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Homogenization of Evolution Problems in a Fiber Reinforced Structure

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We study the homogenization of parabolic or hyperbolic equations like

$$\rho_\varepsilon(x)\frac{\partial^n u_\varepsilon}{\partial t^n} - div(a_\varepsilon(x)\nabla u_\varepsilon) = f \quad \text{on } \ \Omega \times (0,T) + \textit{boundary conditions}, \quad n \in \{1,2\},$$

where the coefficients a_ε and ρ_ε takes values of very different order on a ε-periodic subset $T_\varepsilon \subset \Omega$ (fibered structure) and elsewhere. We find a non local effective equation deduced from a homogenized system of several equations.

1. Introduction

We are concerned with the homogenization of parabolic or hyperbolic boundary-value problems of the type

$$\begin{cases} \rho_\varepsilon(x)\frac{\partial^n u_\varepsilon}{\partial t^n} - \operatorname{div}(a_\varepsilon(x)\nabla u_\varepsilon) = f \quad \text{in } \ \Omega \times (0,T), \quad n \in \{1,2\}, \\ + \ \text{boundary conditions.} \end{cases} \tag{1}$$

In the usual setting of homogenization, the ε-periodic functions a_ε are assumed to satisfy conditions of uniform ellipticity and boundedness like $0 < \delta < a_\varepsilon(x) < \beta < +\infty$. The homogenized equation has then the same form as the initial equation. In the case of elliptic equations ($\rho_\varepsilon = 0$), the precise conditions on the sequence (a_ε) under which the asymptotic behaviour of (1) is still a classical one are established in [7], [8]. However when highly inhomogeneous media take place, like for instance in the theory of temperature superconductivity, these assumptions may not be satisfied anymore and the homogenization can lead to unusual models like non-local ones, as pointed out in [14]. This is well known in the case of scalar linear elliptic equations where the appearance of a non local term in the limit energy can be interpreted in the general context of Dirichlet forms (see [17]). In [4], [9], [12] explicit computations are performed for a fiber reinforced structure. Some extensions of those kinds of results to the framework of elasticity, where the theory of Dirichlet forms breaks down, are given in [5], [13]. However very few results of this type are known when evolution problems are considered.

In this paper, we study the effective properties of a composite made of an ε-periodic frame of parallel disconnected fibers T_ε of conductivity λ_ε and mass density $\rho_{1\varepsilon}$ surrounded by a matrix of conductivity ε^α ($\alpha \geq 0$) and mass density $\rho_{0\varepsilon}$. Hence the functions $a_\varepsilon(x)$ and

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$\rho_\varepsilon(x)$ are defined by

$$\begin{array}{llll} a_\varepsilon(x) = \lambda_\varepsilon & \text{if } x \in T_\varepsilon\ ; & a_\varepsilon(x) := \varepsilon^\alpha & \text{if } x \in \Omega \setminus T_\varepsilon, \\ \rho_\varepsilon(x) = \rho_{1\varepsilon} & \text{if } x \in T_\varepsilon\ ; & \rho_\varepsilon(x) := \rho_{0\varepsilon} & \text{if } x \in \Omega \setminus T_\varepsilon. \end{array}$$

The particular effects due to the lack of uniform boundedness and the lack of uniform ellipticity are brought to the fore by two different choices of scaling.

In the first example, we consider the case of very thin fibers (the Lebesgue measure of T_ε tends to 0) of very high conductivity, surrounded by a matrix of constant conductivity 1. Therefore, the condition of uniform ellipticity is fullfilled, but not the condition of uniform boundedness. More precisely, denoting by r_ε the radius of the fibers, our assumption reads as:

$$\alpha = 0, \quad \lambda_\varepsilon = \frac{|\Omega|}{|T_\varepsilon|} k_\varepsilon, \quad 0 < r_\varepsilon << \varepsilon, \quad k_\varepsilon \to k \in]0, +\infty].$$

Similarly the coefficient $\rho_\varepsilon(x)$ is assumed to satisfy

$$\lim_{\varepsilon \to 0} \frac{|T_\varepsilon|}{|\Omega|} \rho_{1\varepsilon} = \overline{\rho}_1, \quad \lim_{\varepsilon \to 0} \frac{|\Omega \setminus T_\varepsilon|}{|\Omega|} \rho_{0\varepsilon} = \overline{\rho}_0, \quad \overline{\rho}_1, \overline{\rho}_0 \in [0, +\infty[.$$

In order to describe the asymptotic behaviour of (1), we introduce besides the limit u of the solution u_ε, a new variable v which accounts for the macroscopic temperature of the structure T_ε. This temperature v is defined as the weak limit in the sense of measures of the rescaled functions defined by $v_\varepsilon(x,t) = \frac{|\Omega|}{|T_\varepsilon|} u_\varepsilon(x,t)$ if $x \in T_\varepsilon$, $v_\varepsilon(x,t) = 0$ otherwise. The limit problem we derive in Theorem 2.2 depends on k, $\overline{\rho}_0$, $\overline{\rho}_1$ and on the capacitary parameter

$$\gamma := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2 |\ln r_\varepsilon|}.$$

In the case where k and γ are positive and finite we prove (see Theorem 2.2) that (u, v) is the unique solution of a problem of the type

$$\begin{cases} \overline{\rho}_0 \frac{\partial^n u}{\partial t^n} - \Delta u + 2\pi\gamma(u - v) = f & \text{in } \Omega \times (0, T), \\ \overline{\rho}_1 \frac{\partial^n v}{\partial t^n} - k \frac{\partial^2 v}{\partial x_3^2} + 2\pi\gamma(v - u) = 0 & \text{in } \Omega \times (0, T), \\ + \ \text{boundary conditions.} \end{cases}$$

This result, which extends to the setting of linear evolution equations the results obtained in [11], [4] for elliptic equations, has been already partially announced in [3].

