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An Epi-Convergence Result for Bivariate Convex Functions

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We prove that for a large class of convex and lower semi-continuous biavariate functions defined over $I\!R^N$, epi-convergence in one variable implies epi-convergence in both variables. We also show that for closed-valued and graph-convex mappings with domains with non empty interiors, pointwise convergence implies graph convergence. We provide a number of applications for both results.

Keywords: Convex functions, graph-convex mappings, epi-convergence, graph convergence, differential inclusions, parametric optimization

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Introduction

Consider the following sequence of convex functions $f_n : I\!R^N \times I\!R^M \to I\!R$, $n \geq 0$. Suppose, moreover, that $\forall y \in I\!R^M$, $f_n(\cdot, y)$ converge continuously to $f_0(\cdot, y)$. Then, it is well known that $f_n(\cdot,\cdot)$ converge continuously to $f_0(\cdot,\cdot)$ [6, Corollary 7.18]. More specifically, $\forall x_n \to x$ and $\forall y_n \to y$, we have

$$\lim_n f_n(x_n, y_n) = f_0(x, y).$$

A natural question then is to see if, for convex and lsc functions $f_n : I\!R^N \times I\!R^M \to \overline{I\!R}$, $n \geq 0$, the following statement is valid: $\forall y \in I\!R^M$, $f_n(\cdot, y)$ epi-converge to $f_0(\cdot, y)$ implies that $f_n(\cdot,\cdot)$ epi-converge to $f_0(\cdot,\cdot)$.
Moreover, a set valued map from $I\!R^N$ to $I\!R^M$ whose graph is convex possesses strong continuity properties in the interior of its domain [6, corollary 9.38]. This fact leads us to suspect that there is a strong relationship between pointwise convergence and graph convergence of closed-valued, graph-convex maps with domains with non empty interiors.
In this short paper, we show that under very mild conditions the answer to our first question is affirmative. We also show that this result, despite its simplicity, has a wide range of applications. One of these application is a theorem showing that pointwise convergence implies graph convergence for set-valued maps with convex graphs and domains with non empty interiors.
Other applications in areas such as differential inclusion, parametric optimization, and metric regularity also follow from our result.

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1. The main result

Throughout this paper, we will consider functions with values in $\overline{I\!R} = I\!R \cup \{+\infty\}$. (our functions are not allowed to take the value $-\infty$). Therefore, we say a function is *proper* if, and only if, it is not identically equal to $+\infty$. We use *lsc* to indicate a lower semi-continuous function. Recall the definition of epi-convergence [6, Proposition 7.2]: A sequence of functions $\{h^n : I\!R^n \to \overline{I\!R}\}$ epi-converges to $h : I\!R^n \to \overline{I\!R}$, if for all $x \in I\!R^n$,

(i) $\forall x^n \to x, \liminf h^n(x^n) \geq h(x)$,
(ii) $\exists x^n \to x, \limsup h^n(x^n) \leq h(x)$.

When we know a priori that h is lsc, a sequence of functions $\{h^n : I\!R^n \to \overline{I\!R}\}$ epi-converges to an lsc function $h : I\!R^n \to \overline{I\!R}$, if for all $x \in I\!R^n$,

(i) $\forall x^n \to x, \liminf h^n(x^n) \geq h(x)$,
(ii) $\forall x \in D, \exists x^n \to x, \limsup h^n(x^n) \leq h(x)$,

where D is the projection on $I\!R^n$ of some countable and dense subset of $\operatorname{epi} h$ [5, Corollary 2.5].

For a sequence of functions, we will use $\xrightarrow{p}$ and $\xrightarrow{e}$ to indicate pointwise convergence and epi-convergence respectively.

The following theorem is our main result.

Theorem 1.1. *Consider a sequence of convex, lsc functions $f_n : I\!R^N \times I\!R^M \to \overline{I\!R}$, $n \geq 0$. Assume:*

(i) *there exists $\bar{y}$ and a neighborhood $V(\bar{y})$ such that $\forall y' \in V(\bar{y})$ there exists some x such that $f_0(x, y') < +\infty$.*
(ii) *there exists a set $D \subset I\!R^M$ that is the projection of some countable and dense subset of* $\operatorname{epi} f_0$*, such that $\forall y \in D$, $f_n(\cdot, y)$ are uniformly minorized and*

$$f_n(\cdot, y) \xrightarrow{e} f_0(\cdot, y).$$

Then,

$$f_n(\cdot, \cdot) \xrightarrow{e} f_0(\cdot, \cdot).$$

The idea of the proof of the main result is similar to an argument used in [5] in a completely different setting: condition (ii) of epi-convergence is easy to verify. We verify condition (i) of epi-convergence for a sequence of regularized functions f_n^λ, we then use the properties of the regularized sequence to show that condition (i) holds for f_n.

