Journal of Convex Analysis
Volume 11 (2004), No. 1, 69–79

Strong Convergence Theorems for Nonexpansive Nonself-Mappings and Inverse-Strongly-Monotone Mappings

Hideaki Iiduka
Department of Mathematical and Computing Sciences,
Tokyo Institute of Technology, Oh-okayama, Meguro-ku, Tokyo, 152-8522, Japan
Hideaki.Iiduka@is.titech.ac.jp

Wataru Takahashi
Department of Mathematical and Computing Sciences,
Tokyo Institute of Technology, Oh-okayama, Meguro-ku, Tokyo, 152-8522, Japan
wataru@is.titech.ac.jp

Received July 29, 2002
Revised manuscript received August 07, 2003

In this paper, we introduce an iterative scheme for finding a common element of the set of fixed points of a nonexpansive nonself-mapping and the set of solutions of the variational inequality for an inverse-strongly-monotone mapping in a Hilbert space. Then we show that the sequence converges strongly to a common element of two sets. Using this result, we consider the problem of finding a common element of the set of zeros of a maximal monotone mapping and the set of zeros of an inverse-strongly-monotone mapping and the problem of finding a common element of the closed convex set and the set of zeros of the gradient of a continuously Fréchet differentiable convex functional.

Keywords: Metric projection, inverse-strongly-monotone mapping, nonexpansive nonself-mapping, variational inequality, strong convergence

2000 Mathematics Subject Classification: Primary 47H05, 47J05, 47J25

1. Introduction

Let C be a closed convex subset of a real Hilbert space H. A mapping S of C into H is called *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|$$

for all $x, y \in C$. We denote by $F(S)$ the set of fixed points of S. A mapping A of C into H is called *monotone* if for all $x, y \in C$, $\langle x - y, Ax - Ay\rangle \geq 0$. The *variational inequality problem* is to find a $u \in C$ such that

$$\langle v - u, Au\rangle \geq 0$$

for all $v \in C$. The set of solutions of the variational inequality is denoted by $VI(C, A)$. A mapping A of C into H is called *inverse-strongly-monotone* if there exists a positive real number α such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2$$

for all $x, y \in C$; see [5] and [13]. For such a case, A is called α-inverse-strongly-monotone.

ISSN 0944-6532 / \$ 2.50 © Heldermann Verlag

In this paper, we introduce an iterative scheme for finding a common element of the set of fixed points of a nonexpansive nonself-mapping and the set of solutions of the variational inequality for an inverse-strongly-monotone mapping in a real Hilbert space. Then we show that the sequence converges strongly to a common element of two sets. Using this result, we consider the problem of finding a common element of the set of zeros of a maximal monotone mapping and the set of zeros of an inverse-strongly-monotone mapping and the problem of finding a common element of the closed convex set and the set of zeros of the gradient of a continuously Fréchet differentiable convex functional.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and let C be a closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \le \|x - y\|$ for all $y \in C$. P_C is called the *metric projection* of H onto C. We know that P_C is a nonexpansive mapping of H onto C. It is also known that P_C satisfies

$$\langle x - y, P_C x - P_C y\rangle \ge \|P_C x - P_C y\|^2 \tag{1}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the properties: $P_C x \in C$ and $\langle x - P_C x, P_C x - y\rangle \ge 0$ for all $y \in C$. In the context of the variational inequality problem, this implies that

$$u \in VI(C, A) \Longleftrightarrow u = P_C(u - \lambda Au), \quad \forall \lambda > 0. \tag{2}$$

It is also known that H satisfies *Opial's condition* [16], i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \ne x$.

We state some examples for inverse-strongly-monotone mappings. If $A = I - T$, where T is a nonexpansive mapping of C into itself and I is the identity mapping of H, then A is 1/2-inverse-strongly-monotone and $VI(C, A) = F(T)$; see [11]. A mapping A of C into H is called *strongly monotone* if there exists a positive real number η such that $\langle x - y, Ax - Ay\rangle \ge \eta\|x - y\|^2$ for all $x, y \in C$. In such a case, we say that A is η-strongly monotone. If A is η-strongly monotone and κ-*Lipschitz continuous*, i.e., $\|Ax - Ay\| \le \kappa\|x - y\|$ for all $x, y \in C$, then A is η/κ^2-inverse-strongly-monotone. Let f be a continuously Fréchet differentiable convex functional on H and let ∇f be the gradient of f. If ∇f is $1/\alpha$-Lipschitz continuous, then ∇f is α-inverse-strongly-monotone; see [1].

