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Sequential Representation Formulae for G-Subdifferential and Clarke Subdifferential in Smooth Banach Spaces

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We extend to smooth Banach spaces the proximal formula for Clarke subdifferential of lower semicontinuous function. In the process we also extend the representation of G-subdifferential and the singular G-subdifferential in terms of sequential limits of smooth subdifferentials of controlled rank.

1. Introduction

Clarke subdifferential (see [6]) and the smaller G-subdifferential of Lipschitz functions have proved themselves to be useful tools in many problems of non smooth analysis because of their nice topological properties and developed calculus. Another advantage of these two subdifferentials is that they are well behaved in any Banach space. They can be extended to lower semicontinuous functions via the distance to the epigraph. Nevertheless, such construction is implicit and it is therefore useful to have an analytical representation of the Clarke subdifferential as well as G-subdifferential. Rockafellar [21], Ioffe [12], Loewen [17, 18], and Borwein and Ioffe [2] are some works in this direction. The aim of this paper is to prove the full scope of the representation formulae, to be found in the mentioned works, in the setting of smooth Banach spaces.

We now list the main results and provide references for the known parts. This would be cumbersome to do if we are to give all the definitions and therefore we postpone recalling some of them.

We refer to Section 2 for definition of $D^-_{\beta,k}$ - the β-smooth subdifferential of *rank* k and $N_{\beta,k}$ - the cone of β-smooth normals of rank k; see Definition 2.1 and (12). Note also the different but equivalent definition in Fréchet case: (11). Using these one may construct the following limiting objects:

$$\partial_{\beta,k} f(x) = \{w^* - \lim_{n\to\infty} p_n;\ p_n \in D^-_{\beta,k} f(x_n), x_n \to_f x\}, \tag{1}$$

and

$$\widetilde{N}_{\beta,k}(S,x) = \{w^* - \lim_{n\to\infty} p_n;\ p_n \in N_{\beta,k}(S,x_n), x_n \to x\}. \tag{2}$$

As standard throughout the paper, $\partial_\beta f(x) = \bigcup_{k>0} \partial_{\beta,k} f(x)$ and $\widetilde{N}_\beta(S,x) = \bigcup_{k>0} \widetilde{N}_{\beta,k}(S,x)$.

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Limiting subdifferentials of this kind were considered by many authors; see the comments in [5]. In [19] (Theorem 9.5) it is shown that if the space has Fréchet smooth norm and $\beta = F$, the Fréchet bornology, then $\widetilde{N}_F$ coincides with the normal cone N, defined by Kruger and Mordukhovich - Definition 2.7. This was further refined in [3].

Actually, in view of the mentioned equivalence between the two definitions of F-normals (see Section 2, the comments after (11), or [5]) and the fact that each w^* convergent *sequence* is norm bounded (Banach-Steinhaus Theorem, see [22]) Theorem 9.5 of [19] follows from Theorem 2.9 of [19]. It should be mentioned however that the construction used in [8] for establishing the equivalence (and which originates from [16]) is far from being trivial. Note also that most of the results of the present paper, concerning Asplund space, are essentially covered in Theorem 2.9 of [19].

For the sake of completeness we present a proof of

Proposition 1.1. ([19]) *Let X be an Asplund space and S be a closed subset of X. Then for each $x \in S$ we have that $N(S,x) = \widetilde{N}_F(S,x)$.*

The following proposition is an easy corollary of the corresponding result for smooth subdifferentials of Borwein, Mordukhovich and Shao: [3], Proposition 2.3. We present a proof of their result for the sake of completeness, see Lemma 2.2.

Proposition 1.2. *Let X be a Banach space with some bornology β and $f : X \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous, $x \in \mathrm{dom} f$. Then for each $l > k > 1$ we have that*

(i) $p \in \partial_{\beta,k} f(x) \Rightarrow (p,-1) \in \widetilde{N}_{\beta,k}(\mathrm{epi} f, (x, f(x)))$;

(ii) $(p,-1) \in \widetilde{N}_{\beta,k}(\mathrm{epi} f, (x, f(x))) \Rightarrow p \in \partial_{\beta,l} f(x)$.

Of course, Propositions 1.1 and 1.2 show that the subdifferential corresponding to Kruger-Mordukhovich cone coincides with ∂_F when the space is Asplund, see [19, 3] for more details.

In many situations one needs to consider not only the subdifferential of a given function, but also its so called *singular* subdifferential. A prime example of this is when Clarke subdifferential is represented via G-subdifferential, see Proposition 1.7 (for more examples we refer to [5, 19]).

The singular limiting subdifferential ∂^∞_β is defined by

$$p \in \partial^\infty_{\beta,k} f(x) \iff (p,0) \in \widetilde{N}_{\beta,k}(\mathrm{epi} f, (x, f(x))); \tag{3}$$

$\partial^\infty_\beta f(x) = \bigcup_{k>0} \partial^\infty_{\beta,k} f(x)$.

The following proposition is of crucial importance. It allows, in case the space possesses suitable smoothness, computing the singular limiting subdifferential in much the same way as the limiting subdifferential itself, instead of having to deal with the epigraph.

Proposition 1.3. *Assume that X is β-smooth, or X is Asplund space and $\beta = F$. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous and $x \in \mathrm{dom} f$. For any $l > k > 0$ we have that*

(i) $\{w^* - \lim_{n\to\infty} \lambda_n^{-1} p_n;\ p_n \in D^-_{\beta,\lambda_n k} f(x_n),\ x_n \to_f x,\ \lambda_n \to \infty\} \subset \partial^\infty_{\beta,k} f(x)$;

(ıı) $\partial^{\infty}_{\beta,k} f(x) \subset \beta^*\!-\!\mathrm{cl}\{w^*\!-\lim_{n\to\infty} \lambda_n^{-1} p_n;\ p_n \in D^-_{\beta,\lambda_n l} f(x_n),\ x_n \to_f x,\ \lambda_n \to \infty\}$.

If the β^* topology is metrizable on bounded sets (as, for example, when X is separable or $\beta = F$), then taking β^* closure in the right hand side of (ıı) is redundant.

Proposition 1.3 readily follows from the following key fact.

Proposition 1.4. *Let X be β-smooth, or X be Asplund and $\beta = F$. Let $f : X \to \mathbb{R}$ be a proper lower semicontinuous function and $(p, 0) \in N_{\beta,k}(\mathrm{epi} f, (x_0, f(x_0)))$, where $k > 0$. Then for any $\varepsilon > 0$, $l > k$ and β^* neighborhood U^* of 0 in X^* there exist $y \in X$ and $(q, \lambda) \in N_{\beta,l}(\mathrm{epi} f, (y, f(y)))$ such that $\lambda \in (-\varepsilon, 0)$, $\|y - x_0\| < \varepsilon$, $|f(y) - f(x_0)| < \varepsilon$ and*

$$q \in p + U^*.$$

The partial case of Proposition 1.4 when the space has F-smooth norm is proved in [2]. A similar but rougher (i.e. U^* is w^* neighborhood and the rank is not controlled) approximation of the singular smooth normals to the epigraph is obtained in smooth Banach spaces by Zhu in [24]. The case of X Asplund is covered in [19, 23].

We refer to Section 3, or to [2, 3, 12], for the definition of the cone of G-normals $N_G = \bigcup_{k>0} N_{G,k}$. The following representation of N_G is proved in [2, 3] in the case when the norm is β-smooth. For the sake of completeness we present a proof in our slightly more general setting.

Proposition 1.5. *Assume that the space X is β-smooth with respect to some bornology β, or that X is Asplund and $\beta = F$. If S is a closed subset of X and $x \in S$ then*

$$N_G(S, x) = \bigcup_{k>0} \mathrm{cl}^*\{w^*\!- \lim_{n\to\infty} p_n;\ p_n \in N_{\beta,k}(S, x_n),\ x_n \to x\}. \tag{4}$$

As far as functions are concerned, G-subdifferential is defined in the standard way:

$$p \in \partial_{G,k} f(x) \iff (p, -1) \in N_{G,k}(\mathrm{epi} f, (x, f(x))), \tag{5}$$

$\partial_G f(x) = \bigcup_{k>0} \partial_{G,k} f(x)$. The singular G-subdifferential is

$$p \in \partial^{\infty}_{G,k} f(x) \iff (p, 0) \in N_{G,k}(\mathrm{epi} f, (x, f(x))), \tag{6}$$

$\partial^{\infty}_G f(x) = \bigcup_{k>0} \partial^{\infty}_{G,k} f(x)$. Using the representation (4) as well as Propositions 1.2 and 1.3, we obtain

Theorem 1.6. *Assume that the space X is β-smooth, or X is Asplund space and $\beta = F$. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous and $x \in \mathrm{dom} f$. Then*

$$\partial_G f(x) = \bigcup_{k>0} \mathrm{cl}^*\{w^*\!- \lim_{n\to\infty} p_n;\ p_n \in D^-_{\beta,k} f(x_n),\ x_n \to_f x\}, \tag{7}$$

$$\partial^{\infty}_G f(x) = \bigcup_{k>0} \mathrm{cl}^*\{w^*\!- \lim_{n\to\infty} \lambda_n^{-1} p_n;\ p_n \in D^-_{\beta,\lambda_n k} f(x_n),\ x_n \to_f x,\ \lambda_n \to \infty\}. \tag{8}$$

Part (7) of the above Theorem is known for spaces with β-smooth norm, cf. [2, 3], while part (8) was established so far only for spaces with F-smooth norm and $\beta = F$, cf. [2].