In the second example we assume that the lebesgue measure of the fibers is constant of order 1 but the conductivity of the surrounding matrix tends to 0. Thus, the condition of uniform boundedness is fullfilled, but not the condition of uniform ellipticity. More precisely, we assume:

$$\alpha = 2, \quad \lambda_\varepsilon = \frac{|\Omega|}{|T_\varepsilon|} k, \quad r_\varepsilon = c\varepsilon, \quad 0 < c < \frac{1}{2}.$$

The study of this problem (including the case $\alpha \neq 2$ and the case of linear elasticity) has been carried out in [19] by means of asymptotic expansions, for the peculiar behaviour of the mass density $\rho_\varepsilon(x)$ corresponding to $\overline{\rho}_0 = 0$. The limit problem associated with $\overline{\rho}_0 > 0$, more intricate (see Remark 2.6), is given in Theorem 2.5 and has the following form

$$\begin{cases} \frac{1}{1-\pi c^2}\overline{\rho}_0 \frac{\partial^n u}{\partial t^n} + \left(\overline{\rho}_1 - \frac{\pi c^2}{1-\pi c^2}\overline{\rho}_0\right)\frac{\partial^n v}{\partial t^n} - k\frac{\partial^2 v}{\partial x_3^2} = f \quad \text{in } \Omega\times(0,T), \\ u = \Phi_n(v) \quad \text{in } \Omega\times(0,T), \ + \ \text{boundary conditions}, \end{cases}$$

where u and v are described above and the affine correspondance Φ_n is function of the elementary evolution problems (18) on the unit cell.

Let us notice that the same scaling (with $n = 1$) is also used to modelize the weakly compressible single phase flow in a porous medium with cracks and fractures, leading to the so-called double porosity model (see [2]). In this case the porous medium $\Omega \setminus T_\varepsilon$ consists of an union of totaly disconnected open subsets of size ε.

The paper is organized as follows: in Section 2 we give some notations and state our main results in a precise way. In Section 3 some preliminary results are collected. Section 4 is devoted to the proofs.

2. Notations and main results

In the sequel, $\Omega := \omega\times]0,L[$ where ω is a bounded open domain of $\mathbb{R}^2$ with smooth boundary. The Lebesgue measure of a set E is denoted by $|E|$ and the mean value of a function f on E is denoted by $⨍_E f dx$. For any $K \subset \mathbb{R}^n$, $\mathcal{M}_b(K)$ (resp. $C_b(K)$) denotes the set of bounded Radon measure (resp. bounded continuous functions) on K. Given a sequence of positive reals (r_ε), the geometry of the set T_ε will be described in terms of the open disk of $\mathbb{R}^2$

$$D^\varepsilon := \{(x_1,x_2)\in\mathbb{R}^2,\ \sqrt{x_1^2+x_2^2} < \frac{r_\varepsilon}{\varepsilon}\}, \tag{2}$$

by considering first its periodization on all $\mathbb{R}^2$ given by $D^\varepsilon_\sharp := \bigcup_{i\in Z^2}\{i\} + D^\varepsilon$, then by setting

$$T_\varepsilon := (\omega \cap \varepsilon D^\varepsilon_\sharp)\times(0,L). \tag{3}$$

We introduce also

$$Y := \left(-\frac{1}{2},\frac{1}{2}\right)^2, \quad Y^i_\varepsilon := \varepsilon(\{i\}+Y), \quad D^i_\varepsilon := \varepsilon(\{i\}+D^\varepsilon), \quad I_\varepsilon := \{i\in Z^2,\ Y^i_\varepsilon \subset \omega\}. \tag{4}$$

Our aim is to study the asymptotic behaviour of the sequence of evolution problems

$$\rho_\varepsilon(x)\frac{\partial^n u_\varepsilon}{\partial t^n} - \operatorname{div}(a_\varepsilon(x)\nabla u_\varepsilon) = f \quad \text{in } \Omega\times(0,T), \quad u_\varepsilon \in \mathcal{D}_n, \tag{5}$$

where $n \in \{1,2\}$, $0 < T < +\infty$ and

$$\begin{aligned} &\mathcal{D}_1 := \{u \in L^2(0,T;H^1_0(\Omega))\cap C([0,T],L^2(\Omega)),\ u(0)=\varphi_0 \quad \text{in } \Omega\}, \\ &\mathcal{D}_2 := \left\{u\in C([0,T];H^1_0(\Omega))\cap C^1([0,T];L^2(\Omega)),\ u(0)=\varphi_0,\ \frac{\partial u}{\partial t}(0)=v_0 \quad \text{in } \Omega\right\}, \\ &f\in L^2(\Omega\times(0,T)), \quad \varphi_0\in L^2(\Omega), \qquad \text{if } n=1, \\ &f\in L^2(\Omega), \quad \varphi_0\in H^1_0(\Omega), \quad v_0\in L^2(\Omega) \qquad \text{if } n=2. \end{aligned} \tag{6}$$