If A is an α-inverse-strongly-monotone mapping of C into H, then it is obvious that A is $1/\alpha$-Lipschitz continuous. We also have that for all $x, y \in C$ and $\lambda > 0$,

$$\begin{aligned} \|(I - \lambda A)x - (I - \lambda A)y\|^2 &= \|(x - y) - \lambda(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2\lambda\langle x - y, Ax - Ay\rangle + \lambda^2\|Ax - Ay\|^2 \\ &\le \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2. \end{aligned} \tag{3}$$

So, if $\lambda \le 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping of C into H.

A set-valued mapping $T : H \to 2^H$ is called *monotone* if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is *maximal* if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be an inverse-strongly-monotone mapping of C into H and let $N_C v$ be the *normal cone* to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\}$, and define

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases} \tag{4}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$; see Theorem 3 of [17].

3. Strong Convergence Theorem

In this section, we prove the main theorem in this paper. To prove it, we use the following lemma [14]:

Lemma 3.1. *Let C be a closed convex subset of a real Hilbert space H. Let S be a nonexpansive nonself-mapping of C into H such that $F(S) \neq \emptyset$. Then $F(S) = F(P_C S)$.*

Now we can state a strong convergence theorem.

Theorem 3.2. *Let C be a closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly-monotone mapping of C into H and let S be a nonexpansive nonself-mapping of C into H such that $F(S) \cap VI(C, A) \neq \emptyset$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = P_C(\alpha_n x + (1 - \alpha_n) S P_C(x_n - \lambda_n A x_n))$$

for every $n = 1, 2, \ldots$, where $\{\alpha_n\}$ is a sequence in $[0, 1)$ and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\alpha$,

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty,$$

then $\{x_n\}$ converges strongly to $P_{F(S)\cap VI(C,A)} x$.

Proof. Put $y_n = P_C(x_n - \lambda_n A x_n)$ for every $n = 1, 2, \ldots$. Let $u \in F(S) \cap VI(C, A)$. Since $I - \lambda_n A$ is nonexpansive and $u = P_C(u - \lambda_n A u)$ from (2), we have

$$\begin{aligned} \|y_n - u\| &= \|P_C(x_n - \lambda_n A x_n) - P_C(u - \lambda_n A u)\| \\ &\leq \|(x_n - \lambda_n A x_n) - (u - \lambda_n A u)\| \\ &\leq \|x_n - u\| \end{aligned}$$

for every $n = 1, 2, \ldots$. Then we have

$$\begin{aligned}
\|x_2 - u\| &= \|P_C(\alpha_1 x + (1-\alpha_1)Sy_1) - P_C u\| \\
&\leq \|\alpha_1 x + (1-\alpha_1)Sy_1 - u\| \\
&\leq \alpha_1\|x-u\| + (1-\alpha_1)\|Sy_1 - u\| \\
&\leq \alpha_1\|x-u\| + (1-\alpha_1)\|y_1 - u\| \\
&\leq \alpha_1\|x-u\| + (1-\alpha_1)\|x - u\| \\
&= \|x-u\|.
\end{aligned}$$

If $\|x_k - u\| \leq \|x-u\|$ holds for some $k \in \mathbf{N}$, we can similarly show $\|x_{k+1} - u\| \leq \|x-u\|$. Therefore, $\{x_n\}$ is bounded. Hence $\{y_n\}$, $\{Sy_n\}$ and $\{Ax_n\}$ are also bounded. Since $I - \lambda_n A$ is nonexpansive, we also have