Remark 1.2. Recall that a sequence $h_n : I\!R^N \to \overline{I\!R}$ is uniformly minorized, if $\exists \gamma > 0$ such that $\forall n$, $h_n(x) \geq -\gamma(||x|| + 1), \forall x \in I\!R^N$.

Remark 1.3. In assumption (ii), if $f_n(\cdot, y) \xrightarrow{e} f_0(\cdot, y)$ and $f_0(\cdot, y)$ is proper, then the uniform minorization condition is automatically satisfied [6, proposition 7.34].

Remark 1.4. Assumption (i) is weaker than requiring that $\operatorname{int} \operatorname{dom} f(\cdot, \cdot)$ is not empty in $I\!R^N \times I\!R^M$.

Remark 1.5. When $f_n \equiv f_0$, the joint epi-convergence of f can lead to a type of continuity that is related to, but not the same as *epi-continuity* [6, Sec. 7.F].

We start with two simple lemmas.

Lemma 1.6. *Suppose* $f : I\!R^N \times I\!R^M \to \overline{I\!R}$ *is proper, convex and lsc. Then*

$$f^\lambda(x, y) := \inf_{u \in R^N} \{f(u, y) + \tfrac{1}{2\lambda} \parallel u - x \parallel_n^2\}$$

is convex and lsc (jointly in x *and* y*) for all* $\lambda > 0$.

Proof. The fact f^λ is convex and proper is a direct result of Proposition 2.22 in [6] and the fact that $f(u, y) + \frac{1}{2\lambda} \parallel u - x \parallel_n^2$ is convex jointly in x, y, and u. Let $||\cdot||_n$ and $||\cdot||_m$ denote the norms in $I\!R^N$ and $I\!R^M$, respectively, and let B_n and B_m denote balls in $I\!R^N$ and $I\!R^M$ respectively. In light of Theorem 1.17 in [6], we only need to show that $f(u, y) + \frac{1}{2\lambda}||u - x||_n^2$ is level bounded in u locally uniformly in (x, y). Therefore, we need to show that for any $(\bar{x}, \bar{y}) \in I\!R^N \times I\!R^M$, for any $\alpha > 0$, there exist $\rho > 0$ and $\delta > 0$ such that $\forall x' \in B_n(\bar{x}, \rho)$, and $\forall y' \in B_m(\bar{y}, \rho)$, we have

$$u \in \{u | f(u, y') + \frac{1}{2\lambda}||u - x'||_n^2 \leq \alpha\} \implies ||u||_n < \delta.$$

Fix $\alpha > 0$. Since f is convex and proper, $\exists \gamma > 0$ such that $\forall u \in I\!R^N$ and $\forall y' \in B_m(\bar{y}, \rho)$, we have

$$f(u, y') > -\gamma(||u||_n + \rho_1) - \gamma, \tag{1}$$

where $\rho_1 = ||\bar{y}||_m + \rho$. To see that (1) is valid, let $u = (u_1, \cdots, u_n)$ and $y' = (y_1, \cdots, y_m)$. Then, by Ex. 7.34 in [6], we have

$$f(u, y') > -\gamma\sqrt{u_1^2 + \cdots + u_n^2 + y_1^2 + \cdots + y_m^2} - \gamma, \tag{2}$$

However,

$$\sqrt{u_1^2 + \cdots + u_n^2 + y_1^2 + \cdots + y_m^2} < \sqrt{u_1^2 + \cdots + u_n^2} + \sqrt{y_1^2 + \cdots + y_m^2}.$$

Now (2) and the fact that $y' \in B_m(\bar{y}, \rho)$ imply that

$$f(u, y') > -\gamma(||u||_n + \rho_1) - \gamma.$$

Because of (1), we have

$$f(u, y') + \frac{1}{2\lambda}||u - x'||_n^2 > -\gamma(||u||_n + \rho_1) - \gamma + \frac{1}{2\lambda}||u - x'||_n^2. \tag{3}$$

