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On a Class of Multidimensional Optimal Transportation Problems

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We prove existence, uniqueness, duality results and give a characterization of optimal measure-preserving maps for a class of optimal transportation problems with several marginals with compact support in $\mathbb{R}$ under the requirement that the cost function satisfies a so-called monotonicity of order 2 condition. Explicit formulas for the minimizers are given and links with some rearrangement inequalities are sketched.

1. Introduction

Optimal transportation problems (with two marginals) have received a lot of attention in the last decade mainly for two reasons. The first one is that since the seminal paper of Brenier [2] and his polar factorization theorem, our understanding of the geometry of the problem, of existence, uniqueness and duality issues has radically improved. We refer the reader of course to Brenier [2], to Mc Cann and Gangbo [6] and to Gangbo [5] who developed a dual approach that is of particular interest in the present article. The second reason results from the importance of mass transportation problems in very different applied fields: probability theory and statistics [15], [14], fluid mechanics [3], shape optimization [1], interacting gases [12], mathematical economics [10], [4] to name only a few.

In contrast, there are only few known results on existence of solutions to Monge's problem with several marginals. Gangbo and Święch in [7] however completely solved the problem in the case of a particular (quadratic) cost function and with several marginals with support in $\mathbb{R}^n$. In the present article, we restrict ourselves to the case of marginals supported on the real line but consider a wider class of transportation costs than in [7].

1.1. Monge's optimal problem with several marginals in $\mathbb{R}$

We first recall that, given a probability space $(\Omega, \mathcal{A}, \mu)$, a measurable space $(\Omega', \mathcal{A}')$ and a measurable map $s : \Omega \to \Omega'$, the push-forward of μ through s, denoted by $s\sharp\mu$ is the probability measure on $(\Omega', \mathcal{A}')$ defined by:

$$s\sharp\mu(B) := \mu(s^{-1}(B)) \text{ for every } B \in \mathcal{A}'.$$

Assume now that we are given $n+1$ Borel probability measures $(\mu_0, \dots, \mu_n)$ on $\mathbb{R}$ and a cost function f, our aim is to study the following problem (Monge's optimal transportation

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with $n+1$ marginals in $\mathbb{R}$):

$$(\mathcal{M})\ \inf_{s\in\Gamma} C(s) := \int_{\mathbb{R}} f(t, s_1(t), \ldots, s_n(t))d\mu_0(t)$$

where

$$\Gamma := \{s = (s_1, \ldots, s_n) \text{ each } s_i\colon \mathbb{R} \to \mathbb{R} \text{ is Borel, and } s_i\sharp\mu_0 = \mu_i \text{ for } i = 1, \ldots, n\}.$$

In the sequel, given some subset K of $\mathbb{R}^k$, we shall denote by $C^0(K, \mathbb{R})$ the space of continuous real-valued functions on K. In what follows, given a Borel measure, with support $K \subset \mathbb{R}$ and h a real-valued μ-integrable function that is only defined on K, slightly abusing notations, we shall write $\int_{\mathbb{R}} hd\mu$ instead of $\int_K hd\mu$ that is we will extend h by 0 outside K.

Closely following Wilfrid Gangbo's approach developed in [5] to prove Brenier's theorem and extended in [7] for a certain class of transportation problems with several marginals, our analysis is based on the study of the *dual* problem. This dual problem is the optimization program:

$$(\mathcal{D})\ \sup_{h\in\mathcal{H}} F(h) := \sum_{i=0}^{n} \int_{\mathbb{R}} h_i(t_i)d\mu_i(t_i)$$

with

$$\mathcal{H} := \{h = (h_0, \ldots, h_n) : h_i \in C^0(T_i, \mathbb{R}) \text{ for } i = 0, \ldots, n, \text{ and}$$
$$\sum_{i=0}^{n} h_i(t_i) \le f(t_0, \ldots, t_n) \text{ for all } (t_0, \ldots, t_n) \in T_0 \times \ldots \times T_n\}$$

and $T_i := \mathrm{supp}(\mu_i),\ i = 0, \ldots, n$.

1.2. Assumptions and main result

In what follows, we will assume that each measure μ_i has a compact support and that μ_0 does not charge points (simple examples show that otherwise Γ could be empty):

$$T_i \text{ is a compact subset of } \mathbb{R} \text{ for } i = 0, \ldots, n \tag{1}$$

$$\mu_0(\{t\}) = 0, \text{ for all } t \in T_0. \tag{2}$$

The main assumption on the cost function f is the monotonicity of order 2 condition, introduced and discussed in the next section. We further assume that

$$f \in C^0(T, \mathbb{R}) \tag{3}$$

where

$$T := T_0 \times \ldots \times T_n.$$

Let us remark at this early stage that if $s \in \Gamma$ and $h \in \mathcal{H}$ then by the very definitions of Γ and $\mathcal{H}$ we have

