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A Local Selection Theorem for Metrically Regular Mappings

A. L. Dontchev
Mathematical Reviews,
Ann Arbor, MI 48107-8604, USA
ald@ams.org

To Robert G. Bartle in memoriam

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We prove the following extension of a classical theorem due to Bartle and Graves. Let a set-valued mapping $F : X \rightrightarrows Y$, where X and Y are Banach spaces, be metrically regular at $\bar{x}$ for $\bar{y}$ and with the property that the mapping whose graph is the restriction of the graph of the inverse F^{-1} to a neighborhood of $(\bar{y}, \bar{x})$ is convex and closed valued. Then for any function $G : X \to Y$ with $\operatorname{lip} G(\bar{x}) \cdot \operatorname{reg} F(\bar{x}\,|\,\bar{y})) < 1$, the mapping $(F + G)^{-1}$ has a continuous local selection $x(\cdot)$ around $(\bar{y} + G(\bar{x}), \bar{x})$ which is also calm.

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1. Introduction

The classical inverse function theorem stated for a function $f : X \to Y$, with X and Y Banach spaces, assumes that f is continuously differentiable in a neighborhood of a given reference point $\bar{x}$ and, most importantly, the Fréchet derivative $\nabla f(\bar{x})$ has a linear and bounded inverse; then the theorem claims that there exist neighborhoods U of $\bar{x}$ and V of $\bar{y} := f(\bar{x})$ such that the mapping

$$V \ni y \mapsto f^{-1}(y) \cap U \tag{1}$$

is single valued (a function defined on V) which is moreover continuously differentiable ($\mathcal{C}^1$) in V and whose derivative is the inverse of ∇f. The mapping in (1) is obtained by restricting the graph of f^{-1} to a "box" around $(\bar{y}, \bar{x})$, that is, the product of neighborhoods of $\bar{x}$ and $\bar{y}$ respectively, and is called *graphical localization* of f^{-1} around $(\bar{y}, \bar{x})$. The inverse function theorem then says that the invertibility of $\nabla f(\bar{x})$ implies (actually, it is equivalent) to the existence of a single-valued graphical localization of f^{-1} around $(\bar{y}, \bar{x})$ which is $\mathcal{C}^1$.

An inverse function type theorem may be obtained in Hilbert spaces when the Jacobian $\nabla f(\bar{x})$ is merely surjective. Indeed, in this case the mapping (1), although in general set-valued, has a local single-valued selection $x(\cdot)$, that is, a function $x(\cdot)$ exists with $x(y) \in f^{-1}(y) \cap U$ for all $y \in V$, which is continuously differentiable in V. The precise result is as follows:

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Theorem 1.1. *Let X and Y be Hilbert spaces and let $f : X \to Y$ be a function which is $\mathcal{C}^1$ around $\bar{x}$ and such that the derivative $B := \nabla f(\bar{x})$ is surjective. Then there exist a neighborhood V of $\bar{y} := f(\bar{x})$ and a $\mathcal{C}^1$ function $x : V \to X$ such that*

$$x(\bar{y}) = \bar{x} \qquad \text{and} \qquad f(x(y)) = y \text{ for every } y \in V,$$

and moreover $\nabla x(\bar{y}) = (B^*B)^{-1}B^*$.

Proof. In terms of the adjoint operator B^* consider the mapping

$$(x, u) \mapsto g(x, u) := \begin{pmatrix} x + B^*u \\ f(x) \end{pmatrix},$$

which satisfies $g(\bar{x}, 0) = (\bar{x}, \bar{y})$ and whose Jacobian is

$$J = \begin{pmatrix} I & B^* \\ B & 0 \end{pmatrix}.$$

It is well known that, in Hilbert spaces, if B is surjective than the operator J is invertible in the sense that J^{-1} is linear and bounded from $X \times Y$ into itself. Hence, by the classical inverse function theorem, the mapping g^{-1} has a single-valued and continuously differentiable graphical localization $(v, y) \mapsto (\xi(v, y), \eta(v, y))$ around $((\bar{x}, \bar{y}), (\bar{x}, 0))$. In particular, for some neighborhoods U of $\bar{x}$ and V of $\bar{y}$, the function $x(y) := \xi(\bar{x}, y)$ satisfies $y = f(x(y))$ for $y \in V$. It remains to observe that B^*B is invertible and, from the equation $B^*f(x(y)) = B^*y$, the derivative of $x(\cdot)$ with respect to y satisfies $B^*B\nabla x(\bar{y}) = B^*$. □

If X and Y are arbitrary Banach spaces, the surjectivity of the Jacobian implies the existence of a selection of (1) which is merely continuous and calm. This follows from a classical theorem by Bartle and Graves, Theorem 6 in [1]. Up to some minor adjustments and simplifications, the Bartle-Graves theorem in question is as follows:

Theorem 1.2. *Let X and Y be Banach spaces and let $f : X \to Y$ be a function which is strictly differentiable at $\bar{x}$ and such that the strict derivative $\nabla f(\bar{x})$ is surjective. Then there exist a neighborhood V of $\bar{y} := f(\bar{x})$, a continuous function $x : V \to X$ and a constant $\gamma > 0$ such that for every $y \in V$*

$$f(x(y)) = y \quad \text{and} \quad \|x(y) - \bar{x}\| \leq \gamma\|y - \bar{y}\|.$$

In other words, the surjectivity of the strict derivative implies that a graphical localization of f^{-1} around the point $(\bar{y}, \bar{x})$ has a selection which is continuous and calm. A function $g : X \to Y$ is said to be *calm at $\bar{x}$* when there exist a neighborhood

V of $\bar{x}$ and a constant $\gamma > 0$ such that

$$\|g(x) - g(\bar{x})\| \leq \gamma\|x - \bar{x}\| \text{ for every } x \in V. \tag{2}$$

The infimum of γ for which (2) holds is called *modulus of calmness* and is denoted by $\operatorname{clm} g(\bar{x})$.

As noted in [3], p. 300, in contrast to the smooth local selection in Theorem 1.1 for Hilbert spaces, the selection in Bartle-Graves theorem even for a linear and bounded mapping f may be not Lipschitz continuous anywhere near $\bar{x}$.