Since $y' \in B_m(\bar{y}, \rho)$ and $x' \in B_n(\bar{x}, \rho)$, $f(u, y') + \frac{1}{2\lambda}||u - x'||_n^2 \leq \alpha$ implies that

$$-\gamma(||u||_n + \rho_1) - \gamma + \frac{1}{2\lambda}||u - x'||_n^2 \leq \alpha,$$

which implies that $||u||_n$ is bounded by the same bound for all $x' \in B_n(\bar{x}, \rho)$ and $y' \in B_m(\bar{y}, \rho)$. $\square$

Note that, in general, it is possible that a function $f : I\!R^N \times I\!R^M \to \overline{I\!R}$ is proper, convex, and lsc, yet $g(x) = \inf_y f(x, y)$ is not lsc. Take for example the function $f : I\!R^2 \to \overline{I\!R}$ where $f(x, y) = 0$ for $x > 0$, $y > 0$, $xy \geq 1$, and $f(x, y) = +\infty$ otherwise. Clearly, $g(x) = 0$ on $(0, +\infty)$ and $g(x) = +\infty$ otherwise, is not lsc.

Remark 1.7. When $f(\cdot, y)$ is a proper function for every y, the proof of Lemma 1.2 becomes much simpler. In this case, f^λ is convex and finite valued, and hence it is continuous on $I\!R^N \times I\!R^M$.

Lemma 1.8. *Suppose that a sequence $h_n : I\!R^N \to \overline{I\!R}$ of convex lsc functions that epi-converges to a function $h_0 : I\!R^N \to \overline{I\!R}$. Assume further that h_n are uniformly minorized. For $n \geq 0$, let*

$$h_n^\lambda(x) = \inf_{u \in R^N} \{h_n(u) + \frac{1}{2\lambda}||u - x||^2\}.$$

Then $h_n^\lambda \xrightarrow{p} h_0^\lambda$, $\forall \lambda > 0$.

Proof. When h_0 is proper, the lemma holds by a classical result (Theorem 7.37 in [6]). If h_0 is not proper, then $h_0 \equiv +\infty$ implies that $h_0^\lambda \equiv +\infty$. Suppose that the claim of pointwise convergence of our lemma does not hold. Then, there exist x, M, and a subsequence n_k such that

$$h_{n_k}^\lambda(x) < M. \tag{4}$$

Then, the uniform minorization condition implies that $h_{n_k}^\lambda$ are equi-coercive (using equation similar to (3) of the last part of the proof of Lemma 1.2). Hence, $\exists u_{n_k}$ such that $u_{n_k} \to u_0$ for some $u_0 \in I\!R^N$. Hence,

$$\liminf_k h_{n_k}^\lambda(x) = \liminf_k \{h_{n_k}(u_{n_k}) + \frac{1}{2\lambda}||u_{n_k} - x||^2\} \geq h_0(u_0) + \frac{1}{2\lambda}||u_0 - x||^2 = +\infty,$$

which contradicts (4). □

Lemma 1.9. *Consider a convex lsc function $h : I\!R^N \to \overline{I\!R}$. Then,*

$$\lim_{\lambda \to 0^+} h^\lambda = h.$$

Proof. Again, in the case h is proper, this result is classic [6, Theorem 1.25]. If $h \equiv +\infty$, then $h^\lambda \equiv +\infty$, and the conclusion of the lemma still holds.

Finally, the proof of Theorem 1.1:

Proof of Theorem 1.1. Since f_0 is lsc, condition (ii'), i.e. the "limsup" part of the definition of epi-convergence, clearly holds because of assumption (ii) in our theorem. More precisely, let $\tilde{D}$ be a countable dense subset of $epi f_0$. Let D_1 be the projection of $\tilde{D}$ on $I\!R^N \times I\!R^M$ and let D be the projection of $\tilde{D}$ on $I\!R^M$. Assumption (ii) implies that $\forall (x, y) \in D_1$, $\exists x_n \to x$ such that $\limsup f_n(x, y) \leq f(x, y)$. To prove the "liminf" part, consider the following functions:

$$f_n^\lambda(x, y) = \inf_{u \in R^N} \{f_n(u, y) + \frac{1}{2\lambda}||u - x||^2\}$$

$$f_0^\lambda(x,y) = \inf_{u\in R^N}\{f_0(u,y) + \frac{1}{2\lambda}||u-x||^2\},$$