We refer to [6] for the standard definition of Clarke normals N_C and Clarke subdifferential ∂_C. As shown for example in [12], an equivalent definition for the Clarke normal cone N_C is

$$N_C(S, x) = \overline{\mathrm{co}}^* N_G(S, x). \tag{9}$$

Then $p \in \partial_C f(x) \iff (p, -1) \in N_C(\mathrm{epi} f, (x, f(x)))$. It is routine to verify the following

Proposition 1.7. *Let X be a Banach space, $f : X \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous and $x \in \mathrm{dom} f$. Then*

$$\partial_C f(x) = \overline{\mathrm{co}}^* \left(\partial_G f(x) + \partial_G^\infty f(x) \right).$$

Proposition 1.7 together with (7) and (8) provides a sequential representation of Clarke subdifferential. Omitting taking closures twice, we have

Theorem 1.8. *Assume that X is β smooth Banach space; or X is Asplund and $\beta = F$. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. Then*

$$\partial_C f(x) = \overline{\mathrm{co}}^* \left(\widetilde{\partial}_G f(x) + \widetilde{\partial}_G^\infty f(x) \right), \textit{ where}$$

$$\widetilde{\partial}_G f(x) = \bigcup_{k>0} \{ w^*\!- \lim_{n\to\infty} p_n;\ p_n \in D^-_{\beta,k} f(x_n), x_n \to_f x \}, \textit{ and}$$

$$\widetilde{\partial}_G^\infty f(x) = \bigcup_{k>0} \{ w^*\!- \lim_{n\to\infty} \lambda_n^{-1} p_n;\ p_n \in D^-_{\beta,\lambda_n k} f(x_n),\ x_n \to_f x,\ \lambda_n \to \infty \}.$$

The above formula goes back to Rockafellar [21] in finite dimensional space. Loewen [18] had proved it in superreflexive space. Ioffe [12] and Borwein and Ioffe [2] had established this representation in spaces with Fréchet smooth norm.

As was already mentioned, in [2] the sequential representation of the singular G-subdifferential is checked only for Fréchet smooth space. Therefore, the principal import of the present paper is the extension of the latter representation to smooth spaces, which are not Fréchet smooth, as for example $L^1[0, 1]$.

Remark. Since author's interest in this subject was motivated by a theoretical problem, see [14], the choice was to establish Theorem 1.8 in general β-smooth spaces. In this setting it appears to be no way to avoid technicalities related to controlling the rank and, as mentioned in [2], the latter is necessary in order to obtain correct representation.

However, if one is interested only in spaces with β-smooth norm - which are most likely to appear in applications - then significant simplifications are available.

This is so because in that case for $g(x) = \|x\|^2$ the mapping $x \to \|g'(x)\|$ is continuous. By composition one obtains a bump b, which is β-smooth, Lipschitz continuous and such that $x \to \|b'(x)\|$ is continuous. Therefore, the space Y of all $f : X \to \mathbb{R}$ bounded, Lipschitz continuous, β-smooth and such that $x \to \|f'(x)\|$ is continuous, considered with the norm $\|f\|_Y = \sup\{|f(x)|, \|f'(x)\|;\ x \in X\}$ is Banach and contains a bump. Also, it obviously satisfies the other requirements of Lemma I.2.5 of [8], meaning that the Smooth Variational Principle (Theorem 2.4) is valid with perturbations from Y.

Thus, if one would impose in Definition 2.1 the additional requirement '$x \to \|g'(x)\|$ is continuous', one would define a smaller subdifferential, say $D_{\beta}^{=}$, having the same calculus as D_{β}^{-}. The advantage of $D_{\beta}^{=}$ is that, just like in the case $\beta = F$, if $p \in D_{\beta}^{=}$ and $\|p\| < k$ then $p \in D_{\beta,k}^{-}$. This means that one is spared the tedious estimations of the Lipschitz constants of the supporting functions.

We proceed as follows. Section 2 contains the necessary definitions and tools concerning smooth subdifferentials; Propositions 1.1, 1.2, 1.3 and 1.4 are proved therein. In Section 3 we prove Propositions 1.5, 1.7 and Theorem 1.6.

2. Preliminaries. Smooth and limiting subdifferentials

Throughout this paper $(X, \|\cdot\|)$ is a Banach space with its norm. The dual space is denoted by X^*, the dual norm of X^* is denoted also by $\|\cdot\|$ as this does not cause confusion. The w^* topology on X^* is the topology of the pointwise convergence of the linear functionals. The closed unit ball of X is denoted by B_X, i.e. $B_X = \{x \in X;\ \|x\| \le 1\}$. For $x \in X$ and $\varepsilon > 0$ we put $B(x;\varepsilon) = x + \varepsilon B_X$. If $p \in X^*$ and $x \in X$ then $p(x) = \langle p, x\rangle$ is the value of the functional p at x. The product space $X \times \mathbb{R}$ is always endowed with the norm $\|(x,t)\| = \|x\| + |t|$, where $x \in X$ and $t \in \mathbb{R}$. In this way the dual norm in $X^* \times \mathbb{R}$ is $\|(p,t)\| = \max\{\|p\|, |t|\}$, where $p \in X^*$ and $t \in \mathbb{R}$. We denote the convex hull of the set S by co$S$, while the closed convex hull is $\overline{\text{co}}S$ (resp. $\overline{\text{co}}^*S$ if the closure is taken with respect to the w^* topology). The norm closure is cl$S$, whilst the w^* closure is cl$^*S$.

For convenience in the sequel all neighborhoods of 0 are assumed to be convex symmetric and closed in the respective topology. The class of topologies that are of immediate concern for us is defined as follows. A *bornology* β on X is a collection of closed bounded and symmetric subsets of X whose union is X and the union of any two elements of β lies in another element of β; and $aU \in \beta$ whenever $U \in \beta$ and $a > 0$. The β^* topology on X^* is the topology of uniform convergence on the elements of β. It can be described as follows. Let for $U \in \beta$

$$U^* = \{p \in X^* : \ p(U) \le 1\}. \tag{10}$$

Then $\{U^*\}_{U\in\beta}$ forms a base of neighborhoods of 0 for the β^* topology. It is clear from the conditions imposed on β that the β^* topology is stronger than the w^* topology (coinciding with it when, for example, all $U \in \beta$ are finite subsets) and weaker than the norm topology (coinciding with it when some element of β contains a ball with nonzero radius).

The function $g : X \to \mathbb{R}$ is said to be β-differentiable at $x \in X$ if there exists $p \in X^*$ such that for every $U \in \beta$

$$\lim_{t\to 0} \sup_{y\in U} \left| \frac{g(x+ty) - g(x) - tp(y)}{t} \right| = 0.$$

We denote $g'_{\beta}(x) = p$, writing simply $g'(x)$ when there can be no confusion.

The function $g : X \to \mathbb{R}$ is said to be β-smooth if it is β-differentiable on X and the derivative g' is a continuous mapping from $(X, \|\cdot\|)$ to (X^*, β^*).

The most important bornologies from our perspective are the *Fréchet* bornology F consisting of all closed bounded sets and the *Hadamard* bornology H of all norm-compact sets. It is easy to see, using for example finite ε-nets, that the dual topology, generated by the Gâteaux bornology of all finite sets, i.e. w^*, coincides with H^*. Consequently, the Hadamard bornology produces the same collection of smooth functions as the Gâteaux bornology. Thus we have no use of Gâteaux bornology here and may reserve the letter for G-subdifferential. From the perspective of this paper all bornologies are somewhere between H and F.

The Banach space X is said to be *β-smooth* if there is a Lipschitz continuous and β-smooth function with nonempty and bounded support (bump function) from X to $\mathbb{R}$. It is easy to see that any space possessing an equivalent norm, which is β-differentiable on its unit sphere, is β-smooth. The inverse is not true as shown by Haydon in [11]. Each separable space admits Gâteaux smooth renorm and is therefore H-smooth. If the dual is also separable, then the space admits F-smooth renorm, see for example [8, 20].

All β-smooth spaces and all Asplund spaces are *Gâteaux differentiability spaces* and therefore their dual balls are *w^* sequentially compact*, see [15]. In other words, a w^* convergent subsequence can be extracted from each norm bounded *sequence* in the dual space. As mentioned in [1], this is the property that allows obtaining sequential representations instead of topological.

