Journal of Convex Analysis
Volume 10 (2003), No. 2, 445–464

Examples of the Lavrentiev Phenomenon with Continuous Sobolev Exponent Dependence

M. Foss
Kansas State University, Department of Mathematics,
Manhattan, KS 66506-2602, USA
foss@math.ksu.edu

Received November 8, 2002

We construct variational problems with infima that have non-trivial continuous dependence upon the Sobolev space from which the competing functions are taken. It is shown, for each μ in a particular class of continuous functions, that there is a variational integral and boundary conditions such that, for every $p \in [1, \infty]$, the infimum is equal to $\mu(p)$ if the admissible class is a subset of $W^{1,p}$. Thus, the manner in which the infimum depends upon the Sobolev exponent may be prescribed.

1. Introduction

In 1926, M. Lavrentiev [9] demonstrated that there exists a functional of the form $\int_a^b f(x, u(x), u'(x))\, dx$ such that the infimum for the functional over the absolutely continuous functions is strictly less than its infimum over the continuously differentiable functions, when $u : \mathbb{R} \to \mathbb{R}$ is required to meet certain boundary conditions. Thus, the infimum for the functional exhibits a sensitivity to the regularity required of the admissible functions. This phenomenon is called Lavrentiev's phenomenon.

To be more precise, we record the following definitions. Let the domain $\Omega \subset \mathbb{R}^m$, the functional $J : W^{1,1}(\Omega; \mathbb{R}^n) \to \overline{\mathbb{R}}$ and the admissible class $\mathcal{A}^1 \subseteq W^{1,1}(\Omega; \mathbb{R}^n)$ be given. For each $p \in [1, +\infty]$, set $\mathcal{A}^p := \mathcal{A}^1 \cap W^{1,p}(\Omega; \mathbb{R}^n)$.

Definition 1.1. Given $p_* \in (1, +\infty)$, the functional J is said to have *property* (Λ^{p_*}) if

$$\inf_{\mathbf{u}\in\mathcal{A}^p} J[\mathbf{u}] < \inf_{\mathbf{u}\in\mathcal{A}^{p'}} J[\mathbf{u}] \tag{Λ^{p_*}}$$

for each $p, p' \in [1, +\infty]$ satisfying $1 \le p < p_* < p' \le +\infty$.

Definition 1.2. The functional J exhibits *the Lavrentiev phenomenon* if it has the property

$$\inf_{\mathbf{u}\in\mathcal{A}^1} J[\mathbf{u}] < \inf_{\mathbf{u}\in\mathcal{A}^\infty} J[\mathbf{u}]. \tag{Λ}$$

Note that if the functional J has property (Λ^{p_*}) for some $p_* \in (1, +\infty)$, then it has property (Λ) and thus exhibits the Lavrentiev phenomenon.

Remark 1.3. As we have defined it, there exist trivial examples of the Lavrentiev phenomenon. The class $\mathcal{A}^\infty$ may be empty for instance. Although they are not very interesting, technically, this is an example of Lavrentiev's phenomenon by our definitions. One way to exclude many such trivial examples is to require $\mathcal{A}^1$ to be such that $\mathcal{A}^\infty$ is dense in

ISSN 0944-6532 / $ 2.50

$\mathcal{A}^1$. Although this is true for the examples presented in this article, we are not including this requirement in our definition.

Numerous examples of such functionals have been published since Lavrentiev's first. In 1934, B. Manià [11] improved Lavrentiev's example; the Lagrangian f was a simple polynomial. In 1984, J. Ball & V. J. Mizel [2] gave an example with an f that was strictly convex with respect to its third argument. V. J. Mizel & A. C. Heinricher [7], in 1988, provided one dimensional constructions with property (Λ); for these examples, they were also able to give explicit expressions for the pseudo-minimizers. G. Alberti & P. Majer [1] provided a two dimensional example with an autonomous integrand in 1994. An example without boundary conditions was found by K. Dani, W. Hrusa & V. J. Mizel [5] in 1996. W. Hrusa [8] also created an example in one dimension where the functional depends only on u, u' and u''; that is J has the form $\int_a^b f(u(x), u'(x), u''(x))\,dx$, where $u : (a,b) \to \mathbb{R}$. The diversity of examples in the literature demonstrate that the Lavrentiev phenomenon can appear in a wide variety of variational problems; in fact, recent work by W. Hrusa and V. J. Mizel strongly suggests, at least for one dimensional problems, that the phenomenon is generic [12]. For a more extensive survey of the Lavrentiev phenomenon, we refer to G. Buttazzo & M. Belloni [3].

One question that has arisen concerns the possible sensitivity of the infimum for J to the regularity of the class of admissible functions. If we define the function $I : [1,\infty] \to \overline{\mathbb{R}}$ by

$$I(p) := \inf_{u \in \mathcal{A}^p} J[u],$$

how might I depend on the Sobolev exponent p? This is the question we consider in this article.

Until fairly recently, in all examples of the Lavrentiev phenomenon, there existed some $p_* \in (1,\infty)$ and constants c_0, $c_1 \in \overline{\mathbb{R}}$ such that for each $p_0 \in [1, p_*)$ and $p_1 \in [p_*, \infty]$

$$c_0 = I(p_0) < I(p_1) = c_1.$$

In other words, the examples have an increasing function I that was right continuous and piecewise constant. In 1999, however, A. Siegal [14], constructed one dimensional examples of J with smooth integrands such that

$$I(p) = \begin{cases} c_1, & 1 \le p < p_*; \\ c_2, & p = p_*; \\ c_3, & p_* < p \le \infty; \end{cases}$$

where $p_* \in [1,\infty]$ and c_1, c_2, $c_3 \in \mathbb{R}$ satisfy $c_1 < c_2 < c_3$. These examples show that I can have a jump discontinuity at any $p_* \in [1,\infty]$ and that I may be right continuous, left continuous or neither.

We will provide examples where for every p_* in a given interval the functional has property (Λ^{p_*}). Moreover, we will show that, given $p_0, p_1 \in (1,\infty)$ and $c \in [0,\infty)$, if μ is a monotone absolutely continuous function such that

$$\mu(p) = 0 \quad \forall p \in [1, p_0),$$

and

$$\mu(p) = c \quad \forall p \in [p_1, \infty],$$

then there is a functional J such that, for each $p \in [1, \infty]$, one finds $I(p) = \mu(p)$. Thus, the infima for our examples will have a prescribed continuous dependence upon the exponent of the Sobolev space from which the admissible functions are taken. As the exponent decreases and the class of admissible functions expands, we are able to more closely approach the absolute infimum of the functional. This absolute minimum, however, can never be attained until the exponent of the Sobolev space decreases to at least p_1.

2. The Basic Idea

The underlying idea for all the examples to be presented is simple. The purpose of this section is to provide some intuition for what we will prove.

To begin, we extract from Heinricher & Mizel [7] a result which is the principal basis for our constructions. For each $q \in (1, +\infty)$, let us define the integrand $f(\,\cdot\,; q) : (0,1)\times\mathbb{R}\times\overline{\mathbb{R}} \to \overline{\mathbb{R}}$ by

$$f(x, u, \xi; q) := (|u|^q - x)^2 \, |\xi|^{\frac{3q}{q-1}}$$

and the functional $J[\,\cdot\,; q] : W^{1,1}((0,1); \mathbb{R}) \to \overline{\mathbb{R}}$ by

$$J[u; q] := \int_0^1 f(x, u(x), u'(x); q)\, dx.$$

For the admissible classes, we set

$$\mathcal{A}^1 := \left\{ u \in W^{1,1}((0,1); \mathbb{R}) \mid u(0) = 0 \text{ and } u(1) = 1 \right\},$$

and for each $p \in (1, +\infty]$, we set

$$\mathcal{A}^p := \mathcal{A}^1 \cap W^{1,p}((0,1); \mathbb{R}).$$

Each member in our family of integrands has the following homogeneity property: given $q \in (1, +\infty)$, if we select $p_* = \frac{q}{q-1}$, then

$$f(e^\tau x, e^{\frac{p_*-1}{p_*}\tau} u, e^{-\frac{1}{p_*}\tau} \xi; q) = e^{-\tau} f(x, u, \xi; q) \tag{HP}$$

for every $\tau \in \mathbb{R}$. Using the invariance of f under this rescaling the solutions to the Euler-Lagrange equations for J can be determined and used to construct a Mayer field. With this field, it can be shown (see Theorem A in [7]) that

$$1 \le p < \tfrac{q}{q-1} \Rightarrow \inf_{u \in \mathcal{A}^p} J[u; q] = J[x^{\frac{1}{q}}; q] = 0, \tag{1}$$

while

$$\tfrac{q}{q-1} \le p \le +\infty \Rightarrow \inf_{u \in \mathcal{A}^p} J[u; q] = J[x^{\frac{3}{2q+1}}; q] = \tfrac{2}{3}\tfrac{(q-1)^2}{q+2}\left(\tfrac{3}{2q+1}\right)^{\frac{3q}{q-1}} > 0. \tag{2}$$

Thus, there is a jump in the infimum for J as the Sobolev exponent for the admissible class increases past $\frac{q}{q-1}$, i.e. the functional has property $(\Lambda^{\frac{q}{q-1}})$.