$$\begin{aligned} C(s) &= \int_{\mathbb{R}} f(t, s_1(t), \ldots, s_n(t)) d\mu_0(t) \\ &\geq \int_{\mathbb{R}} h_0(t) d\mu_0(t) + \sum_{i=1}^{n} \int_{\mathbb{R}} h_i(s_i(t)) d\mu_0(t) \\ &= \sum_{i=0}^{n} \int_{\mathbb{R}} h_i(t_i) d\mu_i(t_i) = F(h) \end{aligned}$$

hence

$$\inf(\mathcal{M}) \geq \sup(\mathcal{D}). \tag{4}$$

Further, if $s \in \Gamma$ and $h \in \mathcal{H}$ are such that $F(h) = C(s)$ then s solves $(\mathcal{M})$ and h solves $(\mathcal{D})$. Note also that (4) implies that the value of each problem is finite.

The main results of the paper may be summarized as follows. If conditions (1), (2), (3) are satisfied and if f is strictly monotone of order 2 on T (see Definition 2.1), then we have:

- $(\mathcal{M})$ admits a unique (up to μ_0 - a.e. equivalence) solution $s = (s_1, \ldots, s_n)$,
- each component s_i of s is nondecreasing,
- $(\mathcal{D})$ is dual to $(\mathcal{M})$ in the sense that $\inf(\mathcal{M}) = \sup(\mathcal{D})$,
- $(\mathcal{D})$ admits solutions and if $h = (h_0, \ldots, h_n) \in \mathcal{H}$ solves $(\mathcal{D})$ one has for μ_0 - a.e. $t \in T_0$:

$$h_0(t) + \sum_{i=1}^{n} h_i(s_i(t)) = f(t, s_1(t), \ldots, s_n(t)).$$

In the next section, the monotonicity of order 2 assumption on the cost function is introduced and some preliminary results are given. In Section 3, we study the dual problem. In Section 4, we state and prove the announced existence, uniqueness, duality and characterization results. Section 5 is devoted to two concluding remarks, in particular we relate the monotonicity of the solution of $(\mathcal{M})$ to Hardy-Littlewood's rearrangement inequality.

2. Preliminaries

2.1. Monotone functions of order 2

In what follows $(e_1, \ldots, e_p)$ denotes the canonical basis of $\mathbb{R}^p$.

Definition 2.1. A real-valued function g defined on some nonempty set A of $\mathbb{R}^p$ is strictly monotone of order 2 on A if for all $(i, j) \in \{1, \ldots, p\}^2$ with $i \neq j$, for all $x \in A$, for all $t > 0$ and all $s > 0$ such that $(x + te_i + se_j, x + se_j, x + te_i) \in A^3$ one has:

$$g(x + te_i + se_j) - g(x + se_j) < g(x + te_i) - g(x).$$

Note that if g is of class C^2, and A is convex, a sufficient condition for g to be strictly monotone of order 2 on A is that all second order derivatives $\partial_{ij} g$ with $i \neq j$ are negative.

This requirement holds for instance in the case (considered in [7]) $A = \mathbb{R}^p$ and:

$$g(x_1, \ldots, x_p) := \frac{1}{2} \sum_{1 \le i \ne j \le p} (x_i - x_j)^2.$$

Note also that Definition 2.1 does not imply partial concavity of g, that is concavity of the partial functions $x_i \mapsto g(x_1, \ldots, x_i, \ldots, x_n)$ with fixed $(x_j)_{j \ne i}$ (e.g. $\exp(x_1 - x_2)$ in $\mathbb{R}^2$, $\varphi(x_1)\psi(x_2)$ with φ convex increasing, and ψ decreasing...).

We shall use in the sequel two elementary lemmas.

Lemma 2.2. *Let A_1 and A_2 be two nonempty compact subsets of $\mathbb{R}$, let g be a continuous strictly monotone of order 2 function on $A_1 \times A_2$, let $f_2 \in C^0(A_2, \mathbb{R})$ and let*

$$f_1(x_1) := \inf_{x_2 \in A_2} \left(g(x_1, x_2) - f_2(x_2)\right), \text{ for all } x_1 \in A_1.$$

Then there exists a subset N_1 of A_1 which is at most countable and a nondecreasing map γ_1 from A_1 to A_2 such that for all $x_1 \in A_1 \setminus N_1$ and all $x_2 \in A_2$ one has:

$$f_1(x_1) + f_2(x_2) = g(x_1, x_2) \Longleftrightarrow x_2 = \gamma_1(x_1).$$

Proof. Let us firstly note that in the definition of f_1, the inf actually is achieved. Secondly, let us set

$$B := \{(x_1, x_2) \in A_1 \times A_2 : f_1(x_1) + f_2(x_2) = g(x_1, x_2)\}$$

and let $((x_1, x_2), (y_1, y_2)) \in B^2$. We then have

$$\begin{aligned} g(x_1, x_2) - f_2(x_2) &\le g(x_1, y_2) - f_2(y_2) \\ g(y_1, y_2) - f_2(y_2) &\le g(y_1, x_2) - f_2(x_2) \end{aligned}$$

which implies $g(x_1, x_2) - g(y_1, x_2) \le g(x_1, y_2) - g(y_1, y_2)$. This latter inequality together with the strict monotonicity of order 2 of g imply that $(x_1 - y_1)(x_2 - y_2) \ge 0$.