We consider *lower semicontinuous* functions from X to $\mathbb{R} \cup \{+\infty\}$. The function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be lower semicontinuous if $f(x_0) \le \liminf_{x \to x_0} f(x)$ for any $x_0 \in X$. The function f is *proper* if it is not everywhere equal to ∞, that is $\operatorname{dom} f \neq \emptyset$, where $\operatorname{dom} f = \{x \in X : f(x) < \infty\}$. It is often useful to consider the convergence in the graph of f. The sequence $\{x_n\}_{n=1}^{\infty}$ is said to converge to x in the graph of f, denoted by $x_n \to_f x$, if $x_n \to x$ (in the norm topology) and $f(x_n) \to f(x)$.

We say that $x \in X$ is a *strong local minimum* of the function $f : X \to \mathbb{R} \cup \{+\infty\}$ if there is $\delta > 0$ such that for any sequence $\{x_n\}_{n=1}^{\infty} \subset x + \delta B_X$, for which

$$\limsup_{n \to \infty} f(x_n) \le f(x),$$

it follows that $x_n \to x$.

The notion of *β-subdifferential* we are about to recall goes back to Crandall and Lions [7]. It is known also as viscosity, variational or smooth subdifferential. Borwein and Ioffe in [2] demonstrated that it is quite useful to split the β-subdifferential in the following way.

Definition 2.1. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper and lower semicontinuous function, $x_0 \in \operatorname{dom} f$ and $k > 0$. Then

$D^-_{\beta,k} f(x_0) = \{p \in X^*;$ there is a β-smooth function g with local Lipschitz constant k such that $g'(x_0) = p$ and $f - g$ has a local minimum at $x_0\}$ is the set of all β-smooth subdifferentials of *rank* k. We put

$$D^-_{\beta} f(x_0) = \bigcup_{k>0} D^-_{\beta,k} f(x_0).$$

If $x_0 \notin \operatorname{dom} f$ then $D^-_{\beta,k} f(x_0) = D^-_{\beta} f(x_0) = \emptyset$.

Recall that X is an *Asplund* space if each convex and continuous real valued function on X is Fréchet differentiable on a set of second category. Topologically, a Banach space is Asplund if and only if any separable subspace of it has separable dual, cf. [8, 20]. It is known, see for example [8, 20], that if X has F-smooth bump, then X is Asplund. To the best of author's knowledge there is no proved example of Asplund space which is not F-smooth, but it is "likely" that such examples should exist.

In order to proceed simultaneously the important case when X is Asplund space, we define $D^-_{F,k}f(x_0)$ for $x_0 \in \mathrm{dom}F$ in the following manner: $p \in D^-_{F,k}f(x_0)$ if and only if $\|p\| \le k$ and

$$\liminf_{\|h\|\to 0} \frac{f(x_0+h)-f(x_0)-p(h)}{\|h\|} \ge 0. \tag{11}$$

Proposition VIII.1.2 in [8] shows that if the space is F-smooth and $p \in D^-_{F,k}f(x_0)$ then there is a F-smooth function g with $g'(x_0) = p$, such that $f-g$ attains a local minimum at x_0. (In fact, Proposition VIII.1.2 is stated for spaces with Lipschitz and F-differentiable, but not necessarily smooth, bump and correspondingly the function obtained may not be F-smooth. But, if the bump is in addition F-smooth, then the construction in [8] will provide F-smooth g.) It follows that g is locally $\|p\|+\varepsilon$ Lipschitz around x_0 for any $\varepsilon > 0$ and therefore p will be in $D^-_{\beta,k'}$ for each $k' > k$ if the definition is taken as for general bornology with $\beta = F$. Obviously, the definition with F-smooth supporting function implies the other one, and so, if the space is F-smooth, the discrepancy that occurs as a result of having two definitions for $D^-_{F,k}$, is insignificant.

If S is a closed subset of X then the indicator function of S is

$$\mathrm{I}_S(x) = \begin{cases} 0, & x \in S \\ \infty, & x \notin S. \end{cases}$$

If $S \subset X$ is a closed set and $x \in S$ then the β-smooth *normal cone* to S at x is $N_\beta(S,x) = \cup_{k=1}^\infty N_{\beta,k}(S,x)$, where

$$N_{\beta,k}(S,x) = D^-_{\beta,k}\mathrm{I}_S(x). \tag{12}$$

The following relation is established in [3].

Lemma 2.2. ([3], Proposition 2.3) *Let X be a Banach space equipped with some bornology β. Let $f : X \to \mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuous function. Then for any $x \in X$ and $l > k > 1$ the following two assertions are fulfilled.*

(i) *If $p \in D^-_{\beta,k}f(x)$ then $(p,-1) \in N_{\beta,k}(\mathrm{epi}f,(x,f(x)))$;*

(ii) *If $(p,-1) \in N_{\beta,k}(\mathrm{epi}f,(x,f(x)))$ then $p \in D^-_{\beta,l}f(x)$.*

Proof. The case of D^-_F, as defined in (11), is straightforward, so we omit it.

(i) Let $p \in D^-_{\beta,k}f(x_0)$ which means by definition that there is a β-smooth and locally k-Lipschitz $g : X \to \mathbb{R}$ such that $f-g$ attains a local minimum at x_0 and $g'(x_0) = p$. Then the function $\widetilde{g}(x,t) = g(x)-t$ is β-smooth and locally k-Lipschitz: recall that $\|\widetilde{g}'(x,t)\| = \max\{\|g'(x)\|,1\} \le k$. Also, $\mathrm{I}_{\mathrm{epi}f} - \widetilde{g}$ attains a local minimum at $(x_0,f(x_0))$. Indeed, if $\mathrm{I}_{\mathrm{epi}f}(x,t) - \widetilde{g}(x,t) < \infty$ then $t \ge f(x)$ and $-\widetilde{g}(x,t) \ge -g(x)+f(x) \ge f(x_0)-g(x_0) = -\widetilde{g}(x_0,f(x_0))$. Obviously, $\widetilde{g}'(x_0,f(x_0)) = (p,-1)$ and therefore $(p,-1) \in D^-_{\beta,k}\mathrm{I}_{\mathrm{epi}f}(x_0,f(x_0))$.

(ii) Let $g : X\times\mathbb{R} \to \mathbb{R}$ be β-smooth, locally k-Lipschitz and such that $g'(x_0,f(x_0)) = (p,-1)$ and $\mathrm{I}_{\mathrm{epi}f} - g$ attains a local minimum at $(x_0,f(x_0))$. We assume for simplicity that $x_0 = 0$, $f(0) = 0$ and $g(0,0) = 0$. Let $\delta > 0$ be such that $g'_t(x,t) < -k/l$ for $(x,t) \in \delta B_X \times [-\delta, \delta]$. Such δ exists because g is β-smooth. Since g is strictly decreasing on $t \in [-\delta, \delta]$ for each fixed $x \in \delta B_X$, it is straightforward to find an open neighborhood U of $(0,0)$ in $X \times \mathbb{R}$ such that there is $\varphi(x)$ solution to $g(x, \varphi(x)) = 0$ in U and if $(x_1, y_1) \in U$ and $g(x_1, y_1) = 0$ then $y_1 = \varphi(x_1)$. It is easy to see that φ is continuous. At each point $(x, \varphi(x)) \in U$ we can apply Implicit Function Theorem to the restriction of g to arbitrary finite dimensional subspace of X multiplied by $\mathbb{R}$ to obtain that φ is Gâteaux differentiable and

$$\varphi'(x) = -\frac{g'_x(x, \varphi(x))}{g'_t(x, \varphi(x))} .$$

The function g is locally k-Lipschitz and therefore $k \geq \|g'\|_{(X \times \mathbb{R})^*} = \max\{\|g'_x\|, |g'_t|\}$ around $(0,0)$. Thus $\|\varphi'(x)\| \leq k^{-1} l \|g'_x(x, \varphi(x)\| \leq l$, i.e. φ is locally l-Lipschitz.

Since $g'_x(x, \varphi(x))$ tends to p in β^* topology as $x \to 0$ and $g'_t(x, \varphi(x)) \to -1$ as $x \to 0$, we have that $\beta^*\!-\lim_{x \to 0} \varphi'(x) = p$. It is standard to verify that p is the β-derivative of φ at 0. The same argument applied to $(x, \varphi(x)) \in U$ shows that φ is β-smooth.

If $\varphi(x) > f(x)$ for some x such that $(x, \varphi(x)) \in U$ then $\mathrm{I}_{\mathrm{epi}f}(x, \varphi(x) - t) = 0$ for small $t > 0$, whilst $g(x, \varphi(x) - t) > 0$, since g is strictly decreasing on t. This contradiction shows that $f \geq \varphi$ around 0. From the above computations it follows that $p \in D^-_{\beta,l} f(0)$. $\square$

Transferring the above statement to the limiting cone and subdifferential is now easy. But prior to this we put together few simple facts.

Lemma 2.3. *If $(p, \lambda) \in N_{\beta,k}(\mathrm{epi}f, (x_0, t_0))$ then $\lambda \leq 0$.*
If moreover $t_0 > f(x_0)$ then $\lambda = 0$ and $(p, 0) \in N_{\beta,k}(\mathrm{epi}f, (x_0, f(x_0)))$.