For us, the particularly important part of this result is that it gives us the exact value for the infimum of the functional in the different admissible classes and provides us with the functions at which J attains these different infima. For convenience, let us define $I : [1, +\infty] \times (1, +\infty) \to \overline{\mathbb{R}}$ by

$$I(p, q) := \inf_{u \in \mathcal{A}^p} J[u; q].$$

From (1) and (2), we have an explicit formula for I, namely

$$I(p, q) = \begin{cases} 0, & 1 \le p < \frac{q}{q-1}; \\ \frac{2}{3} \frac{(q-1)^2}{q+2} \left(\frac{3}{2q+1}\right)^{\frac{3q}{q-1}}, & \frac{q}{q-1} \le p. \end{cases} \tag{3}$$

We now parameterize the functions that give us each of the above infima. For $q \in (1, +\infty)$ we let $u_{\mathrm{am}}(\,\cdot\,; q) \in \mathcal{A}^1$ and $u_{\mathrm{pm}}(\,\cdot\,; q) \in \mathcal{A}^{\frac{q}{q-1}}$ be given by

$$u_{\mathrm{am}}(x; q) := x^{\frac{1}{q}} \quad \text{and} \quad u_{\mathrm{pm}}(x; q) := x^{\frac{3}{2q+1}}.$$

We may thus write

$$I(p, q) = \inf_{u \in \mathcal{A}^p} J[u; q] = \begin{cases} J[u_{\mathrm{am}}(\,\cdot\,; q); q], & 1 \le p < \frac{q}{q-1}; \\ J[u_{\mathrm{pm}}(\,\cdot\,; q); q], & \frac{q}{q-1} \le p \le +\infty. \end{cases}$$

The essential idea for the examples in this article is to add a dimension to the domain of the admissible functions and use this additional dimension to parameterize q. For the moment, the domain we use is the unit square

$$\Omega := (0, 1) \times (0, 1),$$

and we set

$$\widetilde{\mathcal{A}}^1 := \left\{u \in W^{1,1}(\Omega; \mathbb{R}) \mid u(0, \,\cdot\,) = 0 \text{ and } u(1, \,\cdot\,) = 1\right\}.$$

Letting $q : (0, 1) \to (1, +\infty)$ be a given continuous function, define a new functional $\widetilde{J} : W^{1,1}(\Omega; \mathbb{R}) \to \overline{\mathbb{R}}$ by

$$\widetilde{J}[u] := \int_0^1 J[u(\,\cdot\,, y); q(y)]\, dy = \int_0^1 \int_0^1 f(x, u(\mathbf{x}), u_{,x}(\mathbf{x}); q(y))\, dx dy, \tag{4}$$

where $\mathbf{x} = (x, y) \in \mathbb{R}^2$ and $u_{,x}(\mathbf{x}) := \frac{\partial}{\partial x} u(x, y)$. Define $\widetilde{I} : [1, +\infty] \to \overline{\mathbb{R}}$ by

$$\widetilde{I}(p) := \inf_{u \in \widetilde{\mathcal{A}}^p} J[u].$$

From (3), we have that

$$\begin{aligned} \widetilde{I}(p) = \int_0^1 I(p, q(y))\, dy &= \int_{\left\{y \mid \frac{q(y)}{q(y)-1} \le p\right\}} J[u_{\mathrm{pm}}(\,\cdot\,; q(y)); q(y)]\, dy \\ = \int_{\left\{y \mid 1 \le p < \frac{q(y)}{q(y)-1}\right\}} J[u_{\mathrm{am}}(\,\cdot\,; q(y)); q(y)]\, dy &+ \int_{\left\{y \mid \frac{q(y)}{q(y)-1} \le p\right\}} J[u_{\mathrm{pm}}(\,\cdot\,; q(y)); q(y)]\, dy \end{aligned} \tag{5}$$

and one might expect that the infimum in $\widetilde{\mathcal{A}}^p$ is attained at the function $u_* : \Omega \to \mathbb{R}$ given by

$$u_*(\mathbf{x}) := \begin{cases} u_{\mathrm{am}}(x; q(y)), & 1 \le p < \frac{q(y)}{q(y)-1}; \\ u_{\mathrm{pm}}(x; q(y)), & \frac{q(y)}{q(y)-1} \le p \le +\infty. \end{cases} \tag{6}$$

If u_* were in $\widetilde{\mathcal{A}}^p$ and $q(\,\cdot\,)$ were non-constant, then (5) would imply that we have an example of the Lavrentiev phenomenon where the infimum increases continuously with the exponent of the Sobolev space.

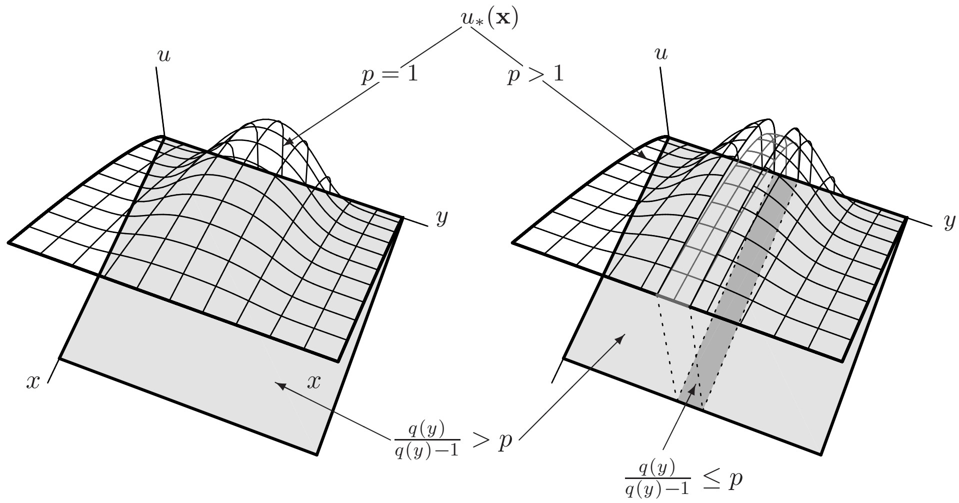

Figure 2.1: Possible discontinuity for u_*

The function given in (6), however, is generally not a member of the admissible space $\widetilde{\mathcal{A}}^p$. One reason we may not find u_* in $\widetilde{\mathcal{A}}^p$ is that it suffers a jump discontinuity whenever $\frac{q(y)}{q(y)-1}$ increases (or decreases) past p (Figure 2.1 illustrates the situation). What we can do though is construct a sequence $\left\{u_*^{(n)}\right\}_{n=1}^{\infty} \subset \widetilde{\mathcal{A}}^p$ such that $\widetilde{J}[u_*^{(n)}] \to \widetilde{J}[u_*]$ as $n \to +\infty$. To see that such a sequence might be found, observe that the integrand for the functional $\widetilde{J}$ has no dependence on $u_{,y}$ and the discontinuity in u_* is with respect to the y argument. Hence, it seems reasonable to expect that $\tilde{J}$ of an appropriate point-wise approximation of u_* will approximate $\widetilde{J}[u^*]$. This would allow us to conclude

$$\widetilde{I}(p) \le \int_0^1 I(p, q(y))\, dy.$$

Meanwhile, the results in [7] show that

$$\widetilde{I}(p) \ge \int_0^1 I(p, q(y))\, dy.$$

It would follow that the guess in (5) is correct.

This is the main idea behind the results and proofs that we will present in this chapter. We will not, however, use the $\widetilde{J}$ defined in (4). Instead, a modified version of that functional will be used. In addition to having properties similar to those of (4), this modified functional will give us greater control on the regularity of its integrand and how its infimum increases with the Sobolev exponent.