For all $x_1 \in A_1$ define then

$$\begin{aligned} \gamma_-(x_1) &:= \inf\{x_2 \in A_2 : (x_1, x_2) \in B\} \\ \gamma_+(x_1) &:= \sup\{x_2 \in A_2 : (x_1, x_2) \in B\}. \end{aligned}$$

From the above, γ_- and γ_+ are non decreasing and if $(x_1, y_1) \in A_1^2$ with $x_1 < y_1$ then $\gamma_-(y_1) \ge \gamma_+(x_1)$. Since A_2 is bounded, the set $\{x_1 \in A_1 : \gamma_+(x_1) - \gamma_-(x_1) > \varepsilon\}$ is finite for every $\varepsilon > 0$, hence $\gamma_- = \gamma_+$ except on an at most countable subset of A_1. Defining γ_1 as a selection of the set-valued map $x_1 \in A_1 \mapsto [\gamma_-(x_1), \gamma_+(x_1)]$ (so that γ_1 is nondecreasing) there exists a subset N_1 of A_1 which is at most countable such that if $x_1 \in A_1 \setminus N_1$, then $(x_1, x_2) \in B$ if and only if $x_2 = \gamma_1(x_1)$. □

Lemma 2.3. *Let $p \in \mathbb{N}$, $p \ge 2$, let $A_1, \ldots, A_p$ be nonempty compact subsets of $\mathbb{R}$, let g be a continuous strictly monotone of order 2 function on $A_1 \times \ldots \times A_p$, let $f_p \in C^0(A_p, \mathbb{R})$ and define for all $(x_1, \ldots, x_{p-1}) \in A_1 \times \ldots \times A_{p-1}$:*

$$V(x_1, \ldots, x_{p-1}) := \inf_{x_p \in A_p} g(x_1, \ldots, x_{p-1}, x_p) - f_p(x_p).$$

Then V is continuous and strictly monotone of order 2 on $A_1 \times \ldots \times A_{p-1}$.

Proof. The continuity of V is straightforward. To prove strict monotonicity of order 2, what one has to prove is: for all $(i,j) \in \{1,\dots,p-1\}^2$ with $i \neq j$, for all $x = (x_1,\dots,x_{p-1}) \in A_1 \times \dots \times A_{p-1}$, all $t > 0$ and all $s > 0$ such that $(x+te_i+se_j, x+se_j, x+te_i) \in (A_1 \times \dots \times A_{p-1})^3$

$$V(x+te_i+se_j) + V(x) < V(x+te_i) + V(x+se_j). \tag{5}$$

For $(y,z) \in (A_1 \times \dots \times A_{p-1})^2$ let us observe that

$$V(y) + V(z) = \inf_{(y_p,z_p)\in A_p^2} \left(g(y_1,\dots,y_p) + g(z_1,\dots,z_p) - f_p(y_p) - f_p(z_p) \right).$$

Using the facts that the infimum is attained in the previous expression and that g is strictly monotone of order 2 on $A_1 \times \dots \times A_p$, we immediately get that (5) holds, hence V is strictly monotone of order 2 on $A_1 \times \dots \times A_{p-1}$. □

2.2. From $\mathcal{H}$ to $\mathcal{H}_0$

Let us denote for all $i = 0,\dots,n$ and all $t := (t_0,\dots,t_n) \in T$, $\pi_i(t) := t_i$ and let us observe that $h \in \mathcal{H}$ if and only if

$$h_i(\tau_i) \le \inf\{f(t) - \sum_{0\le j\le n,\ j\neq i} h_j(t_j) \,:\, t \in T,\, \pi_i(t) = \tau_i\}$$

for all $i = 0,\dots,n$ and all $\tau_i \in T_i$. In view of the maximization problem $(\mathcal{D})$ this suggests to pay special attention to the elements of $\mathcal{H}$ for which there are equalities in the previous system of inequalities. To that end we define:

$$\begin{aligned}\mathcal{H}_0 :=& \{h \in \mathcal{H} \,:\, \forall i = 0,\dots,n,\ \forall \tau_i \in T_i,\\ & h_i(\tau_i) = \inf\{f(t) - \sum_{0\le j\le n,\ j\neq i} h_j(t_j) \,:\, t \in T,\, \pi_i(t) = \tau_i\}\}.\end{aligned}$$

