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On the Regularity of the Convexification Operator on a Compact Set

Rida Laraki*
CNRS, Laboratoire d'Econométrie de l'Ecole Polytechnique,
1 rue Descartes, 75005 Paris, France
laraki@poly.polytechnique.fr

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Let $co_X(\cdot)$ denote the convexification operator on bounded real functions on a convex compact set X. Several necessary and sufficient conditions for the operator $co_X(\cdot)$ to preserve continuity and uniformly Lipschitz continuity are established.

In the special case of a finite dimensional topological vector space, it is shown that (1) the preservation of continuity is equivalent to the closeness of the set of faces of X and (2) the uniform preservation of Lipschitz continuity is equivalent to X being a polytope.

Introduction

Let E denote a real vector space and X be a convex subset of E. Denote by $\mathcal{B}(X)$ the set of bounded real valued functions on X. The *convexification operator* on X, $co_X(\cdot)$, is from $\mathcal{B}(X)$ to itself. It associates to a function f the greatest convex function smaller than f. Explicitly, for f in $\mathcal{B}(X)$ and x in X,

$$co_X(f)(x) := \sup\{g(x) : g \in \mathcal{S}(X), g(\cdot) \leq f(\cdot)\}.$$

where $\mathcal{S}(X)$ denotes the set of bounded real valued convex functions on X and $g(\cdot) \leq f(\cdot)$ means that $g(y) \leq f(y)$ for any y in X.

The main goal of this paper is to establish topological conditions on the geometry of X for the operator $co_X(\cdot)$ to preserve continuity or uniformly Lipschitz continuity.

Two results in the literature are related to the preservation of continuity. In both cases E is assumed to be a finite dimensional topological real vector space:

- Theorem 10.2 and Theorem 20.5 in Rockafellar, 1970, imply that if X is a polytope[1], any bounded and convex function on X is uppersemicontinuous. Since for any compact set X, the function $co_X(f)$ is always lowersemicontinuous when f is continuous (Choquet, 1969, Proposition 26.13, see Proposition 1.3 below), we deduce that the preservation of continuity by $co_X(\cdot)$ holds in the case when X is a polytope. This result could be deduced from our tools (see Example 3 below).
- Kruskal, 1969, gives an example of a 3-dimensional compact set X and a continuous function f on X for which $co_X(f)$ is discontinuous (Example 1 below). This is due to the fact that the set of extreme points[2] of X is not closed.

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[1] X is a polytope if it is the convex hull of finitely many points.
[2] An extreme point of X is a point that cannot be the middle of a non trivial segment in X.

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Throughout this paper, (E,τ) will be assumed to be a Hausdorff topological vector space[3]. Also, (X,τ) will be supposed to be compact, locally convex and metrizable. Let d_τ denote some distance on X compatible with τ. Sometimes, (E,τ) will be supposed to be normed. In such a case, $\|\cdot\|_\tau$ will denote some norm on E compatible with τ. Finally, let d be some distance on X not necessarily related to τ (hence X is not supposed to be d-compact).

For a *topology* τ on E and a *distance* d on X, the operator $co_X(\cdot)$ *preserves* τ*-continuity at* x if the image of any τ*-continuous* function on X is τ-continuous at x. The convexification operator *uniformly preserves* d*-Lipschitz continuity* if there exists a constant $\rho > 0$ such that the image of any d*-Lipschitz function with constant* 1 is d-Lipschitz with constant ρ. Finally, this operator *exactly preserves* d*-Lipschitz continuity* if the image of any d-Lipschitz function with constant 1 is d-Lipschitz with constant 1.

The main results are:

- A necessary and sufficient τ-topological condition on X for the preservation of τ-continuity by $co_X(\cdot)$. It is called τ*-Splitting-Continuous* (Definition 1.12).
  For the special case when E is a finite dimensional vector space, our characterization is simpler and is more close to the intuition behind Example 1 of Kruskal; it is shown that the preservation of τ-continuity by $co_X(\cdot)$ is equivalent to the τ*-Faces-Closed* condition on X[4] (Definition 1.10) which is proved to be strictly stronger than the Kruskal necessary condition (the τ-closeness of the set of extreme points of X, see Example 2 below).
- A sufficient d-condition for the uniform preservation of d-Lipschitz continuity by $co_X(\cdot)$, called d*-Splitting-Lipschitz* (Definition 1.13).
  In particular, it is proved that when X is a polytope and if $\|\cdot\|$ is some norm on $\mathrm{Vect}(X)$[5] then X satisfies the $\|\cdot\|$-Splitting-Lipschitz condition. More precisely, restricted to a finite dimensional normed space E, being a polytope is proved to be a necessary and sufficient condition on X for the uniform preservation of $\|\cdot\|_\tau$-Lipschitz continuity by $co_X(\cdot)$. This Lipschitzian characterization of finite dimensional polytopes is related to one given in Walkup & Wets, 1969, (Section 9.1 below).
- If (X,τ) is a countable product of simplices of Choquet (Definition 1.14), a norm $\|\cdot\|_{X,\tau}$ on $\mathrm{Vect}(X)$ is constructed and is proved that $co_X(\cdot)$ exactly preserves $\|\cdot\|_{X,\tau}$-Lipschitz continuity.

The structure of the paper is as follows. In Section 1 the convexification operator is defined and its equivalence, in our framework, with the τ-closed convexification operator is established. Next, the new concepts (τ-Splitting-Continuous, τ-Faces-Closed and d-Splitting-Lipschitz) are defined and the main results are stated. Sections 2 to 4 are devoted to the preservation of continuity. Section 5 provides several examples showing that our results are tight. Sections 6 to 10 deal with the uniform preservation of Lipschitz continuity. Finally, Section 11 concerns extensions and open questions.

[3]Note that when E is finite dimensional, such topology τ is unique.
[4]Condition τ-Faces-Closed requires that the τ-Kuratowski-limit set of any convergent sequence of faces of X is also a face of X.
[5]The smallest real vector space containing X.

1. Preliminaries

1.1. A geometric formula for the convexification operator

Here is recalled a well known geometric representation formula for the convexification operator. No topological assumption is needed. Let denote the set of *finite-support* probability-measures on X by Δ_X^*. This is the set of all probability-measures σ of the form $\sum_{i=1}^m \alpha_i \delta_{x_i}$ where $\alpha_i \geq 0$, $\sum_{i=1}^m \alpha_i = 1$, $x_i \in X$ and δ_{x_i} denotes the Dirac mass at x_i. For f in $\mathcal{B}(X)$ and $\sigma = \sum_{i=1}^m \alpha_i \delta_{x_i} \in \Delta_X^*$, define

$$\langle \sigma, f \rangle := \sum_{i=1}^m \alpha_i f(x_i)$$

The *barycenter* (or the *resultant*) $r(\sigma) \in X$ of $\sigma = \sum_{i=1}^m \alpha_i \delta_{x_i} \in \Delta_X^*$ is

$$r(\sigma) := \sum_{i=1}^m \alpha_i x_i,$$

hence, $r(\cdot)$ defines a function from Δ_X^* to X. Finally, the set of finite-support probability-measures that are centered at x is

$$\Delta_X^*(x) := r^{-1}(x) = \{\sigma \in \Delta_X^* : r(\sigma) = x\}$$

Proposition 1.1. *For any function f in $\mathcal{B}(X)$ and x in X, $co_X(f)(x)$ satisfies the following formula,*

$$co_X(f)(x) = \inf_{\sigma \in \Delta_X^*(x)} \langle \sigma, f \rangle .$$

Proof. Let us recall the proof of this standard result. Note that for any function g convex and smaller than f, x in X and $\sigma \in \Delta_X^*(x)$,

- $\langle \sigma, g \rangle \leq \langle \sigma, f \rangle$ and
- $g(x) \leq \langle \sigma, g \rangle$ (Jensen's inequality).

Thus any g convex and smaller than f satisfies

$$g(x) \leq \inf_{\sigma \in \Delta_X^*(x)} \langle \sigma, f \rangle .$$

It suffices now to check that the bounded function $h(x) := \inf_{\sigma \in \Delta_X^*(x)} \langle \sigma, f \rangle$ on X is also convex and smaller than f. This is an easy consequence of the fact that $\delta_x \in \Delta_X^*(x)$ and that for any α in $[0,1]$ and (x_1, x_2) in $X \times X$,

$$\alpha \Delta_X^*(x_1) + (1-\alpha) \Delta_X^*(x_2) \subset \Delta_X^*(\alpha x_1 + (1-\alpha) x_2).$$

$\square$

1.2. The τ-closed convexification operator

Here is recalled the well known topological result according to which $co_X(f)$ is τ-lowersemicontinuous for every τ-continuous function f on X (Choquet, 1969, Proposition 26.13).

Define a *regular positive τ-measure* on X as to be a bounded σ-additive positive measure μ on the τ-Borel sets of X that satisfies $\mu(A) = \sup\{\mu(B) : B \subset A$, where B is τ-compact$\}$. The cone of regular positive τ-measures on X is denoted by $\mathcal{M}^+(X,\tau)$ and the real vector space of *regular τ-measures* on X is $\mathcal{M}(X,\tau) := \mathcal{M}^+(X,\tau) - \mathcal{M}^+(X,\tau)$. Finally, denote the set of *regular τ-probability-measures* on X by $\Delta_{X,\tau}$ and the set of τ-continuous functions on X by $\mathcal{C}(X,\tau)$. Note that $\mathcal{M}(X,\tau)$ is in duality with $\mathcal{C}(X,\tau)$, where for $\sigma \in \mathcal{M}(X,\tau)$ and $f \in \mathcal{C}(X,\tau)$, the duality crochet is

$$\langle \sigma, f\rangle := \int_X f(x)\sigma(dx).$$

Hence, let $w(\tau)^*$ denote the associated weak* topology; this is the coarsest topology on $\mathcal{M}(X,\tau)$ for which $\sigma \to \langle \sigma, f\rangle$ is continuous for every function f in $\mathcal{C}(X,\tau)$. For x in X, let $\Delta_{X,\tau}(x)$ denote the closure of $\Delta_X^*(x)$ with respect to the topology $w(\tau)^*$:

$$\Delta_{X,\tau}(x) := cl_{w(\tau)^*}\left[\Delta_X^*(x)\right].$$

Corollary 1.2. *For any function f in $\mathcal{C}(X,\tau)$ and any x in X, $co_X(f)(x)$ satisfies the following formula,*

$$co_X(f)(x) = \inf_{\sigma \in \Delta_{X,\tau}(x)} \langle \sigma, f\rangle.$$

Proof. A direct consequence of Proposition 1.1 and the definition of $w(\tau)^*$ and $\Delta_{X,\tau}(x)$. □

Let the subset of *τ-uppersemicontinuous*[6] (or τ-usc) functions in $\mathcal{B}(X)$ be denoted by $\overline{\mathcal{C}}(X,\tau)$ and let the set of *τ-lowersemicontinuous* (or τ-lsc) functions be denoted by $\underline{\mathcal{C}}(X,\tau) = -\overline{\mathcal{C}}(X,\tau)$. The *$\tau$-closed convexification operator* $\overline{co}_{X,\tau}(\cdot)$ is an operator from $\mathcal{B}(X)$ to $\underline{\mathcal{C}}(X,\tau)$; it associates to a function $f \in \mathcal{B}(X)$, the greatest function in $\underline{\mathcal{C}}(X,\tau) \cap \mathcal{S}(X)$ smaller than f.

