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Pointedness, Connectedness, and Convergence Results in the Space of Closed Convex Cones

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Let $\mathcal{K}(H)$ denote the collection of all closed convex cones in a finite dimensional real Hilbert space H. Painlevé-Kuratowski convergence in this space has been the subject of numerous publications. However, the link between convergence and pointedness has not been explored in its full extent. This note contributes to close this gap. Some results concerning the connected components of $\mathcal{K}(H)$ are also considered.

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1. Introduction

Throughout this paper, H denotes a finite dimensional real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and associated norm $\|\cdot\|$. Although a few results can be stated in an infinite dimensional setting, finite dimensionality of H is an essential assumption in most of this work. The notations B_H and S_H refer, respectively, to the closed unit ball and the unit sphere of H.

The basic ingredient of our discussion is the set

$$\mathcal{K}(H) := \{K \subset H : \ K \ \text{ is a nonempty closed convex cone}\},$$

whose elements can be separated into three different categories:

$$\mathcal{K}(H) = \mathcal{P}(H) \cup \mathcal{R}(H) \cup \mathcal{L}(H). \tag{1}$$

In the above partition,

$$\mathcal{L}(H) := \{K \in \mathcal{K}(H) : \ K = -K \ \}$$

denotes the class of all linear subspaces of H. Such objects can be treated by using the standard tools of linear algebra, and referring to them as cones is somehow an abuse of language. The complement

$$\mathcal{N}(H) := \mathcal{P}(H) \cup \mathcal{R}(H) = \{K \in \mathcal{K}(H) : \ K \neq -K \ \}$$

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is more interesting to deal with and, therefore, will retain most of our attention. The truly "genuine" cones are, however, only those belonging to

$$\mathcal{P}(H) := \{K \in \mathcal{K}(H) : \ K \text{ admits a convex compact shell}\}.$$

Recall that $S \subset H$ is said to be a shell (or base) of $K \in \mathcal{K}(H)$ if

$$0 \notin S \quad \text{and} \quad K = I\!\!R_+ S := \{\alpha x : \ \alpha \in I\!\!R_+, \ x \in S\}.$$

Since the space H is assumed to be finite dimensional, one has the characterization

$$K \in \mathcal{P}(H) \quad \Longleftrightarrow \quad K \text{ is nontrivial and pointed.}$$

Saying that K is nontrivial simply means that $K \neq \{0\}$. Pointedness of K refers to the usual condition $K \cap -K = \{0\}$. Nontrivial pointed cones abound in the literature, be it in connection with theoretical or practical issues. Most authors impose pointedness as a prerequisite for being called a cone, but we will not follow that practice. In fact, we want to take into account also the elements of

$$\mathcal{R}(H) := \{K \in \mathcal{K}(H) : \ K \text{ is not a linear subspace, but it contains at least a line}\}.$$

Members of this set have an ambivalent behavior: they share a bit of pointedness and a bit of linearity. Said it in another way, they are transition (or intermediate) elements between pointed and linear objects.

The partition of $\mathcal{K}(H)$ in the form (1) is dictated primarily by geometric considerations, but there is also a topological background justifying this way of proceeding.

2. The metric space $(\mathcal{K}(H), \delta)$

A natural way of measuring distances between elements of $\mathcal{K}(H)$ is through the expression

$$\delta(K_1, K_2) := \sup_{\|x\| \leq 1} |\operatorname{dist}[x; K_1] - \operatorname{dist}[x; K_2]|,$$

where

$$\operatorname{dist}[x; K] := \inf_{u \in K} \|x - u\|.$$

In this work, all topological properties in $\mathcal{K}(H)$ refer to the metric δ. One of the reasons why this metric is so popular is that convergence relative to δ coincides with convergence in the Painlevé-Kuratowski sense. As indicated in [7, Chapter 4],

$$\{K_n\}_{n \in \boldsymbol{N}} \text{ Painlevé-Kuratowski converges to } K \quad \Longleftrightarrow \quad \delta(K_n, K) \to 0. \tag{2}$$

The left-hand side in (2) simply means that

$$K = \liminf_{n\to\infty} K_n = \limsup_{n\to\infty} K_n,$$

where the lower and upper limits are the sets defined respectively by

$$\liminf_{n\to\infty} K_n := \{x \in H : \lim_{n\to\infty} \operatorname{dist}[x; K_n] = 0\},$$

$$\limsup_{n\to\infty} K_n := \{x \in H : \liminf_{n\to\infty} \operatorname{dist}[x; K_n] = 0\}.$$

There is yet another reason leading to the choice of the metric δ. According to Walkup and Wets [10], the polarity operation

$$K \mapsto K^+ := \{y \in H : \langle x, y\rangle \geq 0 \ \ \forall x \in K\}$$

is an isometry on the space $(\mathcal{K}(H), \delta)$, that is to say,

$$\delta(K_1^+, K_2^+) = \delta(K_1, K_2) \qquad \forall\, K_1, K_2 \in \mathcal{K}(H). \tag{3}$$

Equality (3) is extensively used in this work. We list below some basic properties of the space $(\mathcal{K}(H), \delta)$:

Proposition 2.1. *The metric space* $(\mathcal{K}(H), \delta)$ *is compact and disconnected.*

Proof. According to the celebrated theorem of Zarankiewicz (see, for instance, the books [1],[2] or [7]), any sequence of nonempty closed convex sets in a finite dimensional space admits a subsequence that converges in the Painlevé-Kuratowski sense. Zarankiewicz's theorem does not exclude the possibility of converging toward the empty set. However, this anomalous situation cannot occur if the sequence is formed by nonempty closed convex cones. The limit, if exists, is necessarily a nonempty closed convex cone. This proves the compactness of $(K(H), \delta)$. Let $\theta_H := \{0\}$ denote the trivial cone in H. One can easily show that the singleton $\{\theta_H\}$ is open and closed at the same time. A set with such properties could not exist in $\mathcal{K}(H)$ if this space was connected. □

As mentioned before, we are interested in examining the topological properties of some special subsets of $\mathcal{K}(H)$. The following result is a first step in this direction.

Proposition 2.2. *The set* $\mathcal{P}(H)$ *is open in* $(\mathcal{K}(H), \delta)$.

Proof. We prove that the set $\mathcal{M}(H) := \{K \in \mathcal{K}(H) : \ K \cap -K \neq \{0\}\}$ is closed. Consider a sequence $\{K_n\}_{n\in N}$ in $\mathcal{M}(H)$ converging to $K \in \mathcal{K}(H)$. We must show that K remains in $\mathcal{M}(H)$. Pick up unit vectors $x_n \in K_n$ such that $-x_n \in K_n$. Taking a subsequence if necessary, one may suppose that $\{x_n\} \subset S_H$ converges to some $x \in S_H$. In such a case, $\{-x_n\}$ converges to $-x$. The conclusion is that the limit set K contains both x and $-x$. This proves that $K \in \mathcal{M}(H)$. □

Painlevé-Kuratowski convergence of pointed cones does not produce necessarily a pointed cone. In other words, the set $\mathcal{P}(H)$ is not closed in $(\mathcal{K}(H), \delta)$. Characterizing the topological closure of $\mathcal{P}(H)$ is a delicate matter that is treated next. First, we need to lay down some additional ground material.