Nonemptiness of $\mathcal{H}_0$ and the natural role of $\mathcal{H}_0$ in problem $(\mathcal{D})$ follow from:

Lemma 2.4. *Let $h = (h_0,\dots,h_n) \in \mathcal{H}$. Then there exists $k = (k_0,\dots,k_n) \in \mathcal{H}_0$ such that $k_i \ge h_i$ for $i = 0,\dots,n$.*

Proof. Define for all $\tau_0 \in T_0$:

$$g_0(\tau_0) := \inf\{f(t) - \sum_{j=1}^{n} h_j(t_j) \,:\, t \in T,\, \pi_0(t) = \tau_0\}$$

since $h \in \mathcal{H}$, $g_0 \ge h_0$.

Define then inductively for $i = 1,\dots,n$ and $\tau_i \in T_i$:

$$g_i(\tau_i) := \inf\{f(t) - \sum_{j<i} g_j(t_j) - \sum_{j>i} h_j(t_j) \,:\, t \in T,\, \pi_i(t) = \tau_i\}.$$

Assuming $1 \leq i \leq n-1$, by the definition of g_i, for all $t \in T$ one has

$$g_i(t_i) \leq f(t) - \sum_{j<i} g_j(t_j) - \sum_{j>i} h_j(t_j)$$

which can be rewritten as

$$h_{i+1}(t_{i+1}) \leq f(t) - \sum_{j<i+1} g_j(t_j) - \sum_{j>i+1} h_j(t_j).$$

Taking the infimum in $(t_1, \ldots, t_i, t_{i+2} \ldots t_n)$ of the rightmost member of this inequality, we exactly get $h_{i+1} \leq g_{i+1}$.

This proves that $g_i \geq h_i$ for $i = 0, \ldots, n$. Note also that by construction of g_n one has $g = (g_0, \ldots, g_n) \in \mathcal{H}$.

Define now, for all $\tau_0 \in T_0$:

$$k_0(\tau_0) := \inf\{f(t) - \sum_{j=1}^{n} g_j(t_j) \,:\, t \in T,\, \pi_0(t) = \tau_0\}.$$

Obviously, since $g \in \mathcal{H}$, $k_0 \geq g_0$. Define then inductively for $i = 1, \ldots, n$ and $\tau_i \in T_i$:

$$k_i(\tau_i) := \inf\{f(t) - \sum_{j<i} k_j(t_j) - \sum_{j>i} g_j(t_j) \,:\, t \in T,\, \pi_i(t) = \tau_i\}.$$

As previously, one has $k_i \geq g_i$ ($\geq h_i$) for all i. We aim to prove now that $k_i \leq g_i$ for all i. This actually follows from the definition of k_i, g_i and the fact that for given i, $k_j \geq g_j$ for $j < i$ and $g_j \geq h_j$ for $j > i$. Hence we have $k_i = g_i$ for $i = 0, \ldots, n$, so that for all i and all $\tau_i \in T_i$

$$k_i(\tau_i) := \inf\{f(t) - \sum_{j \neq i} k_j(t_j) \,:\, t \in T,\, \pi_i(t) = \tau_i\},$$

and hence $k = (k_0, \ldots, k_n) \in \mathcal{H}_0$ and $k_i \geq h_i$ for $i = 0, \ldots, n$. □

2.3. Properties of $\mathcal{H}_0$

Lemma 2.5. *Let $h \in \mathcal{H}_0$. Then there exist a subset N of T_0 which is at most countable and n nondecreasing maps $s_1, \ldots, s_n$ from T_0 to $T_1, \ldots, T_n$ such that for all $t_0 \in T_0 \setminus N$ and all $(t_1, \ldots, t_n) \in T_1 \times \ldots \times T_n$ one has:*

$$\sum_{i=0}^{n} h_i(t_i) = f(t_0, t_1, \ldots, t_n) \Longleftrightarrow t_i = s_i(t_0), \textit{for } i = 1, \ldots, n.$$

Proof. Let $(t_0, \ldots, t_n) \in T$ be such that

$$\sum_{i=0}^{n} h_i(t_i) = f(t_0, t_1, \ldots, t_n). \tag{6}$$

Since $h \in \mathcal{H}_0$, (6) can be rewritten as

$$h_0(t_0) = \inf_{t_1' \in T_1} V_1(t_0, t_1') - h_1(t_1') = V_1(t_0, t_1) - h_1(t_1), \tag{7}$$

where

$$V_1(t_0, t_1) := \inf_{(t_2,\dots,t_n) \in T_2 \times \dots \times T_n} f(t_0, t_1, t_2 \dots, t_n) - \sum_{i=2}^{n} h_i(t_i). \tag{8}$$

