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A New System of Generalized Nonlinear Mixed Variational Inequalities in Hilbert Spaces

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In this paper, we introduce and study a new system of generalized nonlinear mixed variational inequalities in Hilbert spaces. We prove the existence and uniqueness of solution for this system of generalized nonlinear mixed variational inequalities. We also give the convergence of the Mann iterative sequences with errors for this system of generalized nonlinear mixed variational inequalities in Hilbert spaces.

Keywords: A system of generalized nonlinear mixed variational inequalities, Mann iterative sequence with errors, nonlinear mapping, existence

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1. Introduction and Preliminaries

It is known that variational inequality theory and complementarity problem are very powerful tools of the current mathematical technology. In recent years, classical variational inequalities and complementarity problems have been extended and generalized to study a large variety of problems arising in mechanics, physics, optimization and control, nonlinear programming, economics and transportation equilibrium, and engineering sciences, etc. (see [1], [2], [4]-[22] and the reference therein).

In 1999, Verma [20] introduced and studied a new system of nonlinear variational inequalities based on a new system of iterative algorithms. Very recently, Verma [21] investigated the approximation-solvability of a new system of nonlinear quasi-variational inequalities in Hilbert spaces.

Motivated and inspired by the above works, in this paper, we introduce and study a new system of generalized nonlinear mixed variational inequalities in Hilbert spaces. We prove the existence and uniqueness of solution for this system of generalized nonlinear mixed variational inequalities. We also give the convergence of the Mann iterative sequences with errors for this system of generalized nonlinear mixed variational inequalities in Hilbert spaces.

Throughout this paper, let H be a real Hilbert space endowed with the inner product

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$\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let $A, B, S, T : H \to H$ be four single-valued mappings and $\varphi : H \to R\bigcup\{+\infty\}$ be a proper convex lower semicontinuous function. We consider the following problem:

Find $x^*, y^* \in H$ such that

$$\begin{cases} \langle \rho Ay^* + Sy^* + x^* - y^*, x - x^*\rangle \geq \rho\varphi(x^*) - \rho\varphi(x), & \forall x \in H, \\ \langle \gamma Bx^* + Tx^* + y^* - x^*, x - y^*\rangle \geq \gamma\varphi(y^*) - \gamma\varphi(x), & \forall x \in H, \end{cases} \tag{1}$$

which is called the system of generalized nonlinear mixed variational inequalities, where $\rho > 0$ and $\gamma > 0$ are two constants.

Special cases of the problem (1) as follows:

(I) If $A = B = 0$, then the problem (1) reduces to finding $x^*, y^* \in H$ such that

$$\begin{cases} \langle \rho Sy^* + x^* - y^*, x - x^*\rangle \geq \rho\varphi(x^*) - \rho\varphi(x), & \forall x \in H, \\ \langle \gamma Tx^* + y^* - x^*, x - y^*\rangle \geq \gamma\varphi(y^*) - \gamma\varphi(x), & \forall x \in H, \end{cases} \tag{2}$$

which is called the system of nonlinear mixed variational inequalities.

(II) If $\varphi = \delta_K$(the indicator function of a nonempty closed convex subset K, then the problem (1) reduces to finding $x^*, y^* \in K$ such that

$$\begin{cases} \langle \rho Ay^* + Sy^* + x^* - y^*, x - x^*\rangle \geq 0, & \forall x \in K, \\ \langle \gamma Bx^* + Tx^* + y^* - x^*, x - y^*\rangle \geq 0, & \forall x \in K, \end{cases} \tag{3}$$

which is called the system of general nonlinear variational inequalities.

(III) If $\varphi = \delta_K$(the indicator function of a nonempty closed convex subset K and $A = B = 0$, then the problem (1) reduces to finding $x^*, y^* \in K$ such that

$$\begin{cases} \langle \rho Sy^* + x^* - y^*, x - x^*\rangle \geq 0, & \forall x \in K, \\ \langle \gamma Tx^* + y^* - x^*, x - y^*\rangle \geq 0, & \forall x \in K, \end{cases} \tag{4}$$

which is called the system of nonlinear variational inequalities.

In the sequel, we give some definitions and lemmas.