Proof. By definition there is a β-smooth and locally k-Lipschitz $g : X \times \mathbb{R} \to R$, such that $g \leq \mathrm{I}_{\mathrm{epi}f}$ around (x_0, t_0); $g(x_0, t_0) = 0$ and $g'(x_0, t_0) = (p, \lambda)$.

Obviously $\mathrm{I}_{\mathrm{epi}f}(x_0, t)$ is a decreasing function with respect to t and therefore $\lambda = g'_t(x_0, t_0) \leq 0$.

If moreover $t_0 > f(x_0)$ then $\mathrm{I}_{\mathrm{epi}f}(x_0, t) = 0$ in a open neighborhood of t_0 and therefore $\lambda = g'_t(x_0, t_0) = 0$. Also, around (x_0, t_0) we have that $\widetilde{g}(x,t) := g(x, t + t_0 - f(x_0)) \leq \mathrm{I}_{\mathrm{epi}f}(x, t + t_0 - f(x_0)) \leq \mathrm{I}_{\mathrm{epi}f}(x,t)$, since $\mathrm{I}_{\mathrm{epi}f}$ is decreasing with respect to t. So, $\widetilde{g} \leq \mathrm{I}_{\mathrm{epi}f}$, $\widetilde{g}(x_0, f(x_0)) = 0 = \mathrm{I}_{\mathrm{epi}f}(x_0, f(x_0))$ and $\widetilde{g}'(x_0, f(x_0)) = g'(x_0, t_0) = (p, 0)$. $\square$

Proof of Proposition 1.2. Let $p \in \partial_{\beta,k} f(x)$. By definition, see (1), this means that $p = w^*\!-\lim_{n\to\infty} p_n$, where $p_n \in D^-_{\beta,k} f(x_n)$ and $(x_n, f(x_n)) \to (x, f(x))$. From Lemma 2.2 it follows that $(p_n, -1) \in N_{\beta,k}(\mathrm{epi}f, (x_n, f(x_n)))$. Therefore, $(p, -1) \in \widetilde{N}_{\beta,k}(\mathrm{epi}f, (x, f(x)))$; see (2).

Let now $(p, -1) \in \widetilde{N}_{\beta,k}(\mathrm{epi}f, (x, f(x)))$. By definition this means that there are $(x_n, t_n) \in \mathrm{epi}f$, such that $(x_n, t_n) \to (x, f(x))$ and $(p_n, -\lambda_n) \in N_{\beta,k}(\mathrm{epi}f, (x_n, t_n))$, such that $p_n \xrightarrow{w^*} p$ and $\lambda_n \to 1$.

According to Lemma 2.3, $(p_n, -\lambda_n) \in N_{\beta,k}(\mathrm{epi}f, (x_n, f(x_n)))$. Also, $f(x_n) \leq t_n$, $t_n \to f(x)$ and f is lower semicontinuous, thus $f(x_n) \to f(x)$, that is, $x_n \to_f x$.

So, we have that $(\lambda_n^{-1}p_n, -1) \in N_{\beta,\lambda_n^{-1}k}(\mathrm{epi} f, (x_n, f(x_n)))$, and since $\lambda_n^{-1}k < l$ for n large enough, Lemma 2.2 implies that $\lambda_n^{-1}p_n \in D^-_{\beta,l}f(x_n)$. As $x_n \to_f x$ and $\lambda_n^{-1}p_n \xrightarrow{w^*} p$, by definition $p \in \partial_{\beta,l}f(x)$. $\square$

The basic tool in dealing with β-smooth spaces is the Smooth Variational Principle of Deville, Godefroy and Zizler: [8], Theorem I.2.3.

Theorem 2.4. (Smooth Variational Principle, [8]) *Let X be a β smooth Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and bounded from below function. Then for any $\varepsilon > 0$ there is a bounded β-smooth and Lipschitz continuous function $g : X \to \mathbb{R}$ such that $f + g$ attains its strong local minimum and*

$$\sup\{\|g(x)\|, \|g'(x)\|; x \in X\} < \varepsilon.$$

It is often easier to use some sum rule for smooth subdifferentials. The one that best suits our purposes is the *Enhanced Fuzzy Sum Rule*, established by Borwein, Mordukhovich and Shao (cf. [3], Theorem 3.1) for spaces with β-smooth norm. In order to extend this sum rule to β-smooth spaces we need to consider a *Leduc function*.

Lemma 2.5. *Let X be β-smooth. There is a constant $\mu \geq 1$ and a β-smooth function $\Lambda : X \to \mathbb{R}$, such that*

$$\Lambda(x) \geq \|x\|^2,$$

and, moreover, if $\|\Lambda'(x)\| < k$ then Λ is locally μk-Lipschitz around x.

Proof. Literally repeating the construction in Proposition II.5.1 of [8], we obtain a Leduc function ψ, which is l-Lipschitz for some l, β-smooth away from the origin, and there is some $a > 0$, so that

$$\|x\| \leq \psi(x) \leq a\|x\| \text{ and } \psi(tx) = t\psi(x),\ \forall x \in X,\ \forall t > 0.$$

Let $\Lambda = \psi^2$. Since $\|\Lambda'(x)\| \leq 2l\psi(x) \to 0$ as $x \to 0$, the function Λ is β-smooth. For $x \neq 0$ we can compute the derivative of Λ in direction $x/\|x\|$, using $\Lambda(tx) = t^2\Lambda(x)$, and obtain that

$$\|\Lambda'(x)\| \geq 2\Lambda(x)/\|x\| \geq 2\|x\|,$$

because $\Lambda(x) \geq \|x\|^2$. Hence, if for some $x_0 \in X$ we have that $\|\Lambda'(x_0)\| < k$, then necessarily $\|x_0\| < k/2$ and $\|\Lambda'(x)\| \leq 2l\psi(x) < lak$ around x_0. So, we can take $\mu = la$. $\square$

Theorem 2.6. ([3], Theorem 3.1) *Assume that X is β-smooth for some bornology β, or that X is Asplund and $\beta = F$. There is a constant $\mu = \mu(X, \beta) \geq 1$, such that $\mu(X, \beta) = 1$ if the norm of X is β-smooth or X is Asplund and $\beta = F$; and the following sum rule is fulfilled:*

If $f : X \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, $g : X \to \mathbb{R}$ is locally k-Lipschitz around x_0 and $f + g$ attains a local minimum at x_0, then for any $\varepsilon > 0$ and $l > \mu k$ there are $x, y \in X$ such that $\|x - x_0\| < \varepsilon$, $\|y - y_0\| < \varepsilon$, $|f(x) - f(x_0)| < \varepsilon$ and $p \in D^-_{\beta,l}f(x)$, $q \in D^-_{\beta,l}g(y)$ such that

$$\|p + q\| < \varepsilon.$$

Proof. The case X Asplund and $\beta = F$ is included here only for easier further reference, because there is essentially nothing new in the above statement in comparison to the standard sum rule: cf. [10], Theorem 3.

Let now X be β-smooth with respect to some bornology β. The constant μ is that of Lemma 2.5, so it is clear that $\mu = 1$ if the norm is β-smooth - simply take $\Lambda = \|\cdot\|^2$.

We follow the proof of Deville and El Haddad, see [9], taking care of the Lipschitz constants of the functions involved.

We can assume without loss of generality that $x_0 = 0$ and $f(0) = g(0) = 0$. Considering instead of g the function $g + \delta\Lambda$ for small enough δ, we can also assume that 0 is a strong local minimum of $f + g$.

Let $r > 0$ be such that f is bounded from below on rB_X and g is k-Lipschitz on rB_X.

Consider the functions

$$w_n(x, y) = \begin{cases} f(x) + g(y) + n\Lambda(x - y), & x, y \in rB_X \\ \infty, & \text{otherwise.} \end{cases}$$

For each $n \in \mathbb{N}$ the function w_n is lower semicontinuous and bounded from below on $X \times X$ and the latter space is β-smooth, so according to the smooth variational principle there exists a function $\varphi_n : X \times X \to \mathbb{R}$ that is Lipschitz continuous, β-smooth and such that $||\varphi_n||_\infty < n^{-1}$, $||\varphi_n'||_\infty < n^{-1}$, and $w_n + \varphi_n$ attains its strong minimum at (x_n, y_n).

We claim that $||x_n - y_n|| \to 0$ and $\{x_n\}$ is a minimizing sequence for $f + g$.

First, observe that

$$(w_n + \varphi_n)(0, 0) \geq (w_n + \varphi_n)(x_n, y_n)$$

and using $w_n(0, 0) = 0$ we obtain $\varphi_n(0, 0) \geq f(x_n) + g(y_n) + n\Lambda(x_n - y_n) + \varphi_n(x_n, y_n)$. Since $\|\varphi\|_\infty < n^{-1}$, we get

$$f(x_n) + g(y_n) + n\Lambda(x_n - y_n) \leq 2n^{-1}. \tag{13}$$

Let $K \leq 0$ be a lower bound of f and g on rB_X. As $\Lambda(x_n - y_n) \geq ||x_n - y_n||^2$ we have that $2n^{-1} > 2K + n||x_n - y_n||^2$, hence $||x_n - y_n|| \leq \sqrt{2(1 - K)n^{-1}} \to 0$.