Applying Lemma 2.3 inductively we obtain that V_1 is strictly monotone of order 2 on $T_0 \times T_1$, and using Lemma 2.2, we obtain the existence of a nondecreasing function $s_1 : T_0 \to T_1$ and of an at most countable set $N_1 \subset T_0$ such that if $t_0 \in T_0 \setminus N_1$ and $t_1 \in T_1$ then condition (7) is equivalent to $t_1 = s_1(t_0)$. A similar argument for variables $t_2, \dots, t_n$ completes the proof. □

Lemma 2.6. *Let $h \in \mathcal{H}_0$, let $N \subset T_0$ and let $(s_1, \dots, s_n)$ be as in Lemma 2.5. Let $i \in \{1, \dots, n\}$, let $\varphi_i \in C^0(T_i, \mathbb{R})$ and define for $\varepsilon > 0$ and all $t_0 \in T_0$:*

$$h_0^\varepsilon(t_0) := \inf_{(t_1,\dots,t_n) \in T_1 \times \dots \times T_n} f(t_0, t_1, \dots t_n) - \sum_{j=1}^{n} h_j(t_j) - \varepsilon \varphi_i(t_i).$$

Then for every $t_0 \in T_0 \setminus N$ one has

$$\lim_{\varepsilon \to 0+} \frac{1}{\varepsilon}[h_0^\varepsilon(t_0) - h_0(t_0)] = -\varphi_i(s_i(t_0)).$$

Proof. Since $h \in \mathcal{H}_0$, it is obvious that $\|h_0^\varepsilon - h_0\|_\infty = 0(\varepsilon)$. Now, let $t_0 \in T_0 \setminus N$, according to Lemma 2.5, we have:

$$h_0(t_0) = f(t_0, s_1(t_0), \dots, s_n(t_0)) - \sum_{i=1}^{n} h_i(s_i(t_0)).$$

Let also $(t_1^\varepsilon, \dots, t_n^\varepsilon)$ in $T_1 \times \dots \times T_n$ be such that

$$h_0^\varepsilon(t_0) = f(t_0, t_1^\varepsilon, \dots, t_n^\varepsilon) - \sum_{i=1}^{n} h_i(t_i^\varepsilon) - \varepsilon \varphi_i(t_i^\varepsilon) \tag{9}$$

and consider a cluster point $(t_1, \dots, t_n)$ of $(t_1^\varepsilon, \dots, t_n^\varepsilon)$. Passing to the limit in (9) yields

$$h_0(t_0) = f(t_0, t_1, \dots, t_n) - \sum_{i=1}^{n} h_i(t_i)$$

and since $t_0 \in T_0 \setminus N$ this implies $t_i = s_i(t_0)$ for all i, hence $(s_1(t_0), \dots, s_n(t_0))$ is the only cluster point of $(t_1^\varepsilon, \dots, t_n^\varepsilon)$. Since T_i is compact for all i, $(t_1^\varepsilon, \dots, t_n^\varepsilon)$ converges to $(s_1(t_0), \dots, s_n(t_0))$.

Further, since $h \in \mathcal{H}_0$, one has

$$h_0(t_0) \leq f(t_0, t_1^\varepsilon, \dots, t_n^\varepsilon) - \sum_{i=1}^{n} h_i(t_i^\varepsilon) = h_0^\varepsilon(t_0) + \varepsilon \varphi_i(t_i^\varepsilon) \tag{10}$$

and by the definition of h_0^ε we have

$$\begin{aligned} h_0^\varepsilon(t_0) &\leq f(t_0, s_1(t_0), \dots, s_n(t_0)) - \sum_{i=1}^n h_i(s_i(t_0)) - \varepsilon\varphi_i(s_i(t_0)) \\ &= h_0(t_0) - \varepsilon\varphi_i(s_i(t_0)). \end{aligned} \tag{11}$$

Rewriting (10) and (11) as

$$-\varphi_i(t_i^\varepsilon) \leq \frac{1}{\varepsilon}[h_0^\varepsilon(t_0) - h_0(t_0)] \leq -\varphi_i(s_i(t_0))$$

and using the convergence of t_i^ε to $s_i(t_0)$, we get the desired result. □

3. The dual problem

3.1. Existence

Proposition 3.1. *Problem $(\mathcal{D})$ admits at least one solution. Moreover if $h \in \mathcal{H}$ is a solution of $(\mathcal{D})$, then there exists $k \in \mathcal{H}_0$ such that $h_i = k_i$ μ_i - a.e. for $i = 0, \dots, n$.*