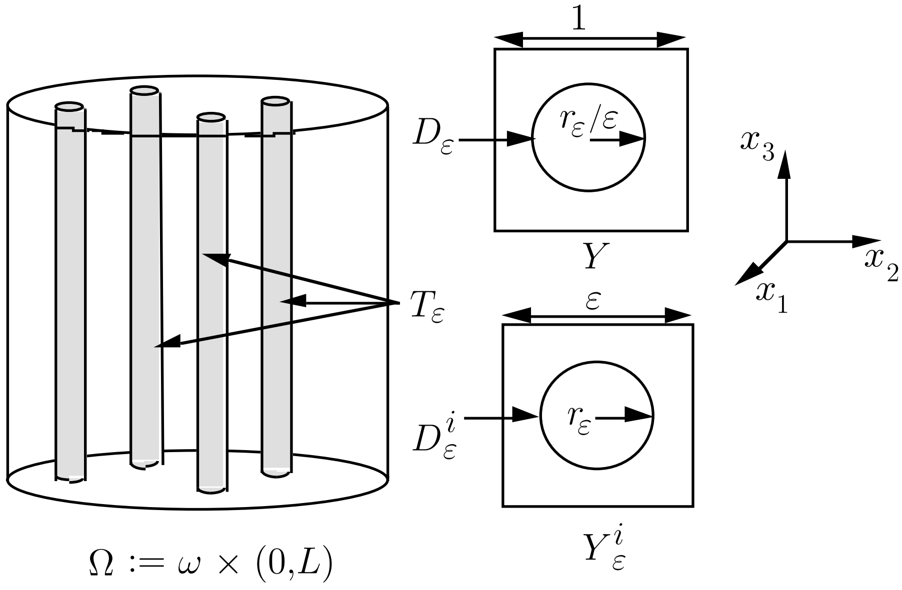

Figure 1.

Notice that if $n = 2$, f is assumed not to depend on t. The sequences a_ε and ρ_ε are defined by

$$a_\varepsilon(x) = \frac{|\Omega|}{|T_\varepsilon|}k_\varepsilon \quad \text{if } \ x \in T_\varepsilon, \quad a_\varepsilon(x) = \varepsilon^\alpha \quad \text{if } \ x \in \Omega \setminus T_\varepsilon,$$

$$\rho_\varepsilon(x) = \rho_{1\varepsilon} \quad \text{if } \ x \in T_\varepsilon, \quad \rho_\varepsilon(x) = \rho_{0\varepsilon} \quad \text{if } \ x \in \Omega \setminus T_\varepsilon,$$

where $\alpha \in \{1, 2\}$ and k_ε, $\rho_{1\varepsilon}$, $\rho_{0\varepsilon}$ are positive reals such that

$$\begin{array}{ccc} \lim\limits_{\varepsilon\to 0} k_\varepsilon = k, & \lim\limits_{\varepsilon\to 0} \frac{|\Omega\setminus T_\varepsilon|}{|\Omega|}\rho_{0\varepsilon} = \overline{\rho}_0, & \lim\limits_{\varepsilon\to 0} \frac{|T_\varepsilon|}{|\Omega|}\rho_{1\varepsilon} = \overline{\rho}_1, \\ k \in]0, +\infty], & \overline{\rho}_1 \in [0, +\infty[, & \overline{\rho}_0 \in [0, +\infty[. \end{array} \tag{7}$$

Notice that by (7), the sequence (ρ_ε) is bounded in $L^1(\Omega)$. The same holds for (a_ε) as soon as $k < +\infty$. The relative compactness of the solutions of (5) requires the following assumption

$$\begin{cases} \sup\limits_{\varepsilon>0} \displaystyle\int_\Omega \rho_\varepsilon(x)|\varphi_0|^2 dx < +\infty, & \text{if } n = 1, \\ \sup\limits_{\varepsilon>0} \left\{ \displaystyle\int_\Omega \rho_\varepsilon(x) \left(|\varphi_0|^2 + |v_0|^2 + |\nabla\varphi_0|^2\right) dx + \int_\Omega a_\varepsilon |\nabla\varphi_0(x)|^2 dx \right\} < +\infty, & \text{if } n = 2, \end{cases} \tag{8}$$

automatically satisfied if φ_0 is bounded (for $n = 1$) (resp. if v_0 and $\nabla\varphi_0$ are bounded and $k < +\infty$ if $n = 2$). The particular effects due respectively to the lack of uniform boundedness and the lack of uniform ellipticity are characterized by two typical choices of scaling. In each situation, the limit problem is expressed in terms of the limit u of the sequence u_ε of the solutions of (5) and the limit v of the sequence

$$v_\varepsilon(x) := \frac{|\Omega|}{|T_\varepsilon|} u_\varepsilon 1_{T_\varepsilon\times(0,T)}. \tag{9}$$

Example 2.1 (High conductivity). The function a_ε is supposed to take very high values on the subset T_ε of fibers, while at the same time the measure of T_ε tends to 0. More precisely we assume

$$0 < r_\varepsilon << \varepsilon, \quad \alpha = 0. \tag{10}$$

The limit problem depends on k, $\overline{\rho}_0$, $\overline{\rho}_1$ defined by (7), and on the parameter

$$\gamma := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2 |\ln(r_\varepsilon)|} \in [0, +\infty]. \tag{11}$$

It is expressed in terms of the limit u of the sequence u_ε of the solutions of (5) and the limit v of the sequence (v_ε) defined by (9) which describes the average behaviour of the restriction of u_ε to the fibers. The effective boundary condition are given by $(u, v) \in \mathcal{D}_n^{\text{eff}}$ defined by

$$\mathcal{D}_n^{\text{eff}} := \Big\{(u,v) \in (L^2(0,T; H_0^1(\Omega)) \times L^2(0,T; L^2(\omega, H_0^1(0,L))) \cap (\mathcal{C}_n(\overline{\rho}_0) \times \mathcal{C}_n(\overline{\rho}_1))\Big\} \tag{12}$$

where $\mathcal{C}_n(0) = L^2(\Omega \times (0,T))$ and if $r > 0$,

$$\mathcal{C}_n(r) := \left\{ g \in C^{n-1}([0,T]; L^2(\Omega)), \begin{array}{c} g(0) = \varphi_0 \;\text{ if } n = 1, \\ \\ g(0) = \varphi_0 \text{ and } \frac{\partial g}{\partial t}(0) = v_0 \text{ if } n = 2 \end{array} \right\}. \tag{13}$$