From (13) we have that $2n^{-1} \geq f(x_n) + g(y_n) > (f+g)(x_n) - |g(x_n) - g(y_n)|$, and therefore

$$2n^{-1} + k\|x_n - y_n\| \geq (f + g)(x_n) \geq 0, \tag{14}$$

hence $(f + g)(x_n) \to 0$ and x_n is a minimizing sequence for $f + g$. As 0 is a strong local minimum of $f + g$, this implies that $x_n \to 0$ and thus $y_n \to 0$ too. Hence, for n large enough the points (x_n, y_n) are interior points for $r(B_X \times B_X)$. Moreover, (14) implies that $f(x_n) \to 0$.

It is clear that for large enough n the function $f(\cdot) + \varphi(\cdot, y_n) + n\Lambda(\cdot - y_n)$ has a local minimum at x_n and from the definition of D^-_β it follows that $p_n = -\varphi'_x(x_n, y_n) - n\Lambda'(x_n - y_n) \in D^-_\beta f(x_n)$. Similarly, the function $g(\cdot) + \varphi(x_n, \cdot) + n\Lambda(x_n - \cdot)$ has a local minimum at y_n and $q_n = -\varphi'_y(x_n, y_n) + n\Lambda'(x_n - y_n) \in D^-_\beta g(y_n)$. Hence, $p_n + q_n = -\varphi'_x(x_n, y_n) - \varphi'_y(x_n, y_n)$ and $||p_n + q_n|| \leq 2n^{-1}$.

Moreover, since g is k-Lipschitz we have that $n\|\Lambda'(x_n - y_n)\| \le k + n^{-1}$ and according to Lemma 2.5, $n\Lambda$ has local Lipschitz constant $(\mu k + 2\mu n^{-1})$ around $x_n - y_n$. For n large enough this means that $p_n \in D^-_{\beta,l} f(x_n)$ and $q_n \in D^-_{\beta,l} g(y_n)$.

Take some large enough n to complete the proof. $\square$

We can now prove what is perhaps the main result of this paper.

Proof of Proposition 1.4. We assume without loss of generality that $x_0 = 0$ and $f(x_0) = 0$.

Let X be β-smooth. Let $g : X \times \mathbb{R} \to \mathbb{R}$ be k-Lipschitz, β-smooth and such that $\mathrm{I}_{\mathrm{epi} f} - g$ has a local minimum at $(0,0)$, $g(0,0) = 0$ and $g'(0,0) = (p,0)$. As in the proof of Theorem 2.6 we may also assume that $(0,0)$ is a strong local minimum of $\mathrm{I}_{\mathrm{epi} f} - g$.

Fix a β^* neighborhood U^* of 0 in X^* of form (10) and $\varepsilon > 0$ such that $\varepsilon < \min\{1, k\}$. Let $\delta_1 \in (0, \varepsilon)$ be such that $\delta_1 B_{X^*} \subset 2^{-1}U^*$ and if the sequence $\{(x_n, t_n)\}_{n=1}^\infty \subset \delta_1 B_X \times [-\delta_1, \delta_1]$ satisfies

$$\limsup_{n\to\infty} (\mathrm{I}_{\mathrm{epi} f}(x_n, t_n) - g(x_n, t_n)) \le 0 \text{ then } (x_n, t_n) \to (0,0). \tag{15}$$

Since g is β-smooth and the β^* topology is stronger than the w^* topology we can find $\delta \in (0, \delta_1)$ such that for arbitrary $(x,t) \in \delta B_X \times [-\delta, \delta]$ it follows that $g'_x(x,t) \in p + 2^{-1}U^*$ and $g'_t(x,t) > -\varepsilon/2$. Then for $x \in \delta B_X$ and $t \in (0, \delta]$ according to the Mean Value Theorem there is $\theta \in (0,1)$ such that $g(x,t) = g(x,0) + g'_t(x,\theta)t$, but $g'_t(x,\theta) > -\varepsilon/2$ and therefore

$$\forall (x,t) \in \delta B_X \times (0, \delta] \Rightarrow g(x,t) > g(x,0) - 2^{-1}\varepsilon t. \tag{16}$$

Consider the function

$$g_0(x,t) = \begin{cases} g(x,0) - 2^{-1}\varepsilon t, & t \ge 0 \\ g(x,0), & t \le 0. \end{cases}$$

We claim that $\mathrm{I}_{\mathrm{epi} f} - g_0$ has a strong local minimum at $(0,0)$. To this end let $\{(x_n, t_n)\}_{n=1}^\infty$ be such that

$$\limsup_{n\to\infty} (\mathrm{I}_{\mathrm{epi} f}(x_n, t_n) - g_0(x_n, t_n)) \le 0. \tag{17}$$

For arbitrary subsequence $\{(x_{n_k}, t_{n_k})\}_{k=1}^\infty$ such that all $t_{n_k} > 0$, (16) implies that $\mathrm{I}_{\mathrm{epi} f}(x_{n_k}, t_{n_k}) - g_0(x_{n_k}, t_{n_k}) > \mathrm{I}_{\mathrm{epi} f}(x_{n_k}, t_{n_k}) - g(x_{n_k}, t_{n_k})$. Therefore (17) and (15) imply that $(x_{n_k}, t_{n_k}) \to (0,0)$. Thus there is no loss of generality if we assume that all $t_n \le 0$. Of course, $\mathrm{I}_{\mathrm{epi} f}(x_n, t_n) = 0$ for n large enough. Thus $\mathrm{I}_{\mathrm{epi} f}(x_n, 0) = 0$ and consequently

$$\limsup_{n\to\infty} (\mathrm{I}_{\mathrm{epi} f}(x_n, 0) - g(x_n, 0)) \le 0.$$

From (15) it follows that $x_n \to 0$. If $t_n \not\to 0$ then we can find a subsequence t_{n_k} such that $t_{n_k} \to a < 0$. But $\mathrm{I}_{\mathrm{epi} f}(0, a) = \infty$ and we get a contradiction with the lower semicontinuity of $\mathrm{I}_{\mathrm{epi} f}$. So, $(x_n, t_n) \to (0,0)$ and therefore $(0,0)$ is a strong local minimum of $\mathrm{I}_{\mathrm{epi} f} - g_0$.

Let now for $n \in \mathbb{N}$

$$\xi_n(t) = \begin{cases} -2^{-1}\varepsilon t, & t \ge 0, \\ \frac{\varepsilon}{2(2n-1)}(1 + (t-1)^{1-2n}), & t \le 0. \end{cases}$$

It is clear that ξ_n is C^1-smooth, $\xi_n'(t) \in [-2^{-1}\varepsilon, 0)$ and $\xi_n(t)$ tends to 0 uniformly with respect to $t \in (-\infty, 0]$ as $n \to \infty$. Put

$$a_n = -\max\{\xi_n'(t);\, t \in [-1,1]\} = -\xi'(-1) = \varepsilon 2^{-2n-1}. \tag{18}$$

Let

$$g_n(x,t) = g(x,0) + \xi_n(t).$$

It is clear that $g_n(x,t) = g_0(x,t)$ for $t \geq 0$ and g_n tends to g_0 as $n \to \infty$ uniformly on $\delta B_X \times [-\delta, \delta]$. Moreover, as a sum of two β-smooth functions g_n is β-smooth. Obviously, $\|g_n'(x,t)\|_{(X\times\mathbb{R})^*} = \max\{\|g_x'(x,0)\|, |\xi_n'(t)|\} \leq k$ for n large enough. That is, g_n is k-Lipschitz. Let

$$F(x,t) = \begin{cases} \mathrm{I}_{\mathrm{epi}f}(x,t), & (x,t) \in \delta B_X \times [-\delta,\delta] \\ \infty, & (x,t) \notin \delta B_X \times [-\delta,\delta]. \end{cases}$$