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \|P_C(x_{n+1} - \lambda_{n+1}Ax_{n+1}) - P_C(x_n - \lambda_n Ax_n)\| \\
&\leq \|(x_{n+1} - \lambda_{n+1}Ax_{n+1}) - (x_n - \lambda_n Ax_n)\| \\
&= \|(x_{n+1} - \lambda_{n+1}Ax_{n+1}) - (x_n - \lambda_{n+1}Ax_n) + (\lambda_n - \lambda_{n+1})Ax_n\| \\
&\leq \|(x_{n+1} - \lambda_{n+1}Ax_{n+1}) - (x_n - \lambda_{n+1}Ax_n)\| + |\lambda_n - \lambda_{n+1}|\|Ax_n\| \\
&\leq \|x_{n+1} - x_n\| + |\lambda_n - \lambda_{n+1}|\|Ax_n\| \qquad (5)
\end{aligned}$$

for every $n = 1, 2, \ldots$. So, we obtain

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|P_C(\alpha_n x + (1-\alpha_n)Sy_n) - P_C(\alpha_{n-1}x + (1-\alpha_{n-1})Sy_{n-1})\| \\
&\leq \|(\alpha_n x + (1-\alpha_n)Sy_n) - (\alpha_{n-1}x + (1-\alpha_{n-1})Sy_{n-1})\| \\
&= \|(\alpha_n - \alpha_{n-1})(x - Sy_{n-1}) + (1-\alpha_n)(Sy_n - Sy_{n-1})\| \\
&\leq |\alpha_n - \alpha_{n-1}|\|x - Sy_{n-1}\| + (1-\alpha_n)\|Sy_n - Sy_{n-1}\| \\
&\leq |\alpha_n - \alpha_{n-1}|\|x - Sy_{n-1}\| + (1-\alpha_n)\|y_n - y_{n-1}\| \\
&\leq |\alpha_n - \alpha_{n-1}|\|x - Sy_{n-1}\| + (1-\alpha_n)(\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|Ax_{n-1}\|) \\
&\leq (1-\alpha_n)\|x_n - x_{n-1}\| + M|\lambda_n - \lambda_{n-1}| + L|\alpha_n - \alpha_{n-1}|
\end{aligned}$$

for every $n = 1, 2, \ldots$, where $L = \sup\{\|x - Sy_n\| : n \in \mathbf{N}\}$ and $M = \sup\{\|Ax_n\| : n \in \mathbf{N}\}$. By mathematical induction, we have

$$\|x_{n+m+1} - x_{n+m}\| \leq \prod_{k=m}^{n+m-1}(1-\alpha_{k+1})\|x_{m+1} - x_m\| + M\sum_{k=m}^{n+m-1}|\lambda_{k+1} - \lambda_k| + L\sum_{k=m}^{n+m-1}|\alpha_{k+1} - \alpha_k|$$

for every $n, m = 1, 2, \ldots$. So, we obtain

$$\begin{aligned}
\limsup_{n\to\infty}\|x_{n+1} - x_n\| &= \limsup_{n\to\infty}\|x_{n+m+1} - x_{n+m}\| \\
&\leq M\sum_{k=m}^{\infty}|\lambda_{k+1} - \lambda_k| + L\sum_{k=m}^{\infty}|\alpha_{k+1} - \alpha_k|
\end{aligned}$$

for every $m = 1, 2, \ldots$. Since $\sum_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$ and $\sum_{n=1}^{\infty}|\lambda_{n+1} - \lambda_n| < \infty$, we obtain

$$\limsup_{n\to\infty}\|x_{n+1} - x_n\| \leq 0$$

and hence

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

From (5) and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, we also obtain $\|y_{n+1} - y_n\| \to 0$. Since

$$\begin{aligned}\|x_n - P_C S y_n\| &\leq \|x_n - P_C S y_{n-1}\| + \|P_C S y_{n-1} - P_C S y_n\| \\ &\leq \alpha_{n-1}\|x - S y_{n-1}\| + \|y_{n-1} - y_n\|,\end{aligned}$$

we have $\|x_n - P_C S y_n\| \to 0$. For $u \in F(S) \cap VI(C,A)$, from (3), we obtain

$$\begin{aligned}\|x_{n+1} - u\|^2 &= \|P_C(\alpha_n x + (1-\alpha_n) S y_n) - P_C u\|^2 \\ &\leq \|\alpha_n x + (1-\alpha_n) S y_n - u\|^2 \\ &\leq \alpha_n \|x - u\|^2 + (1-\alpha_n)\|S y_n - u\|^2 \\ &\leq \alpha_n \|x - u\|^2 + (1-\alpha_n)\|y_n - u\|^2 \\ &\leq \alpha_n \|x - u\|^2 + (1-\alpha_n)\{\|x_n - u\|^2 + \lambda_n(\lambda_n - 2\alpha)\|A x_n - A u\|^2\} \\ &\leq \alpha_n \|x - u\|^2 + \|x_n - u\|^2 + (1-\alpha_n) a(b - 2\alpha)\|A x_n - A u\|^2.\end{aligned}$$