A correspondence $y \to G(y)$ from X to $\Delta_{X,\tau}$ is said to be *$w^*(\tau)$-Kuratowski-limit-uppersemicontinuous* (or *$w^*(\tau)$-KUSC)* at x in X if for any sequence $\{x_n\}_{n.}$ in X τ-converging to x and any sequence $\{\sigma_n\}_n$ in $G(x_n)$ $w(\tau)^*$-converging to σ, $\sigma \in G(x)$.

The correspondence $y \to \Delta_{X,\tau}(y)$ from X to $\Delta_{X,\tau}$ is called the splitting correspondence.

Define the *τ-barycenter*[7] of a regular τ-probability-measure $\sigma \in \Delta_{X,\tau}$ as to be the unique element $r_\tau(\sigma) \in X$ such that for any τ-continuous linear form $l(\cdot)$ on E, $l(x) = \langle l, \sigma\rangle$. Note that $r_\tau(\cdot)$ defines a $w(\tau)^*$-continuous function from $\Delta_{X,\tau}$ to X (Choquet, 1969, Proposition 26.3).

Proposition 1.3 (Choquet, 1969, Proposition 26.13). *For any x in X, $\Delta_{X,\tau}(x) = r_\tau^{-1}(x)$. This implies that the splitting correspondence $y \to \Delta_{X,\tau}(y)$ is $w^*(\tau)$-KUSC. Consequently, the operators $\overline{co}_{X,\tau}(\cdot)$ and $co_X(\cdot)$ coincide on $\mathcal{C}(X,\tau)$.*

Proof. The Theorem of approximation in Choquet, 1969, Lemma 26.14, implies that $r_\tau^{-1}(x)$ coincides with the closure of $\Delta_X^*(x)$ under the topology $w(\tau)^*$ (a consequence of the

[6]A function $f \in \mathcal{B}(X)$ is τ-lowersemicontinuous if for any sequence $\{x_n\}_n$ in X τ-converging to x, $\liminf f(x_n) \geq f(x)$.

[7]Choquet, 1969, Section 26.

assumptions on τ). Since the function $\sigma \to r_\tau(\sigma)$ is $w(\tau)^*$-continuous and $(\Delta_{X,\tau}, w(\tau)^*)$ is metrizable (Choquet, 1969, Section 12)[8], the correspondence $y \to r_\tau^{-1}(y)$ is $w^*(\tau)$-KUSC. Now, let $\{x_n\}_n$ be a sequence in X τ-converging to x. Since f is bounded, switch to a subsequence (of the same name) for which $\lim_{n\to\infty} co_X(f)(x_n)$ exists. Let $\varepsilon > 0$ and $\sigma_n \in \Delta_X^*(x_n)$ be such that

$$co_X(f)(x_n) \geq \langle \sigma_n, f\rangle - \varepsilon$$

Since (X,τ) is compact and metrizable, $(\Delta_{X,\tau}, w(\tau)^*)$ is also compact and metrizable (Choquet, 1969). Hence, switching to a subsequence of the same name if necessary and using the τ-KUSC of $y \to \Delta_{X,\tau}(y)$, it may be assumed that the sequence $\{\sigma_n\}_{n.}$ is $w(\tau)^*$-converging to some $\overline{\sigma} \in \Delta_{X,\tau}(x)$. Consequently,

$$\begin{aligned} \lim_n co_X(f)(x_n) &\geq \lim_n \langle \sigma_n, f\rangle - \varepsilon \\ &= \langle \overline{\sigma}, f\rangle - \varepsilon \\ &\geq \inf_{\sigma\in\Delta_{X,\tau}(x)} \langle \sigma, f\rangle - \varepsilon \\ &= co_X(f)(x) - \varepsilon. \end{aligned}$$

Meaning that $co_X(f)(\cdot)$ is τ-lsc at x. Thus, $\overline{co}_{X,\tau}(\cdot)$ and $co_X(\cdot)$ coincide on $\mathcal{C}(X,\tau)$. □

Remark 1.4.

- The KUSC property is in fact a necessary and sufficient condition for many other optimal reward operators to preserve lsc (Laraki & Sudderth 2002).
- In finite dimension, $\overline{co}_{X,\tau}(f) = co_X(f)$ for every $f \in \underline{\mathcal{C}}(X,\tau)$. This holds even if X is not supposed to be τ-compact (Hiriart-Urruty & Lemarechal, 1993, Chapter X).
- In an infinite dimension, it may be that $\overline{co}_{X,\tau}(\cdot) \neq co_X(\cdot)$ on $\underline{\mathcal{C}}(X,\tau)$. For example[9], take $X = \Delta([0,1])$ to be the set of regular Lebesgue-probability-measures on the interval $[0,1]$ and consider the following τ-lsc function on X; $f(x) = -1$ if x is a Dirac measure on $[0,1]$ and $f(x) = 0$ otherwise. Then $\overline{co}_{X,\tau}(f)(x) = -1$ for any $x \in X$, and $co_X(f)(x) = 0$ for any non-atomic measure x on $[0,1]$.

1.3. Kuratowski-limits and topology

In the previous Section, it was shown that the $w^*(\tau)$-KUSC property of the splitting correspondence $y \to \Delta_{X,\tau}(y)$ implies that the image of a τ-continuous function by $co_X(\cdot)$ is τ-lsc. A main result of this paper is that the $w^*(\tau)$-KLSC of the splitting correspondence $y \to \Delta_{X,\tau}(y)$ at x is a necessary and sufficient condition for the image of a τ-continuous function by $co_X(\cdot)$ to be τ-usc at x. To prove this result, the following background from Kuratowski (1968) is necessary.

Let $(Y,\mathcal{T})$ denote some metrizable topological space (think to (X,τ) or $(\Delta_{X,\tau}, w^*(\tau))$). Let $\mathcal{N}$ be the set of strictly increasing sequence from the set of integers into itself. The sequence $\{y'_n\}_n$ is a subsequence of $\{y_n\}_n$ if there exists $\varphi \in \mathcal{N}$ such that $\{y'_n\}_n = \{y_{\varphi(n)}\}_n$.

[8]Note however that $(\mathcal{M}(X,\tau), w(\tau)^*)$ is not metrizable if X is an infinite set (Choquet, 1969, Theorem 16.9).
[9]Thanks to a referee.

Definition 1.5 (Attouch, 1984, Proposition 1.34). Let $\{Y_n\}_n$ be a sequence[10] of subsets of Y.

The $\mathcal{T}$-Kuratowski-upper-limit of $\{Y_n\}_n$ is

$$\mathcal{T} - K \limsup_n Y_n := \left\{ y \in Y : \exists \varphi \in \mathcal{N} : y_{\varphi(n)} \in Y_{\varphi(n)}, \mathcal{T} - \lim_{n\to\infty} y_{\varphi(n)} = y \right\}$$

The $\mathcal{T}$-Kuratowski-lower-limit of $\{Y_n\}_n$ is

$$\mathcal{T} - K \liminf_n Y_n := \left\{ y \in Y : \exists \left(y_n\right)_{n\in N}, y_n \in Y_n, \mathcal{T} - \lim_{n\to\infty} y_n = y \right\}.$$

The sequence $\{Y_n\}_n$ $\mathcal{T}$-Kuratowski-limit-converges if

$$\mathcal{T} - K \limsup_n Y_n = \mathcal{T} - K \liminf_n Y_n,$$

and in such a case, the $\mathcal{T}$-Kuratowski-limit is denoted $\mathcal{T} - K \lim_n Y_n$.

The main properties of Kuratowski convergence used here are:

(P1) The Kuratowski upper and lower limits are $\mathcal{T}$-closed. The Kuratowski-lower-limit of a sequence of convex sets is a convex set.

(P2) Suppose in addition that Y is $\mathcal{T}$-compact and that the topology $\mathcal{T}$ is Hausdorff. Hence, there exists a topology[11], denoted $2^{\mathcal{T}}$, on $\mathcal{F}(Y, \mathcal{T})$ (the space of $\mathcal{T}$-closed and nonempty subsets of Y), such that a sequence of $\mathcal{T}$-closed sets $\{Y_n\}_n$ in $\mathcal{F}(Y, \mathcal{T})$ $\mathcal{T}$-Kuratowski-limit-converges to Y_0 if and only if $\{Y_n\}_n$ $2^{\mathcal{T}}$-converges to Y_0. The topology $2^{\mathcal{T}}$ corresponds in the literature to the Kuratowski (the exponential or the Vietoris) topology. Moreover, the space $(\mathcal{F}(Y, \mathcal{T}), 2^{\mathcal{T}})$ is also compact, Hausdorff and metrizable[12].

(P3) Let $d_{\mathcal{T}}$ denote some distance on Y compatible with $\mathcal{T}$. Recall that given two $\mathcal{T}$-closed and non-empty sets A and B in Y, their $d_{\mathcal{T}}$-Hausdorff distance is defined by

$$\mathcal{D}_{\mathcal{T}}(A, B) := \max\{\max_{a\in A} d_{\mathcal{T}}(a, B); \max_{b\in B} d_{\mathcal{T}}(b, A)\}.$$

If Y is not supposed to be $\mathcal{T}$-compact, the $d_{\mathcal{T}}$-Hausdorff distance may induce a topology on $\mathcal{F}(Y, \mathcal{T})$ strictly stronger[13] than the Kuratowski topology $2^{\mathcal{T}}$. However, the two topologies turn out to be equivalent since Y is supposed to be $\mathcal{T}$-compact here[14].

[10] This sequential definition of the Kuratowski limits is due to the metrizability of $(Y, \mathcal{T})$.

[11] Attouch, 1984, Theorem 2.76 or Klein & Thompson, 1984, Theorem 3.3.11.

[12] Kuratowski, 1968, Section 42; Attouch, 1984, Theorem 2.76; or Klein and Thompson, 1984, Theorem 2.3.5

[13] Kuratowski, 1968, Section 29.

[14] Kuratowski, 1968, Section 42, or Klein & Thompson, 1984, Corollary 4.4.2. An equivalence between the two topologies when Y is not $\mathcal{T}$-compact but belongs to a finite dimensional space may be founded in Salinetti & Wets, 1979. A comparison with some other set topologies may be founded in Klein & Thompson, 1984, Section 4.2.

1.4. Geometrical definitions

The closed and open segments between x and y are respectively

$$[x, y] := \{\lambda x + (1 - \lambda) y, \lambda \in [0, 1]\},$$

and

$$]x, y[:= [x, y] \setminus \{x, y\}.$$

Let us recall some standard geometric definitions where no topological assumption on X is needed.

Definition 1.6. A point x in X is an extreme point of X (or belongs to $\mathcal{E}(X)$) if there are no points x_1 and x_2 in X such that $x \in]x_1, x_2[$.

Definition 1.7. A subset F of X is a union-of-faces of X if for any σ in Δ_X^* such that $r(\sigma) \in F$, the (finite) support $S(\sigma)$ of σ is included in F.

Definition 1.8. A subset of X is a geometric-face of X if it is convex and is a union-of-faces of X.

Remark 1.9.

(1) In Corollary 3.4 below, it is shown that F is a union-of-faces of X if and only there exists a family of geometric-faces $\{F_t\}_{t \in T}$ such that $F = \cup_{t \in T} F_t$. This justifies our terminology.