3. Seeger-Torki deformation map

For the sake of the exposition, we introduce a map $\Phi : [0, 1] \times \mathcal{N}(H) \to \mathcal{N}(H)$ such that

$$\Phi(\varepsilon, K) \approx K \ \text{ whenever } \ \varepsilon \approx 0. \tag{4}$$

The precise meaning of (4) is explained in a moment. The construction of Φ may seem a little bit obscure, but everything is properly explained in reference [9]. To start with, define

$$K^{\oplus} := \{y \in \mathrm{span}K : \langle x, y\rangle \geq 0 \quad \forall x \in K\},$$

where $\mathrm{span}K := K - K$ denotes the linear hull of K, that is to say, the smallest linear subspace containing K. We use the symbol $\oplus$ to indicate that the polarity operation is relative to the linear hull. Proceed now to enlarge $K^{\oplus}$ by means of a convex conic ε-neighborhood of the form

$$N_\varepsilon(K^{\oplus}) := \{y \in \mathrm{span}K : \mathrm{dist}[y; K^{\oplus}] \leq \varepsilon \langle \pi_K, y\rangle\},$$

with π_K being an arbitrary unit vector belonging to the relative interior of K. By taking the relative polar again, one arrives finally at

$$\Phi(\varepsilon, K) := [N_\varepsilon(K^{\oplus})]^{\oplus} := \{x \in \mathrm{span}K : \langle x, y\rangle \geq 0 \quad \forall y \in N_\varepsilon(K^{\oplus})\}. \tag{5}$$

The basic properties of the map Φ are listed below:

Proposition 3.1. *Let $\Phi : [0,1] \times \mathcal{N}(H) \to \mathcal{N}(H)$ be defined by (5). For each $K \in \mathcal{N}(H)$, one has:*

(*a*) $\Phi(0, K) = K$;
(*b*) $\forall\, \varepsilon > 0, \quad \Phi(\varepsilon, K) \in \mathcal{P}(H)$;
(*c*) $0 \leq \varepsilon_1 \leq \varepsilon_2 \leq 1 \implies \Phi(\varepsilon_1, K) \supset \Phi(\varepsilon_2, K)$;
(*d*) $\varepsilon \in [0,1] \mapsto \delta(\Phi(\varepsilon, K), K)$ *is nondecreasing;*
(*e*) $\forall\, \varepsilon > 0,\ \delta(\Phi(\varepsilon, K), K) \leq \varepsilon$.

Proof. Observe that $N_0(K^{\oplus}) = K^+ \cap \mathrm{span}K$, and therefore

$$\Phi(0, K) = [K^+ \cap \mathrm{span}K]^+ \cap\ \mathrm{span}K = [K + (\mathrm{span}K)^{\perp}] \cap\ \mathrm{span}K\ = K,$$

where the symbol $\perp$ stands for orthogonal complementation. Statements (*b*) and (*e*) are proven in [9]. The monotonicity condition (*c*) is obvious. To prove (*d*), take $0 \leq \varepsilon_1 \leq \varepsilon_2 \leq 1$. By (*a*) and (*c*), one has

$$\mathrm{dist}[x; \Phi(\varepsilon_2, K)] - \mathrm{dist}[x; K] \geq \mathrm{dist}[x; \Phi(\varepsilon_1, K)] - \mathrm{dist}[x; K] \geq 0 \qquad \forall\, x \in H.$$

The very definition of the metric δ yields then the desired conclusion. □

Remark. Proposition 3.1(*e*) indicates that $\Phi(\varepsilon, K)$ converges to K as the approximation parameter ε goes to 0. The notation $\Phi(\varepsilon, K)$ is slightly misleading because it does not reflect the dependence of this set on the vector π_K. In fact, each choice of π_K produces a different deformation map. Perhaps there is an "optimal" way of selecting a unit vector π_K in the relative interior of K, but such a discussion would bring us to far away from our main concern.

4. A general approximation result

Everything is now ready to state a fundamental result establishing that

$$\left\{ \begin{array}{l} \text{a cone in } \mathcal{N}(H) \text{ can always be represented as} \\ \text{limit of a sequence of nontrivial pointed cones.} \end{array} \right.$$

Indeed, with the help of the deformation map Φ, one derives straightforwardly:

Theorem 4.1. *For $K \in \mathcal{K}(H)$, the following implication holds:*

$$K \text{ is not a linear subspace} \Longrightarrow \begin{cases} \forall\, \varepsilon > 0,\ \exists \text{ a nontrivial pointed cone} \\ \quad P \in \mathcal{K}(H) \text{ such that } \delta(P,K) \le \varepsilon. \end{cases} \tag{6}$$

Proof. Given $K \in \mathcal{N}(H)$ and $\varepsilon > 0$, one can choose $P = \Phi(\varepsilon, K)$. □

Our geometric intuition tells us that the reverse of implication (6) is also true. It would be strange having a sequence of nontrivial pointed cones converging to a linear subspace. Relying on geometric intuition is however a very tricky business. Fortunately, everything can be settled properly with the help of the following technical result. The notation $\Pi_K(z)$ refers to the nearest point to z in K.

Lemma 4.2. *Consider an arbitrary $K \in \mathcal{K}(H)$. Then,*

$$K \cap K^+ = \{x + \Pi_K(-x) \ :\ x \in K\}. \tag{7}$$

In particular,

$$K \text{ is a linear subspace} \quad \Longleftrightarrow \quad K \cap K^+ = \{0\}. \tag{8}$$

Proof. Formula (7) clearly implies the equivalence (8). We begin by proving the inclusion

$$M_K := \{x + \Pi_K(-x) :\ x \in K\} \ \subset\ K \cap K^+.$$

Take any $x \in K$. Since the projection $\Pi_K(-x)$ of $-x$ onto K satisfies the variational inequality

$$\langle -x - \Pi_K(-x), u - \Pi_K(-x)\rangle \ \le 0 \qquad \forall\, u \in K,$$

it follows that

$$\langle x + \Pi_K(-x), u\rangle \ \ge \langle x + \Pi_K(-x), \Pi_K(-x)\rangle \qquad \forall\, u \in K.$$

This breaks down into

$$\langle x + \Pi_K(-x), \Pi_K(-x)\rangle \ = 0 \quad \text{and} \quad \langle x + \Pi_K(-x), u\rangle \ \ge 0 \quad \forall\, u \in K.$$

Hence, $x + \Pi_K(-x)$ not only belong to K, but also to K^+. In short, M_K is contained in $K \cap K^+$. On the other hand, $\Pi_K(-x) = 0$ for every $x \in K^+$. Hence,

$$K \cap K^+ = \{x + \Pi_K(-x) :\ x \in K \cap K^+\} \ \subset M_K.$$

This proves the announced formula. □

Remark. Formula (7) seems to be new, but the equivalence (8) can be found in an old paper by Gaddum [3, Theorem 2.1]. The above lemma is true even if the space H is not finite dimensional. By using a duality argument, one sees that K is a linear subspace if and only if $K + K^+$ is dense in H.