3. The Functional

In this section we construct the functionals we wish to examine. A simple remark regarding the regularity of our Lagrangians is also discussed.

First, we define a subclass of the monotone absolutely continuous functions. These functions can be used to prescribe how the infimum of our functionals depends upon the Sobolev exponent. For each $p_1, p_2 \in (1,\infty)$ satisfying $p_1 < p_2$, set

$$\mathcal{M}_{p_1,p_2} := \left\{\mu \in W^{1,1}_{\mathrm{loc}}([1,+\infty];[0,+\infty)) \mid \text{conditions (a)–(c) are satisfied}\right\}, \tag{7}$$

where conditions (a)-(c) are the following:

(a) $\mu \llcorner [1,p_1] = 0$;
(b) $\frac{\partial}{\partial p}\mu \llcorner (p_1,p_2) \geq 0$ a.e.;
(c) $\frac{\partial}{\partial p}\mu \llcorner [p_2,+\infty] = 0$ a.e.

We also put

$$\mathcal{M} := \bigcup_{p_1>1} \bigcup_{p_2>p_1} \mathcal{M}_{p_1,p_2}.$$

If $\mu \in \mathcal{M}_{p_1,p_2}$, then it must be an absolutely continuous nondecreasing function that is 0 on $[1,p_1]$ and constant on $[p_2,+\infty]$. Given a $\mu \in \mathcal{M}$, we will construct a functional whose infimum duplicates μ as the Sobolev exponent p increases from 1 to $+\infty$.

For the sequel, fix $\mu \in \mathcal{M}$. By definition, there exist $\widetilde{p}_1, \widetilde{p}_2 \in (1,+\infty)$ such that $\mu \in \mathcal{M}_{\widetilde{p}_1,\widetilde{p}_2}$. Put

$$p_1 := \sup\left\{p \in (1,+\infty) \mid \mu \in \mathcal{M}_{p,\widetilde{p}_2}\right\} \tag{8}$$

and

$$p_2 := \inf\left\{p \in (1,+\infty) \mid \mu \in \mathcal{M}_{\widetilde{p}_1,p}\right\}. \tag{9}$$

So $\mu \in \mathcal{M}_{p_1,p_2}$, but $\mu \notin \mathcal{M}_{p,p_2}$ and $\mu \notin \mathcal{M}_{p_1,p}$ for any $p \in (p_1,p_2)$.

Now, we will define the integrand for our functional. For each $q \in (1,+\infty)$ and $m \in [3q,+\infty)$, let $h(\,\cdot\,;q,m)$, $f(\,\cdot\,;q,m) : (0,1)\times\mathbb{R}\times\overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be given by

$$h(x,u,\xi;q,m) := |u|^{\frac{m-3q}{q-1}}\left(|u|^{\frac{q}{q-1}} - x\right)^2|\xi|^m \tag{10}$$

and

$$f(x,u,\xi;q,m) := [\mu_{,p}(q)][s(q,m)][h(x,u,\xi;q,m)], \tag{11}$$

where we have put

$$s(q,m) := \tfrac{(m-3)(m-2)}{2}\left[\tfrac{m-1}{m}\tfrac{q}{q-1}\right]^m. \tag{12}$$

Later we will prove that, for an appropriate admissible class, the functional associated with $h(\,\cdot\,;q,m)$ has property (Λ^q). The magnitude of the jump in the infimum for this functional is exactly $\frac{1}{s(q,m)}$. So if f were used as an integrand for a functional, the infimum would be either 0 or $\mu_{,p}(q)$ depending on whether or not the competing functions were allowed to be in the Sobolev space with exponent less than q.

As was described in the previous section, the domain for the competing functions is two-dimensional, and one of these dimensions will be used to parameterize the m and q in (11). As a domain, we use the rectangle

$$\Omega := (0,1) \times (p_1, p_2). \tag{13}$$

For our parameterizations of m and q, we choose $m \in [3p_2, +\infty)$ to be a fixed integer and define $q(y) := y$. Note that these choices satisfy the condition $m \geq 3q(y)$ at each $y \in (p_1, p_2)$, which we required for definitions (10) and (11).

Now we define the functional we wish to examine along with the admissible class $\mathcal{A}^1$. Let $J : W^{1,1}(\Omega;\mathbb{R}) \to \overline{\mathbb{R}}$ be given by

$$J[u] := \int_{\Omega} f(x, u(\mathbf{x}), u_{,x}(\mathbf{x}); y, m)\, d\mathbf{x}. \tag{14}$$

For $\mathcal{A}^1$, set

$$\mathcal{A}^1 := \left\{ u \in W^{1,1}(\Omega;\mathbb{R}) \mid u(0,\,\cdot\,) = 0 \text{ and } u(1,\,\cdot\,) = 1 \right\}. \tag{15}$$

Note that the admissible class $\mathcal{A}^p = \mathcal{A}^1 \cap W^{1,p}(\Omega;\mathbb{R})$ is a dense subset of $\mathcal{A}^1$ for every $p \in [1, +\infty]$. We also define $I(p) : [1, +\infty] \to \overline{\mathbb{R}}$ by

$$I(p) := \inf_{u \in \mathcal{A}^p} J[u]. \tag{16}$$

Our objective is to show that

$$I(p) = \int_{\{y \mid\; y \leq p\}} \mu_{,p}(y)\, dy = \mu(p).$$

Notice that we have only required m to be some fixed integer in $[3p_2, +\infty)$.

Let us make a few remarks regarding the integrand f and the functional J. We first comment on the differentiability of the integrand.

Remark 3.1. The regularity of the integrand f is essentially determined by the regularity of μ over the interval (p_1, p_2) and the value chosen for $m \in [3p_2, +\infty)$. In many cases (particularly when $\mu \in C^\infty((p_1,p_2);\mathbb{R})$), larger values of m provide more differentiability for f. An easy way to see this is to rewrite (11): we have

$$f(x,u,\xi;y,m) = \mu_{,p}(y) M \left[\tfrac{y}{y-1}\right]^m \left[|u|^{\frac{m-y}{y-1}} - 2x|u|^{\frac{m-2y}{y-1}} + x^2 |u|^{\frac{m-3y}{y-1}} \right] |\xi|^m,$$

where $M := \frac{(m-3)(m-2)}{2} \left[\frac{m-1}{m}\right]^m$. If $p_1, p_2 \in (1, +\infty)$ and $\mathbf{x} \in \Omega$, then $y \in (p_1, p_2)$, and it follows that $\mathbf{x} \mapsto \frac{1}{y-1} \in C^\infty(\Omega;\mathbb{R})$. Recalling that m was chosen to be an integer in $[3p_2, +\infty)$, we see that the differentiability of f is completely determined by the differentiability of the terms

$$\tfrac{\partial}{\partial p}\mu \llcorner (p_1, p_2), \quad |u|^{\frac{m-3y}{y-1}} \quad \text{and} \quad |\xi|^m.$$

The last term is infinitely differentiable when m is even, and the second term is k-differentiable when $m \geq k(p_2 - 1) + 3p_2$. Thus if $\mu_{,p}$ is k-differentiable and $m \geq k(p_2 - 1) + 3p_2$ is an even integer, then f is k-differentiable.

Our next remark concerns a homogeneity property of f.

Remark 3.2. For each fixed $y \in (p_1, p_2)$, we claim f has the homogeneity property (HP) with $p_* = y$ (Section 2). Indeed by definition (10), for each $\tau \in \mathbb{R}$ we find

$$\begin{aligned} h\left(e^\tau x, e^{\frac{y-1}{y}\tau}u, e^{-\frac{1}{y}\tau}\xi; y, m\right) &= \left|e^{\frac{y-1}{y}\tau}u\right|^{\frac{m-3y}{y-1}} \left(\left|e^{\frac{y-1}{y}\tau}u\right|^{\frac{y}{y-1}} - e^\tau x\right)^2 \left|e^{-\frac{1}{y}\tau}\xi\right|^m \\ &= e^{\frac{m-3y}{y}\tau} e^{2\tau} e^{-\frac{m}{y}\tau} h(x, u, \xi; y, m) = e^{-\tau} h(x, u, \xi; y, m). \end{aligned}$$

Therefore h has property (HP). It follows from definition (11) that f also has property (HP) for each fixed $y \in (p_1, p_2)$.