The purpose of this paper is to generalize Theorem 2.1 to set-valued mappings that cover in particular systems of inequalities and variational inequalities. First, we describe the notation and terminology we use, which is consistent with the book [9], and briefly discuss some related results. Throughout, unless stated otherwise, X and Y are real Banach spaces with norms $\|\cdot\|$ and closed unit balls $I\!B$; a ball centered at a with radius r is denoted $I\!B_r(a)$. The distance from a point x to a set A is denoted by $d(x, A)$. The notation $F : X \rightrightarrows Y$ means that F is a *set-valued* mapping from X to the subsets of Y; if F is a function, that is, for each $x \in X$ the set of values $F(x)$ consists of no more than one element, then we write $F : X \to Y$. The *domain* of F is defined as $\operatorname{dom} F = \{x \in X \mid F(x) \neq \emptyset\}$ while its *range* as $\operatorname{rge} F = \{y \in Y \mid y \in F(x), x \in \operatorname{dom} F\}$. The *graph* of F is $\operatorname{gph} F = \{(x, y) \mid y \in F(x)\}$ and its *inverse* F^{-1} is defined as $x \in F^{-1}(y) \iff y \in F(x)$. A mapping $F : X \rightrightarrows Y$ with $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ has a *local selection around* $(\bar{x}, \bar{y})$ if there exist neighborhoods U of $\bar{x}$ and V of $\bar{y}$, respectively, and a function $s : U \to V$ such that $s(\bar{x}) = \bar{y}$ and $s(x) \in F(x) \cap V$ for all $x \in U$. Recall that the Lipschitz modulus $\operatorname{lip} g(\bar{x})$ of a function $g : X \to Y$ at a point $\bar{x}$ is defined as

$$\operatorname{lip} g(\bar{x}) := \limsup_{\substack{x, x' \to \bar{x} \\ x \neq x'}} \frac{\|g(x') - g(x)\|}{\|x' - x\|}.$$

A mapping $F : X \rightrightarrows Y$ is said to be *metrically regular* at $\bar{x}$ for $\bar{y}$ if there exists a constant $\kappa > 0$ such that

$$d(x, F^{-1}(y)) \leq \kappa d(y, F(x)) \text{ for all } (x, y) \text{ close to } (\bar{x}, \bar{y}). \tag{3}$$

The infimum of κ for which (3) holds is the *modulus of metric regularity* which we denote by $\operatorname{reg} F(\bar{x}\,|\,\bar{y})$. Metric regularity of F at $\bar{x}$ for $\bar{y}$ is signaled by $\operatorname{reg} F(\bar{y}\,|\,\bar{y}) < \infty$.

The concept of metric regularity has its roots in the Banach open mapping theorem: a linear and bounded mapping $L : X \to Y$ is metrically regular if and only if it is surjective. The modulus of metric regularity of such an L is the same for all points in its graph and

$$\operatorname{reg} L = \sup_{y \in B} d(0, L^{-1}(y)).$$

In particular, if L is invertible, which is the case when $X = Y = I\!R^n$, then $\operatorname{reg} L = \|L^{-1}\|$.

The metric regularity of a set-valued mapping $F : X \rightrightarrows Y$ at $\bar{x}$ for $\bar{y}$ implies the existence of neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that $F^{-1}(y) \cap U \neq \emptyset$ for all $y \in V$. Further, the metric regularity is preserved when F is perturbed by a function with a small Lipschitz constant. Specifically, we have

Theorem 1.3 ([5], Theorem 3.3)**.** *Consider a mapping $F : X \rightrightarrows Y$ with $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ and let $\operatorname{gph} F$ be closed locally around $(\bar{x}, \bar{y})$. Consider also a function $G : X \to Y$. If $\operatorname{reg} F(\bar{x}\,|\,\bar{y}) < \kappa < \infty$ and $\operatorname{lip} G(\bar{x}) < \lambda < \kappa^{-1}$, then*

$$\operatorname{reg}(F + G)(\bar{x}\,|\,\bar{y} + G(\bar{x})) < \frac{\kappa}{1 - \lambda\kappa}.$$

For single-valued mappings that are nonlinear but differentiable, the property of the metric regularity described in Theorem 1.3 goes back to classical theorems by Lyusternik and Graves.

Theorem 1.4 (Lyusternik-Graves). *For a function $f : X \to Y$ which is continuously differentiable near $\bar{x}$, one has*

$$\operatorname{reg} f(\bar{x}\,|\,f(\bar{x})) = \operatorname{reg} \nabla f(\bar{x}).$$

A generalization in a different direction of the Banach open mapping theorem is a result by Robinson and Ursescu:

Theorem 1.5 (Robinson-Ursescu). *For a mapping $F : X \to Y$ and $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, if F has closed and convex graph, then F is metrically regular at $\bar{x}$ for $\bar{y}$ if and only if $\bar{y} \in \operatorname{int} \operatorname{rge} F$.*

In finite dimensions, more can be said about metrically regular mappings. If $f : I\!R^n \to I\!R^n$ is continuously differentiable around $\bar{x}$, then the metric regularity of f at $\bar{x}$ simply means that the Jacobian $\nabla f(\bar{x})$ is a nonsingular matrix and then the graphical localization of f^{-1} around the point $(f(\bar{x}), \bar{x})$ is single-valued and $\mathcal{C}^1$. The equivalence of the metric regularity with the *Lipschitz continuous* single-valued graphical localization of f^{-1} is actually valid for more general set-valued mappings of the form $f + N_C$ where f is a smooth function and N_C is the normal cone mapping to a convex polyhedral set C. This inverse function theorem for variational inequalities was established in [4] together with a formula for the Lipschitz modulus of the localization. Here the theory of inverse function for metrically regular mappings merges with another fundamental result, due to S. Robinson [8], regarding the "stability under linearization" of the property of existence of a Lipschitz continuous single-valued graphical localization. A discussion of various developments around the concept of metric regularity has recently been given by Ioffe [7], details are also available in [5] and [9].

In this paper we prove a generalization of the Battle-Graves theorem (Theorem 3.1) of the following form: Let a set-valued mapping $F : X \rightrightarrows Y$ be metrically regular at $\bar{x}$ for $\bar{y}$ and with the property that a graphical localization of the inverse F^{-1} around $(\bar{y}, \bar{x})$ is convex and closed valued. Then F^{-1} has a continuous local selection $x(\cdot)$ around $(\bar{y}, \bar{x})$ which is calm. Moreover, for any function $G : X \to Y$ with $\operatorname{lip} G(\bar{x}) \cdot \operatorname{reg} F(\bar{x}\,|\,\bar{y})) < 1$, the mapping $(F + G)^{-1}$ has a continuous local selection $x(\cdot)$ around $(\bar{x}, \bar{y} + G(\bar{x}))$ which is calm.