(2) Our definition of a geometric-face is equivalent to the Rockafellar's definition of a face. Recall that F is a face of X, in the sense of Rockafellar, 1970, Section 18, if it is a convex subset of X and satisfies the property that for any $[a, b]$ in X such that $]a, b[\cap F \neq \emptyset$, both endpoints a and b are in F.

(3) Note that a geometric-face F of a finite dimensional τ-compact set X is always τ-closed. That is, there is no conflict between the geometry and the topology in finite dimension. Actually, from Rockafellar, 1970, Theorem 6.2, in finite dimension, a face $F \neq \emptyset$ admits always a nonempty τ-relative interior $ri_\tau(F)$[15]. If the τ-closure of the face F is denoted by $cl_\tau [F]$ and the τ-relative boundary of F is $\partial_\tau [F] := cl_\tau [F] - ri_\tau [F]$, then for any $x \in ri_\tau [F]$ and any $y \in \partial_\tau [F]$ one has $\frac{1}{2}x + \frac{1}{2}y \in ri_\tau [F]$ (Rockafellar, 1970, Theorem 6.1). Consequently, $\partial_\tau [F] \subset F$.

(4) In an infinite dimensional space E, our definition of a union-of-faces is similar but not equivalent to the Choquet's definition of a face. Actually, Choquet, 1969, Problem 26.6, defines a face of X as to be a τ-closed subset of X such that if $\sigma \in \Delta_{X,\tau}$ and $r_\tau(\sigma) \in F$ then *the τ-support* of σ is included in F. Recall that the τ-support of $\mu \in \Delta_{X,\tau}$, denoted $S_\tau(\mu)$, is the smallest τ-closed subset of X with μ-negligible complement.

(5) Note that if a Choquet's face is always τ-closed, a geometric-face may not be. For example, let X be the set of probability-measures over the discrete set of integers $N = \{0, 1, 2, ..., n, ...\}$ and let E denote the space of σ-additive bounded measures over the set of integers. Denote by $\mathcal{C}^1(N)$ the set of bounded functions $g : \{0, 1, 2, ..., n, ...\} \to R$ that satisfies $\lim_{n \to \infty} g(n) = g(0)$. Hence, E and $\mathcal{C}^1(N)$

[15] This is the τ-interior of F relatively to Aff(F) (the affine hull of F).

are in duality; for $x = \sum_{n=0}^{\infty} x_n \delta_n \in \mathrm{E}$ and $g \in \mathcal{C}^1(N)$, the duality crochet is

$$\langle g, x \rangle = \sum_{n=0}^{\infty} g(n) x_n$$

Endow E with the associated weak* topology[16] and denote this topology by τ_1. One can easily prove that τ_1 has all the desired properties required in the introduction for the topology τ. Define now the following geometric-face of X

$$F = \mathrm{Conv}\{\delta_3, \delta_4, \delta_5, ...\},$$

where Conv (Z) denotes the *convex hull* of Z (the smallest convex in E containing Z). Since the sequence $\{\delta_n\}_n$ τ_1-converges to δ_0 ,

$$\delta_0 \in cl_{\tau_1}[F].$$

From $\delta_0 \notin F$, we deduce that F is not τ_1-closed. Note however that $cl_{\tau_1}[F] = \mathrm{Conv}\{\delta_0, \delta_3, \delta_4, \delta_5, ...\}$ is a geometric-face of X. Is the τ_1-closure of a geometric-face always a geometric-face? The answer is in Remark 1.11 below.

(6) In the rest of the paper, only geometric-faces in the sense of Definition 1.8 are considered. Hence, to simplify notations, this will be called a face instead of geometric-face.

1.5. Example 1 of Kruskal

It is well known[17] that the set of extreme points of X may not be τ-closed. Take $X = \mathrm{Conv}\{A; B; C\} \subset R^3$ where

$$C = \left\{x = (a, b, c) : c = 0; (a-1)^2 + b^2 \leq 1\right\},$$

$A = (0, 0, 1)$ and $B = (0, 0, -1)$. Hence $\mathcal{E}(X) = \{A; B; C - \{(0, 0, 0)\}\}$ is not τ-closed. Actually, the sequence $x_n = \left(\frac{1}{n}, \sqrt{1 - \left(1 - \frac{1}{n}\right)^2}, 0\right)$ in $\mathcal{E}(X)$ τ-converges to $x_0 = (0, 0, 0) = \frac{A+B}{2} \notin \mathcal{E}(X)$. Kruskal (1969) considered the following τ-continuous function on X,

$$f(a, b, c) = -c^2.$$

It satisfies $co_X(f)(x_n) = f(x_n) = 0$ (since $x_n \in \mathcal{E}(X)$) and $co_X(f)(x_0) = -1$ ($x_0 = \frac{A+B}{2}$ and $f(A) = f(B) = -1 = \min_{x \in X} f(x)$). Thus, $co_X(f)$ is τ-discontinuous[18].

1.6. Topological conditions on the geometry

Definition 1.10. The set X is *τ-Faces-Closed* (or *τ-FC)* if for any τ-Kuratowski-limit-converging sequence of faces of X, the τ-Kuratowski-limit is also a face of X.

[16] The smallest topology for which $x \to \langle g, x \rangle$ is continuous for every $g \in \mathcal{C}^1(N)$.
[17] Choquet, 1969, Section 25 or Rockafellar, 1970, Section 18.
[18] I discovered this example in Benoist and Hiriart-Urruty (1996).

Remark 1.11. A necessary (but not sufficient) condition for X to be τ-FC is that the τ-closure of a face of X is also a face of X. From Remark 1.9-(4), a face of a convex τ-compact finite dimensional set is always τ-closed and From Remark 1.9-(5), in infinite dimension, some faces may not be τ-closed. Surprisingly, the τ-closure of a face may not be a face. The following example is close to the one given in Remark 1.9-(5). X still denotes the set of probability-measures over the set integers N and E is the space of σ-additive bounded measures over N. Let $\mathcal{C}^2(N)$ denote the set of bounded functions $g : N \to R$ satisfying $\lim_{n\to\infty} g(n) = g(0) = \frac{1}{2}g(1) + \frac{1}{2}g(2)$. Since E and $\mathcal{C}^2(N)$ are in duality, E may be endowed with the associated weak* topology (denoted τ_2). One can easily prove that τ_2 has all the desired properties required in the introduction for the topology τ. Consider the same face as in Remark 1.9-(5):

$$F = \text{Conv}\{\delta_3, \delta_4, \delta_5, ...\}.$$

Hence, by construction of the topology τ_2, $\delta_0 = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_2 \in cl_{\tau_2}[F]$. Since δ_1 and δ_2 are not in $cl_{\tau_2}[F]$, it follows that $cl_{\tau_2}[F]$ is not a face of X.

A correspondence $y \to G(y)$ from X to $\Delta_{X,\tau}$ is $w(\tau)^*$*-Kuratowski-limit-continuous* (or $w(\tau)^*$*-KC)* at x if for any sequence $\{x_n\}_n$ in X that is τ-converging to x, the sequence of sets $\{G(x_n)\}_n$ $w(\tau)^*$-Kuratowski–limit-converges to $G(x)$.

Definition 1.12. Let x be in X. The set X is τ*-Splitting-Continuous (or* τ*-SC)* at x if the splitting correspondence $y \to \Delta_{X,\tau}(y)$ is $w(\tau)^*$-KC at x.

From Proposition 1.3, the splitting correspondence $x \to \Delta_{X,\tau}(x)$ is always $w^*(\tau)$-KUSC. Hence, using Property P2 in Section 1.3, we deduce that (X, τ) is τ-SC at x if and only if the splitting correspondence $y \to \Delta_{X,\tau}(y)$ is $w^*(\tau)$-Kuratowski-limit-lowersemicontinuous (denoted $w^*(\tau)$-KLSC) at x; meaning that for any sequence $\{x_n\}_n$ τ-converging to x, $\Delta_{X,\tau}(x) \subset w^*(\tau) - K \liminf_n \Delta_{X,\tau}(x_n)$.

Definition 1.13. Let $\rho > 0$. The set X is d-Splitting-Lipschitz with constant ρ (or d-SL(ρ)) if for any (x, y) in $X \times X$ and for any $\sum_{i=1}^m \alpha_i\delta_{x_i}$ in $\Delta_X^*(x)$ there exist m points $y_1, ..., y_m$ in X such that $\sum_{i=1}^m \alpha_i\delta_{y_i}$ belongs to $\Delta_X^*(y)$ and $\sum_{i=1}^m \alpha_i d(x_i, y_i) \le \rho d(x, y)$.

Note that if $d(y, z) = \|y - z\|$ where $\|\cdot\|$ is some norm on E, ρ should be greater than 1.

Definition 1.14 (Choquet, 1969, Corollary 28.5). The τ-metrizable set X is a τ-simplex of Choquet if for any x in X, there exists a unique regular τ-probability-measure σ on X such that its barycenter is x and its τ-support belongs to the set of extreme points of X.

Remark 1.15. Hence, X is a finite dimensional simplex if and only if the set of its extreme points is finite and the extreme points are affinely independent.

1.7. The main results

Theorem 1.16. Preservation of continuity

(*a*) *The operator* $co_X(\cdot)$ *preserves* τ*-continuity at* x *if and only if the set* X *is* τ*-SC at* x.

(*b*) *If the operator* co_X *preserves* τ*-continuity, the set* X *is* τ*-FC.*

(c) *If the space* E *is finite dimensional, conditions* τ*-SC and* τ*-FC on* X *are equivalent.*

Theorem 1.17. Preservation of Lipschitz continuity

(d) *If the set* X *is* d*-SL, the operator* $co_X(\cdot)$ *uniformly preserves* d*-Lipschitz continuity.*

(e) *The property SL(*ρ*) is stable under countable product. More precisely, if* $X = \times_n X_n$ *is a countable product of* d_n*-SL(*ρ*)-compact sets* X_n *then there exist many distances for which* X *is compact and SL(*ρ*).*

(f) *Suppose that* X *is a* τ*-simplex of Choquet. Then a norm on* $\mathrm{Vect}(X)$*, denoted* $\|\cdot\|_{X,\tau}$*, is constructed and is proved that the operator* $co_X(\cdot)$ *exactly preserves* $\|\cdot\|_{X,\tau}$*-Lipschitz continuity.*

(g) *If the set* X *is a polytope and* (E,τ) *is normed then* X *is* $\|\cdot\|_\tau$*-SL(*ρ*) for some* $\rho \geq 1$*.*

(h) *If* (E,τ) *is a finite dimensional normed space, the operator* $co_X(\cdot)$ *uniformly preserves* $\|\cdot\|_\tau$*-Lipschitz continuity if and only if the set* X *is a polytope.*

Sections 2 to 4 are devoted to the proof of (a) to (c), Sections 6 to 10 the proof of (d) to (h). Examples in Section 5 and Remark 9.4 show that these results are tight.

2. Proof of part (a) in Theorem 1.16

The following Proposition is to compare with Proposition 1.3.