Therefore, we have

$$\begin{aligned}-(1-\alpha_n)a(b-2\alpha)\|A x_n - A u\|^2 &\leq \alpha_n \|x - u\|^2 + \|x_n - u\|^2 - \|x_{n+1} - u\|^2 \\ &= \alpha_n \|x - u\|^2 + (\|x_n - u\| + \|x_{n+1} - u\|) \\ &\qquad \times (\|x_n - u\| - \|x_{n+1} - u\|) \\ &\leq \alpha_n \|x - u\|^2 + (\|x_n - u\| + \|x_{n+1} - u\|)\|x_n - x_{n+1}\|.\end{aligned}$$

Since $\alpha_n \to 0$ and $\|x_{n+1} - x_n\| \to 0$, we obtain $\|A x_n - A u\| \to 0$. From (1), we have

$$\begin{aligned}\|y_n - u\|^2 &= \|P_C(x_n - \lambda_n A x_n) - P_C(u - \lambda_n A u)\|^2 \\ &\leq \langle (x_n - \lambda_n A x_n) - (u - \lambda_n A u), y_n - u \rangle \\ &= \frac{1}{2}\{\|(x_n - \lambda_n A x_n) - (u - \lambda_n A u)\|^2 + \|y_n - u\|^2 \\ &\qquad - \|(x_n - \lambda_n A x_n) - (u - \lambda_n A u) - (y_n - u)\|^2\} \\ &\leq \frac{1}{2}\{\|x_n - u\|^2 + \|y_n - u\|^2 - \|(x_n - y_n) - \lambda_n(A x_n - A u)\|^2\} \\ &= \frac{1}{2}\{\|x_n - u\|^2 + \|y_n - u\|^2 - \|x_n - y_n\|^2 \\ &\qquad + 2\lambda_n \langle x_n - y_n, A x_n - A u \rangle - \lambda_n^2 \|A x_n - A u\|^2\}.\end{aligned}$$

So, we obtain

$$\|y_n - u\|^2 \leq \|x_n - u\|^2 - \|x_n - y_n\|^2 + 2\lambda_n \langle x_n - y_n, A x_n - A u \rangle - \lambda_n^2 \|A x_n - A u\|^2$$

and hence

$$\begin{aligned}\|x_{n+1} - u\|^2 &= \|P_C(\alpha_n x + (1-\alpha_n) S y_n) - P_C u\|^2 \\ &\leq \|\alpha_n x + (1-\alpha_n) S y_n - u\|^2 \\ &\leq \alpha_n \|x - u\|^2 + (1-\alpha_n)\|S y_n - u\|^2 \\ &\leq \alpha_n \|x - u\|^2 + (1-\alpha_n)\|y_n - u\|^2 \\ &\leq \alpha_n \|x - u\|^2 + \|x_n - u\|^2 - (1-\alpha_n)\|x_n - y_n\|^2 \\ &\qquad + 2(1-\alpha_n)\lambda_n \langle x_n - y_n, A x_n - A u \rangle - (1-\alpha_n)\lambda_n^2 \|A x_n - A u\|^2.\end{aligned}$$

Since $\alpha_n \to 0$, $\|x_{n+1} - x_n\| \to 0$ and $\|Ax_n - Au\| \to 0$, we obtain $\|x_n - y_n\| \to 0$. Since $\|P_C S y_n - y_n\| \le \|P_C S y_n - x_n\| + \|x_n - y_n\|$, we obtain $\|P_C S y_n - y_n\| \to 0$.