2. Aubin continuity and continuous local selections

It is well documented, see [9], Section 9G, that F is metrically regular at $\bar{x}$ for $\bar{y}$ if and only if F^{-1} has the so-called *Aubin property* at $\bar{y}$ for $\bar{x}$: there exist $\kappa \in (0, \infty)$ together with neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$F^{-1}(y') \cap U \subset F^{-1}(y) + \kappa \|y' - y\| I\!B \text{ for all } y, y' \in V; \tag{4}$$

moreover, the modulus $\operatorname{reg} F(\bar{x}, \bar{y})$ is also the infimum of all κ for which (4) holds.

Recall that a mapping $A : T \rightrightarrows X$ is (sequentially) lower semicontinuous on $T \subset Y$ if for every $t \in T$, every $x \in A(t)$ and every sequence $t_k \in T, t_k \to t$, there exist $x_k \in T(t_k)$ for $k = 1, 2, \ldots$, with $x_k \to x$. In our setting, sequential lower semicontinuity and lower semicontinuity coincide.

The Aubin property (4) is a local property of a mapping around a point in its graph, which is preserved for a graphical localization of the mapping around the reference point. However, such a localization may not be lower semicontinuous, in general. In the following lemma we show that if a set-valued mapping A is convex and closed valued locally around the reference point, then the mapping obtained by truncation of A with a ball centered at $\bar{x}$ with radius proportional to the distance to $\bar{y}$ is lower semicontinuous in a neighborhood of $\bar{y}$.

Lemma 2.1. *Consider a mapping $A : Y \rightrightarrows X$ and any $(\bar{y}, \bar{x}) \in \operatorname{gph} A$ and suppose that A is Aubin continuous at $\bar{y}$ for $\bar{x}$ with a constant κ. Let, for some $c > 0$, the sets $A(y) \cap I\!\!B_c(\bar{x})$ be convex and closed for all $y \in I\!\!B_c(\bar{y})$. Then for any $\alpha > \kappa$ there exists $\beta > 0$ such that the mapping*

$$I\!\!B_\beta(\bar{y}) \ni y \mapsto M_0(y) := \{x \in A(y) \mid \|x - \bar{x}\| \le \alpha \|y - \bar{y}\|\}$$

is nonempty, closed and convex valued, and lower semicontinuous.

Proof. Let $I\!\!B_a(\bar{x})$ and $I\!\!B_b(\bar{y})$ be the balls centered at $\bar{x}$ and $\bar{y}$, respectively, that are associated with the Aubin continuity of A (metric regularity of A^{-1}) with a constant κ. Without loss of generality, let $a < c$. Choose $\alpha > \kappa$ and β such that

$$0 < \beta \le \min\{\frac{a}{\alpha}, \frac{c}{2\alpha}, b, c\}.$$

For such a β the mapping M_0 has nonempty closed convex values. It remains to show that M_0 is lower semicontinuous on $I\!\!B_\beta(\bar{y})$.

Let $(x, y) \in \operatorname{gph} M_0$ and $y_k \to y$, $y_k \in I\!\!B_\beta(\bar{y})$. First, let $y = \bar{y}$. Then $M_0(y) = \bar{x}$ and from the Aubin continuity of A there exists a sequence $x_k \in A(y_k)$ such that $\|x_k - \bar{x}\| \le \kappa \|y_k - \bar{y}\|$. Then $x_k \in M_0(y_k)$, $x_k \to x$ as $k \to \infty$ and we are done in this case.

Now let $y \ne \bar{y}$. From the Aubin property of A there exists $\check{x}_k \in A(y_k)$ such that

$$\|\check{x}_k - \bar{x}\| \le \kappa \|y_k - \bar{y}\|$$

and also there exists $\tilde{x}_k \in A(y_k)$ such that

$$\|\tilde{x}_k - x\| \le \kappa \|y_k - y\|.$$

Because of the choice of β, both $\check{x}_k$ and $\tilde{x}_k$ are from $I\!\!B_c(\bar{x})$. Let

$$\epsilon_k = \frac{(\alpha + \kappa)\|y_k - y\|}{(\alpha - \kappa)\|y_k - \bar{y}\| + (\alpha + \kappa)\|y_k - y\|}. \tag{5}$$

Then $0 \le \epsilon_k < 1$ and $\epsilon_k \to 0$ as $k \to \infty$. Let $x_k = \epsilon_k \check{x}_k + (1 - \epsilon_k)\tilde{x}_k$. Then $x_k \in A(y_k)$. Moreover, we have

$$\begin{aligned}
\|x_k - \bar{x}\| &\le \epsilon_k \|\check{x}_k - \bar{x}\| + (1 - \epsilon_k)\|\tilde{x}_k - \bar{x}\| \\
&\le \epsilon_k \kappa \|y_k - \bar{y}\| + (1 - \epsilon_k)(\|\tilde{x}_k - x\| + \|x - \bar{x}\|) \\
&\le \epsilon_k \kappa \|y_k - \bar{y}\| + (1 - \epsilon_k)\kappa \|y_k - y\| + (1 - \epsilon_k)\alpha \|y - \bar{y}\| \\
&\le \epsilon_k \kappa \|y_k - \bar{y}\| + (1 - \epsilon_k)\kappa \|y_k - y\| + (1 - \epsilon_k)\alpha \|y_k - \bar{y}\| + (1 - \epsilon_k)\alpha \|y_k - y\| \\
&\le \alpha \|y_k - \bar{y}\| - \epsilon_k(\alpha - \kappa)\|y_k - \bar{y}\| + (1 - \epsilon_k)(\alpha + \kappa)\|y_k - y\| \le \alpha \|y_k - \bar{y}\|,
\end{aligned}$$

where in the last inequality we take into account the formula (5) for ϵ_k. Thus $x_k \in M_0(y_k)$ and since $x_k \to x$, the proof is complete. □

Adapted to our setting, the Michael selection theorem says that any set-valued mapping acting from a closed ball in Y to X, which is nonempty, closed and convex valued and lower semicontinuous, has a continuous selection. Lemma 2.1 allows us to apply the Michael theorem to the mapping M_0 obtaining, in terms of a metrically regular mapping F, the following result:

Theorem 2.2. *Consider a mapping $F : X \rightrightarrows Y$ which is metrically regular at $\bar{x}$ for $\bar{y}$. Let, for some $c > 0$, the sets $F^{-1}(y) \cap I\!B_c(\bar{x})$ be convex and closed for all $y \in I\!B_c(\bar{y})$. Then the mapping F^{-1} has a continuous local selection $x(\cdot)$ around $(\bar{y}, \bar{x})$ which is calm at $\bar{y}$ with*

$$\operatorname{clm} x(\bar{y}) \leq \operatorname{reg} F(\bar{x}\,|\,\bar{y}). \tag{6}$$

Proof. Choose α and κ such that $\alpha > \kappa > \operatorname{reg} F(\bar{x}\,|\,\bar{y})$ and apply the Michael selection theorem to the mapping M_0 in Lemma 2.1 for $A = F^{-1}$. The obtained continuous local selection is calm with a constant α. Since α could be arbitrarily close to $\operatorname{reg} F(\bar{x}\,|\,\bar{y})$, we obtain (6). □

In the following section we will show that on the same assumptions for a set-valued mapping F, the conclusion of this theorem holds when F is perturbed by a function G with a sufficiently small Lipschitz constant.