where $||\cdot||$ is the norm in $\mathbb{R}^N$. Then, the function f_0^λ is proper and $\operatorname{int}\operatorname{dom} f_0^\lambda \neq \emptyset$ by assumption (i). Moreover, f_0^λ and f_n^λ are convex (Proposition 2.22 in [6]). Furthermore, $f_n^\lambda \xrightarrow{p} f_0^\lambda$ on $\mathbb{R}^N \times D$ by Lemma 1.3. The function f_0^λ is lsc on $\mathbb{R}^N \times \mathbb{R}^M$ by Lemma 1.2. Therefore, $f_n^\lambda \xrightarrow{e} f_0^\lambda$ on $\mathbb{R}^N \times \mathbb{R}^M$ by Theorem 7.17 in [6]. Hence, $\forall(x,y) \in \mathbb{R}^N \times \mathbb{R}^M$, $\forall x_n \to x$ and $\forall y_n \to y$, we have

$$\liminf_n f_n(x_n,y_n) \geq \liminf_n f_n^\lambda(x_n,y_n) \geq f_0^\lambda(x,y). \tag{5}$$

Hence, by taking the limit of (5) as $\lambda \to 0$, and using Lemma 1.4, we obtain

$$\liminf_n f_n(x_n,y_n) \geq f_0(x,y).$$

□

Using Remark 1.3, we obtain

Corollary 1.10. *Consider a sequence of convex lsc functions* $f_n : \mathbb{R}^N \times \mathbb{R}^M \to \overline{\mathbb{R}}$, *for* $n \geq 0$. *Assume:*

(*i*) *there exist* $\bar{y}$ *and a neighborhood* $V(\bar{y})$ *such that* $\forall y \in V(\bar{y})$ *there exists an* x *such that* $f_0(x,y) < \infty$.

(*ii*) *there exists a set* $D \subseteq \mathbb{R}^M$ *that is the projection of some countable dense subset of* epi f_0, *such that* $\forall y \in D$, $f_0(\cdot,y)$ *is proper and*

$$f_n(\cdot,y) \xrightarrow{e} f_0(\cdot,y).$$

Then,

$$f_n(\cdot,\cdot) \xrightarrow{e} f_0(\cdot,\cdot).$$

When $f_0(\cdot,y)$ is proper, for all y, condition (i) of Theorem 1.1 is not needed:

Corollary 1.11. *Consider a sequence of convex lsc functions* $f_n : \mathbb{R}^N \times \mathbb{R}^M \to \overline{\mathbb{R}}$, *for* $n \geq 0$. *Assume:*

(*i*) $\forall y$, $f_0(\cdot,y)$ *is proper.*

(*ii*) *there exists a set* $D \subseteq \mathbb{R}^M$ *that is the projection of some countable dense subset of* epi f_0 *such that* $\forall y \in D$,

$$f_n(\cdot,y) \xrightarrow{e} f_0(\cdot,y).$$

Then,

$$f_n(\cdot,\cdot) \xrightarrow{e} f_0(\cdot,\cdot).$$

Proof. When $f_0(\cdot,y)$ is proper, f_0^λ is continuous by Remark 1.7. Hence, we have, for all $y \in D$,

$$f_n^\lambda(\cdot,y) \xrightarrow{p} f_0^\lambda(\cdot,y).$$

The rest of the proof is the same as the proof of Theorem 1.1. □

The following corollary relates the pointwise convergence to the epi-convergence of bivariate convex functions.

Corollary 1.12. *Consider a sequence of convex lsc functions* $f_n : \mathbb{R}^N \times \mathbb{R}^M \to \overline{\mathbb{R}}$, *converging pointwise to a convex function* f_0. *Assume:*

(*i*) $\forall y$, $f_0(\cdot, y)$ *is lsc and* $\operatorname{int}\operatorname{dom} f_0(\cdot, y) \neq \emptyset$.