Next we show that

$$\limsup_{n\to\infty} \langle x - z_0, x_n - z_0 \rangle \le 0,$$

where $z_0 = P_{F(S)\cap VI(C,A)} x$. To show it, choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle x - z_0, x_n - z_0 \rangle = \lim_{i\to\infty} \langle x - z_0, x_{n_i} - z_0 \rangle.$$

As $\{x_{n_i}\}$ is bounded, we have that a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ converges weakly to z. We may assume without loss of generality that $x_{n_i} \rightharpoonup z$. Since $\|x_n - y_n\| \to 0$, we obtain $y_{n_i} \rightharpoonup z$. Then we can obtain $z \in F(S) \cap VI(C,A)$. In fact, let us first show $z \in VI(C,A)$. Let

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone. Let $(v,w) \in G(T)$. Since $w - Av \in N_C v$ and $y_n \in C$, we have

$$\langle v - y_n, w - Av \rangle \ge 0.$$

On the other hand, from $y_n = P_C(x_n - \lambda_n A x_n)$, we have $\langle v - y_n, y_n - (x_n - \lambda_n A x_n) \rangle \ge 0$ and hence

$$\left\langle v - y_n, \frac{y_n - x_n}{\lambda_n} + Ax_n \right\rangle \ge 0.$$

Therefore, we have

$$\begin{aligned}
\langle v - y_{n_i}, w \rangle &\ge \langle v - y_{n_i}, Av \rangle \\
&\ge \langle v - y_{n_i}, Av \rangle - \left\langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda_{n_i}} + Ax_{n_i} \right\rangle \\
&= \left\langle v - y_{n_i}, Av - Ax_{n_i} - \frac{y_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle \\
&= \langle v - y_{n_i}, Av - Ay_{n_i} \rangle + \langle v - y_{n_i}, Ay_{n_i} - Ax_{n_i} \rangle \\
&\quad - \left\langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle \\
&\ge \langle v - y_{n_i}, Ay_{n_i} - Ax_{n_i} \rangle - \left\langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle.
\end{aligned}$$

Hence we obtain $\langle v - z, w \rangle \ge 0$ as $i \to \infty$. Since T is maximal monotone, we have $z \in T^{-1}0$ and hence $z \in VI(C,A)$. Let us show $z \in F(P_C S)$. Assume $z \notin F(P_C S)$. From Opial's condition, we have

$$\begin{aligned}
\liminf_{i\to\infty} \|y_{n_i} - z\| &< \liminf_{i\to\infty} \|y_{n_i} - P_C S z\| \\
&= \liminf_{i\to\infty} \|y_{n_i} - P_C S y_{n_i} + P_C S y_{n_i} - P_C S z\| \\
&= \liminf_{i\to\infty} \|P_C S y_{n_i} - P_C S z\| \\
&\le \liminf_{i\to\infty} \|y_{n_i} - z\|.
\end{aligned}$$

This is a contradiction. Thus, we obtain $z \in F(P_C S)$. By Lemma 3.1, we obtain $z \in F(S)$. Then we have

$$\begin{aligned} \limsup_{n\to\infty} \langle x - z_0, x_n - z_0 \rangle &= \lim_{i\to\infty} \langle x - z_0, x_{n_i} - z_0 \rangle \\ &= \langle x - z_0, z - z_0 \rangle \leq 0. \end{aligned}$$

Therefore, for any $\varepsilon > 0$, there exists $m \in \mathbf{N}$ such that

$$\langle x - z_0, x_n - z_0 \rangle \leq \varepsilon$$

for all $n \geq m$. On the other hand, since

$$P_C(\alpha_n x + (1 - \alpha_n) S y_n) - P_C(\alpha_n x + (1 - \alpha_n) z_0) = x_{n+1} - z_0 + \alpha_n (z_0 - x),$$

we have

$$\begin{aligned} & \| P_C(\alpha_n x + (1 - \alpha_n) S y_n) - P_C(\alpha_n x + (1 - \alpha_n) z_0) \|^2 \\ \geq \ & \| x_{n+1} - z_0 \|^2 + 2\alpha_n \langle z_0 - x, x_{n+1} - z_0 \rangle. \end{aligned}$$