In their paper [1], Bartle and Graves proved several theorems that are related but different. Perhaps the most known corollary of their work is the following:

Theorem 2.3 ([3], Lemma 3.2, p. 299)**.** *For any bounded linear mapping T from X onto Y, there exists a continuous mapping B such that $TBy = y$ for every $y \in Y$.*

Proof. By the Banach open mapping principle, the mapping T^{-1} is Lipschitz continuous, hence it lower semicontinuous on X. Since it is convex and closed valued, applying the Michael selection theorem completes the proof. □

For an extension of the main Theorem 4 in [1] (also stated on p. 85 of [6]), see the recent paper [2].

3. The local selection theorem

In this section we show that if a mapping F satisfies the assumptions of Theorem 2.2, and hence F^{-1} has a continuous and calm local selection around $(\bar{y}, \bar{x})$, then for any function $G : X \to Y$ with $\operatorname{lip} G(\bar{x}) < 1/\operatorname{reg} F(\bar{x}\,|\,\bar{y})$, the mapping $(F + G)^{-1}$ has a continuous and calm local selection around $(\bar{y} + G(\bar{x}), \bar{x})$. We will prove this result by repeatedly using an argument similar to the proof of Lemma 2.1 in a way which resembles the proofs in the classical works of Lyusternik and Graves, a procedure which goes back to Newton's method.

Theorem 3.1. *Consider a mapping $F : X \rightrightarrows Y$ which is metrically regular at $\bar{x}$ for $\bar{y}$. Let for some $c > 0$ the mapping $I\!B_c(\bar{y}) \ni y \mapsto F^{-1}(y) \cap I\!B_c(\bar{x})$ be closed and convex valued and let $G : X \to Y$ satisfy $\operatorname{lip} G(\bar{x}) \cdot \operatorname{reg} F(\bar{x}\,|\,\bar{y}) < 1$. Then the mapping $(G + F)^{-1}$ has a continuous local selection $x(\cdot)$ around $(\bar{y} + G(\bar{x}), \bar{x})$ which is calm at $\bar{y} + G(\bar{x})$ with*

$$\operatorname{clm} x(\bar{y} + G(\bar{x})) \leq \frac{\operatorname{reg} F(\bar{x}\,|\,\bar{y})}{1 - \operatorname{lip} G(\bar{x}) \cdot \operatorname{reg} F(\bar{x}\,|\,\bar{y})}\,. \tag{7}$$

Proof. The proof consists of two steps. In the first step, we use induction to obtain a Cauchy sequence of continuous functions $z_0, z_1, \cdots$, such that z_n is a continuous and calm selection of $F^{-1}(\cdot - G(z_{n-1}(\cdot)))$. Then we show that this sequence has a limit in the space of continuous functions acting from a fixed ball around $\bar{y}$ to the space X and equipped with the supremum norm, and this limit is the selection whose existence is claimed.

Choose a constant γ that is greater than the right hand side of (7) and let κ, α and λ be such that $\operatorname{reg} F(\bar{x}\,|\,\bar{y}) < \kappa < \alpha < 1/\lambda$, $\lambda > \operatorname{lip} G(\bar{x})$ and $\kappa/(1-\alpha\lambda) \leq \gamma$. Without loss of generality, we assume that $G(\bar{x}) = 0$. Let $I\!B_a(\bar{x})$ and $I\!B_b(\bar{y})$ be the neighborhoods of $\bar{x}$ and $\bar{y}$, respectively, that are associated with the assumed properties of the mapping F and the function G. Specifically,

1) For every $y, y' \in I\!B_b(\bar{y})$ and $x \in F^{-1}(y) \cap I\!B_a(\bar{x})$ there exists $x' \in F^{-1}(y')$ with

$$\|x' - x\| \leq \kappa \|y' - y\|;$$

2) For every $y \in I\!B_b(\bar{y})$ the set $F^{-1}(y) \cap I\!B_a(\bar{x})$ is nonempty, closed and convex;
3) The function G is Lipschitz continuous on $I\!B_a(\bar{x})$ with a constant λ.

From Lemma 2.1 and Theorem 2.2, there exist a constant β, $0 < \beta \leq b$, and a continuous function $z_0 : I\!B_\beta(\bar{x}) \to X$ such that

$$F(z_0(y)) \ni y \quad \text{and} \quad \|z_0(y) - \bar{x}\| \leq \kappa \|y - \bar{y}\|$$

for all $y \in I\!B_\beta(\bar{y})$. Choose a positive τ such that

$$\tau \leq (1 - \alpha\lambda) \min\{a, \frac{a}{2\kappa}, \beta\} \tag{8}$$

and consider the mapping

$$I\!B_\tau(\bar{y}) \ni y \mapsto M_1(y) := \big\{ x \in F^{-1}(y - G(z_0(y))) \mid \|x - z_0(y)\| \leq \alpha\lambda \|z_0(y) - \bar{x}\| \big\}.$$

Clearly, $(\bar{y}, \bar{x}) \in \operatorname{gph} M_1$ and also for any $y \in I\!B_\tau(\bar{y})$, using (8), we have

$$\|y - G(z_0(y)) - \bar{y}\| \leq \tau + \lambda \|z_0(y) - \bar{x}\| \leq \tau + \lambda\kappa\tau \leq \beta \leq b.$$

Then from the Aubin property of F^{-1} there exists $x \in F^{-1}(y - G(z_0(y)))$ with

$$\|x - z_0(y)\| \leq \kappa \|G(z_0(y)) - G(\bar{x})\| \leq \alpha\lambda \|z_0(y) - \bar{x}\|,$$

and hence $x \in M_1(y)$. Thus M_1 is nonempty valued. Further, if $(x, y) \in \operatorname{gph} M_1$, we have

$$\|x - \bar{x}\| \leq \|x - z_0(y)\| + \|z_0(y) - \bar{x}\| \leq (1 + \alpha\lambda)\kappa\tau \leq a.$$

Then, from the choice of τ in (8) and from the property 2) above, since for $y \in I\!B_\tau(\bar{y})$ the set $M_1(y)$ is the intersection of a closed ball with a closed convex set, the mapping M_1 is closed and convex valued in its domain. We will show that this mapping is lower semicontinuous in $I\!B_\tau(\bar{y})$.