Then,

$$f_n(\cdot,\cdot) \xrightarrow{e} f_0(\cdot,\cdot).$$

Proof. By [6, Theorem 7.17], $f_n(\cdot, y) \xrightarrow{e} f_0(\cdot, y)$, for all y. The conclusion of the Lemma follows from Corollary 1.5. □

Recall that a sequence h_n of functions is called *equi-lower semicontinuous*, or equi-lsc, if $\forall x$, $\forall \epsilon >$, there exists a neighborhood V of x such that for all n, we have

$$\inf_{x' \in V} h_n(x') \geq \min\{h_n(x) - \epsilon, \epsilon^{-1}\}$$

Corollary 1.13. *Consider a sequence of convex lsc functions* $f_n : \mathbb{R}^N \times \mathbb{R}^M \to \overline{\mathbb{R}}$, *converging pointwise to a proper and convex function* f_0. *Assume:*

(*i*) $\forall y \in \mathbb{R}^N$, $f_n(\cdot, y)$ *are equi-lsc and proper.*
Then, f_n *are equi-lsc on* $\mathbb{R}^N \times \mathbb{R}^M$.

Proof. Corollary 1.6 and Theorem 7.10 in [6]. □

The following corollary shows that in the case of convex bivariate functions, lower semicontinuity in one variable implies lower semicontinuity in both variables.

Corollary 1.14. *If* $f : \mathbb{R}^N \times \mathbb{R}^M \to \overline{\mathbb{R}}$ *is a convex function such that* $f(\cdot, y)$ *is proper and lsc for all* $y \in \mathbb{R}^M$, *then* f *is lsc.*

Proof. By Remark 1.7, f^λ is convex and finite, and hence it is continuous on $\mathbb{R}^N \times \mathbb{R}^M$. Therefore, $\forall (x, y) \in \mathbb{R}^N \times \mathbb{R}^M$, $\forall x_n \to x$, and $\forall y_n \to y$, we have

$$\liminf_n f(x_n, y_n) \geq \liminf_n f^\lambda(x_n, y_n) \geq f^\lambda(x, y). \tag{6}$$

Since $\forall y \in \mathbb{R}^M$, $f(\cdot, y)$ is proper, convex, and lsc, we have by Lemma 1.3

$$\lim_{\lambda \to 0^+} f^\lambda(x, y) \to f(x, y). \tag{7}$$

Thus, by taking the limit of (6) as $\lambda \to 0^+$ and using (7), we get

$$\liminf_n f(x_n, y_n) \geq f(x, y).$$

Thus, f is lsc. □

Remark 1.15. It is easy to construct examples where the claim of the above corollary is not correct if we have $f(\cdot, y) \equiv +\infty$ for some $y \in \mathbb{R}^M$.

Finally, we emphasize here that Theorem 1.1 is finite dimensional by nature since its proof depends on [6, Theorem 7.17], which does not hold in infinite dimensional spaces.

2. Set-Valued Analysis

In this section, we provide a natural application of Theorem 1.1 in set-valued analysis. We start by reviewing basic notions of convergence for set-valued maps. A sequence C_n of closed sets in $I\!R^N$ converges to a closed subset C of $I\!R^N$, if

$$C = \limsup_n C_n = \liminf_n C_n.$$

For the definition of the limsup and liminf for sets, as well as the for the definitions of various notions of continuity for set valued maps, see [6].
A sequence $S_n : I\!R^N \rightrightarrows I\!R^M$ of set-valued maps converges pointwise to $S : I\!R^N \rightrightarrows I\!R^M$, if for every x, we have $\lim_n S_n(x) = S(x)$. The sequence S_n converges to S graphically, if $\lim \operatorname{gph} S_n = \operatorname{gph} S$, where

$$\operatorname{gph} S = \{(x, y) \in I\!R^N \times I\!R^M | y \in S(x)\}.$$

S is *outer semi-continuous*, or osc, if $\forall x \in I\!R^N$, for any compact set $B \subset I\!R^M$, and for any $\epsilon > 0$, there is a neighborhood $V(x)$ of x such that $\forall x' \in V(x)$, we have

$$S(x') \cap B \subset \epsilon S(x),$$

where $\epsilon C = \{x \in I\!R^N | d(x, C) \leq \epsilon\}$.

Remark 2.1. $S : I\!R^N \rightrightarrows I\!R^M$ has closed graph in $I\!R^N \times I\!R^M$ if, and only if, S is osc [6, Theorems 5.7].