This implies

$$\begin{aligned} \| x_{n+1} - z_0 \|^2 &\leq 2\alpha_n \langle x - z_0, x_{n+1} - z_0 \rangle + (1 - \alpha_n)^2 \| S y_n - z_0 \|^2 \\ &\leq 2\alpha_n \langle x - z_0, x_{n+1} - z_0 \rangle + (1 - \alpha_n) \| x_n - z_0 \|^2 \end{aligned}$$

for every $n = 1, 2, \ldots$. For all $n \geq m$, we have

$$\begin{aligned} \| x_{n+1} - z_0 \|^2 &\leq 2\alpha_n \varepsilon + (1 - \alpha_n) \| x_n - z_0 \|^2 \\ &= 2\varepsilon (1 - (1 - \alpha_n)) + (1 - \alpha_n) \| x_n - z_0 \|^2. \end{aligned}$$

By mathematical induction, we obtain

$$\| x_{n+1} - z_0 \|^2 \leq 2\varepsilon \Big(1 - \prod_{k=m}^{n} (1 - \alpha_k)\Big) + \prod_{k=m}^{n} (1 - \alpha_k) \| x_m - z_0 \|^2.$$

Therefore, we have

$$\limsup_{n\to\infty} \| x_{n+1} - z_0 \|^2 \leq 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $\limsup_{n\to\infty} \| x_{n+1} - z_0 \|^2 \leq 0$ and hence $x_n \to z_0$. □

4. Applications

In this section, we prove some theorems in a real Hilbert space by using Theorem 3.2. Let C be a closed convex subset of a real Hilbert space H. Then a mapping $T : C \to C$ is called *strictly pseudocontractive* if there exists k with $0 \leq k < 1$ such that

$$\| Tx - Ty \|^2 \leq \| x - y \|^2 + k \| (I - T)x - (I - T)y \|^2$$

for all $x, y \in C$. Put $A = I - T$. Then A is $(1 - k)/2$-inverse-strongly-monotone. For the proof, see [23]. Using Theorem 3.2, we first prove a strong convergence theorem for finding a common fixed point of a nonexpansive nonself-mapping and a strictly pseudocontractive mapping.

Theorem 4.1. *Let C be a closed convex subset of a real Hilbert space H. Let T be a k-strictly pseudocontractive mapping of C into itself and let S be a nonexpansive nonself-mapping of C into H such that $F(S) \cap F(T) \neq \emptyset$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = P_C(\alpha_n x + (1-\alpha_n)S((1-\lambda_n)x_n + \lambda_n T x_n))$$

for every $n = 1, 2, \ldots$, where $\{\alpha_n\}$ is a sequence in $[0,1)$ and $\{\lambda_n\}$ is a sequence in $[0, 1-k]$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a,b]$ for some a, b with $0 < a < b < 1-k$,

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty,$$

then $\{x_n\}$ converges strongly to $P_{F(S)\cap F(T)}x$.

Proof. Put $A = I - T$. Then A is $(1-k)/2$-inverse-strongly-monotone. We have $F(T) = VI(C, A)$ and $P_C(x_n - \lambda_n A x_n) = (1-\lambda_n)x_n + \lambda_n T x_n$. So, by Theorem 3.2, we obtain the desired result. □

Using Theorem 3.2, we also have the following theorem which was obtained by [14].

Theorem 4.2. *Let C be a closed convex subset of a real Hilbert space H. Let S be a nonexpansive nonself-mapping of C into H such that $F(S) \neq \emptyset$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = P_C(\alpha_n x + (1-\alpha_n)S x_n)$$

for every $n = 1, 2, \ldots$, where $\{\alpha_n\}$ is a sequence in $[0,1)$. If $\{\alpha_n\}$ is chosen so that

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

then $\{x_n\}$ converges strongly to $P_{F(S)}x$.

