *coerciveness*-like assumption on F would straight forwardly solve the existence problem (see Kawohl [18, Chapter 1]).

Remark 4.2. Let $F : I\!R^k \to I\!R$ be homogeneous of degree one in all the variables. For $D = I\!R^N$, problem

$$\min_{A \text{ convex}} F(\lambda_1(A), .., \lambda_k(A)) + \alpha|A| \tag{28}$$

is equivalent to the isoperimetric problem

$$\min_{|A|=c,\ A \text{ convex}} F(\lambda_1(A), .., \lambda_k(A)), \tag{29}$$

in the sense that the solutions have the same shape, up to a homothety.

Remark 4.3. A natural distance which has the same convergent sequences as the γ-convergence would be the following

$$\tilde{d}(A_1, A_2) = |R_{A_1} - R_{A_2}|_{\mathcal{L}(L^2(D))}.$$

In the metric space $(\mathcal{K}(D), \tilde{d})$, functionals like F_1 or F_2 in relation (2) are trivially Lipschitz, henceforth Theorem 3.4 would follow immediately. Nevertheless, this distance is not suitable for proving the regularity result of Theorem 3.2. Indeed, in two dimensions of the space, we can readily see that $|R_A - R_{A_\varepsilon}|_{\mathcal{L}(L^2(D))}$ in Theorem 3.2 is not asymptotically of the form $\varepsilon^2\epsilon(\varepsilon)$. We can prove that there exists a constant $c > 0$ such that

$$|R_A - R_{A_\varepsilon}|_{\mathcal{L}(L^2(D))} \geq c\varepsilon^2.$$

Indeed, for simplicity, let us assume that A is a sector of a disk, with angle $\omega \in (0, \pi)$ at the origin. By definition, we have that

$$|R_A - R_{A_\varepsilon}|_{\mathcal{L}(L^2(D))} = \sup_{|f|_{L^2(D)}=1} |(R_A - R_{A_\varepsilon})(f)|_{L^2(D)}.$$

Take $f_\varepsilon = \frac{1}{|A \setminus A_\varepsilon|^{\frac{1}{2}}} 1_{A \setminus A_\varepsilon}$; of course $|f_\varepsilon|_{L^2(D)} = 1$. Moreover $R_{A_\varepsilon}(f_\varepsilon) = 0$, hence

$$|R_A - R_{A_\varepsilon}|_{\mathcal{L}(L^2(D))} \geq |u_{A,f_\varepsilon}|_{L^2(D)}.$$

One can estimate this last norm using the usual blow up technique. Let

$$v_\varepsilon(x) := \frac{u_{A,f_\varepsilon}(\varepsilon x)}{\varepsilon}.$$

Then

$$-\Delta v_\varepsilon(x) = \varepsilon f_\varepsilon(\varepsilon x) = \frac{1}{|T_1|} 1_{T_1} \text{ in } \frac{1}{\varepsilon} A,$$

where T_1 is the triangle of edge equal to 1 in the sector ω, and $v_\varepsilon \in H_0^1(\frac{1}{\varepsilon}A)$. We have

$$|u_{A,f_\varepsilon}|_{L^2(A)} = \varepsilon^2 |v_\varepsilon|_{L^2(\frac{1}{\varepsilon}A)}.$$

Using the maximum principle, we get that $v_\varepsilon \geq v_1$, hence the conclusion follows.

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