We now are ready to state the converse of Theorem 4.1:

Corollary 4.3. *For $K \in \mathcal{K}(H)$, the following implication holds:*

$$\left.\begin{array}{r}\forall\, \varepsilon > 0,\ \exists \text{ a nontrivial pointed cone} \\ P \in \mathcal{K}(H) \text{ such that } \delta(P,K) \le \varepsilon\end{array}\right\} \Longrightarrow K \text{ is not a linear subspace.}$$

Proof. Obviously, $\mathcal{P}(H) \subset \mathcal{N}(H)$. By taking the topological closure with respect to the metric δ, one arrives at

$$\mathrm{cl}_\delta[\mathcal{P}(H)] \subset \mathrm{cl}_\delta[\mathcal{N}(H)].$$

So, what we need to prove is that $\mathcal{N}(H)$ is a closed set in $(\mathcal{K}(H), \delta)$. Recall that, by Lemma 4.2, one has the characterization

$$\mathcal{N}(H) = \{K \in \mathcal{K}(H) :\ K \cap K^+ \neq \{0\}\}.$$

Consider a sequence $\{K_n\}_{n\in \mathbb{N}}$ in $\mathcal{N}(H)$ converging to $K \in \mathcal{K}(H)$. Observe, incidentally, that the Walkup-Wets isometry theorem guarantees the convergence of $\{K_n^+\}_{n\in\mathbb{N}}$ toward K^+. Pick up unit vectors x_n such that

$$x_n \in K_n \cap K_n^+ \qquad \forall\, n \in \mathbb{N}.$$

Taking a subsequence if necessary, one may suppose that $\{x_n\}$ converges to some $x \in S_H$. In such a case,

$$x \in \limsup_{n\to\infty} \{K_n \cap K_n^+\} \ \subset K \cap K^+.$$

This proves that the limit K remains in $\mathcal{N}(H)$. □

The combination of Theorem 4.1 and Corollary 4.3 can be stated in a more compact manner, namely

$$\mathcal{N}(H) = \mathrm{cl}_\delta[\mathcal{P}(H)]. \tag{9}$$

This answers the question relative to the topological closure of $\mathcal{P}(H)$. There are many other conclusions that can be drawn from the above discussion. By way of example, we mention:

Corollary 4.4. *The set $\mathcal{R}(H)$ is closed in $(\mathcal{K}(H), \delta)$. In fact, it corresponds to the topological boundary of $\mathcal{P}(H)$.*

Proof. The boundary of $\mathcal{P}(H)$ is defined as the closed set

$$\mathrm{bd}_\delta[\mathcal{P}(H)] \ := \ \mathrm{cl}_\delta[\mathcal{P}(H)] \setminus \mathrm{int}_\delta[\mathcal{P}(H)].$$

In the present context, this expression reduces to $\mathrm{bd}_\delta[\mathcal{P}(H)] = \mathcal{N}(H) \setminus \mathcal{P}(H) = \mathcal{R}(H)$. □

5. On solid cones

We record here a couple of additional topological results which are obtained almost freely with the help of the Walkup-Wets isometry theorem. By applying the polarity mapping to the class of nontrivial pointed cones one recovers

$$\mathcal{G}(H) := \{K \in \mathcal{K}(H) :\ K \text{ has a nonempty interior and } K \neq H\},$$

the class of solid cones strictly contained in H. In fact, $\mathcal{P}(H)$ and $\mathcal{G}(H)$ are mutually polar in the sense that

$$\mathcal{G}(H) = \{K^+ : \ K \in \mathcal{P}(H)\} \quad \text{and} \quad \mathcal{P}(H) = \{K^+ : \ K \in \mathcal{G}(H)\}. \tag{10}$$

The approximation result via pointed cones admits a dual counterpart in which solid cones play the leading role. Without further ado, we state:

Corollary 5.1. *For $K \in \mathcal{K}(H)$, one has:*

$$K \text{ is not a linear subspace} \iff \forall\, \varepsilon > 0,\ \exists\, Q \in \mathcal{G}(H) \text{ such that } \delta(Q, K) \leq \varepsilon.$$

Proof. Obviously, $K \in \mathcal{N}(H)$, if and only if, $K^+ \in \mathcal{N}(H)$. It suffices to apply Theorem 4.1 and Corollary 4.3 to the cone K^+, and then invoke the isometry formula (3). The cone Q serving to approximate K can be constructed with the help of Seeger-Torki deformation map. It is given simply by $Q = [\Phi(\varepsilon, K^+)]^+$. □

Corollary 5.1 can be rephrased by saying that $\mathcal{N}(H) = \mathrm{cl}_\delta[\mathcal{G}(H)]$. We now have two procedures for approximating an element $K \in \mathcal{N}(H)$: from below by means of pointed cones, and from above by means of solid cones. In short,

$$\Phi(\varepsilon, K)] \subset K \subset [\Phi(\varepsilon, K^+)]^+ \qquad \forall\, \varepsilon > 0.$$

Further properties of the set $\mathcal{G}(H)$ are listed in the next corollary:

Corollary 5.2. *The set $\mathcal{G}(H)$ is open in $(\mathcal{K}(H), \delta)$. Its topological boundary is given by*

$$\mathrm{bd}_\delta[\mathcal{G}(H)] = \mathcal{U}(H) := \{K \in \mathcal{K}(H) : K \text{ has empty interior, but it is not a linear subspace}\}.$$

Proof. The polarity mapping $[\ \cdot\ ]^+ : (\mathcal{K}(H), \delta) \to (\mathcal{K}(H), \delta)$ is an homeomorphism. Openness of $\mathcal{G}(H)$ follows from (10) and Proposition 2.2. Observe that $\mathcal{U}(H)$ corresponds to the image of $\mathcal{R}(H)$ under the polarity mapping. It suffices then to invoke Corollary 4.4. □

6. Dimensional issues

Let us start with a preliminary result dealing with Painlevé-Kuratowski convergence of linear subspaces.