Proof. In Theorem 3.2, put $Ax = 0$ for all $x \in C$. Then A is inverse-strongly-monotone. We have $C = VI(C, A)$ and $SP_C(x_n - \lambda_n A x_n) = S x_n$. So, by Theorem 3.2, we obtain the desired result. □

Using Theorem 3.2, we have the following:

Theorem 4.3. *Let H be a real Hilbert space. Let A be an α-inverse-strongly-monotone mapping of H into itself and let $B : H \to 2^H$ be a maximal monotone mapping such that $A^{-1}0 \cap B^{-1}0 \neq \emptyset$. Let J_r^B be the resolvent of B for each $r > 0$. Suppose $x_1 = x \in H$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x + (1-\alpha_n) J_r^B (x_n - \lambda_n A x_n)$$

for every $n = 1, 2, \ldots$, where $\{\alpha_n\}$ is a sequence in $[0,1)$ and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a,b]$ for some a, b with $0 < a < b < 2\alpha$,

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty,$$

then $\{x_n\}$ converges strongly to $P_{A^{-1}0\cap B^{-1}0}x$.

Proof. We have $A^{-1}0 = VI(H, A)$ and $F(J_r^B) = B^{-1}0$. So, putting $P_H = I$, by Theorem 3.2, we obtain the desired result. □

Remark. For finding an element of $(A + B)^{-1}0$, Nakajo and Takahashi [15] considered Mann's type iteration: $x_1 = x \in H$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_{\lambda_n}^B (x_n - \lambda_n A x_n)$$

and obtained that the sequence $\{x_n\}$ converges weakly to $z \in (A + B)^{-1}0$.

Let $A : H \to 2^H$ be a maximal monotone mapping on H, let J_r^A denote the resolvent for $r > 0$ and let $A_r = 1/r(I - J_r^A)$ be the corresponding *Yosida approximation*. Then $A_r : H \to H$ is r-inverse-strongly-monotone. For the proof, see [11]. Using Theorem 4.3, we have the following:

Corollary 4.4. *Let H be a real Hilbert space. Let $A : H \to 2^H$ be a maximal monotone mapping and let $B : H \to 2^H$ be a maximal monotone mapping such that $A^{-1}0 \cap B^{-1}0 \neq \emptyset$. Suppose $x_1 = x \in H$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) J_r^B ((1 - \beta_n) x_n + \beta_n J_r^A x_n)$$

for every $n = 1, 2, \ldots$, where $\{\alpha_n\}$ is a sequence in $[0, 1)$ and $\{\beta_n\}$ is a sequence in $[0, 2]$. If $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen so that $\beta_n \in [c, d]$ for some c, d with $0 < c < d < 2$,

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty,$$

then $\{x_n\}$ converges strongly to $P_{A^{-1}0 \cap B^{-1}0} x$.

Proof. Since A_r is r-inverse-strongly-monotone and $A_r^{-1}0 = F(J_r^A) = A^{-1}0$, we have $x_n - \lambda_n A_r x_n = (1 - \beta_n) x_n + \beta_n J_r^A x_n$, where $\beta_n = \lambda_n / r$. So, by Theorem 4.3, we obtain the desired result. □

Using Theorem 3.2, we have the following:

Theorem 4.5. *Let C be a closed convex subset of a real Hilbert space H. Let f be a continuously Fréchet differentiable convex functional on H and let ∇f be the gradient of f such that $C \cap (\nabla f)^{-1}0 \neq \emptyset$. Suppose ∇f is $1/\alpha$-Lipschitz continuous, $x_1 = x \in H$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) P_C (x_n - \lambda_n \nabla f(x_n))$$

for every $n = 1, 2, \ldots$, where $\{\alpha_n\}$ is a sequence in $[0, 1)$ and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\alpha$,

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty,$$

then $\{x_n\}$ converges strongly to $P_{C \cap (\nabla f)^{-1}0} x$.

Proof. We know from [1] that ∇f is an α-inverse-strongly-monotone mapping and $(\nabla f)^{-1}0 = VI(H, \nabla f)$. We also have $C = F(P_C)$. So, putting $P_H = I$, by Theorem 3.2, we obtain the desired result. □

Note. We consider the initial value problem:

$$\frac{du(t)}{dt} + Tu(t) \ni 0, \quad t > 0, \tag{6}$$

$$u(0) = x,$$

where T is a maximal monotone mapping in H and $x \in \overline{D(T)}$. If $T : H \to 2^H$ is a maximal monotone mapping defined by (4), then (6) implies the following:

$$-\frac{du(t)}{dt} - Au(t) \in N_C(u(t)).$$

We know that if $\mathrm{int} T^{-1}0 \neq \emptyset$, then $\{u(t)\}$ converges weakly to some element of $T^{-1}0 = VI(C, A)$; see [6]. On the other hand, from Theorem 3.2, the sequence $\{x_n\}$ generated by $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) P_C(x_n - \lambda_n A x_n)$$

converges strongly to $P_{T^{-1}0}x$. We do not know the relation between $\{u(t)\}$ and $\{x_n\}$.