Proof. The second statement follows from Lemma 2.4. Now note that $\mathcal{H}_0$ and the linear functional F are invariant under any transformation of the form

$$h = (h_0, \dots, h_n) \mapsto (h_0 + \lambda_0, \dots, h_n + \lambda_n).$$

With

$$(\lambda_0, \dots \lambda_n) \in \mathbb{R}^{n+1} \text{ and } \sum_{i=0}^n \lambda_i = 0$$

together with Lemma 2.4, this in particular implies that for any $h \in \mathcal{H}$ there exists $k \in \mathcal{H}_0^0$ such that $F(k) \geq F(h)$ where

$$\mathcal{H}_0^0 := \{(h_0, \dots, h_n) \in \mathcal{H}_0 : \max_{T_1} h_1 = \dots = \max_{T_n} h_n = 0\}.$$

Let us prove that $\mathcal{H}_0^0$ equipped with the sup norm

$$\|h\|_\infty := \max_{i=0,\dots,n} (\max_{T_i} |h_i|)$$

is compact in $C^0(T_0, \mathbb{R}) \times \dots \times C^0(T_n, \mathbb{R})$ (which will prove the existence result since F is of course continuous with respect to that norm). The fact that $\mathcal{H}_0^0$ is closed in $C^0(T_0, \mathbb{R}) \times \dots \times C^0(T_n, \mathbb{R})$ equipped with the sup norm is obvious.

Let $h \in \mathcal{H}_0^0$. Since $h_i \leq 0$ for $i \geq 1$ one has

$$h_0(t_0) \geq \min_T f \text{ for all } t_0 \in T_0.$$

Now taking $t_i \in T_i$ such that $h_i(t_i) = 0$ for $i \geq 1$ and any $t_0 \in T_0$, we get

$$h_0(t_0) \leq f(t_0, t_1, \dots, t_n) \leq \max_T f.$$

Now let $i \geq 1$ and $t_i \in T_i$. Since $h_j \leq 0$ for $j \geq 1$, we have

$$\begin{aligned} h_i(t_i) &\geq \inf\{f(t_1, \dots, t_n) - h_0(t_0) : t \in T, \ \pi_i(t) = t_i\} \\ &\geq \min_T f - \max_T f. \end{aligned}$$

Hence $\mathcal{H}_0^0$ is bounded.

Now let $(t_0, t_0') \in T_0^2$ and let $(t_1, \dots, t_n)$ and $(t_1', \dots, t_n')$ be in $T_1 \times \ldots \times T_n$ and such that

$$h_0(t_0) = f(t_0, \dots, t_n) - \sum_{i=1}^n h_i(t_i), \ h_0(t_0') = f(t_0', \dots, t_n') - \sum_{i=1}^n h_i(t_i').$$

We have:

$$\begin{aligned} h_0(t_0) \leq & f(t_0, t_1' \dots, t_n') - \sum_{i=1}^n h_i(t_i') \\ = & h_0(t_0') + f(t_0, t_1', \dots, t_n') - f(t_0', t_1', \dots, t_n') \end{aligned}$$

and similarly

$$h_0(t_0') \leq h_0(t_0) + f(t_0', t_1, \dots, t_n) - f(t_0, t_1, \dots, t_n).$$

Consequently

$$|h_0(t_0) - h_0(t_0')| \leq \sup_{(t_1,\dots,t_n) \in T_1 \times \ldots \times T_n} |f(t_0, t_1, \dots, t_n) - f(t_0', t_1, \dots, t_n)|.$$

Similar arguments yield for all $\varepsilon > 0$ and all $i \in \{0, \dots, n\}$

$$\sup\{|h_i(t_i) - h_i(t_i')| : (t_i, t_i') \in T_i^2, \ |t_i - t_i'| \leq \varepsilon\} \leq \omega(\varepsilon),$$

where

$$\omega(\varepsilon) := \sup\{|f(t) - f(t')| : (t, t') \in T^2, \ \|t - t'\| \leq \varepsilon\}.$$

Uniform continuity of f implies that $\omega(\varepsilon)$ goes to 0 as ε goes to 0. This proves that $\mathcal{H}_0^0$ is uniformly equicontinuous. Since $\mathcal{H}_0^0$ is bounded, Ascoli's theorem implies that $\mathcal{H}_0^0$ is compact. $\square$

3.2. Euler-Lagrange equation

Proposition 3.2. *Let $h \in \mathcal{H}_0$ be a solution of $(\mathcal{D})$ and let $s := (s_1, \dots, s_n)$ be as in Lemma 2.5. Then $s \in \Gamma$ and s is a solution of $(\mathcal{M})$.*