In the case $n = 2$, we assume that:

$$k < +\infty. \tag{14}$$

In order to deal with the case $\lim_{\varepsilon \to 0} \rho_{1\varepsilon} = +\infty$, we will make the following assumption:

$$\varphi_0, v_0 \in C(\overline{\Omega}). \tag{15}$$

Theorem 2.2. *Assume* (10), (15) *(and* (14) *if* $n = 2$*), then the sequence* (u_ε) *of the solutions of* (5) *weakly converges in* $L^2(0,T; \mathcal{H}_0^1(\Omega))$ *to* u *and the associated sequence* v_ε *defined by* (9) *weak-star converges in the sense of measures to a function* v*, where* (u,v) *is the unique solution of*

(*i*) *If* $k < +\infty$, $(u,v) \in \mathcal{D}_n^{\text{eff}}$ *(defined by* (12)*) and*

$$\begin{cases} \overline{\rho}_0 \frac{\partial^n u}{\partial t^n} - \Delta u + 2\pi\gamma(u - v) = f & \text{on } \; \Omega \times (0,T), \\ \overline{\rho}_1 \frac{\partial^n v}{\partial t^n} - k \frac{\partial^2 v}{\partial x_3^2} + 2\pi\gamma(v - u) = 0 & \text{on } \; \Omega \times (0,T), \end{cases} \qquad \text{if } \; 0 \leq \gamma < +\infty,$$

$$v = u; \quad (\overline{\rho}_0 + \overline{\rho}_1) \frac{\partial^n u}{\partial t^n} - \Delta u - k \frac{\partial^2 u}{\partial x_3^2} = f \quad \text{on } \; \Omega \times (0,T), \quad \text{if } \; \gamma = +\infty.$$

(*ii*) *If* $k = +\infty$ *and* $n = 1$,

$$\begin{cases} v = 0; \; \overline{\rho}_0 \frac{\partial u}{\partial t} - \Delta u + 2\pi\gamma u = f & \text{on } \; \Omega \times (0,T), \\ u \in L^2(0,T; H_0^1(\Omega)), \; u(0) = \varphi_0, \end{cases} \qquad \text{if } \; \gamma < +\infty,$$

$$u = v = 0, \qquad \text{if } \; \gamma = +\infty.$$

Remark 2.3. In the case where $0 < \gamma < +\infty$, $0 < k < +\infty$ and $\overline{\rho}_1 = 0$, the variable v can be computed in terms of u by solving a one dimensional boundary value problem (see the details in [4]), then substituted in the first equation yielding

$$\overline{\rho}_0 \frac{\partial^n u}{\partial t^n} - \Delta u + c(x_3)u - \int_0^L R(x_3, y_3)u(x_1, x_2, y_3, t)dy_3 = f,$$

where

$$c(x_3) := kc_0^2 \frac{1}{\sinh(c_0 L)} \left(\cosh(c_0(L - x_3)) \sinh(c_0 x_3) + \sinh(c_0(L - x_3)) \cosh(c_0 x_3)\right),$$
$$R(x_3, y_3) := kc_0^3 \frac{1}{\sinh(c_0 L)} \sinh(c_0(L - x_3 \vee y_3)) \sinh(c_0(x_3 \wedge y_3)),$$
$$c_0 := \sqrt{\frac{2\pi\gamma}{k}}.$$

Let $a(.,.)$ the continuous symetric bilinear form on $V := H_0^1(\Omega)$ defined by

$$a(u, w) := \int_\Omega \nabla u \nabla w dx + \int_\Omega c(x_3) u w dx \\ - \int_\omega \left(\iint_{(0,L)^2} R(x_3, y_3) u(x_1, x_2, x_3, t) w(x_1, x_2, y_3, t) dx_3 dy_3 \right) dx_1 dx_2.$$

Denoting $H := L^2(\Omega)$, it is easy to check that $a(w, w) \geq \alpha |w|_V^2 - C|w|_H^2$, $\forall w \in V$ for some suitable constants α, C. We have the following weak formulation of the limit problem in $\mathcal{D}'(0, T; V')$

$$\begin{cases} \overline{\rho}_0 < \dfrac{\mathrm{d}^n u}{\mathrm{d}t^n}, w > + a(u(t, .), w) = < f(t), w >, & \text{for a.e. } t \in [0, T], \quad \forall w \in V, \\ u(0) = \varphi_0, & \text{if } n = 1, \\ u(0) = \varphi_0, \quad \dfrac{\mathrm{d}u}{\mathrm{d}t}(0) = v_0, & \text{if } n = 2. \end{cases}$$

By the J. L. Lions' Theorems (see [16]), the last problem associated to initial conditions characterized by (6) has a unique solution u such that

$$\begin{cases} u \in L^2(0, T; V) \cap C([0, T], H), \ \dfrac{\mathrm{d}u}{\mathrm{d}t} \in L^2(0, T; V'), & \text{if } n = 1, \\ u \in C([0, T], V), \ \dfrac{\mathrm{d}u}{\mathrm{d}t} \in C([0, T], H), \ \dfrac{\mathrm{d}^2 u}{\mathrm{d}t^2} \in L^2(0, T; V'), & \text{if } n = 2. \end{cases}$$

Example 2.4 (Low conductivity). Now the function a_ε takes values of order 1 on T_ε and very small values elsewhere. More precisely we assume:

$$\alpha = 2, \quad r_\varepsilon = c\varepsilon \ (0 < c < \frac{1}{2}), \quad k_\varepsilon = k \in]0, +\infty[. \tag{16}$$

In this case the set $D := D^\varepsilon$ does not depend on ε. We denote by $C_\sharp^\infty(Y)$ the set of Y-periodic functions of $C^\infty(\mathbb{R}^2)$ and by $H_\sharp^1(Y)$ the completion of $C_\sharp^\infty(Y)$ with respect to the norm $w \to \left(\int_Y (|w|^2 + |\nabla w|^2) dx \right)^{\frac{1}{2}}$.