References

[1] J. B. Baillon, G. Haddad: Quelques propriétés des opérateurs angle-bornés et n-cycliquement monotones, Israel J. Math. 26 (1977) 137–150.

[2] F. E. Browder: Nonlinear monotone operators and convex sets in Banach spaces, Bull. Amer. Math. Soc. 71 (1965) 780–785.

[3] F. E. Browder: Nonlinear Operators and Nonlinear Equations of Evolutions in Banach Spaces, Proc. Sympos. Pure Math. 18-2, Amer. Math. Soc. Providence, R.I. (1976).

[4] F. E. Browder: The fixed point theory of multi-valued mappings in topological vector spaces, Math. Ann. 177 (1968) 283–301.

[5] F. E. Browder, W. V. Petryshyn: Construction of fixed points of nonlinear mappings in Hilbert space, J. Math. Anal. Appl. 20 (1967) 197–228.

[6] R. E. Bruck: Asymptotic convergence of nonlinear contraction semigroups in Hilbert space, J. Funct. Anal. 18 (1975) 15–26.

[7] R. E. Bruck: On the weak convergence of an ergodic iteration for the solution of variational inequalities for monotone operators in Hilbert space, J. Math. Anal. Appl. 61 (1977) 159–164.

[8] A. Faik, A. Syam: Differential inclusions governed by a nonconvex sweeping process, J. Nonlinear and Convex Analysis 2 (2001) 381–392.

[9] B. R. Halpern: Fixed points of nonexpanding maps, Bull. Amer. Math. Soc. 73 (1967) 957–961.

[10] H. Iiduka, W. Takahashi: Strong convergence theorems for nonexpansive mappings and inverse-strongly-monotone mappings, Nonlinear Anal., to appear.

[11] H. Iiduka, W. Takahashi, M. Toyoda: Approximation of solutions of variational inequalities for monotone mappings, Panamer. Math. J., to appear.

[12] J. L. Lions, G. Stampacchia: Variational inequalities, Comm. Pure Appl. Math. 20 (1967) 493–517.

[13] F. Liu, M. Z. Nashed: Regularization of nonlinear ill-posed variational inequalities and convergence rates, Set-Valued Anal. 6 (1998) 313–344.

[14] S. Matsushita, D. Kuroiwa: Approximation of fixed points of nonexpansive nonself-mappings, Sci. Math. Jpn. 57 (2003) 171–176.

[15] K. Nakajo, W. Takahashi: Strong and weak convergence theorems by an improved splitting method, Comm. Appl. Nonlinear Anal. 9 (2002) 99–107.

[16] Z. Opial: Weak convergence of the sequence of successive approximation for nonexpansive mappings, Bull. Amer. Math. Soc. 73 (1967) 591–597.

[17] R. T. Rockafellar: On the maximality of sums of nonlinear monotone operators, Trans. Amer. Math. Soc. 149 (1970) 75–88.

[18] R. T. Rockafellar: Monotone operators and the proximal point algorithm, SIAM J. Control Optim. 14 (1976) 877–898.

[19] W. Takahashi: Nonlinear variational inequalities and fixed point theorems, J. Math. Soc. Japan 28 (1976) 168–181.

[20] W. Takahashi: Nonlinear complementarity problem and systems of convex inequalities, J. Optim. Theory Appl. 24 (1978) 493–508.

[21] W. Takahashi: Convex Analysis and Approximation of Fixed Points, Yokohama Publishers, Yokohama (2000) (Japanese).

[22] W. Takahashi: Nonlinear Functional Analysis, Yokohama Publishers, Yokohama (2000).

[23] W. Takahashi, M. Toyoda: Weak convergence theorems for nonexpansive mappings and monotone mappings, J. Optim. Theory Appl. 118 (2003) 417–428.