Let $y \in I\!B_\tau(\bar{y})$ and $x \in M_1(y)$, and let $y_k \in I\!B_\tau(\bar{y})$, $y_k \to y$ as $k \to \infty$. If $z_0(y) = \bar{x}$ then $M_1(y) = \{\bar{x}\}$ and therefore $x = \bar{x}$. Any $x_k \in M_1(y_k) \neq \emptyset$ satisfies

$$\|x_k - z_0(y_k)\| \leq \alpha\lambda \|z_0(y_k)) - \bar{x}\|.$$

From the continuity of the functions z_0 we obtain that $x_k \to z_0(y) = \bar{x} = x$, thus M_1 is lower semicontinuous.

Now let $z_0(y) \neq \bar{x}$. Since $z_0(y_k) \in F^{-1}(y_k - G(\bar{x})) \cap I\!\!B_a(\bar{x})$, the Aubin continuity of F^{-1} yields the existence of $\check{x}_k \in F^{-1}(y_k - G(z_0(y_k)))$ such that

$$\|\check{x}_k - z_0(y_k)\| \leq \kappa \|G(z_0(y_k)) - G(\bar{x})\| \leq \kappa\lambda \|z_0(y_k) - \bar{x}\| \leq \alpha\lambda \|z_0(y_k) - \bar{x}\|. \qquad (9)$$

Then $\check{x}_k \in M_1(y_k)$ and in particular, $\check{x}_k \in I\!\!B_a(\bar{x})$. Further, the inclusion $x \in F^{-1}(y - G(z_0(y))) \cap I\!\!B_a(\bar{x})$ and the Aubin continuity of F^{-1} yield that there exists $\tilde{x}_k \in F^{-1}(y_k - G(z_0(y_k)))$ such that

$$\|\tilde{x}_k - x\| \leq \kappa(\|y_k - y\| + \lambda \|z_0(y_k) - z_0(y)\|) \to 0 \text{ as } k \to \infty. \qquad (10)$$

Let

$$\epsilon_k := \frac{(1 + \alpha\lambda)\|z_0(y_k) - z_0(y)\| + \|\tilde{x}_k - x\|}{\alpha\lambda \|z_0(y) - \bar{x}\| - \kappa\lambda \|z_0(y_k) - \bar{x}\|}.$$

Note that, for $k \to \infty$, the nominator in the definition of ϵ_k goes to zero because of the continuity of z_0 and (10), while the denominator converges to $(\alpha - \kappa)\lambda \|z_0(y) - \bar{x}\| > 0$, therefore $\epsilon_k \to 0$ as $k \to \infty$. Let

$$x_k = \epsilon_k \check{x}_k + (1 - \epsilon_k)\tilde{x}_k.$$

Since $\tilde{x}_k \to x$ and $\epsilon_k \to 0$, we obtain $x_k \to x$ as $k \to \infty$ and also, since F^{-1} is convex valued near $(\bar{x}, \bar{y})$, we have $x_k \in F^{-1}(y_k - G(z_0(y_k)))$ for large k. From (9), (10), the assumption that $x \in M_1(y)$, and the choice of ϵ_k, we have

$$\begin{aligned}
\|x_k - z_0(y_k)\| &\leq \epsilon_k \|\check{x}_k - z_0(y_k)\| + (1 - \epsilon_k)\|\tilde{x}_k - z_0(y_k)\| \\
&\leq \epsilon_k \kappa\lambda \|z_0(y_k) - \bar{x}\| + (1 - \epsilon_k)(\|\tilde{x}_k - x\| + \|x - z_0(y)\| \\
&\quad + \|z_0(y) - z_0(y_k)\|) \\
&\leq \epsilon_k \kappa\lambda \|z_0(y_k) - \bar{x}\| + \|\tilde{x}_k - x\| + (1 - \epsilon_k)\alpha\lambda \|z_0(y) - \bar{x}\| \\
&\quad + \|z_0(y) - z_0(y_k)\| \\
&\leq \alpha\lambda \|z_0(y_k) - \bar{x}\| + \alpha\lambda \|z_0(y_k) - z_0(y)\| \\
&\quad + \|\tilde{x}_k - x\| + \|z_0(y) - z_0(y_k)\| - \epsilon_k \alpha\lambda \|z_0(y) - \bar{x}\| + \epsilon_k \kappa\lambda \|z_0(y_k) - \bar{x}\| \\
&\leq \alpha\lambda \|z_0(y_k) - \bar{x}\| + \|\tilde{x}_k - x\| + (1 + \alpha\lambda)\|z_0(y) - z_0(y_k)\| \\
&\quad - \epsilon_k(\alpha\lambda \|z_0(y) - \bar{x}\| - \kappa\lambda \|z_0(y_k) - \bar{x}\|) \\
&= \alpha\lambda \|z_0(y_k) - \bar{x}\|.
\end{aligned}$$

We obtain that $x_k \in M_1(y_k)$ and since $x_k \to x$, the mapping M_1 is lower semicontinuous in its domain $I\!\!B_\tau(\bar{y})$. Hence, by the Michael selection theorem it has a continuous selection $z_1(\cdot) : I\!\!B_\tau(\bar{y}) \to X$; that is, there exists a continuous function z_1 which satisfies

$$z_1(y) \in F^{-1}(y - G(z_0(y))) \quad \text{and} \quad \|z_1(y) - z_0(y)\| \leq \alpha\lambda \|z_0(y) - \bar{x}\| \text{ for all } y \in I\!\!B_\tau(\bar{y}).$$

Then for $y \in I\!\!B_\tau(\bar{y})$,

$$\|z_1(y) - \bar{x}\| \leq \|z_1(y) - z_0(y\| + \|z_0(y) - \bar{x}\| \leq (1 + \kappa\lambda)\|y - \bar{y}\| \leq \gamma \|y - \bar{y}\|.$$

The induction step is somewhat parallel to the first step. Let z_0 and z_1 be as above and suppose we have also found functions $z_2, z_3, \cdots, z_n$, such that each $z_j, j = 2, \cdots, n$, is a continuous selection of the mapping

$$I\!B_\tau(\bar{y}) \ni y \mapsto M_j(y) := \left\{ x \in F^{-1}(y - G(z_{j-1}(y))) \mid \|x - z_{j-1}(y)\| \le \alpha\lambda \|z_{j-1}(y) - z_{j-2}(y)\| \right\}.$$