Proposition 2.1. *Let* x *be in* X *and* f *in* $\mathcal{C}(X,\tau)$*. If the Splitting correspondence* $y \to \Delta_{X,\tau}(y)$ *is* $w^*(\tau)$*-KLSC at* x*, then* $co_X(f)$ *is* τ*-usc at* x*.*

Proof. Let $\{x_n\}_n$ be τ-converging to x and $\varepsilon > 0$. Suppose that $co_X(f)(x_n)$ converges to some real α (if not switch to a subsequence since f is bounded). Let $\overline{\sigma}$ be such that

$$co_X(f)(x) \geq \langle \overline{\sigma}, f \rangle - \varepsilon$$

By the $w^*(\tau)$-KLSC at x of the Splitting correspondence $y \to \Delta_{X,\tau}(y)$, there exists a sequence $\sigma_n \in \Delta_X(x_n)$ that $w^*(\tau)$-converges to $\overline{\sigma}$. Consequently,

$$\begin{aligned} \alpha &= co_X(f)(x_n) \\ &\leq \lim_n \langle \sigma_n, f \rangle \\ &= \langle \overline{\sigma}, f \rangle \\ &\leq co_X(f)(x) + \varepsilon \end{aligned}$$

Hence, $co_X(f)$ is τ-usc at x. □

Corollary 2.2. *If the set* X *is* τ*-SC at* x *then the operator* $co_X(\cdot)$ *preserves* τ*-continuity at* x*.*

Proof. Since the splitting correspondence $y \to \Delta_{X,\tau}(y)$ is both $w^*(\tau)$-KLSC and $w^*(\tau)$-KUSC at x, the last Proposition and Proposition 1.3 implies that the image of any τ-continuous function by the convexification operator is both τ-usc and τ-lsc at x. □

Proposition 2.3. *Let* x *be in* X *and* f *in* $\mathcal{C}(X,\tau)$*. If the splitting correspondence* $y \to \Delta_{X,\tau}(y)$ *is not* $w^*(\tau)$*-KLSC at* x *then there exists* f *in* $\mathcal{C}(X,\tau)$ *such that* $co_X(f)$ *is* τ*-discontinuous at* x*.*

Proof. Since the splitting correspondence $y \to \Delta_{X,\tau}(y)$ is not $w^*(\tau)$-KLSC at x, there exists a sequence $\{x'_n\}_n$ τ-converging to x and there exists $\sigma \in \Delta_{X,\tau}(x)$ such that

$$\sigma \notin w^*(\tau) - K \liminf_n [\Delta_{X,\tau}(x'_n)] .$$

Considering a subsequence of $\{x'_n\}_n$ and using Property P2 in Section 1.3, we deduce that there exists a sequence $\{x_n\}_n$ τ-converging to x such that (1) $D := w^*(\tau) - K \lim \Delta_{X,\tau}(x_n)$ exists and satisfies $\sigma \notin D$, (2) $\lim co_X f(x_n)$ exists, (3) $co_X f(x_n) = \langle f, \sigma_n \rangle$ for some $\sigma_n \in \Delta_X(x_n)$ and (4) the sequence $\{\sigma_n\}_n$ $w(\tau)^*$-converges to some $\sigma_0 \in D$. Property P1 in Section 1.3 implies that $D \subset \mathcal{M}(X,\tau)$ is convex and $w(\tau)^*$-closed. Since $(\mathcal{M}(X,\tau), w(\tau)^*)$ is a locally convex and Hausdorff real vector space, the Hahn Banach Theorem (Kelley & Namioka, 1963) may be used to deduce the existence of a continuous linear form $l(\cdot)$ on $(\mathcal{M}(X,\tau), w(\tau)^*)$ which separates strictly the singleton $\{\sigma\}$ and D. Since $(\mathcal{M}(X,\tau), w(\tau)^*)$ is the dual of $\mathcal{C}(X,\tau)$ (Cohn 1980, Theorem 7.3.1) the dual of $(\mathcal{M}(X,\tau), w(\tau)^*)$ is $\mathcal{C}(X,\tau)$ (Kelley & Namioka, 1963, Section 17). That is, $l(\cdot)$ has the form $l(\cdot) = \langle f, \cdot \rangle$ for some $f \in \mathcal{C}(X,\tau)$. Finally, from $co_X(f)(x) \leq \langle f, \sigma \rangle$, it may be deduced that $co_X(f)(x) < \min_{\tau \in D} \langle f, \tau \rangle$. Since $\sigma_0 \in D$, $\lim co_X f(x_n) = \langle f, \sigma_0 \rangle \geq \min_{\tau \in D} \langle f, \tau \rangle > co_X(f)(x)$. Consequently, $co_X(f)$ is not τ-usc at x. □

Corollary 2.4. *If the set X is not τ-SC at x then the operator $co_X(\cdot)$ does not preserve τ-continuity at x.*

Remark 2.5. Since (X,τ) is compact and metrizable the Stone-Weierstrass Theorem (Kelley, 1955, Section 7) implies that the space of d_τ-Lipschitz functions (where d_τ is some distance compatible with τ) is dense in $\mathcal{C}(X,\tau)$. Hence, for a strict separation between $\{\sigma\}$ and D in the proof of Proposition 2.3, one can choose f to be d_τ-Lipschitz instead of only τ-continuous. Consequently,

Corollary 2.6. *If the set X is not τ-SC then the operator $co_X(\cdot)$ does not preserve* d_τ-*Lipschitz continuity (meaning that there exists a* d_τ*-Lipschitz function such that its image is not* d_τ*-Lipschitz).*

3. Proof of part (b) in Theorem 1.16

Before proving the main result (b), we start by developing an alternative characterization for the condition τ-FC.

Definition 3.1. For any x in X, let $F_X(x)$ be the face of X generated by x (i.e. the minimal face of X containing x) and call the correspondence $x \to F_X(x)$ from X to X the Minimal-Face correspondence.

Remark 3.2. $F_X(x)$ is well defined since (1) X is a face containing x; (2) the intersection of all faces containing x is a nonempty convex set (since convexity is preserved by intersection and x is always in the intersection); and (3) if y is in the intersection of all faces containing x and σ is a finite-support probability-measure centered at y, then the support of σ is in the intersection of all faces containing x.

Lemma 3.3. *If $\Lambda_X(x)$ denotes the set of points in X which are in the support of σ in $\Delta^*_X(x)$ then $\Lambda_X(x) = F_X(x)$.*

Proof. It suffices to prove that $\Lambda_X(x)$ is a face containing x. Actually, this implies that $F_X(x) \subset \Lambda_X(x)$ ($F_X(x)$ is the minimal face containing x). Now, since (1) $\Lambda_X(x) \subset F_X(x)$ holds by the definition of $F_X(x)$ and (2) $\Lambda_X(x)$ is clearly convex, the equality between the two sets follows.

To prove that $\Lambda_X(x)$ is a face let $y \in \Lambda_X(x)$ be in the support of some $\sigma \in \Delta_X^*(x)$ and let z be in the support of some $\tau \in \Delta_X^*(y)$. Define the finite-support probability-measure μ by

$$\begin{aligned}\mu(\{\widetilde{y}\}) &= \sigma(\{\widetilde{y}\}) + \sigma(\{y\}) \times \tau(\{\widetilde{y}\}) \quad \textit{if} \quad \widetilde{y} \neq y \\ \mu(\{y\}) &= \sigma(\{y\}) \times \tau(\{y\})\end{aligned}$$

Clearly, $\mu \in \Delta_X^*(x)$ and z is in the support of μ. Hence $z \in \Lambda_X(x)$, so that $\Lambda_X(x)$ is a face. □

Corollary 3.4. *A set F is a union-of-faces of X if and only if $\cup_{x\in F} F_X(x) \subset F$ (or equivalently if $\cup_{x\in F} F_X(x) = F$).*

Thus, F is a union-of-faces of X if and only if it is the union of some family of faces of X.

Corollary 3.5. *For any function f in $\mathcal{B}(X)$ and any x in X, $co_X(f)(x) = co_{F_X(x)}(f)(x)$.*

Proof. Since $co_X(f)(x) = \inf_{\sigma\in\Delta_X^*(x)} \langle \sigma, f\rangle$, to compute $co_X(f)(x)$ only the points in X which are in the support of some σ in $\Delta_X^*(x)$ are needed. Lemma 3.3 says that this set is $F_X(x)$. □

Introduce now some equivalent definitions for the τ-FC condition.

Definition 3.6.

- The set X is τ-Minimal-Face-Lowersemicontinuous at x if the Minimal-Face correspondence $x \to F_X(x)$ is τ-KLSC at x, meaning that for any sequence $\{x_n\}_n$ that τ-converges to x, $F_X(x) \subset \tau - K \liminf_n F_X(x_n)$.
- The set X is τ-Union-of-Faces-Closed if for any sequence $\{F_n\}_n$ of union-of-faces of X which τ-Kuratowski-limit–converges, its limit set, $\tau - K \lim F_n$, is also a union-of-faces of X.

Proposition 3.7. *For the set X, the following conditions are equivalent:*

(*a*) *X is τ-Faces-Closed;*
(*b*) *X is everywhere τ-Minimal-Face-Lowersemicontinuous;*
(*c*) *X is τ-Union-of-Faces-Closed.*

Proof. $(b) \Rightarrow (c)$. Let $\{F_n\}_n$be a sequence of union-of-faces of X that τ-Kuratowski–limit-converges to F and let $x \in F$. Since, there exists a sequence $x_n \in F_n$ such that $x_n \to x$, and X is τ-Minimal-Face-Lowersemicontinuous,

$F_X(x) \subset \tau - K \liminf F_n = F$. Hence, $\cup_{x\in F} F_X(x) \subset F$. Consequently, F is a union-of-faces of X.

$(c) \Rightarrow (a)$. Let $\{F_n\}_n$be a τ-Kuratowski-converging sequence of faces of X and let $F := \tau - K \lim F_n$. Since F_n is convex, by P1 in Section 1.3, F is also convex. Since X is τ-Union-of-Faces-Closed, F is a union-of-faces of X. Consequently, F is a face of X.

$(a) \Rightarrow (b)$. Let $\{x_n\}_n$ be τ-converging to x. Let $F := \tau - K \liminf_n F_X(x_n)$. By P2 in Section 1.3, there exists a subsequence $\{x'_n\}_n$ of $\{x_n\}_n$ such that

$$\tau - K \lim F_X(x'_n) = \tau - K \liminf_n F_X(x_n) = F.$$

Since $\{F_X(x'_n)\}_n$ is a sequence of faces and X is τ-Faces-Closed, F is a face and from $x \in F$, it follows that $F_X(x) \subset F$. □

Proposition 3.8. *If the operator $co_X(\cdot)$ preserves τ-continuity at x then X is τ-Minimal-Face-Lowersemicontinuous at x.*

Proof. Suppose that X is not τ-Minimal-Face-Lowersemicontinuous at x. Hence, there exists a sequence $\{x_n\}_n$ τ-converging to x such that $F_X(x)$ is not included in $F :=\tau - K \lim F_X(x_n)$. Choose $y \in F_X(x)$ such that $y \notin F$. From (1) F is τ-closed, (2) $\tau - K \lim cl_\tau [F_X(x_n)] = \tau - K \lim F_X(x_n) = F$ and (3) (X, τ) is metrizable, we deduce that there exists a closed convex neighborhood $V(y)$ of y such that for n large enough $V(y) \cap cl_\tau [F_X(x_n)] = \emptyset$. Urysohn's Lemma (Kelley, 1955) shows the existence of a negative function $f \in \mathcal{C}(X, \tau)$ satisfying $f(y) = -1$ and $f(z) = 0$ for $z \in cl_\tau [F_X(x_n)]$, for n large enough. Thus, $\lim co_X(f)(x_n) = \lim co_{F_X(x_n)}(f)(x_n) = 0$ (from Corollary 3.5). Since $y \in F_X(x)$, $f \leq 0$ and $f(y) < 0$, we deduce that $co_X(f)(x) < 0$. That is, f is τ-discontinuous at x. □

4. Proof of part (c) in Theorem 1.16

Here is established part (c) of our main results.