Definition 1.1. A mapping $T : H \to H$ is said to be k-strongly monotone if there exists a constant $k > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq k\|x - y\|^2, \quad \forall x, y \in H.$$

Definition 1.2. A mapping $T : H \to H$ is said to be s-Lipschitz continuous if there exists a constant $s > 0$ such that

$$\|Tx - Ty\| \leq s\|x - y\|, \quad \forall x, y \in H.$$

Lemma 1.3. *Let $\{a_n\}, \{b_n\}$, and $\{c_n\}$ be three sequences of nonnegative numbers satisfying the following conditions: there exists n_0 such that*

$$a_{n+1} \leq (1 - t_n)a_n + b_n t_n + c_n, \quad \forall n \geq n_0, \tag{5}$$

where

$$t_n \in [0, 1], \quad \sum_{n=0}^{\infty} t_n = +\infty, \quad \lim_{n\to\infty} b_n = 0, \quad \sum_{n=0}^{\infty} c_n < +\infty.$$

Then $a_n \to 0$ as $n \to +\infty$.

Proof. Let $\sigma = \inf\{a_n : n \geq n_0\}$. Then $\sigma \geq 0$. Suppose that $\sigma > 0$ and so $a_n \geq \sigma > 0$ for all $n \geq n_0$. It follows from (5) that, for all $n \geq n_0$,

$$\begin{aligned} a_{n+1} &\leq a_n - \sigma t_n + t_n b_n + c_n \\ &= a_n - (\frac{1}{2}\sigma - b_n)t_n - \frac{1}{2}\sigma t_n + c_n. \end{aligned} \tag{6}$$

Since $b_n \to 0$ as $n \to \infty$, there exists $n_1 \geq n_0$ such that

$$\frac{1}{2}\sigma \geq b_n, \quad \forall n \geq n_1. \tag{7}$$

Combining (6) and (7), we get

$$a_{n+1} \leq a_n - \frac{1}{2}\sigma t_n + c_n, \quad \forall n \geq n_1.$$

This implies that

$$\frac{1}{2}\sigma \sum_{n=n_1}^{\infty} t_n \leq a_{n_1} + \sum_{n=n_1}^{\infty} c_n < +\infty,$$

which is a contradiction. Therefore, $\sigma = 0$ and so there exists a subsequence $\{a_{n_j}\} \subset \{a_n\}$ such that $a_{n_j} \to 0$ as $j \to \infty$. It follows from (5) that

$$a_{n_j+1} \leq a_{n_j} + b_{n_j} t_{n_j} + c_{n_j}$$

and so $a_{n_j+1} \to 0$ as $j \to \infty$. A simple induction leads to $a_{n_j+k} \to 0$ as $j \to \infty$ for all $k \geq 1$ and this means that $a_n \to 0$ as $n \to \infty$. This completes the proof of Lemma 1.3. □

Lemma 1.4 ([3], [22]). *For a given $u \in H$, the point $z \in H$ satisfies the following inequality*

$$\langle u - z, v - u \rangle \geq \rho\varphi(u) - \rho\varphi(v), \quad \forall v \in H$$

if and only if

$$u = J_\varphi^\rho(z),$$

where $J_\varphi^\rho = (I + \rho\partial\varphi)^{-1}$ and $\partial\varphi$ denotes the subdifferential of a proper convex lower semcontinuous function φ.

Lemma 1.5. *For given $x^*, y^* \in H$, (x^*, y^*) is a solution of the problem (1) if and only if*

$$\begin{cases} x^* = J_\varphi^\rho(y^* - \rho(Ay^* + Sy^*)), \\ y^* = J_\varphi^\gamma(x^* - \gamma(Bx^* + Tx^*)). \end{cases}$$

Proof. It is easy to know that Lemma 1.5 follows from Lemma 1.4 and so the proof is omitted. □

2. Existence and Uniqueness

In this section, we shall show the existence and uniqueness of solution for the problems (1), (2), (3) and (4), respectively.

Theorem 2.1. *Let $S : H \to H$ be a k_1-strongly monotone and s_1-Lipschitz continuous mapping, $T : H \to H$ be a k_2-strongly monotone and s_2-Lipschitz continuous mapping, $A : H \to H$ be a l_1-Lipschitz continuous mapping, and $B : H \to H$ be a l_2-Lipschitz continuous mapping. If*

$$\left.\begin{array}{ll} 0 < \rho < \min\{\frac{2(k_1-l_1)}{s_1^2-l_1^2},\ \frac{1}{l_1}\}, & l_1 < k_1, \\ 0 < \gamma < \min\{\frac{2(k_2-l_2)}{s_2^2-l_2^2},\ \frac{1}{l_2}\}, & l_2 < k_2, \end{array}\right\} \tag{8}$$

then the problem (1) has a unique solution (x^*, y^*).