Then for $y \in I\!B_\tau(\bar{y})$ we obtain

$$\|z_j(y) - z_{j-1}(y)\| \le (\alpha\lambda)^{j-1}\|z_1(y) - z_0(y)\| \le (\alpha\lambda)^j \|z_0(y) - \bar{x}\|, \quad j = 2, \cdots, n.$$

Therefore,

$$\begin{aligned} \|z_j(y) - \bar{x}\| &\le \sum_{i=1}^{j} (\alpha\lambda)^i \|z_i(y) - z_{i-1}(y)\| + \|z_0(y) - \bar{x}\| \\ &\le \sum_{i=0}^{j} (\alpha\lambda)^i \|z_0(y) - \bar{x}\| \\ &\le \frac{\kappa}{1 - \alpha\lambda}\|y - \bar{y}\| \le \gamma\|y - \bar{y}\|. \end{aligned}$$

Hence, from (8), for $j = 2, \cdots, n$,

$$\|z_j(y) - \bar{x}\| \le a \tag{11}$$

and also

$$\|y - G(z_j(y)) - \bar{y}\| \le \tau + \lambda\|z_j(y) - \bar{x}\| \le \tau + \frac{\kappa\lambda\tau}{1 - \alpha\lambda} \le \beta \le b. \tag{12}$$

Consider the mapping

$$I\!B_\tau(\bar{y}) \ni y \mapsto M_{n+1}(y) := \left\{ x \in F^{-1}(y - G(z_n(y))) \mid \|x - z_n(y)\| \le \alpha\lambda\|z_n(y) - z_{n-1}(y)\| \right\}.$$

As in the first step, we obtain that M_{n+1} is nonempty, closed and convex valued. Let $y \in I\!B_\tau(\bar{y})$ and $x \in M_{n+1}(y)$, and let $y_k \in I\!B_\tau(\bar{y})$, $y_k \to y$ as $k \to \infty$. If $z_{n-1}(y) = z_n(y)$ then $M_{n+1}(y) = \{z_n(y)\}$ and hence $x = z_n(y)$, and from $z_n(y_k) \in F^{-1}(y_k - G(z_{n-1}(y_k))) \cap I\!B_a(\bar{x})$ and $y_k - G(z_{n-1}(y_k)) \in I\!B_b(\bar{y})$, using the Aubin property of F^{-1}, we obtain that there exists $x_k \in F^{-1}(y_k - G(z_n(y_k)))$ such that

$$\|x_k - z_n(y_k)\| \le \kappa\|G(z_n(y_k)) - G(z_{n-1}(y_k))\| \le \alpha\lambda\|z_n(y_k) - z_{n-1}(y_k)\|.$$

Therefore $x_k \in M_{n+1}(y_k)$, $x_k \to z_1(y) = x$ as $k \to \infty$, and hence M_{n+1} is lower semicontinuous for the case considered.

Let $z_n(y) \ne z_{n-1}(y)$. From (11) and (12) for $y = y_k$, since

$$z_n(y_k) \in F^{-1}(y_k - G(z_{n-1}(y_k))) \cap I\!B_a(\bar{x}),$$

the Aubin continuity of F^{-1} implies the existence of $\check{x}_k \in F^{-1}(y_k - G(z_n(y_k)))$ such that

$$\|\check{x}_k - z_1(y_k)\| \le \kappa\|G(z_n(y_k)) - G(z_{n-1}(y_k))\| \le \kappa\lambda\|z_n(y_k) - z_{n-1}(y_k)\|.$$

Similarly, since $x \in F^{-1}(y - G(z_n(y))) \cap \mathbb{B}_a(\bar{x})$, there exists $\tilde{x}_k \in F^{-1}(y_k - G(z_n(y_k)))$ such that

$$\begin{aligned}\|\tilde{x}_k - x\| &\le \kappa(\|y_k - y\| + \|G(z_n(y_k)) - G(z_n(y))\|) \\ &\le \kappa(\|y_k - y\| + \lambda\|z_n(y_k) - z_n(y)\|) \to 0 \text{ as } k \to \infty.\end{aligned}$$

Let

$$\epsilon_k := \frac{\alpha\lambda\|z_{n-1}(y) - z_{n-1}(y_k)\| + (1+\alpha\lambda)\|z_n(y) - z_n(y_k)\| + \|\tilde{x}_k - x\|}{\alpha\lambda\|z_n(y) - z_{n-1}(y)\| - \kappa\lambda\|z_n(y_k) - z_{n-1}(y_k)\|}.$$

Then $\epsilon_k \to 0$ as $k \to \infty$. Taking

$$x_k = \epsilon_k \check{x}_k + (1 - \epsilon_k)\tilde{x}_k,$$

we obtain that $x_k \in F^{-1}(y_k - G(z_n(y_k)))$ for large k. Further, we estimate $\|x_k - z_n(y_k)\|$ in the same way as in the first step, that is,

$$\begin{aligned}\|x_k - z_n(y_k)\| &\le \epsilon_k\|\check{x}_k - z_n(y_k)\| + (1-\epsilon_k)\|\tilde{x}_k - z_n(y_k)\| \\
&\le \epsilon_k\kappa\lambda\|z_n(y_k) - z_{n-1}(y_k)\| \\
&\quad +(1-\epsilon_k)(\|\tilde{x}_k - x\| + \|x - z_n(y)\| + \|z_n(y) - z_n(y_k)\|) \\
&\le \epsilon_k\kappa\lambda\|z_n(y_k) - z_{n-1}(y_k)\| + \|\tilde{x}_k - x\| \\
&\quad +(1-\epsilon_k)\alpha\lambda\|z_n(y) - z_{n-1}(y)\| + \|z_n(y) - z_n(y_k)\| \\
&\le \alpha\lambda\|z_n(y_k) - z_{n-1}(y_k)\| + \alpha\lambda\|z_n(y_k) - z_n(y)\| + \alpha\lambda\|z_{n-1}(y_k) - z_{n-1}(y)\| \\
&\quad +\|\tilde{x}_k - x\| + \|z_n(y) - z_n(y_k)\| - \epsilon_k\alpha\lambda\|z_n(y) - z_{n-1}(y)\| \\
&\quad +\epsilon_k\kappa\lambda\|z_n(y_k) - z_{n-1}(y_k)\| \\
&\le \alpha\lambda\|z_n(y_k) - z_{n-1}(y_k)\| \\
&\quad +\|\tilde{x}_k - x\| + (1+\alpha\lambda)\|z_n(y) - z_n(y_k)\| + \alpha\lambda\|z_{n-1}(y) - z_{n-1}(y_k)\| \\
&\quad -\epsilon_k(\alpha\lambda\|z_n(y) - z_{n-1}(y)\| - \kappa\lambda\|z_n(y_k) - z_{n-1}(y_k)\|) \\
&= \alpha\lambda\|z_n(y_k) - z_{n-1}(y_k)\|.\end{aligned}$$