Proposition 4.1. *If the space* E *is finite dimensional, the two conditions τ-SC and τ-FC on X are equivalent.*

From the last Sections, it needs only be shown that the τ-KLSC of the Minimal-Face correspondence $x \to F_X(x)$ implies the $w^*(\tau)$-KLSC of the splitting correspondence $y \to \Delta_{X,\tau}(y)$. The proof is given in three steps. The closed and open half-lines starting at a and containing b are respectively denoted by $[a \to b) := \{a + \mu(b-a), \mu \geq 0\}$, and $]a \to b) := [a \to b) \backslash \{a\}$.

Step 1. Is shown here that if the Minimal-Face correspondence $x \to F_X(x)$ is τ-KLSC then for any $x \in X$, $\alpha\delta_a + (1-\alpha)\delta_b \in \Delta^*_X(x)$ and $\{x_n\}_n \to x$, there exists $\alpha_n\delta_{a_n} + (1-\alpha_n)\delta_{b_n} \in \Delta^*_X(x_n)$ such that $\alpha_n\delta_{a_n} + (1-\alpha_n)\delta_{b_n} \to \alpha\delta_a + (1-\alpha)\delta_b$.

Lemma 4.2. *If the space* E *is finite dimensional and the set X is τ-Minimal-Face-Lowersemicontinuous then X satisfies the following property, denoted $[P]$:*

$[P]$: for any x, a and b in X such that $x \in]a, b[$ and any sequence $\{x_n\}_n \to x$, there exist y, $\{a_n\}_n$ and $\{y_n\}_n$ such that $y \in]x \to b)$, $x_n \in [a_n, y_n]$, $a_n \to a$ and $y_n \to y$.

Proof. Suppose that X is τ-Minimal-Face-Lowersemicontinuous and let a and b in X be such $x \in]a, b[$ and $\{x_n\}_n \to x$. For each integer $k \leq \dim(\mathrm{E})$ for which $H_k = \{n : dim(F_X(x_n)) = k\}$ is infinite, define the sequence $\{x^k_n\}_n = \{x_n\}_{n \in H_k}$. This is a subsequence of $\{x_n\}_n$, hence converges to x. For simplicity, denote such a sequence $\{x_n\}_n$ also.

$[P]$ is now proved by induction on k.

$k = 1$. Since X is τ-Minimal–Face-Lowersemicontinuous at x, there exists a sequence $\{a_n\}_n$ in $F_X(x_n)$ such that $a_n \to a$. Also, there exists a sequence $\{b_n\}_n$ in $F_X(x_n)$ such $b_n \to b$. Since $dim(F_X(x_n)) = k = 1$, necessarily $x_n \in [a_n, b_n]$.

Suppose now that $[P]$ is true for all $l \leq k-1$. Since X is τ-Minimal–Face-Lowersemicontinuous at x and $x_n \to x$, there exists a sequence $\{a'_n\}_n$ in $F_X(x_n)$ such that $\{a'_n\} \to a$. Define

$$\{z_n\} := \arg \max_{q \in X : x_n \in]a'_n, q[} \|x_n - q\| .$$

Where $\|\cdot\|$ is the euclidean norm. z_n is the farthest point in X from x_n when one follows the direction $\overrightarrow{a'_n x_n}$. z_n is clearly well defined; It exists because X is compact and is unique since the norm function $\|\cdot\|$ is positively homogeneous. Hence, $z_n \in \partial_\tau F_X(x_n)$[19]. Since $\{a'_n\} \to a$ and $\{x_n\} \to x$, we deduce that $\{z_n\} \to z$ and that $x \in]a, z]$. If $z \neq x$, taking $a_n = a'_n$ and $y_n = z_n$, $[P]$ is proved. Now, suppose that $z = x$. Since $z_n \in \partial_\tau F(x_n)$, $\{z_n\} \to x$ and $dim\,[\partial_\tau F_X(x_n)] < dim\,[F_X(x_n)] = k$, the induction hypothesis implies that there exist two sequences $\{a^*_n\}_n$ and $\{y_n\}_n$ in $F_X(z_n) \subset F_X(x_n)$ such that $z_n \in]a^*_n, y_n[$, $\{a^*_n\} \to a$ and $\{y_n\} \to y \in]x \to b]$. Since, $x_n \in]a'_n, z_n[$ and $z_n \in]a^*_n, y_n[$, there exists a sequence $\{a_n\}_n$ such that $a_n \in [a'_n, a^*_n]$ and $x_n \in]a_n, y_n[$. From $a'_n \to a$ and $a^*_n \to a$ it follows that $\{a_n\} \to a$. Hence y, $\{a_n\}_n$ and $\{y_n\}_n$ satisfies $[P]$. □

Now is proved the main result of Step 1.

Lemma 4.3. *Suppose that the space* E *is finite dimensional and that the set X is τ-Minimal-Face-Lowersemicontinuous. Let x, a and b be in X with $x \in]a, b[$ and let $\{x_n\}_n \to x$. Then, there exist two sequences $\{a_n\}_n$ and $\{b_n\}_n$ in X such that $x_n \in [a_n, b_n]$, $a_n \to a$ and $b_n \to b$.*

Proof. Suppose that X is τ-Minimal-Face-Lowersemicontinuous and let x, a and b be in X such that $x \in]a, b[$ and let $\{x_n\}_n \to x$. By the property $[P]$, there exists y satisfying the following property $\left[P_{x,a,b,\{x_n\}}\right]$.

$\left[P_{x,a,b,\{x_n\}}\right]$: $y \in]x \to b)$ and there exist two sequences $\{a_n\}$ and $\{y_n\}$ in X such that $x_n \in [a_n, y_n]$, $a_n \to a$ and $y_n \to y$.

The following two stability properties hold:

(i) If y satisfies $\left[P_{x,a,b,\{x_n\}}\right]$ and if $z \in]a, y[$ then z satisfies $\left[P_{x,a,b,\{x_n\}}\right]$.
Actually, if $z = \beta a + (1-\beta) y$ and if $\{a_n\}_n$ and $\{y_n\}_n$ are some sequences associated with y then the sequences $\{a_n\}_n$ and $z_n := \beta a_n + (1-\beta) y_n$ are associated with z.

(ii) If for all $m \in \{1, 2, ..\}$, y^m satisfies $\left[P_{x,a,b,\{x_n\}}\right]$ and if $\lim_{m\to\infty} y^m = y \neq x$ then y satisfies $\left[P_{x,a,b,\{x_n\}}\right]$.
Actually, let $\{y^m_n\}_n$ and $\{a^m_n\}_n$ be two sequences associated with y^m, meaning that $\lim_n y^m_n = y^m$, $\lim_n a^m_n = a$ and $x_n \in [a^m_n, y^m_n]$. Hence, a diagonal extraction implies that there exists $\varphi(\cdot) \in \mathcal{N}$ such that the sequences $\left\{y^{\varphi(n)}_n\right\}_n$ and $\left\{a^{\varphi(n)}_n\right\}_n$ satisfies the desired property for y.

[19]Recall that $\partial_\tau F$ denotes the relative τ-boundary of the face F.

Denote by $D_{x,a,b,\{x_n\}}$ the set of points y in X satisfying $\left[P_{x,a,b,\{x_n\}}\right]$. By $[P]$, (i) and (ii) above, there exists $y_0 \in]x \to b)$ such that $D_{x,a,b,\{x_n\}} =]x, y_0]$. We claim that $b \in D_{x,a,b,\{x_n\}}$ (which will end the proof of Lemma 4.3). Indeed, assume that $y_0 \in]x, b[$. Since $y_0 \in D_{x,a,b,\{x_n\}}$, there exist two sequences $\{a_n\}_n$ and $\{y_n\}_n$ in X such that $x_n \in [a_n, y_n]$, $a_n \to a$ and $y_n \to y_0$. Since $y_0 \in]a, b[$ and $y_n \to y_0$, property $[P]$ implies the existence of $z_0 \in]y_0 \to b)$ and the existence of sequences $\{a_n^*\}_n$ and $\{z_n\}_n$ in X such that $y_n \in [a_n^*, z_n]$, $a_n^* \to a$ and $z_n \to z_0$. Hence $z_0 \in]x \to b)$, $x_n \in co\{a_n, a_n^*, z_n\}$, $\lim a_n = \lim a_n^* = a$ and $\lim a_n = z_0$. Consequently $z_0 \in D_{x,a,b,\{x_n\}}$. A contradiction with (i), $D_{x,a,b,\{x_n\}} =]x, y_0]$ and $z_0 \in]y_0 \to b)$. □

Step 2. Suppose that X is τ-Minimal-Face-Lowersemicontinuous. Let $m \geq 2$ and consider the following property $[P_m]$

$[P_m]$ For any x in X, any sequence $\{x_n\}_n \to x$ and any $\sum_{i=1}^m \alpha_i \delta_{x_i}$ in $\Delta_X^*(x)$, there exists a sequence $\sum_{i=1}^m \alpha_i(n) \delta_{x_i(n)}$ in $\Delta_X^*(x_n)$ that converges to $\sum_{i=1}^m \alpha_i \delta_{x_i}$.

From Step 1, $[P_2]$ holds. Suppose now $[P_m]$ holds for $m \geq 2$ and prove that $[P_{m+1}]$ holds.

Let $\sum_{i=1}^m \alpha_i \delta_{x_i}$ be in $\Delta_X^*(x)$. If $\alpha_1 = 1$ then we are done. Suppose then that $\alpha_1 < 1$ and define $x_2' = \frac{1}{1-\alpha_1} \sum_{i=1}^m \alpha_{i+1} x_{i+1}$. Hence, $\alpha_1 \delta_{x_1} + [1 - \alpha_1] \delta_{x_2'} \in \Delta_X^*(x_n)$. Using Step 1, we deduce that there exists a sequence $\alpha_1(n) \delta_{x_1(n)} + [1 - \alpha_1(n)] \delta_{x_2'(n)}$ in $\Delta_X^*(x_n)$ that converges to $\alpha_1 \delta_{x_1} + [1 - \alpha_1] \delta_{x_2'}$. Also, using the induction hypothesis, we obtain that there exists a sequence

$\sum_{i=1}^m \alpha_{i+1}(n) \delta_{x_{i+1}(n)}$ in $\Delta_X^*(x_2'(n))$ that converges to $\sum_{i=1}^m \alpha_{i+1} \delta_{x_{i+1}}$. Hence, the sequence $\alpha_1(n) \delta_{x_1(n)} + [1 - \alpha_1(n)] \sum_{i=1}^m \alpha_{i+1}(n) \delta_{x_{i+1}(n)}$ is in $\Delta_X^*(x_n)$ and converges to $\sum_{i=1}^m \alpha_i \delta_{x_i}$.