Finally, the sequence S_n is *equi-outer semi-continuous*, or equi-osc, if $\forall x \in I\!R^N$, for any compact set $B \subset I\!R^M$, and for any $\epsilon > 0$, there is a neighborhood $V(x)$ of x such that $\forall x' \in V(x)$, we have

$$S_n(x') \cap B \subset \epsilon S_n(x).$$

Theorem 2.2. *Consider a sequence of set-valued functions $S_n : I\!R^N \rightrightarrows I\!R^M$ that converge pointwise to a set-valued map S with a closed and convex graph. Assume $\forall x$,*

$$\lim_n S_n(x) = S(x),$$

and assume $\operatorname{int} \operatorname{dom} S \neq \emptyset$.
Then,

$$\lim_n \operatorname{gph} S_n = \operatorname{gph} S.$$

Proof. Apply Theorem 1.1 to the indicator functions of the graphs of S_n and S. Hence, for a fixed x,

$$\delta_{S_n(x)}(\cdot) \xrightarrow{e} \delta_{S(x)}(\cdot).$$

Moreover, by our assumption on the $\operatorname{dom} S$, there is $\bar{x}$ and $V(\bar{x})$ such that $\forall x' \in V(\bar{x})$, there exists y such that $\delta_{S(x')}(x', y)$ is finite. Note also that the indicator functions are uniformly minorized. Hence, by Theorem 1.1, we have

$$\delta_{S_n(\cdot)}(\cdot) \xrightarrow{e} \delta_{S(\cdot)}(\cdot),$$

which implies that S_n converges to S graphically. $\square$

Corollary 2.3. *Consider a sequence of set valued maps S_n. Suppose that S_n converges pointwise to S and S_n, S are closed-valued and graph-convex. If we further assume* $\operatorname{int}\operatorname{dom} S \neq \emptyset$, *then S_n are equi-osc.*

Proof. Theorem 2.1 and Theorem 3.3 in [3]. □

Theorem 2.4. *Consider a set valued map $S : \mathbb{R}^N \rightrightarrows \mathbb{R}^M$ that is closed-valued, graph-convex, and* $\operatorname{dom} S = \mathbb{R}^N$. *Then,* $\operatorname{gph} S$ *is closed.*

Proof. Same as Theorem 2.1, except that we use Corollary 1.9 instead of Theorem 1.1. □

The above theorem has many immediate applications. In all the following results, the set-valued maps are closed-valued, have convex graphs, and their domain is the entire space. Therefore, these maps are osc by Theorem 2.3 and Remark 2.1.
Recall that a set-valued map is locally bounded at $\bar{x}$, if there exists a neighborhood V of $\bar{x}$ such the set $S(V) = \cup_{x' \in V} S(x')$ is bounded.

Corollary 2.5. *Let S be as in Theorem 2.3. Suppose that there is $\bar{x}$ such that $S(\bar{x})$ is bounded, then S is locally bounded on $\mathbb{R}^N$.*

Proof. Theorem 2.3, [6, Theorem 5.18], and Remark 2.1. □

Corollary 2.6. *Let S be as in Theorem 2.3. Suppose that there is $\bar{x}$ such* $\operatorname{int} S(\bar{x}) \neq \emptyset$, *then* $\operatorname{int}\operatorname{gph} S \neq \emptyset$.

Proof. Theorem 2.3, [6, Theorem 5.9], and Remark 2.1. □

We finally mention the following metric regularity result

Corollary 2.7. *Let S be as in Corollary 2.3. Let $\bar{y} \in \operatorname{int} S(\bar{x})$ for some $\bar{x}$. Then, $\exists \epsilon > 0$ along with coefficients α, β in $\mathbb{R}_+$ such that*

$$d(x, S^{-1}(u)) \leq (\alpha|x - \bar{x}| + \beta) d(u, S(x)),$$

when $|u - \bar{u}| \leq \epsilon$.

Proof. Corollary 2.4 and [6, Theorems 5.7, 9.48]. □

3. Differential Inclusions

We now apply the results of the previous section to problems in differential inclusions. Consider the following differential inclusion DI over $[0,1]$:

$$x'(t) \in_{a.e.} A(x(t))$$

$$x(t) \in_{a.e.} K$$
$$x \in W^{1,1}([0,1]; \mathbb{R}^n).$$

Consider also the approximating inclusions DI_n over $[0,1]$

$$x'(t) \in_{a.e.} A_n(x(t))$$

$$x(t) \in_{a.e.} K_n$$

$$x \in W^{1,1}([0,1]; I\!R^N),$$

where K_n are closed convex subset in $I\!R^N$, and $A_n : I\!R^n \rightrightarrows I\!R^N$ are set-valued maps that are closed and convex valued.