We conclude that $x_k \in M_{n+1}(y_k)$ and since $x_k \to x$ as $k \to \infty$, the mapping M_{n+1} is lower semicontinuous in $\mathbb{B}_\tau(\bar{y})$. Hence, the mapping M_{n+1} has a continuous selection $z_{n+1}(\cdot) : \mathbb{B}_\tau(\bar{y}) \to X$, that is,

$$z_{n+1}(y) \in F^{-1}(y - G(z_n(y))) \quad \text{and} \quad \|z_{n+1}(y) - z_n(y)\| \le \alpha\lambda\|z_n(y) - z_{n-1}(y)\|.$$

Thus

$$\|z_{n+1}(y) - z_n(y)\| \le (\alpha\lambda)^{(n+1)}\|z_0(y) - \bar{x}\|.$$

The induction step is complete. We obtain an infinite sequence of bounded continuous functions $z_0, \cdots, z_n, \cdots$ such that for all $y \in \mathbb{B}_\tau(\bar{y})$ and for all n,

$$\|z_n(y) - \bar{x}\| \le \sum_{i=0}^{n}(\alpha\lambda)^i\|z_0(y) - \bar{x}\| \le \frac{\kappa}{1-\alpha\lambda}\|y - \bar{y}\| \le \gamma\|y - \bar{y}\|$$

and moreover,

$$\sup_{y \in B_\tau(\bar{y})}\|z_{n+1}(y) - z_n(y)\| \le (\alpha\lambda)^n \sup_{y \in B_\tau(\bar{y})}\|z_0(y) - \bar{x}\| \le (\alpha\lambda)^n\kappa\tau \quad \text{for } n \ge 1.$$

The sequence $\{z_n\}$ is a Cauchy sequence in the space of functions that are continuous and bounded on $\mathbb{B}_\tau(\bar{y})$ equipped with the supremum norm. Then this sequence has a limit $x(\cdot)$ which is a continuous function in $\mathbb{B}_\tau(\bar{y})$ and satisfies

$$x(y) \in F^{-1}(y - G(x(y))) \text{ and } \|x(y) - \bar{x}\| \leq \frac{\kappa}{1-\alpha\lambda}\|y - \bar{y}\| \leq \gamma\|y - \bar{y}\|$$

for all $y \in \mathbb{B}_\tau(\bar{y})$. Hence x is a continuous local selection of $(G+F)^{-1}$ and has the calmness property (7). □

Proof of Theorem 1.2. Apply Theorem 3.1 with $F = \nabla f(\bar{x})$ and $G(x) = f(x) - \nabla f(\bar{x})x$. Metric regularity of F is equivalent to the surjectivity of $\nabla f(\bar{x})$ and F^{-1} is convex and closed valued. The mapping G has $\operatorname{lip} G(\bar{x}) = 0$ and finally $F + G = f$. □

Note that Theorem 2.2 follows from Theorem 3.1 with G the zero function.

4. Applications

Theorems 3.1 can be also stated in a corresponding "implicit function" form as follows:

Theorem 4.1. *Let X, Y be Banach spaces and Z be a metric space. Consider a mapping $F : X \rightrightarrows Y$ and $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ which satisfies the conditions in Theorem 3.1. Consider also a function $G : X \times Z \to Y$ which satisfies $G(\bar{x}, \bar{p}) = 0$ for some $\bar{p} \in Z$ and $\operatorname{lip}_x G(\bar{x}, \bar{p}) \cdot \operatorname{reg} F(\bar{x}\,|\,\bar{y}) < 1$, and is continuous in a neighborhood of $(\bar{x}, \bar{p})$ (here the Lipschitz modulus of $G(x,p)$ is with respect to x where $\limsup$ is also with respect to $p \to \bar{p}$). Then there exist neighborhoods U of $\bar{x}$ and P of $\bar{p}$, a continuous function $x(\cdot) : P \to U$, and a constant γ such that*

$$\bar{y} \in G(x(p), p) + F(x(p)) \qquad \text{and} \qquad \|x(p) - \bar{x}\| \leq \gamma\|G(\bar{x}, p)\| \text{ for every } p \in P.$$

Sketch of proof. The proof is parallel to the proof of Theorem 3.1. First we choose κ, α and λ such that $\operatorname{reg} F(\bar{x}\,|\,\bar{y}) < \kappa < \alpha < 1/\lambda$ and $\lambda > \operatorname{lip}_x G(\bar{x}, \bar{p})$ and neighborhoods of $\bar{x}$, $\bar{y}$ and $\bar{p}$ that are associated with the metric regularity of F at $\bar{x}$ for $\bar{y}$ with constant κ and G is Lipschitz continuous with respect to x with constant λ uniformly in p. By appropriately choosing a sufficiently small radius τ of a ball around $\bar{p}$, we construct an infinite sequence of continuous and bounded functions $z_j : \mathbb{B}_\tau(\bar{p}) \to X$, $j = 0, 1, \cdots$, that is uniformly in $\mathbb{B}_\tau(\bar{p})$ convergent to a function $x(\cdot)$ satisfying the conclusion of the theorem. The first z_0 satisfies

$$z_0(p) \in F^{-1}(\bar{y} - G(\bar{x}, p)) \qquad \text{and} \qquad \|z_0(p) - \bar{x}\| \leq \kappa\|G(\bar{x}, p)\|.$$

For $j = 1, 2, \cdots$, the functions z_j is a continuous selection of the mapping

$$\begin{aligned}&\mathbb{B}_\tau(\bar{p}) \ni p \mapsto M_j(p)\\ &\qquad := \left\{ x \in F^{-1}(\bar{y} - G(z_{j-1}(p), p)) \mid \|x - z_{j-1}(p)\| \leq \alpha\lambda\|z_{j-1}(p) - z_{j-2}(p)\| \right\},\end{aligned}$$

where $z_{-1}(p) = G(\bar{x}, p)$. Then for all $p \in \mathbb{B}_\tau(\bar{p})$ we obtain

$$z_j(p) \in F^{-1}(\bar{y} - G(z_{j-1}(p), p)) \qquad \text{and} \qquad \|z_j(p) - z_{j-1}(p)\| \leq (\alpha\lambda)^j\|z_0(p) - G(\bar{x}, p)\|,$$

hence,

$$\|z_j(y) - \bar{x}\| \leq \frac{\kappa}{1-\alpha\lambda}\|G(\bar{x}, p)\|.$$

We obtain a Cauchy sequence of continuous and bounded function which is convergent with respect to the supremum norm. Passing to the limit with $j \to \infty$ we obtain a selection with the desired properties. □