The limit problem, expressed as before in terms of the limit u of the sequence (u_ε) of the solutions of (5) and the limit v of the associated sequence (v_ε), is the following coupled system of equations

$$\begin{cases} \frac{1}{1-|D|}\overline{\rho}_0\frac{\partial^n u}{\partial t^n} + \left(\overline{\rho}_1 - \frac{|D|}{1-|D|}\overline{\rho}_0\right)\frac{\partial^n v}{\partial t^n} - k\frac{\partial^2 v}{\partial x_3^2} = f \quad \text{in} \quad \Omega\times(0,T), \\ u = \Phi_n(v), \\ (u,v)\in\mathcal{D}_n^{\mathrm{eff}}. \end{cases} \tag{17}$$

The affine correspondance Φ_n is defined as follows: given a function $z\in L^2(\Omega\times(0,T))$ and a fixed $x\in\Omega$, we denote by $w_n^{z,x}$ the unique solution (if possible) of the following elementary evolution problem on the unit cell Y:

$$\begin{aligned} &\begin{cases} \frac{1}{1-|D|}\overline{\rho}_0\frac{\partial w_1}{\partial t} - \Delta w_1(y,t) = f(x,t) \quad \text{in} \ (Y\setminus D)\times(0,T), \\ w_1\in L^2(0,T;H^1_\sharp(Y)),\ w_1(y,t)=z(x,t) \quad \text{a.e. in} \ D\times(0,T), \\ w_1\in C([0,T],L^2(Y)), \quad w_1(0)=\varphi_0, \ \text{if} \ \overline{\rho}_0>0, \end{cases} && \text{if } n=1, \\ &\begin{cases} \frac{1}{1-|D|}\overline{\rho}_0\frac{\partial^2 w_2}{\partial t^2} - \Delta w_2(y,t) = f(x) \quad \text{in} \ (Y\setminus D)\times(0,T), \\ w_2\in L^2(0,T;H^1_\sharp(Y)), w_2(y,t)=z(x,t) \quad \text{a.e. in} \ D\times(0,T), \\ w_2\in C([0,T];H^1_\sharp(Y))\cap C^1([0,T];L^2(Y)), w_2(0)=\varphi_0, \frac{\partial w_2}{\partial t}(0)=v_0 \ \text{if} \ \overline{\rho}_0>0, \end{cases} && \text{if } n=2. \end{aligned} \tag{18}$$

Then

$$(\Phi_n(z))(x,t) := \int_Y w_n^{z,x}(y,t)dy. \tag{19}$$

The domain $\mathcal{D}_n^{eff}$ is given by

$$\begin{aligned} \mathcal{D}_n^{eff} := \Big\{ &(u,v)\in\mathcal{C}_n(\overline{\rho_0},\overline{\rho_1})\times\mathcal{E}_n(\overline{\rho_1}), \\ &\frac{\partial v}{\partial x_3}\in L^2(\Omega\times(0,T)), v=0 \ \text{ on } \ \omega\times\{0,L\}\times(0,T)\Big\}, \end{aligned} \tag{20}$$

where

$$\mathcal{E}_n(\overline{\rho_1}) := \left\{ v\in C^{n-1}([0,T],L^2(\Omega)),\ v(0)=\varphi_0, \left(\text{and } \frac{\partial v}{\partial t}(0)=v_0 \text{ if } n=2\right)\right\}, \ \text{if } \overline{\rho_1}>0,$$

$$\begin{aligned} \mathcal{C}_n(\overline{\rho_0},\overline{\rho_1}) := \Bigg\{ &u\in C^{n-1}([0,T],L^2(\Omega)), \\ &u(0)=0\left(\text{ and } \frac{\partial u}{\partial t}(0)=v_0 \ \text{ if } \ n=2\right)\Bigg\}, \ \text{if } \overline{\rho_0}>0 \text{ and } \overline{\rho_1}>0, \end{aligned}$$

$$\mathcal{C}_n(0,r)=\mathcal{C}_n(r,0)=\mathcal{E}_n(0) := L^2(\Omega\times(0,T)), \quad \text{for } r\geq 0.$$

Theorem 2.5. *Assume* (16) *then the sequence* $(u_\varepsilon, v_\varepsilon)$ *where* u_ε *is the solution of* (5) *and* v_ε *is defined by* (9) *weakly converges in* $(L^2(\Omega\times(0,T)))^2$ *to a couple* (u,v) *solution of* (17)*.*

Remark 2.6. Example 2.4 is studied in [19] assuming $\overline{\rho}_0 = 0$. This describes a situation where the component of the matrix is at the same time very insulating and very light, or, in the framework of linear elasticity, where the fibers are simultaneously very heavy and very rigid (the function f in (5) is then multiplied by ρ_ε). In this case the limit problem is simplified and has the following form:

$$\begin{cases} \overline{\rho}_1 \frac{\partial^n v}{\partial t^n} - k \frac{\partial^2 v}{\partial x_3^2} = f, & \text{in } \ \Omega \times (0,T), \\ C(v-u) = f, & \text{in } \ \Omega \times (0,T), \\ + \ \text{boundary conditions}, & \end{cases}$$

where

$$C := \inf \left\{ \int_Y |\nabla w|^2 dy, \ w \in H^1_\sharp(Y), \ w = 1 \quad \text{on} \quad D, \ \ -\!\!\!\!\!\!\int_Y w dy = 0 \right\}.$$

3. Preliminary results

Let us introduce the sequence of measures μ_ε on $\Omega \times (0,T)$ defined by

$$\mu_\varepsilon := \frac{|\Omega|}{|\mathbf{T}_\varepsilon|} 1_{T_\varepsilon}(x) dx\, dt. \tag{21}$$