Proposition 6.1. *The set $\mathcal{L}(H)$ is open and closed in $(\mathcal{K}(H), \delta)$.*

Proof. One can easily show that the Painlevé-Kuratowski limit of a sequence of linear subspaces must be a linear subspace. This takes care of closedness. Openness of $\mathcal{L}(H)$ has been implicitly proved in Corollary 4.3. □

The above proposition is very rough in the sense that it says nothing on the dimension of the linear subspaces that are involved. Let us proceed then to partition $\mathcal{L}(H)$ in the form

$$\mathcal{L}(H) = \bigcup_{d=0}^{\dim H} \mathcal{L}_d(H), \tag{11}$$

where

$$\mathcal{L}_d(H) := \{K \in \mathcal{K}(H) : \ K \text{ is a } d\text{-dimensional linear subspace}\},$$

with the usual convention $\dim\{0\} = 0$. The two propositions stated below address the dimensional aspect of Painlevé-Kuratowski convergence of linear subspaces.

Proposition 6.2. *Each set $\mathcal{L}_d(H)$ is closed in $(\mathcal{K}(H), \delta)$.*

Proof. The case $d = 0$ is trivial, so take $1 \le d \le \dim H$. Consider a sequence $\{K_n\}_{n\in \mathbb{N}}$ in $\mathcal{L}_d(H)$ converging to $K \in \mathcal{K}(H)$. By Proposition 6.1, one knows that K is a linear subspace. We will prove that the dimension of K is exactly d. For each space K_n, construct a basis $\{a_n^1, \cdots, a_n^d\}$ formed by unit vectors such that

$$\langle a_n^i, a_n^j\rangle = 0 \qquad \forall\, i \neq j.$$

By compactness of the unit sphere, one may suppose that each sequence $\{a_n^i\}_{n\in\mathbb{N}}$ converges to some $a^i \in S_H$. In this way, one obtains a collection $\{a^1, \cdots, a^d\}$ of mutually orthogonal unit vectors lying in K. Thus, $\dim K \ge d$. On the other hand, the orthogonal spaces $\{K_n^\perp\}_{n\in\mathbb{N}}$ converge to $K^\perp$, and

$$\dim K_n^\perp = \dim H - d \qquad \forall n \in \mathbb{N}.$$

Hence, $\dim K^\perp \ge \dim H - d$. This yields $\dim K = \dim H - \dim K^\perp \le d$, and completes the proof. □

Proposition 6.3. *Each set $\mathcal{L}_d(H)$ is open in $(\mathcal{K}(H), \delta)$.*

Proof. The complement of $\mathcal{L}_d(H)$ is given by

$$\mathcal{K}(H)\backslash\mathcal{L}_d(H) \ = \mathcal{N}(H) \ \cup \ [\bigcup_{k\neq d} \mathcal{L}_k(H)],$$

that is to say, it is a finite union of closed sets. □

The dimensional analysis of Painlevé-Kuratowski limits can be carried out in the more general context of the space $\mathcal{K}(H)$. For an arbitrary cone $K \in \mathcal{K}(H)$, one must distinguish between the "inner" dimension

$$\underline{\dim} K := \dim[K \cap -K]$$

and the "outer" dimension

$$\overline{\dim} K := \dim[K - K].$$

Of course, for linear subspaces both integers coincide with the usual dimension. Pointedness, solidity, linearity, and many other concepts, can be expressed in terms of the functions

$$\underline{\dim} \ : \mathcal{K}(H) \to \mathbb{N} \quad \text{and} \quad \overline{\dim} \ : \mathcal{K}(H) \to \mathbb{N}.$$

The first examples that come to mind are

$$K \text{ is pointed} \iff \underline{\dim} K = 0,$$

$$K \text{ is solid} \iff \overline{\dim} K = \dim H.$$

One can think also of more elaborate concepts in which intermediate dimensions play an important role. For instance, the sublevel set

$$\{\overline{\dim} \ \leq d\} := \{K \in \mathcal{K}(H) : \ \overline{\dim} K \leq d\}$$

is formed by the cones whose spans are at most d-dimensional. Sets of the form $\{\overline{\dim} \ \geq d\}$, $\{\overline{\dim} \ = d\}$, $\{\underline{\dim} \ \leq d\}$, $\cdots$, are defined and interpreted in a corresponding way.

As indicated in the next lemma, inner and outer dimensions are related through polarity.

Lemma 6.4. *Let $P, Q \in \mathcal{K}(H)$ be polar to each other. Then,*

$$\underline{dim} P + \overline{dim} Q = dim H. \tag{12}$$

Proof. Formula (12) can be derived from a more general result (see [8, Theorem 1]). □

The following result generalizes Proposition 2.2 and the first part of Corollary 5.2. Not only the extreme dimensions $d = 0$ and $d = \dim H$ are being taken into account, but also the intermediate ones.

Proposition 6.5. *Let $d \in \{0, 1, \cdots, dim H\}$. Then,*

(a) *the sets $\{\underline{dim} \ \geq d\}$ and $\{\overline{dim} \ \leq d\}$ are closed;*
(b) *the sets $\{\underline{dim} \ \leq d\}$ and $\{\overline{dim} \ \geq d\}$ are open.*

Proof. Consider a sequence $\{K_n\}_{n \in \boldsymbol{N}}$ in $\{\underline{\dim} \geq d\}$ converging to $K \in \mathcal{K}(H)$. In each linear subspace $K_n \cap -K_n$, one can find a collection $\{a_n^1, \cdots, a_n^d\}$ of d orthonormal vectors. By passing to the limit as in Proposition 6.2, one produces d orthonormal vectors lying in $K \cap -K$. Thus, $\underline{\dim} K \geq d$. In short, $\{\underline{\dim} \geq d\}$ is a closed set. Openness of $\{\underline{\dim} \leq d\}$ is obtained by complementation:

$$\{\underline{\dim} \ \leq d\} = \begin{cases} \mathcal{K}(H) \backslash \{\underline{\dim} \ \geq d+1\} & \text{if } d \neq \dim H, \\ \mathcal{K}(H) & \text{if } d = \dim H. \end{cases}$$

Closedness of $\{\overline{\dim} \ \leq d\}$ follows from the polarity formula established in Lemma 6.4. Indeed,

$$\begin{aligned} \{\overline{\dim} \ \leq d\} &= \{K \in \mathcal{K}(H) : \ \underline{\dim} K^+ \geq \dim H - d\} \\ &= \{K^+ : \ \underline{\dim} K \geq \dim H - d\}. \end{aligned}$$

In other words, $\{\overline{\dim} \ \leq d\}$ is the image of a closed set under the polarity map. Openness of $\{\overline{\dim} \ \geq d\}$ is obtained by complementation. □

Painlevé-Kuratowski convergence can't neither decrease the inner dimension, nor increase the outer dimension. More precisely:

Corollary 6.6. *Let $\{K_n\}_{n \in \boldsymbol{N}}$ be a sequence in $\mathcal{K}(H)$ converging to $K \in \mathcal{K}(H)$. Then,*

$$\forall\, n \ \textit{large enough}, \qquad \underline{dim} K_n \leq \underline{dim} K \quad \textit{and} \quad \overline{dim} K \leq \overline{dim} K_n. \tag{13}$$

Proof. It is immediate from Proposition 6.5. □

Both inequalities in (13) can be strict. In contrast with the $\mathcal{L}_d(H)$'s,

$$\begin{cases} \text{for intermediate dimensions } d = 1, \cdots, \dim H - 1, \text{ the level} \\ \text{sets } \{\underline{\dim} = d\} \text{ and } \{\overline{\dim} = d\} \text{ are neither open nor closed.} \end{cases} \tag{14}$$

This can be checked by constructing suitable counter-examples.