If a mapping $F : X \rightrightarrows Y$ has convex and closed graph, then, by the Robinson-Ursescu theorem (Theorem 1.5), the metric regularity of F at $\bar{x}$ for $\bar{y}$ is equivalent to the condition $\bar{y} \in \operatorname{int}\operatorname{rge} F$. For such mapping we obtain the following corollary of Theorem 3.1:

Corollary 4.2. *Let $F : X \rightrightarrows Y$ have convex and closed graph, let $f : X \to Y$ be strictly differentiable at $\bar{x}$ and let $(\bar{x}, \bar{y}) \in \operatorname{gph}(f + F)$. Let the strict derivative $\nabla f(\bar{x})$ together with F satisfy the condition*

$$\bar{y} \in \operatorname{int}\operatorname{rge}(f(\bar{x}) + \nabla f(\bar{x})(\cdot - \bar{x}) + F(\cdot)). \tag{13}$$

Then there exist neighborhoods U of $\bar{x}$ and V of $\bar{y}$, a continuous function $x(\cdot) : V \to U$, and a constant γ such that

$$(f + F)(x(y)) \ni y \qquad \text{and} \qquad \|x(y) - \bar{x}\| \leq \gamma\|y - \bar{y}\| \text{ for every } y \in V.$$

An implicit function version of the above corollary easily follows from Theorem 4.1.

As a more specific application we consider the following controlled boundary value problem:

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = 0,\ x(1) = b, \tag{14}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a smooth function, the control $u(t) \in \mathcal{U}$ where $\mathcal{U}$ is convex and compact subset of $\mathbb{R}^m$. The pair (x, u) is a *feasible* solution of (14) when it satisfies the differential equation and $u(t) \in \mathcal{U}$ for almost every $t \in [0, 1]$, and also $x \in W_0^{1,\infty}([0, 1], \mathbb{R}^n)$, the space of all Lipschitz continuous functions x with values in $\mathbb{R}^n$ and with $x(0) = 0$, and $u \in L^\infty([0, 1], \mathbb{R}^m)$, the space of all essentially bounded and measurable functions with values in $\mathbb{R}^m$. We equip L^∞ with the essential supremum norm $\|u\|_\infty$ and $W_0^{1,\infty}$ with the norm $\|x\|_{1,\infty} = \|\dot{x}\|_\infty$. For simplicity, we assume that $f(0, 0) = 0$ and $0 \in \mathcal{U}$ and take $(0, 0)$ as the reference solution.

We apply Corollary 4.2 with the following specifications: $X = W_0^{1,\infty}([0, 1], \mathbb{R}^n) \times L^\infty([0, 1], \mathbb{R}^m)$ and $Y = L^\infty([0, 1], \mathbb{R}^n) \times \mathbb{R}^n$,

$$F(x, u) = \begin{cases} (Ax + Bu - \dot{x}, x(1)) & \text{for } u \in L^\infty, u(t) \in \mathcal{U} \text{ a.e.} \\ \emptyset & \text{otherwise,} \end{cases}$$

where $A = \nabla_x f(0, 0), B = \nabla_u f(0, 0)$, $G(x, u) = (f(x, u) - Ax - Bu, 0)$. Then $(G + F)(x, u) = (f(x, u) - \dot{x}, x(1))$ for $u \in L^\infty, u(t) \in \mathcal{U}$ a.e. Clearly, F has convex and closed graph. The condition (13) is equivalent to the following: there exists an $\epsilon > 0$ such that for any (y, b), $y \in L^\infty([0, 1], \mathbb{R}^n)$ and $b \in \mathbb{R}^n$ with $\|y\|_\infty + \|b\| < \epsilon$, there exists a feasible solution (x, u) of the linearized boundary value problem

$$\dot{x}(t) = Ax(t) + Bu(t) - y(t), \quad x(0) = 0, x(1) = b$$

The latter condition in turn is equivalent to the existence of a feasible solution of

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = 0,\, x(1) = b \tag{15}$$

for all b with sufficiently small norm. This property of the linear control system (15) is the so-called *null-controllability* and can be equivalently written as

$$0 \in \operatorname{int} \int_0^1 e^{At} B\mathcal{U} dt,$$

where the integral is in the sense of Aumann. If $0 \in \operatorname{int}\mathcal{U}$, the null-controllability is equivalent to the rank condition $\operatorname{rank}[B, AB, \cdots, A^{n-1}B] = n$.

Summarizing, if the linearization $\dot{x}(t) = Ax(t) + Bu(t)$ of (14) is null-controllable for L^∞ controls with values in $\mathcal{U}$, then there is a continuous function $b \mapsto (x(b), u(b))$ from a neighborhood V of zero in $I\!R^n$ to the product $W_0^{1,\infty}([0,1], I\!R^n) \times L^\infty([0,1], I\!R^m)$ such that for each $b \in V$, $(x(b), u(b))$ is a solution of the controlled boundary value problem (14) and moreover the function $(x(\cdot), u(\cdot))$ is calm at zero.

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This paper was largely inspired by Robert G. Bartle who passed away Sept. 18, 2002. He was able to see the previous paper [2] published and read a preliminary version of the present paper. Shortly before he died he sent the author a letter where he, among other things, wrote the following:

"Your results are inded, an impressive and far-reaching extension of the theorem that Professor Graves and I published over a half-century ago. I was a student in a class of Graves in which he presented the theorem in the case that the parameter domain is the interval $[0,1]$. He expressed the hope that it could be generalized to a more general domain, but said that he didn't see how to do so. By a stroke of luck, I had attended a seminar a few month before given by André Weil, which he titled "On a theorem of Stone." I (mis)understood that he was refereeing to M. H. Stone, rather than A. H. Stone, and attended. Fortunately, I listened carefully enough to learn about para compactness and continuous partitions of unity (which were totally new to me) and which I found to be useful in extending Graves' proof. So the original theorem was entirely due to Graves; I only provided an extension of his proof, using methods that were not known to him. However, despite the fact that I am merely a "middleman", I am pleased that this result has been found to be useful."

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