When (10) is assumed, we denote by (R_ε) a sequence of positive reals such that

$$r_\varepsilon << R_\varepsilon << \varepsilon, \tag{22}$$

and consider the following subsets of $\mathbb{R}^2$

$$C^{R_\varepsilon} := \left\{ (x_1,x_2) \in \mathbb{R}^2, \ \sqrt{x_1^2 + x_2^2} = \frac{R_\varepsilon}{\varepsilon} \right\}, \quad C^i_{R_\varepsilon} := \varepsilon(\{i\} + C^{R_\varepsilon}),$$
$$C^{r_\varepsilon} := \left\{ (x_1,x_2) \in \mathbb{R}^2, \ \sqrt{x_1^2 + x_2^2} = \frac{r_\varepsilon}{\varepsilon} \right\}, \quad C^i_{r_\varepsilon} := \varepsilon(\{i\} + C^{r_\varepsilon}),$$

and the functions $\tilde{u}_\varepsilon$ and $\tilde{v}_\varepsilon$ defined on $\Omega \times (0,T)$ by

$$\tilde{u}_\varepsilon(x_1,x_2,x_3,t) := \sum_{i \in I_\varepsilon} \left(-\!\!\!\!\!\!\int_{C^i_{R_\varepsilon}} u_\varepsilon(s_1,s_2,x_3,t) d\mathcal{H}^1(s_1,s_2) \right) 1_{Y^i_\varepsilon}(x_1,x_2), \tag{23}$$

$$\tilde{v}_\varepsilon(x_1,x_2,x_3,t) := \sum_{i \in I_\varepsilon} \left(-\!\!\!\!\!\!\int_{C^i_{r_\varepsilon}} u_\varepsilon(s_1,s_2,x_3,t) d\mathcal{H}^1(s_1,s_2) \right) 1_{Y^i_\varepsilon}(x_1,x_2), \tag{24}$$

where Y^i_ε, I_ε are given by (4). The following proposition characterizes the asymptotic behaviour of several sequences associated to the solution u_ε of (5) under the assumption (10) of Example 2.1. In the sequel, the letter "C" denotes a suitable positive constant independent of ε and which may vary from line to line.

Proposition 3.1 (Example 2.1). *Assume* (10) *and let* u_ε *be a sequence in* $L^2(0,T;\, H_0^1(\Omega))$ *such that*

$$\int_{t=0}^{T}\int_{\Omega}|u_\varepsilon|^2 dxdt \le C, \qquad \int_{t=0}^{T}\int_{\Omega} a_\varepsilon|\nabla u_\varepsilon|^2 dxdt \le C. \tag{25}$$

Then,

$$\int |u_\varepsilon|^2 d\mu_\varepsilon \le C, \qquad \int \left|\frac{\partial u_\varepsilon}{\partial x_3}\right|^2 d\mu_\varepsilon \le C, \tag{26}$$

and there exists two functions u, v *such that, up to a subsequence,*

$$u_\varepsilon \rightharpoonup u \quad \textit{weakly in} \quad L^2(0,T;\, H_0^1(\Omega)), \tag{27}$$

$$\begin{gathered} u_\varepsilon\mu_\varepsilon \overset{\star}{\rightharpoonup} v dxdt, \quad \frac{\partial u_\varepsilon}{\partial x_3}\mu_\varepsilon \overset{\star}{\rightharpoonup} \frac{\partial v}{\partial x_3}dxdt \quad \star\textit{-weakly in} \quad \mathcal{M}_b(\Omega\times(0,T)),\\ v, \quad \frac{\partial v}{\partial x_3} \in L^2(\Omega\times(0,T)), \end{gathered} \tag{28}$$

$$v = 0 \quad \textit{on} \quad \omega\times\{0,L\}\times(0,T), \tag{29}$$

$$\tilde{u}_\varepsilon - u_\varepsilon \to 0 \quad \textit{strongly in} \quad L^2(\Omega\times(0,T)), \tag{30}$$

$$\tilde{v}_\varepsilon \rightharpoonup v \quad \textit{weakly in} \quad L^2(\Omega\times(0,T)). \tag{31}$$

Proof. We derive easily (27) from (25). On the other hand, by the Fubini's theorem, (7) and (21) we have

$$\int |u_\varepsilon|^2 d\mu_\varepsilon = \int \left|\int_0^{x_3}\frac{\partial u_\varepsilon}{\partial x_3}(x_1,x_2,z,t)dz\right|^2 d\mu_\varepsilon \le C\int_{t=0}^{T}\int_\Omega a_\varepsilon|\nabla u_\varepsilon|^2 dxdt. \tag{32}$$

From (32) and (25) we deduce (26). We apply then the next lemma (see [6], [10]) to the sequence of measures $\nu_\varepsilon := \mu_\varepsilon$, the compact $K := \overline{\Omega\times(0,T)}$ and the sequence of functions $f_\varepsilon = u_\varepsilon$ (resp. $f_\varepsilon = \frac{\partial u_\varepsilon}{\partial x_3}$):

Lemma 3.2. *Let* ν_ε *and* ν *bounded Radon measures on a compact* $K \subset \mathbb{R}^N$ *such that* $\nu_\varepsilon \overset{\star}{\rightharpoonup} \nu$. *Let* (f_ε) *a sequence of* ν_ε*-measurable functions such that* $\sup_\varepsilon \int |f_\varepsilon|^2 d\nu_\varepsilon < +\infty$. *Then the sequence of measures* $f_\varepsilon\nu_\varepsilon$ *is sequentially relatively compact in the weak-star topology* $\sigma(\mathcal{M}_b(K), C_b(K))$ *and every cluster point* m *is of the form* $m = f\nu$ *with* $f \in L^2_\nu$. *Moreover, if* $f_\varepsilon\nu_\varepsilon \overset{\star}{\rightharpoonup} f\nu$, *then* $\liminf_\varepsilon \int |f_\varepsilon|^2 d\nu_\varepsilon \ge \int |f|^2 d\nu$.