Obviously $F - g_n$ is lower semicontinuous and bounded from below, so by Theorem 2.4 there exists a Lipschitz continuous and β-smooth function $\varphi_n : X \to \mathbb{R}$ such that

$$\max\{\|\varphi_n\|_\infty,\ \|\varphi_n'\|_\infty\} < (2n)^{-1}\min\{a_n, \delta\}$$

and $F - g_n + \varphi_n$ attains its minimum at (x_n, t_n). Indeed, $(x_n, t_n) \in \delta B_X \times [-\delta, \delta]$. Since $\{g_n - \varphi_n\}_{n=1}^\infty$ converges to g_0 uniformly on $\delta B_X \times [-\delta,\delta]$, it is easy to verify that $\{(x_n,t_n)\}_{n=1}^\infty$ is a minimizing sequence of $\mathrm{I}_{\mathrm{epi}f} - g_0$ on the latter set. By the first part of the proof it follows that $(x_n, t_n) \to (0,0)$. Choose $n_0 \in \mathbb{N}$ such that $\|x_{n_0}\| < \delta$, $|t_{n_0}| < \delta$ and $\|\varphi_{n_0}'\|_\infty < l - k$. Then the function $\mathrm{I}_{\mathrm{epi}f} - g_{n_0} + \varphi_{n_0}$ has a local minimum at (x_{n_0}, t_{n_0}) and since $g_{n_0} - \varphi_{n_0}$ is l-Lipschitz on $\delta B_X \times [-\delta,\delta]$ we obtain by definition that

$$(q, \lambda) \in D^-_{\beta,l}\mathrm{I}_{\mathrm{epi}f}(x_{n_0}, t_{n_0}),$$

where $q = (g_{n_0} - \varphi_{n_0})_x'(x_{n_0}, t_{n_0}) \in g_x'(x_{n_0}, 0) + \delta B_{X^*} \subset p + U^*$
and $\lambda = (g_{n_0} - \varphi_{n_0})_t'(x_{n_0}, t_{n_0}) \leq \xi_{n_0}'(t_{n_0}) + 2^{-1}a_{n_0} \leq -a_{n_0} + 2^{-1}a_{n_0} < 0$ by (18). Thus, Lemma 2.3 implies that $t_{n_0} = f(x_{n_0})$, meaning that $|f(x_{n_0}) - f(0)| = |t_{n_0}| < \delta$. In the same way we see that $\lambda \geq \xi_{n_0}'(t_{n_0}) - 2^{-1}\delta \geq -2^{-1}\varepsilon - 2^{-1}\delta > -\varepsilon$ and the proof of the case when X is β-smooth, is completed.

If X is Asplund and $\beta = F$ we may consider instead of above g the function $g(x,t) = p(x) - \nu\|x\| - \nu|t|$ for small enough $\nu > 0$ and go through the same steps, using Ekeland's Variational Principle (cf. for example [20]) and Theorem 2.6 where needed. $\square$

The proof of the sequential representation for the singular limiting subdifferential is now straightforward.

Proof of Proposition 1.3. If $p = w^*\!-\lim \lambda_n^{-1}p_n$, where $p_n \in D^-_{\beta,\lambda_n k}f(x_n)$, $x_n \to_f x$ and $\lambda_n \to \infty$; then by Lemma 2.2 we have that $(p_n, -1) \in N_{\beta,\lambda_n k}(\mathrm{epi}f, (x_n, f(x_n)))$. Therefore, $(\lambda_n^{-1}p_n, -\lambda_n^{-1}) \in N_{\beta,k}(\mathrm{epi}f, (x_n, f(x_n)))$. As $(\lambda_n^{-1}p_n, -\lambda_n^{-1}) \xrightarrow{w^*} (p, 0)$, the latter means that $(p,0) \in \widetilde{N}_{\beta,k}(\mathrm{epi}f, (x, f(x)))$, or $p \in \partial^\infty_{\beta,k}f(x)$. The proof of (i) is completed.

Let now $A = \{w^*\!-\lim_{n\to\infty} \lambda_n^{-1}p_n;\ p_n \in D^-_{\beta,\lambda_n l}f(x_n),\ x_n \to_f x,\ \lambda_n \to \infty\}$.

Let U^* be an arbitrary β^* neighborhood of 0 in X^* of form (10). Let $p \in \partial^\infty_{\beta,k}f(x)$, that is, $(p, 0) = w^*\!-\lim(q_n, -\mu_n)$, where $(q_n, -\mu_n) \in N_{\beta,k}(\mathrm{epi}f, (y_n, t_n))$ with $(y_n, t_n) \to (x, f(x))$.

Note that $\mu_n \geq 0$ according to Lemma 2.3. As in the proof of Proposition 1.2 we use Lemma 2.3 to show that $y_n \to_f x$ and we may assume that $t_n = f(y_n)$.

Fix $l' \in (k, l)$.

If $\mu_n > 0$ for some $n \in \mathbb{N}$ then we put $x_n = y_n$, $\lambda_n = \mu_n^{-1}$ and $p_n = \lambda_n q_n$.

If, on the other hand, $\mu_n = 0$ then by Proposition 1.4 there exist $(\overline{p}_n, -\lambda_n^{-1}) \in N_{\beta,l'}(\mathrm{epi}f, (x_n, f(x_n)))$, such that $\overline{p}_n \in q_n + U^*$, $\lambda_n > n$ and $\|y_n - x_n\| < n^{-1}$, $|f(y_n) - f(x_n)| < n^{-1}$. Let $p_n = \lambda_n \overline{p}_n$.

Having constructed the sequences $\{p_n\}$, $\{\lambda_n\}$ and $\{x_n\}$, we see that $l \geq \|(\lambda_n^{-1}p_n, -\lambda_n^{-1})\| \geq \|\lambda_n^{-1}p_n\|$. As B_{X^*} is w^* sequentially compact, we can extract from the bounded $\{\lambda_n^{-1}p_n\}$ a convergent subsequence. Let for simplicity this subsequences be denoted again by $\{\lambda_n^{-1}p_n\}$. Since $\lambda_n^{-1}p_n \in q_n + U^*$, $q_n \xrightarrow{w^*} p$ and U^* is w^* closed, we have that

$$w^*\!-\lim \lambda_n^{-1}p_n \in p + U^*. \tag{19}$$

As $(\lambda_n^{-1}p_n, -\lambda_n^{-1}) \in N_{\beta,l'}(\mathrm{epi}f, (x_n, f(x_n)))$, we have that $(p_n, -1) \in N_{\beta,\lambda_n l'}(\mathrm{epi}f, (x_n, f(x_n)))$, that is $p_n \in D_{\beta,\lambda_n l}f(x_n)$ by Lemma 2.2. But $x_n \to_f x$ and $\lambda_n \to \infty$, so (19) implies that $A \cap \{p + U^*\} \neq \emptyset$. Since U^* was arbitrary from the local base of β^* topology, we get that $p \in \beta^*\!-\mathrm{cl}A$. $\square$

We finish this section by demonstrating Proposition 1.1. Recall the following

Definition 2.7. ([19]) The Kruger-Mordukhovich normal cone $N(S, x)$ to the set $S \subset X$ at $x \in S$ is

$$N(S, x) = \{w^*\!- \lim_{n\to\infty} p_n;\ p_n \in \widehat{N}_{\varepsilon_n}(S, x_n),\ x_n \to x,\ \varepsilon_n \to 0,\ \varepsilon_n > 0\},$$

where for $\varepsilon \geq 0$

$$\widehat{N}_\varepsilon(S, x) = \{p \in X^*;\ \limsup_{y\to x,\ y\in S} \frac{p(y-x)}{\|y-x\|} \leq \varepsilon\}.$$

Proof of Proposition 1.1. Straight from the definition it follows that $N_F(S, x) = \widehat{N}_0(S, x)$, $\forall x \in S$. Therefore $\widetilde{N}_F(S, x) \subset N(S, x)$.

Note that if $\varepsilon > 0$ and $p \in \widehat{N}_\varepsilon(S, x)$ then there is $\delta > 0$ such that $p(y - x) \leq 2\varepsilon\|y - x\|$ for all $y \in S \cap \delta B_X$. In other words $g(y) = \mathrm{I}_S(y) - p(y - x) + 2\varepsilon\|y - x\|$ attains a local minimum at x.

Therefore, $p \in N(S, x) \iff \exists p_n \xrightarrow{w^*} p$, $\{x_n\} \subset S$, $x_n \to x$ and $\varepsilon_n \to 0^+$, such that $g_n(y) = \mathrm{I}_S(y) - p_n(y - x_n) + \varepsilon_n\|y - x_n\|$ has a local minimum at x_n.

Fix now $p \in N(S, x)$ and a sequence $p_n \xrightarrow{w^*} p$ as above. Since any w^* convergent sequence is bounded, there is $k > 0$, such that $\|p_n\| + \varepsilon_n < k$, $\forall n \in \mathbb{N}$.

Then the function $y \to p_n(y - x_n) - \varepsilon_n\|y - x_n\|$ has Lipschitz constant $< k$ and, as $D_F^-(-p_n(\cdot + x_n) + \varepsilon_n\|\cdot - x_n\|) \subset -p_n + \varepsilon_n B_{X^*}$, Theorem 2.6 ensures that there are $z_n \in S$ and $q_n \in D_{F,k}^-\mathrm{I}_S(z_n) = N_{F,k}(S, z_n)$ such that $\|z_n - x_n\| < \varepsilon_n$ and $\|q_n - p_n\| < 2\varepsilon_n$. This means that $z_n \to x$ and $q_n \xrightarrow{w^*} p$; or $N(S, x) \subset \widetilde{N}_F(S, x)$. $\square$

3. G-subdifferential and Clarke subdifferential

In the beginning of this section we give a definition for G-normal normal cone to a closed subset and prove a sequential representation of it in terms of β-smooth normals in β-smooth space. This representation is well known - see [2, 3, 12], but the existing proofs concern spaces with β-smooth norm.