7. Connectedness

As seen in Propositions 6.1-6.3, closedness and openness are topological properties shared by $\mathcal{L}(H)$ and the different $\mathcal{L}_d(H)$'s. As far as connectedness is concerned, only the later sets enjoy this property. Before we record this fact in the next proposition, recall that:

Lemma 7.1. *For a map* $\Gamma : [0,1] \to \mathcal{K}(H)$, *the following two conditions are equivalent:*

(a) Γ *is continuous;*

(b) $\forall\, z \in H$, *the real-valued function* $t \in [0,1] \mapsto dist[z;\Gamma(t)]$ *is continuous.*

The above lemma is known and can be stated in a more general context. For instance, the interval $[0,1]$ can be replaced by an arbitrary metric space. Without further ado, we write:

Proposition 7.2. *Each set* $\mathcal{L}_d(H)$ *is path-connected (in particular, connected).*

Proof. The cases $d = 0$ and $d = \dim H$ are trivial, so take $1 \le d \le \dim H - 1$. Let P and Q be two different d-dimensional linear subspaces. We must construct a continuous function $\Gamma : [0,1] \to \mathcal{K}(H)$ such that

$$\Gamma(0) = P, \quad \Gamma(1) = Q, \quad \text{and} \quad \Gamma(t) \in \mathcal{L}_d(H) \quad \forall\, t \in [0,1].$$

Denote by r the dimension of the linear subspace $P \cap Q$. Let $\{u_1, \cdots, u_d\}$ and $\{v_1, \cdots, v_d\}$ be orthonormal bases of P and Q respectively. Without loss of generality, one may suppose that

$$\{u_1, \cdots, u_r\} \text{ is a basis of } P \cap Q \qquad \text{and} \qquad u_j = v_j \quad \forall\, j \in \{1, \cdots, r\}.$$

For each $t \in [0,1]$, let $\Gamma(t)$ denote the linear subspace spanned by $\{w_1(t), \cdots, w_d(t)\}$, with

$$w_j(t) := (1-t)u_j + t v_j \qquad \forall\, j \in \{1, \cdots, d\}.$$

Clearly, $\Gamma(0) = P$ and $\Gamma(1) = Q$. To make sure that each $\Gamma(t)$ is d-dimensional, one needs to check that the vectors $\{w_1(t), \cdots, w_d(t)\}$ are linearly independent. Take arbitrary coefficients $\alpha_1, \cdots, \alpha_d$ such that

$$\sum_{j=1}^{d} \alpha_j w_j(t) = 0. \tag{15}$$

In such a case,

$$(1-t)\tilde{u} + t\tilde{v} = 0, \quad \text{with} \quad \tilde{u} := \sum_{j=1}^{d} \alpha_j u_j \quad \text{and} \quad \tilde{v} := \sum_{j=1}^{d} \alpha_j v_j.$$

Hence, $\tilde{u} \in P \cap Q$. It follows that

$$\hat{u} := \sum_{j=r+1}^{d} \alpha_j u_j = \tilde{u} - \sum_{j=1}^{r} \alpha_j u_j$$

is in $P \cap Q$. But, by construction, $\hat{u}$ is in the orthogonal of $P \cap Q$. The conclusion is that

$$\alpha_j = 0 \qquad \forall\, j \in \{r+1, \cdots, d\}.$$

Plugging this information in (15), one deduces that also the first coefficients $\alpha_1, \cdots, \alpha_r$ are equal to zero. This takes care of the linear independence requirement. Finally, it remains to check that the path Γ is continuous. For any $z \in H$, one has

$$\mathrm{dist}[z; \Gamma(t)] = \inf_{\alpha \in \boldsymbol{R}^d} \|z - W(t)\alpha\| \;=\; \left\|z - W(t)[W^*(t)W(t)]^{-1}W^*(t)z\right\|,$$

where $W^*(t)$ denotes the transpose of the matrix $W(t) := [w_1(t) \cdots w_d(t)]$. Lemma 7.1 and the continuity of $t \in [0,1] \mapsto \mathrm{dist}[z; \Gamma(t)]$ yield the desired conclusion. □

The next result is in the same vein as Proposition 7.2, but its proof is more elaborate.

Proposition 7.3. *The set $\mathcal{N}(H)$ is path-connected.*

Proof. For the sake of the exposition, it is convenient to split the proof in two parts:

Step 1. Consider an arbitrary $K \in \mathcal{N}(H)$. We will show that

$$t \in [0,1] \mapsto \Gamma(t) := \{(1-t)v + t\,\|v\|\, w_K : \; v \in K\} \tag{16}$$

is continuous path lying in $\mathcal{N}(H)$, and joining K with the ray

$$\mathbb{R}_+ w_K := \{\mu\, w_K : \; \mu \in \mathbb{R}_+\}.$$

Here w_K is any unit vector selected from $K \cap K^+$. The existence of such a vector is guaranteed by Lemma 4.2. That Γ is continuous can be proven with the help of Lemma 7.1. For any fixed $z \in H$, the infimal-value function

$$t \in [0,1] \mapsto \mathrm{dist}[z; \Gamma(t)] = \inf_{v \in K} \left\|(1-t)v + t\,\|v\|\, w_K - z\right\|$$

is clearly upper-semicontinuous. Lower-semicontinuity is a more delicate matter, but one can invoke the standard tools of parametric optimization (apply, for instance, [5, Theorem 5.1]). The most interesting part of the proof is checking that

$$\Gamma(t) \in \mathcal{N}(H) \qquad \forall\, t \in]0,1[.$$

That $\Gamma(t)$ is a cone is fairly clear. Closedness of $\Gamma(t)$ is also easy to prove. Indeed, if one considers a sequence $\{x_n\}_{n\in\boldsymbol{N}} \to x$ of the form

$$x_n := (1-t)v_n + t\,\|v_n\|\, w_K \qquad \text{with} \quad v_n \in K, \tag{17}$$