We also wish to record

Remark 3.3. The integrand f is nonnegative and convex with respect to ξ. Clearly, we have $\frac{\partial^2}{\partial \xi^2} f(x, u, \xi; y, m) \geq 0$ at every $(\mathbf{x}, u, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}$ since $m \geq 3p_2 > 3$ is an integer.

Our final remark concerns the functional J.

Remark 3.4. We obviously have $h \geq 0$. Since $\mu \in \mathcal{M}_{p_1,p_2}$ implies $\mu_{,p} \geq 0$, we also have $f \geq 0$ and therefore $J \geq 0$ (14). Let us define $u_{\mathrm{am}} \in \mathcal{A}^1$ by

$$u_{\mathrm{am}}(\mathbf{x}) := x^{\frac{y-1}{y}}. \tag{17}$$

We see that

$$J[u_{\mathrm{am}}] = \int_\Omega \mu_p(y) s(y, m) h\big(x, x^{\frac{y-1}{y}}, \tfrac{y-1}{y} x^{-\frac{1}{y}}; y, m\big)\, d\mathbf{x},$$

but $h\big(x, x^{\frac{y-1}{y}}, \xi; y, m\big) = 0$ at any $\xi \in \mathbb{R}$. It follows that $J[u_{\mathrm{am}}] = 0$, and hence u_{am} is an absolute minimizer for J over $\mathcal{A}^1$.

For a concrete example illustrating our definitions to this point, let us take $\mu \in \mathcal{M}$ to be

$$\mu(p) = \begin{cases} 0, & 1 \leq p < \frac{3}{2}; \\ p - \frac{3}{2}, & \frac{3}{2} \leq p < \frac{5}{2}; \\ 1, & \frac{5}{2} \leq p. \end{cases}$$

So $\mu_{,p} \neq 0$ only over the interval $(\frac{3}{2}, \frac{5}{2})$. Therefore $\mu \in \mathcal{M}_{\widetilde{p}_1, \widetilde{p}_2}$ for any $\widetilde{p}_1 \leq \frac{3}{2}$ and $\widetilde{p}_2 \geq \frac{5}{2}$. By definitions (8) and (9), we put $p_1 = \frac{3}{2}$ and $p_2 = \frac{5}{2}$. By (13), the domain Ω is the rectangle $(0,1) \times (\frac{3}{2}, \frac{5}{2})$. Selecting $m = 14$, our integrand (11) becomes

$$f(x, u, \xi; y, 14) = 66 \left(\tfrac{13}{14}\right)^{14} \left(\tfrac{y}{y-1}\right)^{14} |u|^{\frac{14-3y}{y-1}} \big(|u|^{\frac{y}{y-1}} - x\big)^2 \xi^{14},$$

and the functional (14) is

$$J[u] = \int_\Omega f(x, u(\mathbf{x}), u_{,x}(\mathbf{x}); y, 14)\, d\mathbf{x}.$$

The convexity of f with respect to ξ is obvious. Furthermore, for this example, it is clear that f is C^∞ with respect to x, y and ξ. The exponent on $|u|$ is always larger than 4.

Therefore f is C^3 with respect to u. In fact, the integrand $f(\,\cdot\,,\,\cdot\,,\,\cdot\,;\,\cdot\,,14)$ is jointly C^3 with respect to all its arguments. To obtain a Lagrangian that has more differentiability, we may simply chose m to be a larger integer. Finally, we note that for any $\xi \in \mathbb{R}$ we have $f\big(x, x^{\frac{y-1}{y}}, \xi; y, m\big) = 0$. Thus u_{am} given in (17) is the absolute minimizer for J.

With the functional J in hand, we now proceed to prove that it has the desired property, namely that $I(p) = \mu(p)$. Our proof splits naturally into two sections. The first of which establishes that $I(p) \geq \mu(p)$. In the second section we prove the counterpart: $I(p) \leq \mu(p)$. With these two inequalities, the proof will be completed.

4. $I(p) \geq \mu(p)$

The goal for this section is to prove the inequality $I(p) \geq \mu(p)$.

Lemma 4.1. *For each $p \in [1, +\infty]$ we have $I(p) \geq \mu(p)$.*

Proof. The proof is split into three parts. The first is the case $1 \leq p \leq p_1$. In the second case, we prove the lemma for $p_1 < p < p_2$. The last case covers $p_2 \leq p \leq +\infty$.

Case 1. $1 \leq p \leq p_1$:

Since $p \in [1, p_1]$ and $\mu \in \mathcal{M}_{p_1,p_2}$, we have $\mu(p) = 0$. Hence

$$I(p) = \inf_{u \in \mathcal{A}^p} \left[\int_{\Omega} f(x, u(\mathbf{x}), u_{,x}(\mathbf{x}); y, m)\, d\mathbf{x} \right] \geq 0,$$

because $f \geq 0$. Therefore $I(p) \geq \mu(p)$.

For the remaining two cases, we will reduce our problem to a family of one-dimensional problems. To this end, it will be useful to define the following one-dimensional admissible classes. For each $p \in [1, +\infty]$, we set

$$\mathcal{B}^p := \left\{ v \in W^{1,p}((0,1); \mathbb{R}) \mid v(0) = 0 \text{ and } v(1) = 1 \right\}. \tag{18}$$

We now proceed to Case 2.

Case 2. $p_1 < p < p_2$:

Of the three cases, this is the most complicated. The idea is to treat the two-dimensional problem as a one-dimensional variational problem at each of the $y \in (p_1, p]$. Employing techniques that Heinricher & Mizel use in [7], a lower bound can be established for each of these one-dimensional problems. This will lead to the desired lower bound for the two-dimensional problem.

Let $u \in \mathcal{A}^p$ be given. We first reduce the two-dimensional problem to a family of one-dimensional problems. Define $\Omega_* := (0,1) \times (p_1, p] \subset \Omega$. Using the non-negativity of f, we have

$$\begin{aligned} J[u] = \int_{\Omega} f(x, u(\mathbf{x}), u_{,x}(\mathbf{x}); y, m)\, d\mathbf{x} &\geq \int_{\Omega_*} f(x, u(\mathbf{x}), u_{,x}(\mathbf{x}); y, m)\, d\mathbf{x} \\ &= \int_{\Omega_*} \mu_p(y) s(y, m) h(x, u, u_{,x}; y, m)\, d\mathbf{x}. \end{aligned}$$

By Fubini's theorem and the definition for Ω_*, we may write

$$J[u] \geq \int_{(p_1,p]} \mu_{,p}(y) s(y,m) \left[\int_{(0,1)} h(x,u,u_{,x};y,m)\,dx \right] dy.$$

Since $u \in \mathcal{A}^p \subset W^{1,p}(\Omega;\mathbb{R})$, for almost every $y \in (p_1,p]$ we find $u(\,\cdot\,,y)$ in $\mathcal{B}^p$. It follows that

$$\begin{aligned} J[u] &\geq \int_{\{y|p_1<y\leq p\}} \mu_{,p}(y)s(y,m) \left[\int_{(0,1)} h(x,u,u_{,x};y,m)\,dx \right] dy \\ &\geq \int_{\{y|p_1<y\leq p\}} \mu_{,p}(y)s(y,m) \left[\inf_{v\in\mathcal{B}^p} \int_{(0,1)} h(x,v(x),v'(x);y,m)\,dx \right] dy \\ &\geq \int_{\{y|p_1<y\leq p\}} \mu_{,p}(y)s(y,m) \left[\inf_{v\in\mathcal{B}^y} \int_{(0,1)} h(x,v(x),v'(x);y,m)\,dx \right] dy, \end{aligned} \tag{19}$$

where $v'(x) = \frac{d}{dx}v(x)$. In the last step we have used the fact that $\mathcal{B}^p \subseteq \mathcal{B}^y$ for any $y \in (p_1,p]$. Let us define $G : W^{1,1}((0,1);\mathbb{R}) \to L^1((p_1,p);\overline{\mathbb{R}})$ by

$$G[v](y) := \int_{(0,1)} h(x,v(x),v'(x);y,m)\,dx. \tag{20}$$

We now rewrite (19) as

$$J[u] \geq \int_{\{y|p_1<y\leq p\}} \mu_{,p}(y)s(y,m) \left[\inf_{v\in\mathcal{B}^y} G[v](y) \right] dy. \tag{21}$$

For the proof of this case, it is clear that it suffices to show $\inf_{v\in\mathcal{B}^y} G[v](y)$ is bounded below by $\frac{1}{s(y,m)}$ for each $y \in (p_1,p)$. We will use exactly the same argument presented in [7] to do this.