Proof. First, we prove the existence of the solution. Define a mapping $F : H \to H$ as follows:

$$F(x) = J_\varphi^\rho[J_\varphi^\gamma(x - \gamma(Bx + Tx)) - \rho(A + S)(J_\varphi^\gamma(x - \gamma(Bx + Tx)))], \quad \forall x \in H. \tag{9}$$

Since J_φ^ρ is a nonexpansive mapping [3], [22], for any $x, y \in H$, we have

$$\begin{aligned} & \|F(x) - F(y)\| \\ = \;& \|J_\varphi^\rho[J_\varphi^\gamma(x - \gamma(Bx + Tx)) - \rho(A + S)(J_\varphi^\gamma(x - \gamma(Bx + Tx)))] \\ & -J_\varphi^\rho[J_\varphi^\gamma(y - \gamma(By + Ty)) - \rho(A + S)(J_\varphi^\gamma(y - \gamma(By + Ty)))]\| \\ \le \;& \|J_\varphi^\gamma(x - \gamma(Bx + Tx)) - J_\varphi^\gamma(y - \gamma(By + Ty)) \\ & -\rho(S(J_\varphi^\gamma(x - \gamma(Bx + Tx))) - S(J_\varphi^\gamma(y - \gamma(By + Ty))))\| \\ & +\rho\|A(J_\varphi^\gamma(x - \gamma(Bx + Tx))) - A(J_\varphi^\gamma(y - \gamma(By + Ty)))\|. \end{aligned} \tag{10}$$

Since S, T are strongly monotone and A, B, S, T are all Lipschitz continuous, it follows that

$$\begin{aligned} & \|J_\varphi^\gamma(x - \gamma(Bx + Tx)) - J_\varphi^\gamma(y - \gamma(By + Ty)) \\ & -\rho(S(J_\varphi^\gamma(x - \gamma(Bx + Tx))) - S(J_\varphi^\gamma(y - \gamma(By + Ty))))\|^2 \\ = \;& \|J_\varphi^\gamma(x - \gamma(Bx + Tx)) - J_\varphi^\gamma(y - \gamma(By + Ty))\|^2 \\ & +\rho^2\|S(J_\varphi^\gamma(x - \gamma(Bx + Tx))) - S(J_\varphi^\gamma(y - \gamma(By + Ty)))\|^2 \\ & -2\rho\langle J_\varphi^\gamma(x - \gamma Tx) - J_\varphi^\gamma(y - \gamma Ty), S(J_\varphi^\gamma(x - \gamma Tx)) - S(J_\varphi^\gamma(y - \gamma Ty))\rangle \\ \le \;& \|J_\varphi^\gamma(x - \gamma(Bx + Tx)) - J_\varphi^\gamma(y - \gamma(By + Ty))\|^2 \\ & +\rho^2 s_1^2\|J_\varphi^\gamma(x - \gamma(Bx + Tx)) - J_\varphi^\gamma(y - \gamma(By + Ty))\|^2 \\ & -2\rho k_1\|J_\varphi^\gamma(x - \gamma(Bx + Tx)) - J_\varphi^\gamma(y - \gamma(By + Ty))\|^2 \\ \le \;& (1 - 2\rho k_1 + \rho^2 s_1^2)\|J_\varphi^\gamma(x - \gamma(Bx + Tx)) - J_\varphi^\gamma(y - \gamma(By + Ty))\|^2 \\ \le \;& (1 - 2\rho k_1 + \rho^2 s_1^2)\|x - y - \gamma((Bx + Tx) - (By + Ty))\|^2 \\ = \;& (1 - 2\rho k_1 + \rho^2 s_1^2) \\ & \cdot[\sqrt{\|x - y\|^2 - 2\gamma\langle x - y, Tx - Ty\rangle + \gamma^2\|Tx - Ty\|} + \gamma l_2\|x - y\|]^2 \\ \le \;& (1 - 2\rho k_1 + \rho^2 s_1^2)[\sqrt{1 - 2\gamma k_2 + \gamma^2 s_2^2} + \gamma l_2]^2\|x - y\|^2 \end{aligned} \tag{11}$$