Proof. Let $h \in \mathcal{H}_0$ be a solution of $(\mathcal{D})$ and $s := (s_1, \dots, s_n)$ and N be as in Lemma 2.5, that is, for all $t_0 \in T_0$

$$h_0(t_0) + \sum_{i=1}^n h_i(s_i(t_0)) = f(t_0, s_1(t_0), \dots, s_n(t_0)) \tag{12}$$

and $N \subset T_0$ is at most countable and such that for all $t_0 \in T_0 \setminus N$ and all $(t_1, \dots, t_n) \in T_1 \times \dots \times T_n$

$$\sum_{i=0}^{n} h_i(t_i) = f(t_0, t_1, \dots, t_n) \Longleftrightarrow t_i = s_i(t_0), \text{for } i = 1, \dots, n.$$

Since μ_0 does not charge points, by (2) we have

$$\mu_0(N) = 0. \tag{13}$$

Let $i \in \{1, \dots, n\}$, let $\varphi_i \in C^0(T_i, \mathbb{R})$ and define for $\varepsilon > 0$

$$h_i^\varepsilon := h_i + \varepsilon\varphi_i, \ h_j^\varepsilon := h_j \text{ for all } j \in \{1, \dots, n\} \text{ with } i \neq j$$

and for all $t_0 \in T_0$

$$h_0^\varepsilon(t_0) := \inf_{(t_1,\dots,t_n)\in T_1\times\dots\times T_n} f(t_0, t_1, \dots t_n) - \sum_{j=1}^{n} h_j(t_j) - \varepsilon\varphi_i(t_i).$$

It is immediate to check that $h^\varepsilon := (h_0^\varepsilon, h_1^\varepsilon, \dots, h_n^\varepsilon) \in \mathcal{H}$. Since h solves $(\mathcal{D})$, we have for all $\varepsilon > 0$

$$\frac{1}{\varepsilon}[F(h^\varepsilon) - F(h)] \leq 0,$$

equivalently

$$\int_{T_i} \varphi_i(t_i)d\mu_i(t_i) + \int_{T_0} \frac{1}{\varepsilon}[h_0^\varepsilon(t_0) - h_0(t_0)]d\mu_0(t_0) \leq 0.$$

Lemma 2.6 and (13) yield then

$$\lim_{\varepsilon\to 0_+} \frac{1}{\varepsilon}[h_0^\varepsilon(t_0) - h_0(t_0)] = -\varphi_i(s_i(t_0)) \ \mu_0 \text{ - a.e. } t_0 \in T_0$$

this, in turn, with Lebesgue's dominated convergence theorem and the fact that $\|h_0^\varepsilon - h_0\|_\infty \leq \varepsilon\|\varphi_i\|_\infty$ yields

$$\int_{T_i} \varphi_i(t_i)d\mu_i(t_i) \leq \int_{T_0} \varphi_i(s_i(t_0))d\mu_0(t_0).$$

Changing φ_i into $-\varphi_i$ we indeed obtain equality in the previous inequality. This proves that for all $i \in \{1, \dots, n\}$

$$\int_{T_i} \varphi_i(t_i)d\mu_i(t_i) = \int_{T_0} \varphi_i(s_i(t_0))d\mu_0(t_0), \text{ for all } \varphi_i \in C^0(T_i, \mathbb{R}).$$

Hence $s_i\sharp\mu_0 = \mu_i$ for all $i \in \{1, \dots, n\}$, i.e., $s = (s_1, \dots, s_n) \in \Gamma$.

To prove that s solves $(\mathcal{M})$, note that by (12), (4) and the fact that $s \in \Gamma$ we have

$$\begin{aligned} C(s) &= \int_{T_0} f(t_0, s(t_0)) d\mu_0(t_0) \\ &= \int_{T_0} \left(h_0(t_0) + \sum_{i=1}^{n} h_i(s_i(t_0)) \right) d\mu_0(t_0) \\ &= \sum_{i=0}^{n} \int_{T_i} h_i d\mu_i = F(h) \\ &= \sup(\mathcal{D}) \leq \inf(\mathcal{M}). \end{aligned}$$

This proves that s is a solution of $(\mathcal{M})$ and that the duality relation $\inf(\mathcal{M}) = \sup(\mathcal{D})$ holds. □

4. Existence and uniqueness of optimal measure preserving maps

Finally, our main results can be summarized as

Theorem 4.1. *If f is strictly monotone of order 2 on T and if (1), (2), (3) hold, then we have:*

1) *problems $(\mathcal{M})$ and $(\mathcal{D})$ admit at least one solution;*
2) *$(\mathcal{D})$ is dual to $(\mathcal{M})$ in the sense:*

$$\sup(\mathcal{D}) = \inf(\mathcal{M});$$

3) *the minimum in $(\mathcal{D})$ is attained by some $h = (h_0, \dots, h_n) \in \mathcal{H}_0$;*
4) *there exists a Borel map $s = (s_1, \dots, s_n)$ from T_0 to $T_1 \times \dots \times T_n$ which satisfies:*

$$h_0(t) + \sum_{i=1}^{n} h_i(s_i(t)) = f(t, s(t)), \text{ for all } t \in T_0,$$