Step 3. From Step 2, for any $\{x_n\} \to x$ and σ in $\Delta_X^*(x)$, there exists a sequence $\{\sigma_n\}_n$ in $\{\Delta_X^*(x_n)\}_n$ converging to σ. Since, for any y in X, $cl_{w(\tau)^*}[\Delta_X^*(y)] = \Delta_{X,\tau}(y) = r_\tau^{-1}(y)$ and since the barycenter application $\sigma \to r_\tau(\sigma)$ from $\Delta_{X,\tau}$ to X is $w(\tau)^*$-continuous, we conclude that any σ in $\Delta_{X,\tau}(x)$ may be $w(\tau)^*$-approximated by some sequence in $\{\Delta_X^*(x_n)\}_n$. Hence, the splitting correspondence is $w(\tau)^*$-KLSC.

5. Examples

5.1. Example 2

Proposition 5.1. *Condition τ-Faces-Closed for X is strictly stronger than the τ-closeness of the set of extreme point of X.*

Proof. A small modification of Kruskal's example (Example 1) leads us to the following example in R^4:

$$X = co \left\{ \begin{array}{c} A = (0,0,-1,0); B = (0,0,1,0) \\ D = \left\{(a,b,0,d) : (a-1)^2 + b^2 = 1; |d| \leq 1 - |b|\right\} \end{array} \right\}.$$

Hence the set of extreme points of X

$$\mathcal{E}(X) = \left\{ \begin{array}{c} A = (0,0,-1,0); B = (0,0,1,0) \\ \left\{(a,b,0,d) : (a-1)^2 + b^2 = 1; |d| = 1 - |b|\right\} \end{array} \right\},$$

is clearly τ-closed. For $x_n = \left(\frac{1}{n}, \sqrt{1-\left(1-\frac{1}{n}\right)^2}, 0, 0\right)$, $x_n \to x = (0,0,0,0)$. A simple computation shows that

$$F_X(x_n) = \left\{ \left(\frac{1}{n}, \sqrt{1-\left(1-\frac{1}{n}\right)^2}, 0, d\right) : |d| \leq 1 - \sqrt{1-\left(1-\frac{1}{n}\right)^2} \right\}.$$

But

$$\tau - K \lim F_X(x_n) = co\left\{A^1 = (0,0,0,-1), B^1 = (0,0,0,1)\right\}$$

and

$$F_X(x) = co\left\{A, B, A^1, B^1\right\}.$$

Hence, X is not τ-Minimal–Face-Lowersemicontinuous at x. □

5.2. Example 3

Proposition 5.2. *If X is a polytope then it is τ-SC.*

Proof. Since X is the convex hull of finitely many points, it is included in a finite dimensional space and contains finitely many faces. Let $\{F_n\}_n$be a τ-Kuratowski-limit converging sequence of faces of X. Since the sequence contains a finite number of distinct sets, its limit, $\tau{-}K \lim F_n$, is necessarily the τ-closure of some face of X. By Remark 1.9-(4) above, the τ-closure of a face of X it is also a face of X. Hence, X is τ-FC. Finally, by part (c) of Theorem 1.16, X is τ-SC. □

5.3. Example 4

Here is a class of τ-SC sets that does not contain polytopes.

Definition 5.3. The set X is strongly convex if any face of it that differs from X is reduced to a point (which is necessarily an extreme point).

Proposition 5.4. *A strongly convex set X is τ-FC if and only if the set of its extreme points $\mathcal{E}(X)$ is τ-closed.*

Proof. Let X be a strongly convex set and suppose that the set of its extreme points is a τ-closed set. Let $\{F_n\}_n$ be a sequence of faces of X that is τ-Kuratowski–limit-converging to some $F \subset X$. If $F = X$ then we are done. If not, then for n large we have necessarily $F_n \neq X$. Since X is strongly convex, for n large, $F_n = \{x_n\}$ where x_n is some extreme points of X. Since F_n τ-Kuratowski-limit-converges, x_n τ-converges to some x in X, which is necessarily an extreme point. □

Remark 5.5.

- If X is the unit ball of an Hilbert space then X is strongly convex. In such a case, the set of extreme points of X is the unit sphere. Hence if τ is a topology for which the unit sphere is τ-closed, then the unit ball would be τ-FC.
- Suppose that X is included in a finite dimensional space. Then X is strongly convex if and only if $\mathcal{E}(X) = \partial_\tau X$. Since $\partial_\tau X$ is always τ-closed, we deduce that a finite dimensional strongly convex set is always τ-FC (hence τ-SC by part (c) of Theorem 1.16).

5.4. Example 5

Corollary 2.6 shows that condition τ-SC on X is necessary for the operator $co_X(\cdot)$ to preserve d_τ-Lipschitz continuity. Is this condition also sufficient? The answer is no.

Let $X = \{(a,b) \in R^2 : a^2 + b^2 \leq 1\}$ be the unit ball in R^2. Remark 5.5 above shows that X is τ-SC. Let $x_0 = (0,1)$ and define the 1-Lipschitz function f on X by: $f(x) = -\|x - x_0\|$ where $\|\cdot\|$ denotes the euclidean norm. The idea is to construct a sequence (x_0^n) converging to the extreme point x_0 such that there exists a convex decomposition of x_0^n supported by two extreme points x_1^n and x_2^n that satisfies $\frac{\|x_i^n - x_0\|}{\|x_0^n - x_0\|} \to \infty$ for $i \in \{1,2\}$. That is, let $x_0^n = \left(0, \cos(\frac{1}{n})\right)$, $x_1^n = \left(\sin(\frac{1}{n}), \cos\left(\frac{1}{n}\right)\right)$ and $x_2^n = \left(-\sin(\frac{1}{n}), \cos\left(\frac{1}{n}\right)\right)$. Since points x_1^n and x_2^n are extreme points of X and $x_0^n = \frac{1}{2}x_1^n + \frac{1}{2}x_2^n$

$$co_X(f)(x_0^n) \leq \frac{1}{2}f(x_1^n) + \frac{1}{2}f(x_2^n) = -2\sin\left(\frac{1}{2n}\right).$$

Since x_0 is an extreme point of X,

$$co_X(f)(x_0) = f(x_0) = 0.$$

Thus

$$co_X(f)(x_0) - co_X(f)(x_0^n) = -co_X(f)(x_0^n) \geq 2\sin\left(\frac{1}{2n}\right),$$

so

$$\frac{co_X(f)(x_0) - co_X(f)(x_0^n)}{\|x_0 - x_0^n\|} \geq 2\frac{\sin\left(\frac{1}{2n}\right)}{1 - \cos\left(\frac{1}{n}\right)} \to +\infty,$$

$co_X(f)$ is not $\|\cdot\|$-Lipschitz.

6. Proof of part (d) in Theorem 1.17

Proposition 6.1. *If the set X is* d-*SL(ρ) and if the function f on X is* d-*Lipschitz with constant L then $co_X(f)$ is* d-*Lipschitz with constant ρL.*

Proof. Let x, y be in X and suppose that $co_X(f)(x) = \lim_{n\to\infty} \sum_{i=1}^{m_n} \alpha_i^n f(x_i^n)$ with $\sum_{i=1}^{m_n} \alpha_i^n \delta_{x_i^n} \in \Delta_Z^*(x)$. Since X is d-SL with constant ρ, there exists a sequence of vectors

$(y_i^n)_{i=1}^{m_n}$ such that $\sum_{i=1}^{m_n} \alpha_i^n \delta_{y_i^n} \in \Delta_Z^*(y)$ and $\sum_{i=1}^{m_n} \alpha_i^n d(x_i^n, y_i^n) \leq \rho d(x,y)$. Thus,

$$\begin{aligned}
co_X(f)(y) - co_X(f)(x) &= co_X(f)(y) - \lim_{n\to\infty} \sum_{i=1}^{m_n} \alpha_i^n f(x_i^n) \\
&= \lim_{n\to\infty} \left[co_X(f)(y) - \sum_{i=1}^{m_n} \alpha_i^n f(x_i^n) \right] \\
&\leq \limsup_{n\to\infty} \left[\sum_{i=1}^{m_n} \alpha_i^n f(y_i^n) - \sum_{i=1}^{m_n} \alpha_i^n f(x_i^n) \right] \\
&\leq \limsup_{n\to\infty} \left[\sum_{i=1}^{m_n} \alpha_i^n \left| f(x_i^n) - f(y_i^n) \right| \right] \\
&\leq L \limsup_{n\to\infty} \left[\sum_{i=1}^{m_n} \alpha_i^n d(x_i^n, y_i^n) \right] \\
&\leq \rho \times L \times d(x,y)
\end{aligned}$$

□

7. Proof of part (e) in Theorem 1.17

Here is shown that the class of $\|\cdot\|$-SL(ρ) sets is stable under countable product. Using parts (f) and (g) of Theorem 1.17, one may construct an infinite dimensional compact set (for example the Hilbert cube) for which the convexification operator preserves uniformly (or exactly) Lipschitz continuity.

Proposition 7.1. *Let $\{X_n\}_n$ be a sequence of d_n-compact convex sets that are supposed to be d_n-SL(ρ). Assume also that there exists a constant $M > 0$ such that,* $\sup_n \sup_{x_n \in X_n, y_n \in X_n} d_n(x_n, y_n) \leq M$. *Let $\theta = \{\theta_n\}_n$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \theta_n < +\infty$. Define $X := \times_n X_n$ and let d_θ denote the following distance on X,*

$$d_\theta(x,y) := \sum_{n=1}^{\infty} \theta_n d_n(x_n, y_n).$$

Then X is d_θ-compact and d_θ-SL(ρ).

Proof. Clearly, d_θ is well defined and X is d_θ-compact. Let show that X is also d_θ-SL(ρ). Let $x = \sum_{k=1}^{K} \alpha_k x^k$ and y be in X. Since X_n is a d_n-SL(ρ), there exists a vector $(y_n^k)_{k=1,\ldots K}$ in X_n^K such that $y_n = \sum_k \alpha_k y_n^k$ and $\sum_k \alpha_k d_n(x_n^k, y_n^k) \leq \rho d_n(x_n, y_n)$. Define now $y^k := \left\{y_n^k\right\}_n$ in X. Thus

$$\begin{aligned}\sum_k \alpha_k d_\theta\left(x^k, y^k\right) &= \sum_k \alpha_k \sum_n \theta_n d_n\left(x_n^k, y_n^k\right) \\ &= \sum_n \theta_n \sum_k \alpha_k d_n\left(x_n^k, y_n^k\right) \\ &\leq \sum_n \theta_n \rho d_n\left(x_n, y_n\right) \\ &= \rho d_\theta(x, y)\end{aligned}$$

□

8. Proof of part (f) in Theorem 1.17

8.1. Simplices

Let Y be a measurable space. Denote by $\mathcal{M}(Y)$ the set of σ-additive bounded measures on Y and by $\mathcal{M}^+(Y)$ the set of σ-additive bounded positive measures on Y. Then any $\eta \in \mathcal{M}(Y)$ can be decomposed uniquely as the difference of two measures with disjoint supports η^+ and η^- in $\mathcal{M}^+(Y)$ (the Hahn-Jordan decomposition Theorems, Cohn, 1980, Theorem 4.1.4 and Corollary 4.1.5). The total variation norm of $\eta \in \mathcal{M}(Y)$ is defined by

$$\|\eta\|_{\mathcal{M}(Y)} := \eta^+(Y) + \eta^-(Y).$$

Note that if X is a τ-simplex of Choquet, the barycenter linear transformation $r_\tau(\cdot)$ that associates to each $\sigma \in \Delta_{\mathcal{E}(X),\tau}$, $r_\tau(\sigma) \in X$ defines an affine bijection between X and $\Delta_{\mathcal{E}(X),\tau}$ (the set of regular τ-probability-measures over $\mathcal{E}(X)$). This may be extended into an affine bijection between Vect(X) and $\mathcal{M}(\mathcal{E}(X),\tau)$ (the set of σ-additive regular τ-measure on $\mathcal{E}(X)$). Denoting this extension also by $r_\tau(\cdot)$, one may define a norm $\|\cdot\|_{X,\tau}$ on Vect(X) as follows:

$$\|x\|_{X,\tau} := \left\|r_\tau^{-1}(x)\right\|_{\mathcal{M}(\mathcal{E}(X),\tau)}$$

Proposition 8.1. *Suppose that the set X is a τ-simplex of Choquet. Hence, X is $\|\cdot\|_{X,\tau}$-SL(1). In addition, if $\mathcal{E}(X)$ is infinite, the topology induced by $\|\cdot\|_{X,\tau}$ is strictly stronger than the topology τ.*

Before proving Proposition 8.1, a useful Lemma is established.