and

$$\begin{aligned} & \rho\|A(J_\varphi^\gamma(x-\gamma(Bx+Tx)))-A(J_\varphi^\gamma(y-\gamma(By+Ty)))\| \\ \leq\ & \rho l_1\|x-y-\gamma(Bx+Tx-(By+Ty))\| \\ \leq\ & \rho l_1[\|x-y-\gamma(Tx-Ty)\|+\gamma l_2\|x-y\|] \\ \leq\ & \rho l_1[\sqrt{1-\gamma k_2+\gamma^2 s_2}+\gamma l_2]\|x-y\|. \end{aligned} \tag{12}$$

From (10)-(12), we have

$$\|Fx-Fy\| \leq \theta_1\theta_2\|x-y\|, \quad \forall x,y \in H, \tag{13}$$

where

$$\theta_1 = \sqrt{1-2\rho k_1+\rho^2 s_1^2}+\rho l_1, \qquad \theta_2 = \sqrt{1-2\rho k_2+\rho^2 s_2^2}+\rho l_2. \tag{14}$$

It follows from (8) that $\theta_1 < 1$ and $\theta_2 < 1$. Thus, (13) implies that F is a contractive mapping and so, there exists a point $x^* \in H$ such that $x^* = F(x^*)$. Let

$$y^* = J_\varphi^\gamma(x^*-\gamma(Bx^*+Tx^*)).$$

From the definition of F, we have

$$\begin{cases} x^* = J_\varphi^\rho(y^*-\rho(Ay^*+Sy^*)), \\ y^* = J_\varphi^\gamma(x^*-\gamma(Bx^*+Tx^*)). \end{cases}$$

By Lemma 1.5, we know that (x^*, y^*) is a solution of the problem (1).

Next, we show the uniqueness of the solution. Let (u^*, v^*) be another solution of the problem (1). It follows from Lemma 1.5 that

$$\begin{cases} u^* = J_\varphi^\rho(v^*-\rho(Av^*+Sv^*)), \\ v^* = J_\varphi^\gamma(u^*-\gamma(Bu^*+Tu^*)). \end{cases}$$

As the proof of (13), we have

$$\|x^*-u^*\|^2 \leq \theta_1\theta_2\|x^*-u^*\|^2. \tag{15}$$

Since $\theta_1 < 1$ and $\theta_2 < 1$, it follows that $u^* = x^*$ and so $v^* = y^*$. This completes the proof. □

From Theorem 2.1, we can obtain the following theorems.

Theorem 2.2. *Let S and T be the same as in Theorem 2.1. If $0 < \rho < 2k_1/s_1^2$ and $0 < \gamma < 2k_2/s_2^2$, then the problem (2) has a unique solution (x^*, y^*).*

Theorem 2.3. *Let K be a nonempty closed convex set of a Hilbert space H. Suppose that A, B, S, and T are the same in Theorem 2.1. If the condition (8) holds, then the problem (3) has a unique solution (x^*, y^*).*

Theorem 2.4. *Let K be a nonempty closed convex set of a Hilbert space H. Suppose that S and T are the same as in Theorem 2.1. If $0 < \rho < 2k_1/s_1^2$ and $0 < \gamma < 2k_2/s_2^2$, then the problem (4) has a unique solution (x^*, y^*).*

Remark 2.5. If $S = T$, then the problem (4) reduces to finding $x^*, y^* \in K$ such that

$$\begin{cases} \langle \rho T y^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in K, \\ \langle \gamma T x^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in K, \end{cases}$$

which is defined by Verma [20], and is called the new system of nonlinear variational inequalities. Hence Theorem 2.1, 2.2, 2.3 and 2.4 are the extensions of the result of Verma [20].

3. Algorithms and Convergence

In this section, we construct some new iterative algorithms for the problems (1)-(4). We also give the convergence analysis of the iterative sequences generated by the algorithms.