Our objective is to determine

$$\inf_{v\in\mathcal{B}^y} G[v](y)$$

at each $y \in (p_1,p]$. As stated before, we will treat this as a one-dimensional variational problem at each $y \in (p_1,p]$ and use a field theory argument essentially taken from [7]. The key ingredients are the solutions to the Euler-Lagrange equations associated with functional $G[\,\cdot\,](y)$ (which we will not write out explicitly) and the convexity of h with respect to ξ.

We fix $y \in (p_1,p]$ for now. To find the solutions of these Euler-Lagrange equations, we recall the homogeneity property (HP) of h: for any $\tau \in \mathbb{R}$

$$h(e^{\tau}x, e^{\frac{y-1}{y}\tau}u, e^{-\frac{1}{y}\tau}\xi; y,m)e^{\tau} = h(x,u,\xi;y,m). \tag{22}$$

This property gives us a variational symmetry for $G[\,\cdot\,](y)$, and a theorem of E. Noether's tells us that this symmetry has a corresponding conservation law (see Olver [13]). The conservation law provides a means of reducing the order of the Euler-Lagrange equations by one. One way of finding the conservation law (see Logan [10]) is to observe that for every $\tau \in \mathbb{R}$ (at $\tau = 0$ in particular) we have

$$\frac{\partial}{\partial \tau}\left[h(e^{\tau}x, e^{\frac{y-1}{y}\tau}v(x), e^{-\frac{1}{y}\tau}v'(x); y, m)e^{\tau}\right] = 0.$$

It follows that for every $v \in \mathcal{B}^y$

$$xh_{,x} + \frac{(y-1)v}{y}h_{,u} - \frac{v'}{y}h_{,\xi} + h = 0. \tag{23}$$

Suppose $v \in C^2((0,1);\mathbb{R})$. Then we can use (23) to write

$$\frac{d}{dx}\left[xh + \frac{(y-1)v}{y}h_{,\xi} - xv'h_{,\xi}\right] = \left(h_{,u} - \frac{d}{dx}h_{,\xi}\right)\left(xv' - \frac{(y-1)v}{y}\right). \tag{24}$$

Now, let us further suppose that v satisfies the Euler-Lagrange equations

$$h_{,u} - \frac{d}{dx}h_{,\xi} = 0. \tag{E-L}$$

Then (24) reduces to

$$xh + \frac{(y-1)v}{y}h_{,\xi} - xv'h_{,\xi} = K, \tag{25}$$

where K is a constant. This is the conservation law corresponding to the variational symmetry (22): if $v \in C^2((0,1);\mathbb{R})$ is a solution to (E-L), then the quantity on the left hand side of (25) must remain constant along that solution. Moreover, it is clear that if (25) holds for some K, then (24) implies that either the Euler-Lagrange equations (E-L) are satisfied, or there is a $\lambda \in \mathbb{R}$ such that $v(x) = \lambda x^{\frac{y-1}{y}}$ for all $x \in (0,1)$. Hence, one way to find solutions to (E-L) is to find solutions to the conservation law that are not in the family of functions $x \mapsto \lambda x^{\frac{y-1}{y}}$. We do this for $K = 0$.

We seek a v that satisfies

$$xh + \frac{(y-1)v}{y}h_{,\xi} - xv'h_{,\xi} = 0 \tag{26}$$

and that is not in the family $x \mapsto \lambda x^{\frac{y-1}{y}}$. From the definition for h, we easily find

$$h_{,\xi} = \frac{m}{\xi}h.$$

With $\xi = v'$, the conservation law (26) can be reformulated as

$$h\left[\frac{(y-1)m}{y}\frac{v}{v'} - (m-1)x\right] = 0. \tag{27}$$

For v to satisfy (27) it is sufficient that it satisfy

$$\frac{v(x)}{v'(x)} = \frac{(y-1)m}{(m-1)y}\frac{1}{x}.$$

Thus for any $\lambda \in \mathbb{R}$

$$v(x;\lambda) := \lambda x^{\frac{(y-1)m}{(m-1)y}} \tag{28}$$

is a solution to (27) which implies it is a solution to the conservation law (25). As argued above, the mapping $v(\,\cdot\,;\lambda)$ must also satisfy the Euler-Lagrange equations (E-L) for each $\lambda \in \mathbb{R}$; for this last statement, we have used the observation that, as with any constant function, the function $v = 0$ is a trivial solution for (E-L) since $m \geq 3$.

Of course, only $v(\cdot\,;1)$ is actually a member of $\mathcal{B}^y$, but we are going to use all the solutions we have found to construct a point-slope field. Observe that the family of curves $v(\cdot\,;\lambda)$ covers the slab $(0,1)\times\mathbb{R}$ simply. So for each point in $(0,1)\times\mathbb{R}$ there is exactly one $\lambda \in \mathbb{R}$ such that $v(\cdot\,;\lambda)$ passes through this point. In fact, by defining $\lambda : (0,1)\times\mathbb{R} \to \mathbb{R}$ as

$$\lambda(\widetilde{x},\widetilde{v}) := \widetilde{v}\,\widetilde{x}^{\frac{(1-y)m}{(m-1)y}}, \tag{29}$$

we have

$$v(\widetilde{x};\lambda(\widetilde{x},\widetilde{v})) = \widetilde{v}.$$

Thus $\lambda(\widetilde{x},\widetilde{v})$ provides the unique member of our family $v(\,\cdot\,;\lambda)$ with a graph that includes the point $(\widetilde{x},\widetilde{v})$. To each point $(\widetilde{x},\widetilde{v}) \in (0,1)\times\mathbb{R}$, we assign the slope of the curve $v(\,\cdot\,;\lambda(\widetilde{x},\widetilde{v}))$ at that point: define $\pi : (0,1)\times\mathbb{R} \to \mathbb{R}$ by

$$\pi(\widetilde{x},\widetilde{v}) := \frac{d}{dx}v(x;\lambda(\widetilde{x},\widetilde{v}))\Big|_{x=\widetilde{x}} = \frac{(y-1)m}{(m-1)y}\frac{\widetilde{v}}{\widetilde{x}}.$$

The mapping π constitutes the point-slope field we wanted to construct. Observe that one finds $\pi \in C^\infty_{\mathrm{loc}}((0,1)\times\mathbb{R};\mathbb{R})$.

Since we constructed the map π using the solutions of the Euler-Lagrange equations and π is differentiable away from the line $\{0\}\times\mathbb{R}$, it follows that the mapping $h_* : (0,1)\times W^{1,y}((0,1);\mathbb{R}) \to \mathbb{R}$ given by

$$\begin{aligned} h_*(x,v) &:= h(x,v(x),\pi(x,v(x));y,m) \\ &\qquad + [v'(x)-\pi(x,v(x))]h_{,\xi}(x,v(x),\pi(x,v(x));y,m) \end{aligned}$$

is a total derivative. Indeed, one may compute

$$h_*(x,v) := \frac{d}{dx}S(x,v(x))$$

with

$$\begin{aligned} S(x,v) := \frac{1}{(m-3)(m-2)}&\left[\frac{(y-1)m}{(m-1)y}\right]^m \left(\frac{v^{\frac{y}{y-1}}}{x}\right)^{m-3} \\ \times &\left[(m-2)(m-1) - 2(m-3)(m-1)\left(\frac{v^{\frac{y}{y-1}}}{x}\right) + (m-3)(m-2)\left(\frac{v^{\frac{y}{y-1}}}{x}\right)^2\right]. \end{aligned}$$

Thus h_* is an integrand for a path independent Hilbert integral [4, 6], provided that x remains away from zero. Actually, if $\widetilde{v}$ is a member of $\mathcal{B}^y$, then it satisfies

$$\lim_{x\to0^+}\frac{\widetilde{v}(x)}{x^{\frac{y-1}{y}}} = \lim_{x\to0^+}\frac{\widetilde{v}^{\frac{y}{y-1}}}{x} = 0,$$

and since $m > 3$, we have that $\lim_{x\to 0^+} S(x,\widetilde{v}(x)) = 0$. It follows that

$$\int_{(0,1)} h_*(x,\widetilde{v}(x))\,dx = S(1,\widetilde{v}(1)) \tag{30}$$

for any $\widetilde{v} \in \mathcal{B}^y$. This property of h_* is vital to our proof.