Proposition 3.1. (Proposition 10.1 in [3]) *Suppose that the inclusions DI and DI_n are defined as above. Suppose that A_n converges pointwise to A, $K = \lim K_n$. Suppose further that A_n are equi-osc. Let x_n be sequence of solutions for DI_n. Let x_0 be a cluster point in $W^{1,1}$ of such sequence, then x_0 is a solution to DI.*

We now obtain a result that generalizes Corollary 10.3 and 10.5 in [3].

Theorem 3.2. *Suppose the inclusions DI and DI_n are defined as above. Suppose A_n converges pointwise to A, $K = \lim K_n$. Suppose that A_n and A are closed-valued and graph-convex. Assume further that* $\operatorname{int}\operatorname{dom} A \neq \emptyset$*. Let x_n be sequence of solutions for DI_n. Let x_0 be a cluster point in $W^{1,1}$ of such sequence, then x_0 is a solution to DI.*

Proof. Corollary 2.2 and Proposition 3.1. □

Remark 3.3. Note that in the results of [3], A_n are assumed to be sublinear (convex processes). Moreover, they are also assumed to satisfy the following condition:

$$\forall x, \ \sup_n d(0, A_n(x)) < +\infty, \tag{8}$$

which implies that $\operatorname{dom} A \neq \emptyset$. Therefore, it is clear that the assumptions of Theorem 3.2 are weaker than those of Corollary 10.3 in [3]. In fact, (6) and the convexity of the graphs of each A_n can be used to show that A_n are equi-lipschitzian, which is much stronger than the equi-osc requirement of Proposition 3.1.

If we weaken the requirement that x_n converges to x_0 in $W^{1,1}([0,1])$, we can still obtain the following result.

Theorem 3.4. *Suppose the inclusions DI and DI_n are defined as above. Suppose A_n converge pointwise to A, $K = \lim K_n$. Suppose further that A_n and A are locally bounded, closed valued and graph-convex. Moreover, assume* $\operatorname{int}\operatorname{dom} A \neq \emptyset$*. Let x_n be a sequence of solutions for DI_n. Suppose there is a subsequence x_{n_k} such that x_{n_k} converges to x_0 in L^1 and that x'_{n_k} converges to x'_0 weakly in L^1. Then, x_0 is a solution to DI.*

Proof. Theorem 10.4 in [3] yields the same conclusion provided that A_n are equi-lsc, which is in fact the case due to Corollary 2.2. □

Remark 3.5. When $\operatorname{int}\operatorname{dom} A = I\!R^N$, and $\exists \bar{x}$ such that $A(\bar{x})$ is bounded, Corollary 2.4. can be used to to show that A is locally bounded. In this case, the graphic convergence of A_n to A and the the fact that A and A_n are graph-convex, along with Corollary 2.4, will imply that A_n are eventually locally bounded.

4. Level sets of convex functions

There are many results in the literature that relate the epi-convergence of functions to the convergence of their level sets. For example, see [2], Chapter 1 in [4], and Proposition 7.7 in [6]. In this section, we use Theorem 2.1 to provide a simple proof for a result along the same lines.
Recall that given a function $h : I\!R^N \to I\!R$ and $\alpha \in I\!R$, the level set $Lev^\alpha h$ is the set

$$Lev^\alpha h = \{x \in I\!R^N | h(x) \leq \alpha\}.$$

Theorem 4.1. *Consider a sequence of proper, convex, and lsc functions $h_n : I\!R^N \to \overline{I\!R}$. The following are equivalent:*

(i) $h_n \xrightarrow{e} h_0$

(ii) *there is a set D that is the projection of a countable dense subset of* $\operatorname{epi} h_0$*, such that $\forall \alpha \in D$ and*

$$Lev^\alpha h_n \to Lev^\alpha h_0.$$

Proof. The fact that (i) implies (ii) is a well known result [6, Theorems 7.7]. To show that (ii) implies (i) we use Theorem 2.1 by considering the level set as images of a set-valued map $\gamma_n : I\!R \rightrightarrows I\!R^N$, where

$$\gamma_n(\alpha) = Lev^\alpha h_n.$$

Note that the graphs of such maps are the epigraphs of h_n. Now the conclusion follows again from Theorem 1.1 and Theorem 2.1. $\square$

The above theorem is similar to Theorem 2.3 in [4]. On one hand, the underlying space of Soueycatt's result is any metric space. On the other hand, his assumptions on the sequence h_n and the corresponding level sets are stronger.