8.2. A new splitting of probabilities

Lemma 8.2. [20]*Let μ be some probability-measure on some measurable space Y and let $(\alpha_k, \mu_k)_{k=1}^m$ be such that, $\forall k = 1, ..., m$:*

(i) μ_k *is a probability-measure on* Y;
(ii) $\alpha_k \geq 0$, $\sum_{k=1}^m \alpha_k = 1$ *and* $\sum_{k=1}^m \alpha_k \mu_k = \mu$.

Then, for each probability-measure ν on Y, there exists $\nu_1, ..., \nu_m$ such that $\forall k = 1, ..., m$

[20]In a previous version, this lemma was proved only when Y is countable. Jean-Francois Mertens (whom I thank) remarked that my previous proof was generalizable.

(*i'*) ν_k *is a probability-measure on* Y;
(*ii'*) $\nu = \sum_{k=1}^m \alpha_k \nu_k$; *and*
(*iii'*) $\|\mu - \nu\|_{\mathcal{M}(Y)} = \sum_{k=1}^m \alpha_k \|\mu_k - \nu_k\|_{\mathcal{M}(Y)}$.

Proof. Let $\eta := \nu - \mu$ and let $\eta^+$and η^-be its Hahn-Jordan decomposition. Since $\eta(Y) = \nu(Y) - \mu(Y) = 0$, it follows that

$$\|\eta\|_{\mathcal{M}(Y)} = 2\eta^+(Y) = 2\eta^-(Y).$$

If $\eta^+(Y) = 0$ then $\mu = \nu$ and the Lemma holds (take $\nu_k = \mu_k$). Suppose now that $\eta^+(Y) > 0$.

Since (1) $\nu = \eta^+ + \mu - \eta^-$ is in $\mathcal{M}^+(Y)$, (2) $\eta^+$and η^-have disjoint supports and (3) μ is in $\mathcal{M}^+(Y)$, we deduce that $\xi := \mu - \eta^-$is also an element of $\mathcal{M}^+(Y)$.

$\sum_{k=1}^m \alpha_k \mu_k = \mu$ implies that for any $k = 1, ...m$, the measure μ_k is absolutely continuous with respect to μ. By Cohn, 1980, Theorem 4.2.2, it admits a Radon-Nikodym derivative with respect to μ

$$f_k(y) := \frac{\mu_k(dy)}{\mu(dy)} \geq 0, \qquad y \in Y.$$

$f_k(\cdot)$ is unique up to μ-almost everywhere equality. Now, if the positive measure ν_k is defined by

$$\nu_k(dy) := \left[\frac{\int_Y f_k(y)\eta^-(dy)}{\eta^+(Y)}\right] \eta^+(dy) + f_k(y)\xi(dy),$$

then a simple computation shows

$$\int_Y \nu_k(dy) = \int_Y f_k(y)\eta^-(dy) + \int_Y \mu_k(dy) - f_k(y)\eta^-(dy) = 1$$

Hence, ν_k is a probability-measure. The uniqueness of Radon-Nikodym derivative implies that $\sum_{k=1}^m \alpha_k f_k(y) = 1$, $\mu-$almost everywhere. Replacing each function f_k by some other positive function (of the same name) equals to f_k $\mu-$almost everywhere, it may be assumed that $\sum_{k=1}^m \alpha_k f_k(y) = 1$ everywhere in Y. Since $\eta^+(Y) = \eta^-(Y)$,

$$\sum_{k=1}^m \alpha_k \nu_k = \nu.$$

A simple calculation yields

$$\nu_k(dy) - \mu_k(dy) = \left[\frac{\int_Y f_k(y)\eta^-(dy)}{\eta^+(Y)}\right] \eta^+(dy) - f_k(y)\eta^-(dy),$$

which is the Hahn-Jordan decomposition of $\nu_k - \mu_k$. This implies

$$\|\nu_k - \mu_k\|_{\mathcal{M}(Y)} = 2\int_Y f_k(y)\eta^-(dy)$$

from which we obtain that

$$\sum_{k=1}^{m} \alpha_k \|\nu_k - \mu_k\|_{\mathcal{M}(Y)} = 2\int_Y \eta^-(dy)$$
$$= \|\nu - \mu\|_{\mathcal{M}(Y)}$$

□

Remark 8.3. The ν_k constructed in the last proof depends, for a fixed k, only on μ,ν and μ_k. In particular, it is not depending on α_k nor on $\mu_{k'}$,$k' \neq k$. More precisely, ν_k is such that $[\nu_k - \mu_k]^+$is proportional to $[\nu - \mu]^+$.

8.3. Proof of Proposition 8.1.

Let X be a τ-simplex, and x, y be two points in X. Suppose that $\sum_{k=1}^m \alpha_k \delta_{x_k} \in \Delta_X^*(x)$. Lemma 8.2 implies that there exist m points $y_k \in X$, $k = 1, ..., m$, such that $\sum_{k=1}^m \alpha_k \delta_{y_k} \in \Delta_X^*(y)$ and $\sum_{k\in K} \alpha_k \|r^{-1}(x_k) - r^{-1}(y_k)\|_{\mathcal{M}(\mathcal{E}(X),\tau)} = \|r^{-1}(x) - r^{-1}(y)\|_{\mathcal{M}(\mathcal{E}(X),\tau)}$. Since by definition, $\|\cdot\|_{X,\tau} = \|r^{-1}(\cdot)\|_{\mathcal{M}(\mathcal{E}(X),\tau)}$, it follows that X is $\|\cdot\|_{X,\tau}$-SL with constant 1.

When X and $\mathcal{E}(X)$ are endowed with the topology τ and $\Delta_{\mathcal{E}(X),\tau}$ is endowed with the weak* topology $w^*(\tau)$, the affine bijection $r_\tau(\cdot)$ from $\Delta_{\mathcal{E}(X),\tau}$ to X is continuous (Choquet, 1969, Proposition 26.3). Also, it is well known that the topology induced by the total variation norm on $\mathcal{M}(\mathcal{E}(X),\tau)$ is stronger than the weak* topology $w^*(\tau)$. Hence, the topology induced by $\|\cdot\|_{X,\tau}$ (the image, by $r_\tau(\cdot)$, of $\|\cdot\|_{\mathcal{M}(\mathcal{E}(X),\tau)}$) is stronger than the topology τ on X. Since X is τ-compact, $\|\cdot\|_{X,\tau}$ and τ are equivalent if and only if X is $\|\cdot\|_{X,\tau}$-compact. This is true only if $\mathcal{E}(X)$ is finite. Actually, if $\mathcal{E}(X)$ is infinite and if $\{x_n\}_n$ is a sequence of different extreme points that τ-converges to some point x_0 then one has $\|x_n - x_0\|_{X,\tau} = \|\delta_{x_n} - \delta_{x_0}\|_{\mathcal{M}(\mathcal{E}(X),\tau)} = 2$. If $\mathcal{E}(X)$ is finite, then X is included in a finite dimensional space and in this case all the vectorial topologies are equivalent.

9. Proof of part (g) in Theorem 1.17

Let X be a polytope. Then there exist m points e_i, $i = 1, ..., m$ such that

$$X = \text{Conv}\{e_1, ..., e_m\}.$$

Suppose in addition that $\{e_1, ..., e_m\} = \mathcal{E}(X)$ and define the (quotient) norm[21] on Vect(X), denoted $\|\cdot\|_X$, as follows

$$\|z\|_X = \inf_{\left\{\alpha\in R^m:\sum_{i=1}^m \alpha_i e_i = z\right\}} \sum_{i=1}^m |\alpha_i|.$$

Note that when the polytope X is a finite dimensional simplex (i.e. $e_1, ..., e_m$ are affinely independent) the norm defined above for a simplex of Choquet coincides with the one defined here.

[21]It is the quotient norm on X, considered as the quotient of l_n^1 by the kernel of the mapping $\alpha \to \sum_{i=1}^n \alpha_i e_i$.

Proposition 9.1. *If X is a polytope then there exists a constant ρ, depending only on X, such that X is $\|\cdot\|_X$-SL(ρ).*

Consequently, by the equivalence of norms property in finite dimension, we deduce that a similar result holds for any norm on Span(X). To prove Proposition 9.1, we use the following Lipschitzian characterization of polyhedra, due to Walkup & Wets (1969).

9.1. The Walkup & Wets result

Let $l(\cdot)$ be an affine transformation between two finite dimensional normed vector spaces A and B. For a subset K of A, define the set valued function $\kappa(\cdot)$ which associates to each $b \in l(K)$, the following subset of K

$$\kappa(b) := l^{-1}(b) \cap K.$$

The Hausdorff distance $D(\kappa(b), \kappa(b'))$ is a metric on the collection of all nonempty sections $\kappa(b)$:

$$D(\kappa(b), \kappa(b')) = \max\left[r(b, b'), r(b', b)\right]$$

with,

$$r(b, b') = \max_{a \in \kappa(b)} \min_{a' \in \kappa(b')} \|a - a'\|_A,$$

where $\|\cdot\|_A$ denotes the norm of the vector space A.

The set valued function $\kappa(\cdot)$ is Lipschitz if there exists a constant ρ such that for any b and b' in $l(K)$

$$D(\kappa(b), \kappa(b')) \leq \rho \|b - b'\|_B.$$

where $\|\cdot\|_B$ denotes the norm of B and the constant ρ depends only on $l(\cdot)$, K, $\|\cdot\|_A$ and $\|\cdot\|_B$.

Theorem 9.2 (Walkup & Wets (1969)). *Suppose that K is convex and compact. Then K is a polytope if and only if for any affine transformation $l(\cdot)$, $\kappa(\cdot)$ is Lipschitz.*

9.2. A consequence of Walkup & Wets

Here is a very useful consequence of the characterization of Walkup & Wets (1969).