Another property that we need is the convexity of h with respect to the argument ξ. This convexity implies that Weierstrass' excess function is nonnegative. That is

$$E(x,v,\xi,\widetilde{\xi}) := h(x,v,\xi;y,m) - h(x,u,\widetilde{\xi};y,m) - (\xi-\widetilde{\xi})h_{,\xi}(x,u,\widetilde{\xi};y,m) \geq 0.$$

We are now prepared to compare the values of $G[v(\,\cdot\,;1)](y)$ with the value of $G[\,\cdot\,](y)$ for other competing functions. Let $\widetilde{v} \in \mathcal{B}^y$. Using (30) and the nonnegativity of the excess function, we may write

$$\begin{aligned} G[\widetilde{v}](y) &= \int_{(0,1)} [h(x,\widetilde{v},\widetilde{v}';y,m) - h_*(x,\widetilde{v})]\;dx\; + \int_{(0,1)} h_*(x,\widetilde{v})\,dx \\ &= \int_{(0,1)} E(x,\widetilde{v},\widetilde{v}',\pi(x,\widetilde{v});y,m)\,dx\; + \int_{(0,1)} h_*(x,v(x;1))\,dx \\ &\geq \int_{(0,1)} h(x,v(x;1),v'(x;1);y,m)\,dx = G[v(\,\cdot\,;1)]. \end{aligned}$$

Hence

$$\inf_{\widetilde{v}\in\mathcal{B}^y} G[\widetilde{v}](y) = G[v(\,\cdot\,;1)](y) = \frac{1}{s(y,m)}$$

for each $y \in (p_1, p]$.

Recalling (21)

$$J[u] \geq \int_{\{y|p_1<y\leq p\}} \mu_{,p}(y)s(y,m)\left[\inf_{v\in\mathcal{B}^y} G[v](y)\right] dy,$$

we may now write

$$J[u] \geq \int_{\{y|p_1<y\leq p\}} \mu_{,p}(y)\,dy = \mu(p) - \mu(p_1).$$

Therefore $I(p) \geq \mu(p)$ since $\mu \in \mathcal{M}_{p_1,p_2}$ implies $\mu(p_1) = 0$. The lemma has thus been proven for this case.

There remains only the last case to prove. The proof for this case is brief as it essentially follows from Case 2.

Case 3. $p_2 \leq p$:

Since $p \geq p_2$ and $\mu \in \mathcal{M}_{p_1,p_2}$, we have $\mu(p) = \mu(p_2)$. Using what we have done in Case 2 above, we find

$$J[u] \geq \int_{\{y|p_1<y\leq p_2\}} \mu_{,p}(y)s(y,m)\left[\inf_{v\in\mathcal{B}^y}\int_{(0,1)} h(x,v,v';y,m)\,dx\right]dy$$
$$= \int_{\{y|p_1<y\leq p_2\}} \mu_{,p}(y)s(y,m)\left[\inf_{v\in\mathcal{B}^y} G[v](y)\right]dy \geq \int_{\{y|p_1<y\leq p_2\}} \mu_{,p}(y)\,dy = \mu(p_2)-\mu(p_1).$$

Again, we have $I(p)\geq\mu(p)$ and the lemma is proven. □

In this section, we have shown $I(p)\geq\mu(p)$. In the next section we will prove the opposing inequality.

5. $I(p)\leq\mu(p)$

We wish to prove

Lemma 5.1. *For each $p\in[1,+\infty]$ we have $I(p)\leq\mu(p)$.*

Proof. As in Lemma 4.1, the proof splits naturally into three cases. The first of which is $1\leq p\leq p_1$; the second $p_1<p<p_2$; and the third $p_2\leq p$.

Case 1. $1\leq p\leq p_1$:

For this case, since $p\in[1,p_1]$ we may simply choose $u\in\mathcal{A}^p$ to be the absolute minimizer given in Remark 3.4. Recall in (17), we defined u_{am} by

$$u_{\mathrm{am}}(\mathbf{x}) := x^{\frac{y-1}{y}},$$

and $J[u_{\mathrm{am}}]=0$. Since $\mathbf{x}\in\Omega$, we have $y>p_1$. Therefore $u_{\mathrm{am}}\in\mathcal{A}^p$. Furthermore $\mu(p)=0$, because $\mu\in\mathcal{M}_{p_1,p_2}$ and $p\leq p_1$. Thus $I(p)\leq J[u_{\mathrm{am}}]=0=\mu(p)$ which proves this case.

To prove the remaining cases, for each $p\in(p_1,+\infty]$ we will construct a sequence $\left\{u^{(k)}\right\}_{k=1}^{\infty}\subset\mathcal{A}^p$ such that $\lim_{k\to+\infty}J[u^{(k)}]=\mu(p)$. This, of course, implies that $I(p)\leq\mu(p)$.

Case 2. $p_1<p<p_2$:

Let K be some integer such that both $K>\frac{1}{p_2-p}$ and $K>\frac{m-p}{p(p-1)}$. For each of the $k\in\{K,K+1,\dots\}$, we will divide the domain Ω into 5 subdomains and construct a sequence of mappings $\left\{u^{(k)}\right\}_{k=K}^{\infty}\subset\mathcal{A}^p$ over Ω using these subdomains (see Figure 5.1). The sequence we construct will have the desired properties.

For each $k\in\{K,K+1,\cdots\}$, set

$$\Omega_1^{(k)} := \left\{(x,y)\in\mathbb{R}^2 \mid p+\tfrac{1}{k}<y<p_2 \text{ and } 0<x<1\right\};$$
$$\Omega_2^{(k)} := \left\{(x,y)\in\mathbb{R}^2 \mid p\leq y\leq p+\tfrac{1}{k} \text{ and } 0<x\leq p-y+\tfrac{1}{k}\right\};$$
$$\Omega_3^{(k)} := \left\{(x,y)\in\mathbb{R}^2 \mid p\leq y\leq p+\tfrac{1}{k} \text{ and } p-y+\tfrac{1}{k}<x<1\right\};$$
$$\Omega_4^{(k)} := \left\{(x,y)\in\mathbb{R}^2 \mid p_1<y<p \text{ and } 0<x\leq\tfrac{1}{k}\right\};$$
$$\Omega_5^{(k)} := \left\{(x,y)\in\mathbb{R}^2 \mid p_1<k<p \text{ and } \tfrac{1}{k}<x<1\right\}.$$

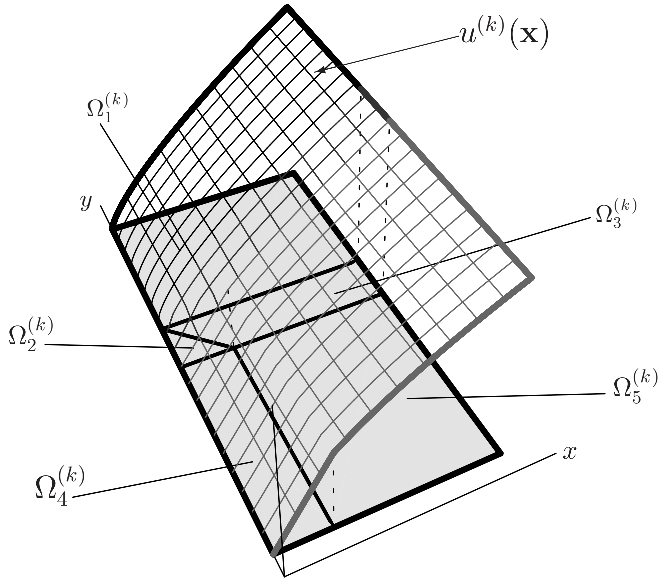

Figure 5.1: The decomposition of Ω and an example of $u^{(k)}$

Also let $l^{(k)} : \mathbb{R} \to \mathbb{R}$ be given by

$$l^{(k)}(y) := \frac{(p-1)m}{(m-1)p} + k\left[\frac{p+\frac{1}{k}-1}{p+\frac{1}{k}} - \frac{(p-1)m}{(m-1)p}\right](y-p).$$