$s \in \Gamma$, each s_i is nondecreasing and s is a solution of $(\mathcal{M})$;

5) *uniqueness also holds: if $\overline{s}$ is a solution of $(\mathcal{M})$ then $s = \overline{s}$ μ_0 - a.e..*

Proof. The statements 1), 2), 3) and 4) have already been proved. The last thing to prove is uniqueness. Assume that $\overline{s} = (\overline{s}_1, \dots, \overline{s}_n) \in \Gamma$ also solves $(\mathcal{M})$. We then have

$$\begin{aligned} \inf(\mathcal{M}) = C(\overline{s}) &= \int_{T_0} f(t_0, \overline{s}(t_0)) d\mu_0(t_0) \\ &= \sup(\mathcal{D}) = F(h) = \sum_{i=0}^{n} \int_{T_i} h_i d\mu_i \\ &= \int_{T_0} \left(h_0(t_0) + \sum_{i=1}^{n} h_i(\overline{s}_i(t_0)) \right) d\mu_0(t_0). \end{aligned}$$

But since $h \in \mathcal{H}$ for all $t_0 \in T_0$ we have

$$f(t_0, \overline{s}(t_0)) \geq h_0(t_0) + \sum_{i=1}^{n} h_i(\overline{s}_i(t_0))$$

hence

$$f(t_0, \overline{s}(t_0)) = h_0(t_0) + \sum_{i=1}^{n} h_i(\overline{s}_i(t_0)) \ \mu_0 \text{ - a.e. } t_0 \in T_0.$$

Using Lemma 2.5 this finally implies $\overline{s} = s$ μ_0 - a.e.. □

5. Concluding remarks

We end this paper with two remarks. The first one is related to the Monge-Kantorovich problem associated with $(\mathcal{M})$. The second one gives the explicit form of the solution via rearrangement formulas.

The Monge-Kantorovich problem which is naturally associated with $(\mathcal{M})$ is

$$(\mathcal{MK}) \inf_{\mu \in \Delta} \int_{T_0} \int_{T_1} \cdots \int_{T_n} f(t_0, \dots, t_n) d\mu(t_0, \dots, t_n)$$

where Δ is the set of all Borel probability measures, μ, on T such that

$$\pi_i \sharp \mu = \mu_i \text{ for } i = 0, \dots, n.$$

$(\mathcal{MK})$ is a natural linear relaxation of $(\mathcal{M})$. Theorem 4.1 actually implies that the infimum of $(\mathcal{MK})$ equals the supremum of $(\mathcal{D})$. More interesting is the fact that the infimum of $(\mathcal{MK})$ is achieved at a measure μ whose support essentially is one dimensional. More precisely the graph of the map $t_0 \in T_0 \mapsto (t_0, s_1(t_0), \dots, s_n(t_0))$ (where we recall that each s_i is nondecreasing) has full measure for μ. Otherwise stated, one has

$$\sum_{i=0} h_i(t_i) = f(t) \ \mu \text{ - a.e. } t,$$

where h is a solution of $(\mathcal{D})$.

Theorem 4.1 states that each component of the solution s of $(\mathcal{M})$ is nondecreasing. Together with the requirement that s belongs to the constrained set Γ, this actually allows us to compute s in terms of rearrangements. Indeed, the condition $s_i \sharp \mu_0 = \mu_i$ and the fact that s_i is nondecreasing on T_i yield the rearrangement-like explicit formulas

$$s_i(t_0) = \inf\{t_i \in T_i : u_i(t_i) > t_0\} \ \mu_0 \text{ - a.e. } t_0 \tag{14}$$

where

$$u_i(t_i) := \inf\{t_0 \in T_0 : \mu_0([t_0, +\infty)) \leq \mu_i([t_i, +\infty))\}. \tag{15}$$

The fact that u_i is well-defined comes from assumption (2). Now note that the fact that s given by formulas (14), (15) is a solution of $(\mathcal{M})$ can be intrepreted as a form of Hardy-Littlewood's inequality. Indeed, assume that we are given n bounded Borel real-valued functions on T_0, $(x_1, \dots, x_n)$, and set $\mu_i := x_i \sharp \mu_0$ for $i = 1, \dots, n$. If we define s_i by formulas (14), (15) for $i = 1, \dots, n$, we can see that each s_i is a monotone rearrangement of x_i and the optimality of s tells us that

$$\int f(t_0, x_1(t_0), \dots, x_n(t_0)) d\mu_0(t_0) \geq \int f(t_0, s_1(t_0), \dots, s_n(t_0)) d\mu_0(t_0),$$

which is nothing but a slight refinement of Hardy-Littlewood's inequality. Finally, we notice that the uniqueness part in Theorem 4.1 implies that the previous inequality is strict unless each x_i is nondecreasing.

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