We denote the *distance* function to a closed $S \subset X$ by $\rho_S(x) = \inf\{\|x-y\|;\ y \in S\}$.

The lower *Dini-Hadamard* directional derivative of $f : X \to R$ at $x \in X$ at direction $h \in X$ is

$$d^- f(x,h) = \liminf_{t \to 0^+, h' \to h} \frac{f(x+th') - f(x)}{t}.$$

Definition 3.1. ([2], Definition 2) Let S be a closed subset of the Banach space X and $x \in S$. Then $p \in X^*$ is G-normal of rank k to S at x if for any $\varepsilon > 0$ and any finite dimensional subspace $Y \subset X$, there are $y \in B(x;\varepsilon)$ and $q \in X^*$ such that

$$\langle q - p, h\rangle \le \varepsilon\|h\|,\ \forall h \in Y \text{and } q(h) \le kd^-\rho_S(y,h),\ \forall h \in Y.$$

The set of all G-normals of rank k to S at x is denoted by $N_{G,k}(S,x)$, whilst $N_G(S,x) = \cup_{k>0} N_{G,k}(S,x)$.

Proof of Proposition 1.5. With μ given by Theorem 2.6 we will show that for each $l > k\mu$

$$\mathrm{cl}^* \widetilde{N}_{\beta,k}(S,x) \subset N_{G,k}(S,x) \subset \mathrm{cl}^* \widetilde{N}_{\beta,l}(S,x), \tag{20}$$

which, of course, implies (4).

Assume that $x = 0$.

Let $p = w^*\!-\lim_{n\to\infty} p_n$, where $p_n \in N_{\beta,k}(S,x_n)$ and $x_n \to 0$. Obviously, $x_n \in S$, for otherwise the normal cone would be empty. We check Definition 3.1 in order to show that $p \in N_{G,k}(S,0)$.

Fix $\varepsilon > 0$ and a finite dimensional subspace Y of X. Fix n large enough, so that $\|x_n\| < \varepsilon$ and $\langle p - p_n, h\rangle \le \varepsilon\|h\|$, $\forall h \in Y$. This is possible due to w^* convergence.

Since $p_n \in N_{\beta,k}(S,x_n)$, there is β-smooth φ, which is k-Lipschitz around x_n, such that $\mathrm{I}_S - \varphi$ attains a local minimum at x_n and $\varphi'(x_n) = p_n$. We can indeed assume that $\varphi(x_n) = 0$, which in effect means that there is $\delta > 0$ such that $\varphi(y) \le 0$ for $y \in S \cap B(x_n;\delta)$. Let also φ be k-Lipschitz on $B(x_n;\delta)$.

If $y \in B(x_n; 2^{-1}\delta)$ and $y_0 \in S \setminus B(x_n;\delta)$ then $\|y_0 - y\| \ge \|y_0 - x_n\| - \|x_n - y\| \ge \delta/2$. Since $x_n \in S$ and $\|y - x_n\| \le \delta/2$, we have that $\rho_S(y) = \inf\{\|y - y_1\|;\ y_1 \in S \cap B(x_n;\delta)\}$. Fix any $y_1 \in S \cap B(x_n;\delta)$. Since $\varphi(y_1) \le 0$, we have that $\varphi(y) \le \varphi(y) - \varphi(y_1) \le k\|y - y_1\|$. Taking infimum over $y_1 \in B(x_n;\delta)$, we get $\varphi(y) \le k\rho_S(y)$, $\forall y \in B(x_n; 2^{-1}\delta)$.

Fix $h \in Y$ and let $t_m \to 0^+$ and $h_m \to h$ be such that $d^-\rho_S(x_n,h) = \lim_{m\to\infty} t_m^{-1}\rho_S(x_n + t_m h_m)$. Since the set $\{h\} \cup \{h_m\}_{m=1}^\infty$ is norm compact and φ is H-smooth, $\sup_m |\langle \varphi'(y) - \varphi'(x_n), h_m\rangle| \to 0$ as $y \to x_n$. Thus $kt_m^{-1}\rho_S(x_n + t_m h_m) \ge t_m^{-1}\varphi(x_n + t_m h_m) = t_m^{-1}(\varphi(x_n + t_m h_m) - \varphi(x_n)) = \langle \varphi'(x_n + \xi_m h_m), h_m\rangle \to \langle \varphi'(x_n), h\rangle$ as $m \to \infty$. Therefore, $p_n(h) \le kd^-\rho_S(x_n,h)$.

We have shown that $p \in N_{G,k}(S,0)$. It is not difficult to see that $N_{G,k}(S,0)$ is w^* closed. Therefore, the left hand side inclusion of (20) is verified. Although we have computed only the case of β-smooth space, the case of X Asplund and $\beta = F$ can be tackled similarly, using instead of φ the function $\psi(y) = p_n(y - x_n) - \varepsilon\|y - x_n\|$, or by direct computation; see [19].

Let now $p \in N_{G,k}(S, 0)$ and $l > \mu k$. We want to show that $p \in \mathrm{cl}^*\widetilde{N}_{\beta,l}(S, 0)$. This will be done once we manage to verify the following

Claim. For each w^* neighborhood W^* of 0 in X^* and each $\varepsilon > 0$ there is $q \in N_{\beta,l}(S, y)$, such that $p - q \in W^*$ and $\|y\| \leq 2\varepsilon$.

Of course, we simply fix W^* and apply the claim with $\varepsilon = n^{-1}$ to obtain $x_n \to 0$ and $q_n \in N_{\beta,l}(S, x_n) \cap \{p + W^*\}$. Since W^* is w^* closed, the limit q of the w^* convergent subsequence, that we can extract from bounded $\{q_n\}_{n=1}^{\infty}$ due to the w^* sequential compactness of B_{X^*}, will be in $\widetilde{N}_{\beta,l}(S, 0) \cap \{p + W^*\}$, showing that the latter intersection is nonempty. Since W^* is arbitrary, p is in the w^* closure of $\widetilde{N}_{\beta,l}(S, 0)$.

Proof of the Claim. Fix W^* and ε. We can assume without loss of generality that $W^* = \{x^* \in X^*;\ x^*(h) \leq 3\delta\|h\|,\ \forall h \in Y\}$ for some finite dimensional subspace Y of X.

According to Definition 3.1 there are $x_0 \in \varepsilon B_X$ and $p_0 \in X^*$ such that $\langle p - p_0, h\rangle \leq \delta\|h\|$ and $p_0(h) \leq kd^-\rho_S(x_0, h)$ for all $h \in Y$.

Consider the function

$$g(x) = \inf_{h \in Y}\{-p_0(h) + \delta\|h\| + k\|x_0 + h - x\|\}.$$

Since ρ_S is 1-Lipschitz, $d^-\rho_S(x_0, h) \leq \|h\|$. Therefore, $p_0(h) \leq k\|h\|$, $\forall h \in Y$. Thus $g(x_0) = 0$ and also $-p_0(h) + \delta\|h\| + k\|x_0 + h - x\| \geq -k\|h\| + \delta\|h\| + k(\|h\| - \|x_0 - x\|) \geq k\|x_0 - x\|$ for $\|h\| > \delta^{-1}2k\|x_0 - x\|$. Since, taking $h = 0$, $g(x) \leq k\|x_0 - x\|$, this means that the infimum in the definition of g is taken over cB_Y, where $c = \delta^{-1}2k\|x_0 - x\|$, and is therefore minimum.

As a minimum of k-Lipschitz functions of x, the function g is k-Lipschitz. Let $h_0 \in Y$ and note that $g(x + h_0) = \min_{h \in Y}\{-p_0(h) + \delta\|h\| + k\|x_0 + h - x - h_0\|\} \leq \min_{h \in Y}\{-p_0(h - h_0) + \delta\|h - h_0\| + k\|x_0 + (h - h_0) - x\|\} - p_0(h_0) + \delta\|h_0\|$. Since Y is linear space, this means that $g(x + h_0) \leq g(x) - p_0(h_0) + \delta\|h_0\|$. That is, $g(x + h) - g(x) + p_0(h) \leq \delta\|h\|$, $\forall h \in Y$. In effect this means that for any $q \in D^-_\beta g(x) \Rightarrow \langle q + p_0, h\rangle \leq \delta\|h\|$, $\forall h \in Y$.