then $\{v_n\}_{n\in\boldsymbol{N}}$ must be bounded. Taking a subsequence if necessary, and passing to the limit in (17), one obtains

$$x = (1-t)v + t\,\|v\|\, w_K \qquad \text{with} \quad v \in K.$$

To prove convexity of $\Gamma(t)$, consider the points

$$x = (1-t)v + t\,\|v\|\,w_K \qquad \text{and} \qquad y = (1-t)u + t\,\|u\|\,w_K,$$

with $u, v \in K$. Pick up any $a \in]0,1[$ and form the convex combination

$$q = (1-a)x + ay = (1-t)\{au + (1-a)v\} + t\{a\,\|u\| + (1-a)\,\|v\|\}w_K.$$

We need to show that $q \in \Gamma(t)$. To do this, write

$$q = (1-t)\{au + (1-a)v + bw_K\} + t\{a\,\|u\| + (1-a)\,\|v\| - (t^{-1}-1)b\}w_K,$$

where $b \in I\!R_+$ is selected in such a way that

$$\|au + (1-a)v + bw_K\| = a\,\|u\| + (1-a)\,\|v\| - (t^{-1}-1)b.$$

The existence of such a positive number b can be proven by using the following argument. If one defines

$$f(b) := \|au + (1-a)v + bw_K\| - a\,\|u\| - (1-a)\,\|v\| + (t^{-1}-1)b,$$

then f is a continuous real-valued function verifying

$$f(0) \le 0 \qquad \text{and} \quad f(b) \to \infty \;\text{ as }\; b \to \infty.$$

Thus, there is necessarily some $b \in I\!R_+$ for which $f(b) = 0$. Finally, we must check that $\Gamma(t)$ is not a linear subspace. As a matter of fact, one can prove a stronger result, namely

$$\Gamma(t) \text{ is a nontrivial pointed cone!}$$

Obviously $\Gamma(t)$ is nontrivial because it contains the vector w_K. Take $x \in \Gamma(t) \cap -\Gamma(t)$, that is to say,

$$x = (1-t)v + t\,\|v\|\,w_K \qquad \text{and} \qquad -x = (1-t)u + t\,\|u\|\,w_K, \tag{18}$$

with $v, u \in K$. By summing up both expressions in (18) and rearranging, one obtains

$$(1-t)(v+u) = -t(\|v\| + \|u\|)w_K.$$

Hence,

$$(1-t)\langle v+u, w_K\rangle = -t(\|v\| + \|u\|). \tag{19}$$

Observe that $\langle v+u, w_K\rangle \ge 0$ because $v+u \in K$ and $w_K \in K^+$. So, the equality (19) yields $v = u = 0$. Thus $x = 0$, showing in this way that $\Gamma(t)$ is pointed.

Step 2. Let P and Q be two different members of $\mathcal{N}(H)$. Pick up unit vectors w_P and w_Q from $P \cap P^+$ and $Q \cap Q^+$ respectively. As in Step 1, construct a continuous path joining P with the ray $I\!R_+ w_P$. Next, construct a continuous path $\gamma : [0,1] \to H$ joining w_P and w_Q. Make sure that γ does not pass through the origin, that is to say, $0 \notin \{\gamma(t) : t \in [0,1]\}$. In such a case,

$$t \in [0,1] \mapsto I\!R_+ \gamma(t)$$

allows us to move continuously from $I\!R_+ w_P$ to $I\!R_+ w_Q$, while remaining in $\mathcal{P}(H)$. Finally, we proceed again as in Step 1, but this time reversing the order of the movement, that is to say, we go from $I\!R_+ w_Q$ to Q. $\square$

Corollary 7.4. *The sets $\mathcal{P}(H)$ and $\mathcal{G}(H)$ are path-connected.*

Proof. Two different members P and Q of $\mathcal{P}(H)$ can always be joined by a continuous path lying in $\mathcal{P}(H)$. To prove this, one follows exactly the same steps as in Proposition 7.3. Path-connectedness of $\mathcal{G}(H)$ follows by homeomorphism. □

The proof technique of Proposition 7.3 is based on the construction of a continuous path connecting an element of $\mathcal{N}(H)$ to a certain ray. A similar methodology can be used to prove the connectedness of $\mathcal{R}(H)$. The key observation now is that an element of $\mathcal{R}(H)$ can be joined continuously to a set of the form

$$M(a,w) := \{\lambda a + \mu w : \lambda \in \mathbb{R}, \mu \in \mathbb{R}_+\}.$$

To prove this fact, we rely on the following decomposition lemma:

Lemma 7.5. *Consider an arbitrary $K \in \mathcal{K}(H)$. Suppose that the span of $Q \in \mathcal{K}(H)$ is contained in K (equivalently, both Q and $-Q$ are contained in K). Then, K can be decomposed in the form*

$$K = Q + K \cap Q^-,$$

where $Q^- := -Q^+$ stands for the negative polar cone of Q. In particular, if L is a (closed) linear subspace contained in K, then

$$K = L + K \cap L^\perp.$$

Proof. Since $Q \subset K$ and $K \cap Q^- \subset K$, it follows that $Q + K \cap Q^- \subset K$. In order to prove the reverse inclusion, take any $x \in K$. According to the Moreau orthogonal decomposition theorem [6], x can be written in the form

$$x = \Pi_Q(x) + \Pi_{Q^-}(x)\ .$$

Obviously, $\Pi_Q(x) \in Q$. The point $\Pi_{Q^-}(x) = x - \Pi_Q(x)$ not only belongs to Q^-, but also to $K - Q = K$. □

Remark. Lemma 7.5 remains true even if the underlying space H is not finite dimensional. The second part of the lemma appears in the book by Jacobson [4, Lemma 5.3.2], and has an interesting application in control theory.