5. Epi-Convergence of perspective functions

Let $f : I\!R^N \to \overline{I\!R}$ be a proper convex, lsc function. Let $u > 0$, and $d \in I\!R^N$, and $x_0 \in \operatorname{int}\operatorname{dom} f$. Consider the difference quotient :

$$\Delta f(d,u) = u[f(x_0 + \frac{d}{u}) - f(x_0)].$$

We use the above quotient to define a *perspective function* $\tilde{f} : I\!R^N \times I\!R \to \overline{I\!R}$ of f (Section 2.2 in [1]):

$$\tilde{f}(d,u) = \begin{cases} u[f(x_0 + \frac{d}{u}) - f(x_0)]\,, u > 0 \\ f^{+\infty}(d)\,, u = 0 \\ +\infty\,, u < 0 \end{cases}$$

where

$$f^{+\infty}(d) = \lim_{u \to 0^+} u[f(x_0 + \frac{d}{u}) - f(x_0)]$$

is the horizon function of f. Note that $\tilde{f}$ is convex, proper, and lsc (see Example 3.2.3. in [1]).

Now consider a sequence of proper convex lsc functions f_n that converges to a proper convex lsc function f. Let

$$\tilde{f}_n(d,u) = \begin{cases} u[f_n(x_0 + \frac{d}{u}) - f_n(x_0)]\,, u > 0 \\ f_n^{+\infty}(d)\,, u = 0 \\ +\infty\,, u < 0 \end{cases}$$

Then, we have the following result:

Proposition 5.1. *Let $\tilde{f}_n$ and $\tilde{f}$ be a sequence of functions defined as above. Suppose $f_n \xrightarrow{e} f$, then*

$$\tilde{f}_n \xrightarrow{e} \tilde{f}.$$

Proof. We know that $f_n(x_0) \to f(x_0)$ since $x_0 \in \operatorname{dom} f$. Hence, it is clear that $\forall u \neq 0$, we have $\tilde{f}_n(\cdot, u) \xrightarrow{e} \tilde{f}(\cdot, u)$. Thus, the conclusion of this proposition follows from Theorem 1.1. $\square$

The above theorem allows us to approximate $f^{+\infty}$ using difference quotients of f_n. The definition of epi-convergence yields the following corollary:

Corollary 5.2. *For all $d \in I\!\!R^n$, $\exists u_n \to 0^+$ and $\exists d_n \to d$ such that*

$$\lim_n u_n[f_n(x_0 + \frac{d_n}{u_n}) - f_n(x_0)] = f^{+\infty}(d).$$

6. Convex parametric minimization

Corollary 1.5 has further consequences in parametric minimization:

Theorem 6.1. *Consider a sequence of convex lsc functions $f_n : I\!\!R^N \times I\!\!R^M \to \overline{I\!\!R}$, for $n \geq 0$. Assume:*

(*i*) *there exist $\bar{y}$ and a neighborhood $V(\bar{y})$ such that $\forall y \in V(\bar{y})$ there exists an x such that $f_0(x,y) < \infty$,*

(*ii*) *there exists a dense set $D \subseteq I\!\!R^M$ that is the projection of a countable dense subset of* epi f_0*, such that $\forall y \in D$, $f_0(\cdot, y)$ is proper and*

$$f_n(\cdot, y) \xrightarrow{e} f_0(\cdot, y),$$

(*iii*) *there exists an $\alpha > 0$ and a real number β such that for any $x \neq 0$, we have*

$$f_0(x, 0) \geq \alpha ||x|| + \beta.$$

Then,

$$p_n(y) = \inf_x f_n(x,y) \xrightarrow{e} p(y) = \inf_x f_0(x,y).$$

Moreover, if $P_n(y) = \operatorname*{argmin}_x f_n(x,y)$ for $n \geq 0$, then

$$\limsup_n P_n(y) \subset P_0(y).$$

Proof. Corollary 1.5 and Proposition 7.57 in [6] imply $p_n \xrightarrow{e} p_0$. The statement about P_n follows from the epi-convergence of p_n and the definition of P_n. $\square$

The above theorem slightly generalizes Theorem 7.41 in [6]. We also note that condition (ii) can be expressed as $f^{+\infty}(x, 0) > 0$, for all $x \neq 0$ [6, Corollary 7.43].

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