For each $k \in \{K, K+1, \cdots\}$, the union $\bigcup_{1\le i\le 5} \Omega_i^{(k)} = \Omega$. Furthermore, since $K > \frac{m-p}{p(p-1)}$ we have

$$\frac{\left(p+\frac{1}{k}\right)-1}{\left(p+\frac{1}{k}\right)} \le 1 - \frac{K}{Kp+1} < 1 - \frac{m-p}{p(m-1)} = \frac{(p-1)m}{(m-1)p};$$

so the function $l^{(k)}$ is linear and satisfies

$$l^{(k)}(p) = \frac{(p-1)m}{(m-1)p} > \frac{\left(p+\frac{1}{k}\right)-1}{\left(p+\frac{1}{k}\right)} = l^{(k)}\left(p+\tfrac{1}{k}\right). \tag{31}$$

Now, we define the sequence $\left\{u^{(k)}\right\}_{k=K}^{\infty} \subset \mathcal{A}^p$ as

$$u^{(k)}(x,y) := \begin{cases} x^{\frac{y-1}{y}}, & (x,y) \in \Omega_1^{(k)}; \\ \left(p-y+\frac{1}{k}\right)^{l^{(k)}(y)-1} x, & (x,y) \in \Omega_2^{(k)}; \\ x^{l^{(k)}(y)}, & (x,y) \in \Omega_3^{(k)}; \\ \left(\frac{1}{k}\right)^{\frac{(y-1)m}{(m-1)y}-1} x, & (x,y) \in \Omega_4^{(k)}; \\ x^{\frac{(y-1)m}{(m-1)y}}, & (x,y) \in \Omega_5^{(k)}. \end{cases}$$

It is a straightforward computation to verify that $\{u^{(k)}\}_{k=K}^{\infty}$ is in fact a subset of $\mathcal{A}^p$. First, we establish that $\{u^{(k)}\}_{k=K}^{\infty} \subset W^{1,p}(\Omega;\mathbb{R})$. It is obvious that the sequence is contained in $L^p(\Omega;\mathbb{R})$, hence we need only show

$$\int_{\Omega_i^{(k)}} \left|u_{,x}^{(k)}(\mathbf{x})\right|^p d\mathbf{x} < +\infty \quad \text{and} \quad \int_{\Omega_i^{(k)}} \left|u_{,y}^{(k)}(\mathbf{x})\right|^p d\mathbf{x} < +\infty$$

for each $i = 1,2,3,4,5$. We do this for $i = 1,2$; the remaining inequalities being similar. Let $k \in \{K, K+1, \cdots\}$ be given. For $\Omega_1^{(k)}$, we have (here and in the sequel, we allow the constant C to change from line to line)

$$\int_{\Omega_1^{(k)}} \left|u_{,x}^{(k)}\right|^p d\mathbf{x} = \int_{p+\frac{1}{k}}^{p_2} \int_0^1 \left|\tfrac{y-1}{y} x^{-\frac{1}{y}}\right|^p dxdy \le C \int_0^1 x^{\frac{-kp}{kp+1}}\, dx < +\infty, \tag{32}$$

and

$$\int_{\Omega_1^{(k)}} \left|u_{,y}^{(k)}\right|^p d\mathbf{x} = \int_{p+\frac{1}{k}}^{p_2} \int_0^1 \left|\tfrac{1}{y^2} x^{\frac{y-1}{y}} \ln x\right|^p dxdy \le C \int_0^1 x^{\frac{-kp}{kp+1}}\, dx < +\infty. \tag{33}$$

For $\Omega_2^{(k)}$, we write

$$\begin{aligned}
\int_{\Omega_2^{(k)}} \left|u_{,x}^{(k)}\right|^p d\mathbf{x} &= \int_p^{p+\frac{1}{k}} \int_0^{p-y+\frac{1}{k}} \left|\left(p-y+\tfrac{1}{k}\right)^{l^{(k)}(y)-1}\right|^p dxdy \\
&= \int_p^{p+\frac{1}{k}} \left(p-y+\tfrac{1}{k}\right)^{p\left[l^{(k)}(y)-1\right]+1} dy \le \int_p^{p+\frac{1}{k}} \left(p-y+\tfrac{1}{k}\right)^{\frac{1}{kp+1}} dy < +\infty;
\end{aligned} \tag{34}$$

in the last line we used (31). Also

$$\int_{\Omega_2^{(k)}} \left|u_{,y}^{(k)}\right|^p d\mathbf{x} = \int_p^{p+\frac{1}{k}} \int_0^{\alpha(y)} \left|\left[\widetilde{C} \ln \alpha(y) - \frac{1}{\alpha(y)}\right] \alpha(y)^{l^{(k)}(y)-1} x\right|^p dxdy,$$

where for convenience, we have put $\widetilde{C} := k\left[l^{(k)}(p+\frac{1}{k}) - l^{(k)}(p)\right] = \frac{d}{dy} l^{(k)}(y)$ and $\alpha(y) := p - y + \frac{1}{k}$. Continuing, we have

$$\begin{aligned}
\int_{\Omega_2^{(k)}} \left|u_{,y}^{(k)}\right|^p d\mathbf{x} &= \frac{1}{p+1} \int_p^{p+\frac{1}{k}} \left|\widetilde{C} \ln \alpha(y) - \frac{1}{\alpha(y)}\right|^p \alpha(y)^{pl^{(k)}(y)+1}\, dy \\
&\le C \int_p^{p+\frac{1}{k}} \left(p-y+\tfrac{1}{k}\right)^{p\left[l^{(k)}(y)-1\right]+1} dy \le C \int_p^{p+\frac{1}{k}} \left(p-y+\tfrac{1}{k}\right)^{\frac{1}{kp+1}} dy < +\infty;
\end{aligned} \tag{35}$$

again we used (31) to reach the last line. In a similar fashion, one proves

$$\int_{\Omega_i^{(k)}} \left|u_{,x}^{(k)}\right|^p \, d\mathbf{x} < +\infty \quad \text{and} \quad \int_{\Omega_i^{(k)}} \left|u_{,y}^{(k)}\right|^p \, d\mathbf{x} < +\infty$$

for $i = 3, 4, 5$. This in conjunction with (32)–(35) implies $u^{(k)} \in W^{1,p}(\Omega; \mathbb{R})$. We additionally find that $u^{(k)}(\,\cdot\,, 0) = 0$ and $u^{(k)}(\,\cdot\,, 1) = 1$. Thus $\left\{u^{(k)}\right\}_{k=K}^{\infty} \subset \mathcal{A}^p$.

We now show $\lim_{k\to+\infty} J[u^{(k)}] = \mu(p)$. For each element of the sequence, we have $u^{(k)} \llcorner \Omega_1^{(k)} = u_{\mathrm{am}}$. Using Remark 3.4, we may write

$$\begin{aligned} J[u^{(k)}] &= \int_{\Omega} f(x, u^{(k)}, u_{,x}^{(k)}; y, m) \, d\mathbf{x} = \int_{\Omega\setminus\Omega_1^{(k)}} f(x, u^{(k)}, u_{,x}^{(k)}; y, m) \, d\mathbf{x} \\ &= \int_{\Omega_2^{(k)}} f(x, u^{(k)}, u_{,x}^{(k)}; y, m) \, d\mathbf{x} + \int_{\Omega_3^{(k)}} f(x, u^{(k)}, u_{,x}^{(k)}; y, m) \, d\mathbf{x} \\ &\quad + \int_{\Omega_4^{(k)}} f(x, u^{(k)}, u_{,x}^{(k)}; y, m) \, d\mathbf{x} + \int_{\Omega_5^{(n)}} f(x, u^{(k)}, u_{,x}^{(k)}; y, m) \, d\mathbf{x} \end{aligned} \tag{36}$$