Lemma 9.3. *Let $X = \operatorname{Conv}\{e_1, ..., e_m\}$ be a polytope where $\{e_1, ..., e_m\} = \mathcal{E}(X)$. For x and y in X, define the following distance on X*

$$\mathrm{d}_X(x, y) = \max[c(x, y), c(y, x)]$$

with

$$c(x, y) = \max_{\alpha \in B_X(x)} \min_{\beta \in B_X(y)} \sum_{i=1}^{m} |\alpha_i - \beta_i|$$

where, for $z \in X$

$$B_X(z) = \left\{ \alpha \in K : \sum_{i=1}^{m} \alpha_i \delta_{e_i} \in \Delta_X^*(z) \right\}$$

and

$$K = \{(\alpha_i)_{i=1}^m \in R^m : \alpha_i \geq 0, \forall i = 1, ..., m, \sum_{i=1}^m \alpha_i = 1\}.$$

Then there exists a constant ρ depending only on X such that for any x and y in X

$$\|x - y\|_X \leq \mathrm{d}_X(x, y) \leq \rho \|x - y\|_X .$$

Proof. Endow Vect(X) with the norm $\|\cdot\|_X$ defined above and endow R^m with the standard L_1 norm $\|\cdot\|_1$:

$$\|\alpha\|_1 = \sum_{i=1}^m |\alpha_i| .$$

Let $l(\cdot)$ be the affine transformation from R^m to Vect(X) which associates to each $\alpha \in K$, $l(\alpha) = \sum_{i=1}^m \alpha_i e_i \in X$. Since $X = \text{Conv}\{e_1, ..., e_m\}$, $X = l(K)$. Define as above the set valued function $\kappa(\cdot)$ from X into subset of K,

$$\kappa(x) := l^{-1}(x) \cap K.$$

This may be interpreted as the set of all possible convex decompositions of x over the set of extreme points $\mathcal{E}(X)$. By Theorem 9.2 (Walkup & Wets) there exists a constant $\rho > 0$ such that for any x and y in X

$$D(\kappa(x), \kappa(y)) \leq \rho \|x - y\|_X$$

where

$$D(\kappa(x), \kappa(y)) = \max[r(x, y), r(y, x)]$$

and

$$r(x, y) = \max_{\alpha \in K : l(\alpha) = x} \min_{\beta \in K : l(\beta) = y} \sum_{i=1}^m |\alpha_i - \beta_i| .$$

Thus for any x and y in X, $r(x, y) = c(x, y)$ and $D(\kappa(x), \kappa(y)) = d_X(x, y)$. The fact that $\|x - y\|_X \leq d_X(x, y)$ is trivial. Since $l(\cdot)$, K and $\|\cdot\|_X$ are functions of X, the constant ρ depends only on X. Finally, $d_X(\cdot, \cdot)$ is a distance since $D(\kappa(\cdot), \kappa(\cdot))$ is a distance. □

9.3. Proof of Proposition 9.1.

Now we are ready to prove Proposition 9.1. Let $X = \text{Conv}\{e_1, ..., e_m\}$ be a polytope where $\{e_1, ..., e_m\} = \mathcal{E}(X)$. Let x and y be in X and let $(\alpha_i)_{i=1}^m$ be such that $\sum_{i=1}^m \alpha_i \delta_{e_i} \in \Delta_X^*(x)$. By Lemma 9.3 there exists $\sum_{i=1}^m \beta_i \delta_{e_i} \in \Delta_X^*(y)$ such that

$$\sum_{i=1}^m |\alpha_i - \beta_i| \leq \rho \|x - y\|_X .$$

Let $\sum_{k=1}^l \lambda^k \delta_{x^k} \in \Delta_X^*(x)$ and $\sum_{i=1}^m \alpha_i^k \delta_{e_i} \in \Delta_X^*(x^k)$. Identifying the m-dimensional vectors $\alpha = (\alpha_i)_{i=1}^m$, $\beta = (\beta_i)_{i=1,...,m}$ and $\alpha^k = \left(\alpha_i^k\right)_{i=1}^m$ with the corresponding probability-measures over the finite set $\mathcal{E}(X) = \{e_1, ..., e_m\}$ and applying Lemma 8.2, we deduce that there exist l vectors of probability-measures over $\mathcal{E}(X)$, $\beta^k = \left(\beta_i^k\right)_{i=1}^m$, $k = 1..., l$, such that

$$\sum_{k=1}^{l} \lambda^k \beta^k = \beta,$$

and

$$\sum_{i=1}^{m} |\alpha_i - \beta_i| = \sum_{k=1}^{l} \lambda^k \sum_{i=1}^{m} \left|\alpha_i^k - \beta_i^k\right|.$$

Defining $y^k := \sum_{i=1}^m \beta_i^k e_i$ in X, it follows that

$$\sum_{i=1}^{m} |\alpha_i - \beta_i| \geq \sum_{k=1}^{l} \lambda^k \left\|x^k - y^k\right\|_X.$$

Since $\sum_{i=1}^m |\alpha_i - \beta_i| \leq \rho \|x - y\|_X$, we deduce that X is $\|\cdot\|_X$-SL(ρ).

Remark 9.4. One may ask if it is possible to construct for any polytope X (as for simplices) a norm $N_X(\cdot)$ such that X is $N_X(\cdot)$-SL(1). The answer is no[22]. Indeed, let $\{X_n\}_n$ be the sequence of the n-symmetric polygon of R^2 with diameter 1 with respect to the euclidean norm. Suppose that for each $n = 1, ...,$ there exists a norm $N_{X_n}(\cdot)$ such that X is $N_{X_n}(\cdot)$-SL(1). Without loss of generality, suppose that $N_{X_n}(\cdot)$ has the same symmetric points with respect to the center than the n-symmetric polygon X_n. Consequently, as $n \to \infty$, $\{N_{X_n}(\cdot)\}_n$ uniformly converges to the euclidean norm, $\|\cdot\|$. Since X_n uniformly converges to the euclidean ball in R^2 and since X_n is $N_{X_n}(\cdot)$-SL(1), we deduce that the euclidean ball in R^2 is also $\|\cdot\|$-SL(1). This is in contradiction with Example 5.

10. Proof of part (h) in Theorem 1.17

Proposition 10.1. *If* $(\mathrm{E}, \|\cdot\|_\tau)$ *is a finite dimensional real vector normed space, the operator* $co_X(\cdot)$ *uniformly preserves* $\|\cdot\|$*-Lipschitz continuity if and only if* X *is a polytope.*

Proof. Using the fact that all the norms are equivalent in finite dimension, and using results (d) and (f) in Theorem 1.17, we deduce that when X is a polytope, $co_X(\cdot)$ uniformly preserves $\|\cdot\|_\tau$-Lipschitz continuity. To show the other implication, suppose, without loss of generality, that $\|\cdot\|_\tau$ is the euclidean norm and suppose that the convexification operator $co_X(\cdot)$ uniformly preserves $\|\cdot\|_\tau$-Lipschitz continuity. Clearly, if $\mathcal{E}(X)$ is not τ-closed, X will not be τ-SC and hence one could construct a $\|\cdot\|_\tau$-Lipschitz function for which $co_X(f)$ is not $\|\cdot\|_\tau$-Lipschitz (Corollary 2.6). Now suppose that $\mathcal{E}(X)$ is τ-closed but not finite. Then there exists an accumulation extreme point (because $\mathcal{E}(X)$ is τ-compact since it is supposed to be τ-closed and included in the τ-compact X). That is, there exists a sequence $\{e_n\}_n$ of different extreme points of X converging to an extreme point e. The idea of the proof is now very similar to the one in Example 5. It may be supposed that (a)$\frac{e_n - e}{\|e_n - e\|_\tau}$ converges to some direction v (here is used explicitly the hypothesis of finite dimension) and (b) $\frac{\|e_{n+1} - e\|_\tau}{\|e_n - e\|_\tau} \to 0$. Define α_n and β_n in $]0, \pi[$ such that

$$\cos(\alpha_n) = \frac{\langle e - e_n, e - e_{n+1}\rangle_{\mathrm{E}}}{a_n \times a_{n+1}}$$

[22]Thanks to Jean-Francois Mertens for this remark.

and

$$\cos(\beta_n) = \frac{\langle e - e_n, e_{n+1} - e_n \rangle}{a_n \times b_n}$$

where

$$a_n = \|e - e_n\|_\tau , \qquad b_n = \|e_n - e_{n+1}\|_\tau .$$

Of course we use the fact that the norm is the euclidean norm; $\langle a, b \rangle_{\mathrm{E}}$ denotes the standard scalar product in E. A simple calculation shows that (a) implies that α_n goes to 0 and (b) implies that β_n goes to 0. Since the extreme points are different, α_n and β_n are different from 0. Now, let x_n be the orthogonal projection of e_{n+1} on $[e, e_n]$. Then, a simple triangle calculation shows that

(i) $\|x_n - e_{n+1}\|_\tau = a_{n+1} \times \sin(\alpha_n) = b_n \times \sin(\beta_n)$;

(ii) $x_n = \frac{\cos(\alpha_n)\times a_{n+1}}{\cos(\alpha_n)\times a_{n+1}+\cos(\beta_n)\times b_n} e + \frac{\cos(\beta_n)\times b_n}{\cos(\alpha_n)\times a_{n+1}+\cos(\beta_n)\times b_n} e_n$.

Define the sequence of $\|\cdot\|_\tau$-Lipschitz functions with constant 1, $f_n(\cdot)$, by $f_n(z) = -\|z - e_{n+1}\|$. By the above convex decomposition, we deduce that

$$co_X(f_n)(x_n) \leq -\frac{\cos(\alpha_n) \times a_{n+1}^2 + \cos(\beta_n) \times b_n^2}{\cos(\alpha_n) \times a_{n+1} + \cos(\beta_n) \times b_n}$$

implying

$$\frac{co_X(f_n)(e_{n+1}) - co_X(f_n)(x_n)}{\|x_n - e_{n+1}\|_\tau} \geq \frac{\min(a_{n+1}, b_n)}{2\|x_n - e_{n+1}\|_\tau}$$

since $\|x_n - e_{n+1}\|_\tau = a_{n+1} \times \sin(\alpha_n) = b_n \times \sin(\beta_n)$, it follows that

$$\frac{co_X(f_n)(e_{n+1}) - co_X(f_n)(x_n)}{\|x_n - e_{n+1}\|_\tau} \geq \frac{1}{2}\min\left(\frac{1}{\sin(\alpha_n)}, \frac{1}{\sin(\beta_n)}\right) \to +\infty.$$

□

11. Extensions and open questions

In Laraki 2001 (which was in the origin of this paper) the convexification operator is extended to a zero-sum-game operator called the splitting operator; the regularity of this operator is used to prove the existence of the asymptotic value of a stochastic game where each player controls a discrete-time martingale. In a work in progress, the author is studying the relation between the modulus of continuity of a continuous function and its image by the convexification operator. It is shown in particular that the modulus of continuity is preserved by co_X when X is Splitting-Lipschitz with constant 1. Finally, in Laraki & Sudderth, 2002, some result of this paper are extended to a large class of optimal reward operators. In addition, a general necessary and sufficient condition for the preservation of Holder-continuity is given.

Two open questions arise in infinite dimension: is τ-FC equivalent to τ-SC ? and is $\|\cdot\|_\tau$-SL always equivalent to the uniform preservation of $\|\cdot\|_\tau$-Lipschitz continuity?

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