Now we give the algorithm for solving the problem (1) as follows:

Algorithm 3.1. *For any given $x_0 \in H$, define the Mann iterative sequences $\{x_n\}$ and $\{y_n\}$ with errors as follows:*

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n J_\varphi^\rho(y_n - \rho(Ay_n + Sy_n)) + \alpha_n u_n + w_n, \\ y_n = J_\varphi^\gamma(x_n - \gamma(Bx_n + Tx_n)) + v_n, \end{cases} \tag{16}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ and $\{u_n\}, \{w_n\}$ and $\{v_n\}$ are three sequences in H satisfying the following conditions:

$$\sum_{n=0}^{\infty} \alpha_n = +\infty, \quad \sum_{n=0}^{\infty} \|w_n\| < +\infty, \quad \lim_{n\to\infty} \|u_n\| = \lim_{n\to\infty} \|v_n\| = 0. \tag{17}$$

If $A = B = 0$, then Algorithm 3.1 reduces to the following algorithm for solving the problem (2).

Algorithm 3.2. *For any given $x_0 \in H$, define the Mann iterative sequences $\{x_n\}$ and $\{y_n\}$ with errors as follows:*

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n J_\varphi^\rho(y_n - \rho S y_n) + \alpha_n u_n + w_n, \\ y_n = J_\varphi^\gamma(x_n - \gamma T x_n) + v_n, \end{cases}$$

where $\{\alpha_n\}$, $\{u_n\}, \{w_n\}$ and $\{v_n\}$ are the same as in Algorithm 3.1.

If $\varphi = \delta_K$, then Algorithms 3.1 and 3.2 reduce to the following algorithms for solving the problems (3) and (4), respectively.

Algorithm 3.3. *For any given $x_0 \in H$, define the iterative sequences $\{x_n\}$ and $\{y_n\}$ as follows:*

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n P_K(y_n - \rho(Ay_n + Sy_n)) + \alpha_n u_n + w_n, \\ y_n = P_K(x_n - \gamma(Bx_n + Tx_n)) + v_n, \end{cases}$$

where $\{\alpha_n\}, \{u_n\}, \{w_n\}$, and $\{v_n\}$ are the same as in Algorithm 3.1.

Algorithm 3.4. *For any given $x_0 \in H$, define the iterative sequences $\{x_n\}$ and $\{y_n\}$ as follows:*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_K(y_n - \rho S y_n) + \alpha_n u_n + w_n, \\ y_n = P_K(x_n - \gamma T x_n) + v_n, \end{cases}$$

where $\{\alpha_n\}, \{u_n\}, \{w_n\}$, and $\{v_n\}$ are the same as in Algorithm 3.1.

Theorem 3.5. *Let A, B, S, T be the same as in Theorem* 2.1*, and $\{x_n\}$ be the iterative sequence generated by Algorithm* 3.1*. If the condition* (8) *holds, then (x_n, y_n) converges strongly to the unique solution (x^*, y^*) of the problem* (1)*.*

Proof. By Theorem 2.1, we know that the problem (1) has a unique solution (x^*, y^*). It follows from Lemma 1.5 that

$$\begin{cases} x^* = J_\varphi^\rho(y^* - \rho(Ay^* + Sy^*)), \\ y^* = J_\varphi^\gamma(x^* - \gamma(Bx^* + Tx^*)). \end{cases} \tag{18}$$

From (16) and (18), we have

$$\begin{aligned} & \|x_{n+1} - x^*\| \\ = \ & \|(1-\alpha_n)x_n + \alpha_n J_\varphi^\rho(y_n - \rho(Ay_n + Sy_n)) + \alpha_n u_n + w_n - x^*\| \\ \leq \ & (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|J_\varphi^\rho(y_n - \rho(Ay_n + Sy_n)) \\ & - J_\varphi^\rho(y^* - \rho(Ay^* + Sy^*))\| + \alpha_n\|u_n\| + \|w_n\| \\ \leq \ & (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|y_n - y^* - \rho(Ay_n + Sy_n - (Ay^* + Sy^*))\| \\ & + \alpha_n\|u_n\| + \|w_n\| \\ \leq \ & (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|y_n - y^* - \rho(Sy_n - Sy^*)\| \\ & + \alpha_n\rho l_1\|y_n - y^*\| + \alpha_n\|u_n\| + \|w_n\|. \end{aligned} \tag{19}$$

Since S is k_1-strongly monotone and s_1-Lipschitz continuous, we get

$$\|y_n - y^* - \rho(Sy_n - Sy^*)\| \leq \sqrt{1 - 2\rho k_1 + \rho^2 s_1^2}\|y_n - y^*\|. \tag{20}$$