It is fairly straightforward to show that the first three integrals in (36) go to zero as $k \to +\infty$. We do this for the first integral, the others being similar. Let $k \in \{K, K+1, \cdots\}$ be given. We first observe, from the definition (10) for h, that

$$|u| \le x^{\frac{y-1}{y}} \Longrightarrow h(x, u, \xi; y, m) \le x^{\frac{m-y}{y}} |\xi|^m \tag{37}$$

when $x \in (0, 1)$. Now it is clear from the definition of $u^{(k)}$ that $u^{(k)}(\mathbf{x}) \le x^{l^{(k)}(y)}$ for each $\mathbf{x} \in \Omega_2^{(k)}$. Furthermore, by (31) and since $l^{(k)}$ is linear, we see that $l^{(k)}(y) \ge \frac{y-1}{y}$ for each $y \in [p, p + \frac{1}{k}]$. Hence

$$u^{(k)}(\mathbf{x}) \le x^{\frac{y-1}{y}}$$

for all $\mathbf{x} \in \Omega_2^{(k)}$. Using this last inequality and (37), we have

$$h(x, u^{(k)}(\mathbf{x}), u_{,x}^{(k)}(\mathbf{x}); y, m) \le x^{\frac{m-y}{y}} |u_{,x}(\mathbf{x})|^m = x^{\frac{m-y}{y}} \left(p - y + \tfrac{1}{k}\right)^{m\left[l^{(k)}(y)-1\right]},$$

when $\mathbf{x} \in \Omega_2^{(k)}$. From definition (11) for f, we may now write

$$\begin{aligned} &\int_{\Omega_2^{(k)}} f(x, u^{(k)}, u_{,x}^{(k)}; y, m) \, d\mathbf{x} \\ &\qquad \le \int_p^{p+\frac{1}{k}} \mu_{,p}(y) s(y, m) \left(p - y + \tfrac{1}{k}\right)^{m\left[l^{(k)}(y)-1\right]} \left[\int_0^{p-y+\frac{1}{k}} x^{\frac{m-y}{y}} \, dx \right] dy \\ &\qquad = \int_p^{p+\frac{1}{k}} \frac{y}{m} \mu_{,p}(y) s(y, m) \left(p - y + \tfrac{1}{k}\right)^{m\left[l^{(k)}(y)-1+\frac{1}{y}\right]} dy. \end{aligned}$$

For $y \in [p, p+\frac{1}{k}]$, we find that $l^{(k)}(y) - 1 + \frac{1}{y} \geq 0$ by (31). Therefore

$$\int_{\Omega_2^{(k)}} f(x, u^{(k)}, u^{(k)}_{,x}; y, m)\, d\mathbf{x} \leq \int_p^{p+\frac{1}{k}} \frac{y}{m} \mu_{,p}(y) s(y,m)\, dy \leq \frac{C}{k} \|\mu_{,p}\|_{L^1((p_1,p_2);\mathbb{R})},$$

where C is a constant not dependent upon k. So the first integral in (36) approaches zero as $k \to +\infty$. In a somewhat similar manner, the second and third integrals in (36) may also be shown to tend to zero as $k \to +\infty$.

It only remains to examine the last integral from (36). For this integral, we have

$$\int_{\Omega_5^{(k)}} f(x, u^{(k)}, u^{(k)}_{,x}; y, m)\, d\mathbf{x} = \int_{p_1}^{p} \mu_{,p}(y) s(y,m) \left[\int_{\frac{1}{k}}^{1} h(x, u^{(k)}, u^{(k)}_{,x}; y, m)\, dx \right] dy. \tag{38}$$

At each $y \in (p_1, p)$ we see that $u^{(k)}(\,\cdot\,, y) = v(\,\cdot\,; 1)$, where $v(\,\cdot\,; 1)$ was the solution to the Euler-Lagrange equations that we found earlier in (28). We thus have that as $k \to +\infty$, the inner integral in (38) approaches $\frac{1}{s(y,m)}$ at each $y \in (p_1, p_2)$. Thus

$$\lim_{k\to+\infty} J[u^{(k)}] = \int_{p_1}^{p} \mu_{,p}(y)\, dy = \mu(p) - \mu(p_1) = \mu(p).$$

This implies $\inf_{u\in\mathcal{A}^p} J[u] \leq \mu(p)$. Therefore $I(p) \leq \mu(p)$ for this case.

The final case for this lemma is analogous to but simpler than the previous case.

Case 3. $p_2 \leq p$:

For this case, we will only divide the domain into two subdomains. For every $k \in \{1, 2, \cdots\}$, set

$$\Omega_1^{(k)} := \left\{ (x,y) \in \mathbb{R}^2 \mid p_1 < y < p_2 \text{ and } 0 < x \leq \tfrac{1}{k} \right\}$$
$$\Omega_2^{(k)} := \left\{ (x,y) \in \mathbb{R}^2 \mid p_1 < y < p_2 \text{ and } \tfrac{1}{k} < x < 1 \right\}.$$

So $\Omega_1^{(n)} \cup \Omega_2^{(n)} = \Omega$ for each $k \in \{1, 2, \cdots\}$.

We let the sequence $\left\{u^{(k)}\right\}_{k=1}^{\infty} \subset \mathcal{A}^\infty \subset \mathcal{A}^p$ be defined as

$$u^{(k)}(x,y) := \begin{cases} \left(\frac{1}{k}\right)^{\frac{(y-1)m}{(m-1)y} - 1} x, & (x,y) \in \Omega_1^{(k)}; \\ x^{\frac{(y-1)m}{(m-1)y}}, & (x,y) \in \Omega_2^{(k)}. \end{cases}$$

Clearly, each element of this sequence is in $\mathcal{A}^\infty$.

For any $k \in \{1, 2, \cdots\}$ we have

$$\begin{aligned} J[u^{(k)}] &= \int_{\Omega} f(x, u^{(k)}, u^{(k)}_{,x}; y, m)\, d\mathbf{x} \\ &= \int_{\Omega_1^{(k)}} f(x, u^{(k)}, u^{(k)}_{,x}; y, m)\, d\mathbf{x} + \int_{\Omega_2^{(k)}} f(x, u^{(k)}, u^{(k)}_{,x}; y, m)\, d\mathbf{x}. \end{aligned} \tag{39}$$

The first integral in (39) approaches zero as $k \to +\infty$. As before, for the second integral we have

$$\int_{\Omega_2^{(k)}} f(x, u^{(k)}, u^{(k)}_{,x}; y, m)\, d\mathbf{x} = \int_{p_1}^{p_2} \mu_{,p}(y) s(y,m) \left[\int_{\frac{1}{k}}^{1} h(x, u^{(k)}, u^{(k)}_{,x}; y, m)\, dx \right] dy.$$

For each $y \in (p_1, p_2)$, the inner integral above approaches $\frac{1}{s(y,m)}$ as $k \to +\infty$ (see Case 2). It follows that

$$\lim_{k \to +\infty} J[u^{(k)}] = \int_{p_1}^{p_2} \mu_{,p}(y) dy = \mu(p_2).$$

Since $\mu \in \mathcal{M}_{p_1,p_2}$ and $p \geq p_2$, by definition of $\mathcal{M}_{p_1,p_2}$ we must have $\mu(p) = \mu(p_2)$. Therefore $\inf_{u \in \mathcal{A}^p} J[u] \leq \mu(p)$ and $I(p) \leq \mu(p)$ as desired. Thus the lemma is proven. □

6. Main Result and Summary

Lemma 4.1 and Lemma 5.1, from the previous two sections, imply the following result.

Theorem 6.1 (Main Result)**.** *For each $p \in [1, +\infty]$, we have $I(p) = \mu(p)$.*

What we have shown is that there are functionals that exhibit the Lavrentiev phenomenon and have an infimum that depends continuously upon the amount of regularity required of the competing functions. In fact, for each $\mu \in \mathcal{M}$, we have provided functionals and boundary conditions such that if we require the competing functions to be in $W^{1,p}$, then the infima for the functionals are equal to $\mu(p)$. Moreover, if $\mu \in \mathcal{M}_{p_1,p_2}$ and $\mu_{,p}$ is smooth over the interval (p_1, p_2), then for each non-negative integer k, we have constructed (Remark 3.1) an integrand that is in $C^k(\Omega \times \mathbb{R} \times \mathbb{R}; \mathbb{R})$.

There are many ways to generalize the result we have presented. In our construction of the functionals (Section 3), we restricted our attention to particular parameterizations of the exponents on u and ξ in (10). By modifying this parametrization, one can also construct a functional with an infimum that matches a μ with a finite number of jump discontinuities. By using an approximation similar to that in Lemma 5.1 (Case 2), it can be shown that one may impose full Dirichlet boundary conditions such that Theorem 6.1 remains true. Another possible generalization is to increase the dimensions for the domain and range of the admissible mappings. It is still unknown, however, whether or not there exists a one-dimensional version of the result we have presented. In other words, we do not know if there exists any non-constant $\mu \in \mathcal{M}$ and a one-dimensional variational problem such that the infimum for the functional is $\mu(p)$ when the admissible functions are required to be in $W^{1,p}$.

Acknowledgements. I wish to express my gratitude to W. Hrusa and V. J. Mizel for their roles as my Ph.D. advisors. I am also grateful for E. Acerbi's and G. Mingione's interest in these examples. This research was partially supported under the US National Science Foundation's VIGRE program.

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