We deduce that, up to a subsequence,

$$u_\varepsilon\mu_\varepsilon \overset{\star}{\rightharpoonup} v dxdt, \quad \frac{\partial u_\varepsilon}{\partial x_3}\mu_\varepsilon \overset{\star}{\rightharpoonup} g dxdt, \quad \text{for some } v, g \in L^2(\Omega\times(0,T)).$$

Testing the latter convergences with a function $\varphi \in C^\infty(\overline{\Omega\times(0,T)})$, we obtain

$$\int_{t=0}^{T}\int_\Omega g\varphi dxdt = \lim_{\varepsilon\to0}\int \frac{\partial u_\varepsilon}{\partial x_3}\varphi d\mu_\varepsilon = -\lim_{\varepsilon\to0}\int \frac{\partial \varphi}{\partial x_3}u_\varepsilon d\mu_\varepsilon = -\int_{t=0}^{T}\int_\Omega \frac{\partial\varphi}{\partial x_3} v dxdt, \tag{33}$$

and infer (28). Then by integrating by parts the right-hand member of (33) we get

$$\int_{\omega\times(0,T)} (\varphi v(x_1,x_2,L,t) - \varphi v(x_1,x_2,0,t))dx_1dx_2dt = 0,$$

and by the arbitrary choice of φ obtain (29). The following estimates are proved in [4, p. 420]:

$$\int_{t=0}^{T}\int_{\Omega} |u_\varepsilon - \tilde{u}_\varepsilon|^2 dxdt \le C\varepsilon^2(1 + \left|\ln \frac{R_\varepsilon\sqrt{2}}{\varepsilon}\right| \int_{t=0}^{T}\int_{\Omega} |\nabla u_\varepsilon|^2 dxdt, \tag{34}$$

$$\int |u_\varepsilon - \tilde{v}_\varepsilon|^2 d\mu_\varepsilon \le C r_\varepsilon^2 \int |\nabla u_\varepsilon|^2 d\mu_\varepsilon = C\frac{r_\varepsilon^2}{k_\varepsilon}\int_0^T \int_{T_\varepsilon} a_\varepsilon |\nabla u_\varepsilon|^2 dxdt. \tag{35}$$

From (22), (25) and (34) we deduce (30). Applying Lemma 3.2 with $f_\varepsilon = u_\varepsilon - \tilde{v}_\varepsilon$ and $\nu_\varepsilon = \mu_\varepsilon$, taking into account (7), (25), (26), (28), (35) we infer

$$\int |\tilde{v}_\varepsilon|^2 d\mu_\varepsilon \le C, \quad \tilde{v}_\varepsilon \mu_\varepsilon \stackrel{\star}{\rightharpoonup} v dxdt \quad \text{in } \mathcal{M}(\Omega \times (0,T)). \tag{36}$$

On the other hand, in accordance with the definitions (21) and (24), there holds

$$\int |\tilde{v}_\varepsilon|^2 d\mu_\varepsilon = \int_{t=0}^{T}\int_{\Omega} |\tilde{v}_\varepsilon|^2 dxdt.$$

Therefore by (36), $(\tilde{v}_\varepsilon)$ is bounded in L^2 and up to a subsequence weakly converges to some $h \in L^2$. The proof of Proposition 3.1 is achieved provided we show that $h = v$ which results by passing to the limit as $\varepsilon \to 0$ in the following estimate

$$\left| \int \varphi\, \tilde{v}_\varepsilon d\mu_\varepsilon - \int_{\Omega\times(0,T)} \varphi\, \tilde{v}_\varepsilon dxdt \right| \le C\varepsilon,$$

holding in view of the definitions (24) and (21), for any $\varphi \in C(\overline{\Omega \times (0,T)})$. □

The next results will be used in the proof of Theorem 2.5 and rely on the two-scale approach (see [1], [18]). For the reader's convenience, let us recall that a sequence (f_ε) in $L^2(\Omega \times (0,T))$ two-scale converges with respect to the variable $x' := (x_1, x_2)$ to $f_0 \in L^2(\Omega \times (0,T) \times Y)$ (this property will be denoted simply: $f_\varepsilon \rightharpoonup\!\!\rightharpoonup f_0$) if for each $\Psi \in \mathcal{D}(\Omega \times (0,T), C^\infty_\sharp(Y))$,

$$\lim_{\varepsilon\to 0} \int_{t=0}^{T}\int_{\Omega} f_\varepsilon(x,t)\Psi\left(x,t,\frac{x'}{\varepsilon}\right) dxdt = \int_{t=0}^{T}\int_{\Omega\times Y} f_0(x,t,y)\Psi(x,t,y)dxdydt. \tag{37}$$

It is well known that any bounded sequence in $L^2(\Omega \times (0,T))$ has a two-scale converging subsequence. Proposition 3.3 specifies several relations between different limits associated with a sequence of functions (u_ε) and their gradients:

Proposition 3.3. *Let (u_ε) be a sequence in $L^2(0,T;H_0^1(\Omega))$ such that the sequence $(\nabla u_\varepsilon 1_{T_\varepsilon\times(0,T)})$ is bounded in $L^2(\Omega\times(0,T))$. Assume (16) and the following convergences:*

$$\begin{aligned} &u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^2(\Omega\times(0,T)), \quad v_\varepsilon \rightharpoonup v \quad \text{weakly in } L^2(\Omega\times(0,T)),\\ &u_\varepsilon \rightharpoonup\!\!\rightharpoonup u_0, \qquad \varepsilon\nabla u_\varepsilon 1_{(\Omega\setminus T_\varepsilon)\times(0,T)} \rightharpoonup\!\!\rightharpoonup \chi_m. \end{aligned} \tag{38}$$