Finally, $\mathrm{I}_S(x) + g(x)$ attains a local minimum at x_0, or, equivalently, $g(x) \geq 0$ for $x \in S$ and close enough to x_0. If this was not the case, then we would find a sequence $\{x_n\}_{n=1}^{\infty} \subset S$, converging to x_0, such that $g(x_n) < 0$. The definition of g and the estimate of the norm of the point of minimum in it, tell us that there are $h_n \in Y$, $h_n \to 0$, such that

$$-p_0(h_n) + \delta\|h_n\| + k\|x_0 + h_n - x_n\| < 0. \tag{21}$$

Clearly, $h_n \neq 0$ and since the unit sphere of Y is norm compact, we may assume that $\|h_n\|^{-1}h_n \to h \in Y$, $\|h\| = 1$, as $n \to \infty$. Since $\rho_S(x_0) = 0$, we have by definition that $kd^-\rho_S(x_0, h) \leq \liminf_{n\to\infty} k\|h_n\|^{-1}\rho_S(x_0 + h_n) \leq \liminf_{n\to\infty} k\|h_n\|^{-1}\|x_0 + h_n - x_n\|$, since $x_n \in S$. Applying (21), we obtain that $kd^-\rho_S(x_0, h) \leq \liminf_{n\to\infty} \|h_n\|^{-1}(p_0(h_n) - \delta\|h_n\|) = p_0(h) - \delta$. This contradicts $p_0(h) \leq kd^-\rho_S(x_0, h)$.

We are now able to apply Theorem 2.6 to $\mathrm{I}_S(x)$ and the k-Lipschitz g at x_0 and obtain $x, y \in B(x_0; \varepsilon) \subset 2\varepsilon B_X$, $q_1 \in D^-_{\beta,l}g(x)$ and $q \in N_{\beta,l}(S, y)$, such that $\|q_1 + q\| < \delta$. But for each $h \in Y$, as we have shown, $\langle q_1 + p_0, -h\rangle \le \delta\|h\|$, meaning that $\langle q - p, h\rangle = \langle q + q_1, h\rangle - \langle q_1 + p_0, h\rangle + \langle p_0 - p, h\rangle \le 3\delta\|h\|$. Thus y and q satisfy the claim, thereby completing the proof. □

Lemma 3.2. *Let $A \subset X^*$ be w^* sequentially compact, $x \in X$ and $\lambda \in \mathbb{R}$. Then*

$$\mathrm{cl}^* A \cap x^{-1}(\lambda) = \mathrm{cl}^*\{A \cap x^{-1}(\lambda)\},$$

where $x^{-1}(\lambda) = \{p \in X^*;\ p(x) = \lambda\}$.

Proof. Since $x^{-1}(\lambda)$ is w^* closed, it is clear that $\mathrm{cl}^*\{A \cap x^{-1}(\lambda)\} \subset \mathrm{cl}^* A \cap x^{-1}(\lambda)$.

Let $p \in \mathrm{cl}^* A \cap x^{-1}(\lambda)$ and U^* be a w^* neighborhood of 0 in X^*. Since $p \in \mathrm{cl}^* A \cap x^{-1}(\lambda)$ there are $p_n \in A$ such that

$$p_n - p \in U^* \cap x^{-1}([-n^{-1}, n^{-1}]).$$

In other words, $p_n \in p + U^*$ and $|p_n(x) - \lambda| \le n^{-1}$, since $p(x) = \lambda$.

Because A is w^* sequentially compact, $\{p_n\}$ has a w^* convergent subsequence; let its w^* limit be q. Obviously, $q \in A$, $q(x) = \lambda$ and $q \in p + U^*$. That is, $\{p + U^*\} \cap \{A \cap x^{-1}(\lambda)\} \ne \emptyset$. Since U^* was arbitrary, this means that $p \in \mathrm{cl}^*\{A \cap x^{-1}(\lambda)\}$. □

Proof of Theorem 1.6. Let μ be the one given by Theorem 2.6. If we can prove that for each $k > 1$ and $l > \mu k$

$$\mathrm{cl}^* \partial_{\beta,k} f(x) \subset \partial_{G,k} f(x) \subset \mathrm{cl}^* \partial_{\beta,l} f(x); \tag{22}$$

$$\mathrm{cl}^* \partial^\infty_{\beta,k} f(x) \subset \partial^\infty_{G,k} f(x) \subset \mathrm{cl}^* \partial^\infty_{\beta,l} f(x); \tag{23}$$

we will be done because (22) implies (7), while (23) and Proposition 1.3 imply (8).

Denote $e_2 = (0, 1) \in X \times \mathbb{R}$.

Let $l' \in (\mu k, l)$. Since $N_{G,k}(\mathrm{epi} f, (x, f(x))) \subset \mathrm{cl}^* \widetilde{N}_{\beta,l'}(\mathrm{epi} f, (x, f(x)))$, see (20), and $\widetilde{N}_{\beta,l'}(\mathrm{epi} f, (x, f(x)))$ is w^* sequentially compact as w^* sequentially closed subset of the w^* sequentially compact $l' B_{X^* \times \mathbb{R}}$; Lemma 3.2 shows that

$$(\partial_{G,k} f(x), -1) \subset \mathrm{cl}^*\{\widetilde{N}_{\beta,l'}(\mathrm{epi} f, (x, f(x))) \cap e_2^{-1}(-1)\}.$$

But Proposition 1.2 may be reformulated as

$$\widetilde{N}_{\beta,l'}(\mathrm{epi} f, (x, f(x))) \cap e_2^{-1}(-1) \subset (\partial_{\beta,l} f(x), -1).$$

Taking w^* closures we obtain the right hand side inclusion of (22). The other inclusion of (22) is obtained similarly.

For (23) note that (6), (20), Lemma 3.2 and (3) imply:
$(\partial^\infty_{G,k} f(x), 0) = N_{G,k}(\mathrm{epi} f, (x, f(x))) \cap e_2^{-1}(0) \subset \mathrm{cl}^*\{\widetilde{N}_{\beta,l}(\mathrm{epi} f, (x, f(x))) \cap e_2^{-1}(0)\} = \mathrm{cl}^*(\partial^\infty_{\beta,l}, 0)$. □

The following elementary fact is included here only for the sake of completeness. It is stated in the form used in the subsequent proof.

Lemma 3.3. *Let A and B be subsets of X^*. Assume also that B is a cone. Then*

$$\mathrm{co}A + \mathrm{co}B \subset \overline{\mathrm{co}}^*(A+B).$$

Proof. Assume the contrary: there are $p \in \mathrm{co}A$ and $q \in \mathrm{co}B$ such that $p+q \notin \overline{\mathrm{co}}^*(A+B)$. By the Strong Separation Theorem there is $x \in X$ such that

$$\langle p+q, x\rangle > \sup\langle A+B, x\rangle. \tag{24}$$

If there was $q_1 \in B$ such that $q_1(x) > 0$ then since $\lambda q_1 \in B$, $\forall \lambda \geq 0$ and $\langle p + \lambda q_1, x\rangle \to \infty$ as $\lambda \to \infty$, (24) would have been impossible. Thus, $\sup\langle B, x\rangle = 0$

Note also that $p(x) \leq \sup\langle A, x\rangle$ and $q(x) \leq \sup\langle B, x\rangle = 0$.

Since $0 \in B$ we have that
$\sup\langle A, x\rangle \leq \sup\langle A+B, x\rangle < p(x) + q(x) \leq p(x) \leq \sup\langle A, x\rangle$, contradiction. □

Proof of Proposition 1.7. In order to shorten the annotation we fix x and omit $f(x)$, epi$f$, etc.

By (5) and (6) we have that $(\partial_G, -1) + (\partial_G^\infty, 0) \subset N_G + N_G \subset \mathrm{co}N_G$. Therefore, $\overline{\mathrm{co}}^*(\partial_G + \partial_G^\infty, -1) \subset (\overline{\mathrm{co}}^* N_G, -1) = (\partial_C, -1)$, that is $\overline{\mathrm{co}}^*(\partial_G + \partial_G^\infty) \subset \partial_C$.

For the opposite inclusion, let $(p, -1) \in \mathrm{co}N_G$. There are $(p_i, -\lambda_i) \in N_G$ and $\alpha_i > 0$ with $\sum_{i=1}^n \alpha_i = 1$, such that

$$\sum_{i=1}^n \alpha_i p_i = p \text{ and } \sum_{i=1}^n \alpha_i \lambda_i = 1.$$

As in Lemma 2.3 it is easy to show that $\lambda_i \geq 0$. Let $\lambda_i > 0$ for $i = 1..k$ and $\lambda_i = 0$ for $i = (k+1)..n$. Let $r = \sum_{i=k+1}^n \alpha_i p_i$ if $k < n$ and $r = 0$ otherwise. In any case $r \in \mathrm{co}\partial_G^\infty$, since ∂_G^∞ is a cone.

Also, $\sum_{i=1}^k \alpha_i \lambda_i = \sum_{i=1}^n \alpha_i \lambda_i = 1$, $\lambda_i^{-1} p_i \in \partial_G$, since $(\lambda_i^{-1} p_i, -1) \in N_G$, and therefore $q = \sum_{i=1}^k \alpha_i p_i = \sum_{i=1}^k (\alpha_i \lambda_i)(\lambda_i^{-1} p_i) \in \mathrm{co}\partial_G$. We have that

$$p = q + r \in \mathrm{co}\partial_G + \mathrm{co}\partial_G^\infty.$$

We apply Lemma 3.3 with $A = \partial_G$ and $B = \partial_G^\infty$ in order to obtain that $p \in \overline{\mathrm{co}}^*(\partial_G + \partial_G^\infty)$ and complete the proof. □

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