We now are ready to state:

Proposition 7.6. *$\mathcal{R}(H)$ is path-connected.*

Proof. Let $K \in \mathcal{R}(H)$. Take unit vectors $a \in K \cap -K$ and $w \in K \cap K^+$. Note that a exists because K is not pointed, and w because K is not a subspace. We shall construct a continuous path joining K to the two-dimensional half-space $M(a,w)$. Observe that $L := \mathbb{R}a$ is contained in K. Define $\Gamma : [0,1] \to \mathcal{K}(H)$ by means of the expression

$$\Gamma(t) := \{\lambda a + (1-t)v + t\,\|v\|\,w : v \in K \cap L^\perp, \lambda \in \mathbb{R}\}.$$

Checking the continuity of Γ is a routine work. Since $w \in K^+$ and $a \in K \cap -K$, it follows that $\langle w, a\rangle = 0$, and therefore $w \in K \cap L^\perp$. By Lemma 7.4, $K \cap L^\perp$ is not a linear subspace. Moreover,

$$\Gamma(0) = K \quad \text{and} \quad \Gamma(1) = M(a,w)\ .$$

We claim that $\Gamma(t)$ belongs to $\mathcal{R}(H)$ for all $t \in [0,1]$. First we establish that $\Gamma(t)$ is a closed convex cone. The fact that

$$\tilde{\Gamma}(t) := \{(1-t)v + t\,\|v\|\,w : v \in K \cap L^{\perp}\}$$

is a closed convex cone has been already established in the proof of Proposition 7.3, and then the same holds for

$$\Gamma(t) = L + \tilde{\Gamma}(t)\;.$$

Obviously, $\Gamma(t)$ is not pointed because it contains the line L. Taking $\lambda = 0$ and $v = w$ in the definition of $\Gamma(t)$, one sees that $w \in \Gamma(t)$. Now, we show that $-w$ does not belong to $\Gamma(t)$. Otherwise, we have

$$-w = \lambda a + (1-t)v + t\,\|v\|\,w$$

for some $\lambda \in I\!R$ and some $v \in K \cap L^{\perp}$. Multiplying both sides of this equality by w, we get $-1 = (1-t)\langle v, w\rangle + t\,\|v\|$, yielding a contradiction because $\langle v, w\rangle \geq 0$. This proves that $\Gamma(t)$ is not a linear subspace. Hence, $\Gamma(t) \in \mathcal{R}(H)$ as claimed.

In order to complete the proof of Proposition 7.6, observe that two half-planes, say $M(a,w)$ and $M(a',w')$, can always be connected by a continuous path lying in $\mathcal{R}(H)$. By way of example, one can consider a path of the form

$$t \in [0,1] \mapsto M(\eta(t), \theta(t))\;,$$

where $\eta : [0,1] \to H$ and $\theta : [0,1] \to H$ are continuous curves joining a to a' and w to w', respectively. These curves can (and must) be selected in such a way that, for each $t \in [0,1]$, the condition

$$\left.\begin{array}{l} \lambda \in I\!R,\ \mu \in I\!R_+ \\ 0 = \lambda\eta(t) + \mu\theta(t) \end{array}\right\} \Longrightarrow \lambda = \mu = 0$$

holds. This requirement ensures that $M(\eta(t), \theta(t)) \in \mathcal{R}(H)$ for every $t \in [0,1]$. The proof is then complete. □

Corollary 7.7. *The boundary of $\mathcal{G}(H)$ is path-connected.*

Proof. $\mathrm{bd}_\delta[\mathcal{G}(H)]$ is the image of $\mathcal{R}(H)$ under the homeomorphism $[\,\cdot\,]^+$. □

8. The canonical deformation map

While studying the connectedness of $\mathcal{N}(H)$, we came across an expression of the form

$$\Psi(\varepsilon, K) := \{(1-\varepsilon)v + \varepsilon\,\|v\|\,w_K : v \in K\}, \tag{20}$$

where w_K is an arbitrary unit vector selected from $K \cap K^+$. Such a construction makes sense for any $K \in \mathcal{N}(H)$. For the sake of convenience, we refer to

$$\Psi : [0,1] \times \mathcal{N}(H) \to \mathcal{N}(H)$$

as the canonical deformation map. To facilitate the comparison of Ψ with the Seeger-Torki deformation map Φ, we use the approximation index ε instead of the "time" parameter t. Analogously as in Proposition 3.1, one can write:

Proposition 8.1. *Let $\Psi : [0,1] \times \mathcal{N}(H) \to \mathcal{N}(H)$ be defined by (20). For each $K \in \mathcal{N}(H)$, one has:*

(a) $\Psi(0,K) = K$;
(b) $\forall\, \varepsilon > 0, \quad \Psi(\varepsilon, K) \in \mathcal{P}(H)$;
(c) $0 \leq \varepsilon_1 \leq \varepsilon_2 \leq 1 \implies \Psi(\varepsilon_1, K) \supset \Psi(\varepsilon_2, K)$;
(d) $\varepsilon \in [0,1] \mapsto \delta(\Phi(\varepsilon, K), K)$ *is nondecreasing.*

Proof. Parts (a) and (b) have been established along the proof of Proposition 7.3. In contrast with the case of $\Phi(\,\cdot\,, K)$, the monotonicity of $\Psi(\,\cdot\,, K)$ is not obvious. Suppose $0 < \varepsilon_1 \leq \varepsilon_2 \leq 1$ (the value $\varepsilon_1 = 0$ can be treated afterward by using a continuity argument). Take $z \in \Psi(\varepsilon_2, K)$, so that

$$z = (1-\varepsilon_2)v + \varepsilon_2 \|v\| \, w_K \quad \text{for some } \; v \in K.$$

Let $g : \mathbb{R}_+ \to \mathbb{R}$ be the continuous function defined by

$$g(\alpha) = \alpha + \frac{\varepsilon_1}{1-\varepsilon_1} \|(1-\varepsilon_2)v + \alpha w_K\| - \varepsilon_2 \|v\| .$$

Thus,

$$g(0) = \left[\left(\frac{1-\varepsilon_2}{1-\varepsilon_1} \right) \varepsilon_1 - \varepsilon_2 \right] \|v\| = \left(\frac{\varepsilon_1 - \varepsilon_2}{1-\varepsilon_1} \right) \|v\| \leq 0.$$

Since $g(\alpha) \geq \alpha - \varepsilon_2 \|v\|$, it follows that $g(\alpha) \to \infty$ as $\alpha \to \infty$. Hence, there exists $\bar\alpha \in \mathbb{R}_+$ such that $g(\bar\alpha) = 0$, i.e. such that

$$\frac{\varepsilon_1}{1-\varepsilon_1} \|(1-\varepsilon_2)v + \bar\alpha w_K\| = \varepsilon_2 \|v\| - \bar\alpha.$$

Let

$$v' := \frac{1}{1-\varepsilon_1} [(1-\varepsilon_2)v + \bar\alpha w_K].$$

Observe that $v' \in K$ and

$$(1-\varepsilon_1)v' + \varepsilon_1 \|v'\| \, w_K = (1-\varepsilon_2)v + \bar\alpha w_K + \left(\frac{\varepsilon_1}{1-\varepsilon_1} \right) \|(1-\varepsilon_2)v + \bar\alpha w_K\| \, w_K$$

$$= (1-\varepsilon_2)v + \bar\alpha w_K + \varepsilon_2 \|v\| \, w_K - \bar\alpha w_K = (1-\varepsilon_2)v + \varepsilon_2 \|v\| \, w_K = z.$$

So, $z \in \Psi(\varepsilon_1, K)$, proving in this way the monotonicity condition (b). Part (d) follows from (a) and (c). □