Combining (19) and (20), we obtain

$$\|x_{n+1} - x^*\| \leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta_1\|y_n - y^*\| + \alpha_n\|u_n\| + \|w_n\|. \tag{21}$$

where $\theta_1 = \sqrt{1 - 2\rho k_1 + \rho^2 s_1^2} + \rho l_1$. Again from (16) and (18), we have

$$\begin{aligned} \|y_n - y^*\| &= \|J_\varphi^\gamma(x_n - \gamma(Bx_n + Tx_n)) + v_n - J_\varphi^\gamma(x^* - \gamma(Bx^* + Tx^*))\| \\ &\leq \|x_n - x^* - \gamma(Bx_n + Tx_n - (Bx^* + Tx^*))\| + \|v_n\| \\ &\leq \|x_n - x^* - \gamma(Tx_n - Tx^*)\| + \gamma l_2\|x_n - x^*\| + \|v_n\|. \end{aligned} \tag{22}$$

Since T is k_2-strongly monotone and s_2-Lipschitz continuous, we have

$$\|x_n - x^* - \gamma(Tx_n - Tx^*)\| \leq \sqrt{1 - 2\gamma k_2 + \gamma^2 s_2^2}\|x_n - x^*\|. \tag{23}$$

Letting $\theta_2 = \sqrt{1 - 2\gamma k_2 + \gamma^2 s_2^2} + \gamma l_2$, it follows from (21)-(23) that

$$\begin{aligned}\|x_{n+1} - x^*\| &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta_1\theta_2\|x_n - x^*\| + \alpha_n\theta_1\|v_n\| \\ &\quad +\alpha_n\|u_n\| + \|w_n\| \\ &= (1 - \alpha_n(1-\theta_1\theta_2))\|x_n - x^*\| + \|w_n\| \\ &\quad +\alpha_n(1-\theta_1\theta_2)\cdot\frac{1}{1-\theta_1\theta_2}(\theta_1\|v_n\| + \|u_n\|). \end{aligned} \tag{24}$$

Setting

$$\begin{aligned} a_n &= \|x_n - x^*\|, \\ b_n &= \frac{1}{1-\theta_1\theta_2}(\theta_1\|v_n\| + \|u_n\|), \\ c_n &= \|w_n\|, \\ t_n &= \alpha_n(1-\theta_1\theta_2), \end{aligned}$$

then (24) can be rewritten as follows:

$$a_{n+1} \leq (1-t_n)a_n + b_n t_n + c_n.$$

From the assumption, we know that $\{a_n\}, \{b_n\}, \{c_n\}$, and $\{t_n\}$ satisfy the conditions of Lemma 1.3. Thus, $a_n \to 0$ and so $x_n \to x^*$ as $n \to \infty$. Since $x_n \to x^*$, it follows from (17), (22) and (23) that $y_n \to y^*$ as $n \to \infty$. This completes the proof. □

From Theorem 3.5, we have the following results.

Theorem 3.6. *Let S, T be the same as in Theorem 2.2, and $\{x_n\}, \{y_n\}$ be the iterative sequences generated by Algorithm 3.2. If $0 < \rho < 2k_1/s_1^2$ and $0 < \gamma < 2k_2/s_2^2$, then (x_n, y_n) converges strongly to the unique solution (x^*, y^*) of the problem (2).*

Theorem 3.7. *Let K, A, B, S, T be the same as in Theorem 2.3, and $\{x_n\}, \{y_n\}$ be the iterative sequences generated by Algorithm 3.3. If the condition (8) holds, then (x_n, y_n) converges strongly to the unique solution (x^*, y^*) of the problem (3).*

Theorem 3.8. *Let K, S, T be the same as in Theorem 2.4, and $\{x_n\}, \{y_n\}$ be the iterative sequences generated by Algorithm 3.4. If $0 < \rho < 2k_1/s_1^2$ and $0 < \gamma < 2k_2/s_2^2$, then (x_n, y_n) converges strongly to the unique solution (x^*, y^*) of the problem (4).*

Remark 3.9. If $S = T$ in Algorithm 3.4 and Theorem 3.8, then we can easily get the result of Verma [20] for the convergence theorem, as a special case.

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