There is yet another property of Ψ that deserves to be mentioned. As complement to Proposition 8.1, one can write:

Proposition 8.2. *For each $K \in \mathcal{N}(H)$, one has:*

$$\delta(\Psi(\varepsilon_1, K), \Psi(\varepsilon_2, K)) \leq 2\, |\varepsilon_1 - \varepsilon_2| \qquad \forall\, \varepsilon_1, \varepsilon_2 \in [0,1]. \tag{21}$$

Proof. Fix $K \in \mathcal{N}(H)$ and, of course, a corresponding unit vector w_K in $K \cap K^+$. We shall prove that $\Psi(\,\cdot\,, K)$ satisfies the Lipschitz condition

$$\delta(\Psi(\varepsilon_1, K), \Psi(\varepsilon_2, K)) \leq c\,|\varepsilon_1 - \varepsilon_2| \qquad \forall\, \varepsilon_1, \varepsilon_2 \in [0,1], \tag{22}$$

where the Lipschitz constant

$$c := \sup\left\{ \frac{\|v - w_K\|}{\|(1-\alpha)v + \alpha w_K\|} \,:\, \alpha \in [0,1], v \in K \cap S_H \right\} \tag{23}$$

is a nonnegative number depending on K (and the vector w_K). Observe that the term on the right-hand side of (23) is well defined because

$$1/\sqrt{2} \;\leq \|(1-\alpha)v + \alpha w_K\| \quad \text{for all } \; \alpha \in [0,1] \; \text{ and } \; v \in K \cap S_H.$$

Take $0 \leq \varepsilon_1 \leq \varepsilon_2 \leq 1$. For any $x \in H$, one can write

$$\Delta(x) := |\mathrm{dist}[x; \Psi(\varepsilon_2, K)] - \mathrm{dist}[x; \Psi(\varepsilon_1, K)]| = \mathrm{dist}[x; \Psi(\varepsilon_2, K)] - \mathrm{dist}[x; \Psi(\varepsilon_1, K)].$$

If y_x denotes the orthogonal projection of x onto $\Psi(\varepsilon_1, K)$, then

$$\Delta(x) \leq \|x - u\| - \|x - y_x\| \leq \|u - y_x\| \qquad \forall\, u \in \Psi(\varepsilon_2, K).$$

Since y_x belongs to $\Psi(\varepsilon_1, K)$, there exists $v_x \in K$ such that

$$y_x = (1 - \varepsilon_1)v_x + \varepsilon_1 \|v_x\|\, w_K.$$

Without loss of generality, one can suppose that $v_x \neq 0$. The particular choice

$$u = (1 - \varepsilon_2)v_x + \varepsilon_2 \|v_x\|\, w_K$$

leads to

$$\Delta(x) \leq (\varepsilon_2 - \varepsilon_1)\, \|[v_x - \|v_x\|\, w_K]\|\,. \tag{24}$$

On the other hand,

$$\|[(1 - \varepsilon_1)v_x + \varepsilon_1 \|v_x\|\, w_K]\| = \|y_x\| \leq \|x\|\,, \tag{25}$$

the inequality being due to the fact that orthogonal projections onto cones are norm decreasing. By combining (24) and (25), one arrives at

$$\Delta(x) \leq \left[\sup_{\alpha \in [0,1]} \; \sup_{v \in K \setminus \{0\}} \; \frac{\|[v - \|v\|\, w_K]\|}{\|[(1-\alpha)v + \alpha\, \|v\|\, w_K]\|} \right] (\varepsilon_2 - \varepsilon_1)\, \|x\|\,.$$

By homogeneity, the term between big square brackets reduces to the expression (23). To conclude the proof of (21), it suffices to observe that

$$\|v - w_K\| \leq \sqrt{2} \qquad \forall\, v \in K \cap S_H.$$

Thus, the Lipschitz constant c can not exceed the value 2. This completes the proof. $\square$

Two comments on Proposition 8.2 are in order. First of all, observe that for particular members of $\mathcal{K}(H)$, the Lipschitz constant (23) may be strictly smaller than 2. However, if K contains a unit vector that is orthogonal to w_K, then (23) attains the value 2. Secondly, Proposition 8.2 yields, in particular, the estimate

$$\delta(\Psi(\varepsilon, K), K) \le \min\{1, 2\varepsilon\},$$

confirming that $\Psi(\varepsilon, K)$ converges to K as the approximation parameter ε goes to 0. The above estimate is not as good as

$$\delta(\Phi(\varepsilon, K), K) \le \varepsilon,$$

but, on the other hand, we ignore whether $\Phi(\,\cdot\,, K)$ behaves or not in a Lipschitz manner. In short, each deformation mapping has its own merits and disadvantages.

9. By way of conclusion

In this work we have addressed two issues related to the metric space $(\mathcal{K}(H), \delta)$. The first one has to do with the link between pointedness, solidity and convergence. It has been shown that

$$\mathcal{N}(H) = \mathrm{cl}_\delta[\mathcal{P}(H)] = \mathrm{cl}_\delta[\mathcal{G}(H)].$$

These formulas have deep theoretical implications, but, from a practical point of view, what is worth mentioning is that

$$\left\{ \begin{array}{l} \text{any member of } \mathcal{N}(H) \text{ can be approximated monotonically from below by a} \\ \text{sequence of nontrivial pointed cones. It can also be approximated monotonically} \\ \text{from above by a sequence of solid cones that are strictly contained in } H. \end{array} \right.$$

The approximating sequences are constructed with the help of the Seeger-Torki deformation map Φ. An alternative construction mechanism is provided by the canonical deformation map Ψ. The later map enjoys some additional Lipschitzian properties that enable us to estimate the convergence rate.

The second issue discussed in this work has to do with connectedness. Among other things, it has been shown that

$$\mathcal{K}(H) = \mathcal{N}(H) \cup [\bigcup_{d=0}^{\dim H} \mathcal{L}_d(H)] \tag{26}$$

corresponds to a partition of $\mathcal{K}(H)$ into its different connected components. In fact, all the sets appearing in the right-hand side of (26) are path-connected. Special parts of $\mathcal{N}(H)$ that are path-connected include $\mathcal{P}(H)$, $\mathcal{R}(H)$, $\mathcal{G}(H)$, and $\mathcal{U}(H)$. The table below summarizes the situation and speaks by itself:

	open	closed	path-connected
$\mathcal{P}(H)$	yes	no	yes
$\mathcal{R}(H)$	no	yes	yes
$\mathcal{L}(H)$	yes	yes	no
$\mathcal{N}(H)$	yes	yes	yes
$\mathcal{G}(H)$	yes	no	yes
$\mathcal{U}(H)$	no	yes	yes
$\mathcal{L}_d(H)$	yes	yes	yes

Table 1: Special parts of $(\mathcal{K}(H), \delta)$

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