Then,

$$\begin{aligned} &u_0 \in L^2(\Omega\times(0,T),H^1_\sharp(Y)), \quad \chi_m = \nabla_y u_0 \quad \text{a.e. on} \quad \Omega\times(0,T)\times Y\\ &u(x,t) = \int_Y u_0(x,t,y)dy, \quad v(x,t) = u_0(x,t,y) \quad \text{a. e. on} \quad \Omega\times(0,T)\times D. \end{aligned} \tag{39}$$

If u_ε satisfies furthermore

$$u_\varepsilon \in L^2(\Omega,H^1(0,T)), \quad \frac{\partial u_\varepsilon}{\partial t} \rightharpoonup\!\!\rightharpoonup g, \quad u_\varepsilon(0) = \varphi_0, \tag{40}$$

then besides (39) there holds

$$\begin{aligned} &u_0 \in L^2(\Omega\times Y,H^1(0,T)), \quad g = \frac{\partial u_0}{\partial t}, \quad u_0(x,0,y) = \varphi_0(x) \quad \text{a.e. on} \quad \Omega\times Y,\\ &u \in L^2(\Omega,H^1(0,T)), \quad \frac{\partial u}{\partial t}(x,t) = \int_Y \frac{\partial u_0}{\partial t}(x,t,y)dy \quad \text{a.e. on} \quad \Omega\times(0,T),\\ &v \in L^2(\Omega,H^1(0,T)), \quad \frac{\partial v}{\partial t}(x,t) = \fint_D \frac{\partial u_0}{\partial t}(x,t,y)dy \quad \text{a.e. on} \quad \Omega\times(0,T),\\ &v(x,0) = \varphi_0(x) \quad \text{a.e. on} \quad \Omega. \end{aligned} \tag{41}$$

Proof. Very similar properties are proved in ([1], [5]). However, for the convenience of the reader we give the proof of (41): by choosing a test function $\varphi \in C^\infty(\overline{\Omega\times(0,T)},C^\infty_\sharp(Y))$ such that $\varphi = 0$ on $\Omega\times\{T\}\times Y$, and passing to the limit as $\varepsilon\to 0$ in the following integration by parts formula

$$\int_0^T\int_\Omega \frac{\partial u_\varepsilon}{\partial t}(x,t)\varphi(x,t,\frac{x'}{\varepsilon})dxdt = -\int_0^T\int_\Omega u_\varepsilon(x,t)\frac{\partial\varphi}{\partial t}(x,t\frac{x'}{\varepsilon}) - \int_\Omega \varphi_0(x,0)\varphi(x,0,\frac{x'}{\varepsilon})dx,$$

we obtain by (38), (40)

$$\begin{aligned} &\int_0^T\iint_{\Omega\times Y} g(x,t,y)\varphi(x,t,y)dxdt\\ &\qquad = -\int_0^T\iint_{\Omega\times Y} u_0(x,t,y)\frac{\partial\varphi}{\partial t}(x,t,y) - \iint_{\Omega\times Y}\varphi_0(x,0)\varphi(x,0,y)dxdy. \end{aligned}$$

As the latter equation holds for arbitrary $\varphi\in\mathcal{D}(\Omega\times(0,T)\times Y)$, we deduce that $\frac{\partial u_0}{\partial t} = g$ in $\mathcal{D}'(\Omega\times(0,T)\times Y)$ and $u_0\in L^2(\Omega\times Y,H^1(0,T))$. Then by integration by parts we infer that

$$\iint_{\Omega\times Y}\varphi_0(x,0)\varphi(x,0,y)dxy = \iint_{\Omega\times Y} u_0(x,0,y)\varphi(x,0,y)dxy,$$

and deduce from the arbitrary choice of φ that $u_0(x,0,y) = \varphi_0(x)$ a.e. on $\Omega\times Y$. We conclude the proof of (41) by straightforward localization arguments. $\square$

4. Proof of the Theorems 2.2 and 2.5

Proof of Theorem 2.2. In order to apply Proposition 3.1, we will prove that the solution u_ε of (5) satisfies (25). To that aim, assuming first $n = 1$, we multiply the equation (5) by u_ε and integrate it over $\Omega \times (0,T)$. After integration by parts we obtain

$$\frac{1}{2}\int_\Omega \rho_\varepsilon(x)|u_\varepsilon(x,T)|^2 dx - \frac{1}{2}\int_\Omega \rho_\varepsilon(x)|\varphi_0(x)|^2 dx + \int_{t=0}^T\int_\Omega a_\varepsilon|\nabla u_\varepsilon|^2 dxdt = \int_{t=0}^T\int_\Omega f u_\varepsilon dxdt.$$

In particular the following inequality holds:

$$\int_{t=0}^T\int_\Omega a_\varepsilon|\nabla u_\varepsilon|^2 dxdt \le ||f||_{L^2}||u_\varepsilon||_{L^2} + \frac{1}{2}\int_\Omega \rho_\varepsilon(x)|\varphi_0(x)|^2 dx, \qquad (n=1) \tag{42}$$

from which, thanks to (7) and (8) one can easily derive (25).

If $n = 2$ we multiply (5) by $\frac{\partial u_\varepsilon}{\partial t}$ and integrate it over Ω. After integration by parts with respect to the space variables we obtain

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\int_\Omega \rho_\varepsilon(x)\left|\frac{\partial u_\varepsilon}{\partial t}(x,t)\right|^2 dx + \frac{1}{2}\int_\Omega a_\varepsilon|\nabla u_\varepsilon|^2 dx - \int_\Omega f(x)u_\varepsilon(x,t)dx\right) = 0.$$

By (6), (7), (8) and (14) we infer

$$\begin{aligned}&\frac{1}{2}\int_\Omega \rho_\varepsilon(x)\left|\frac{\partial u_\varepsilon}{\partial t}(x,t)\right|^2 dx + \frac{1}{2}\int_\Omega a_\varepsilon|\nabla u_\varepsilon|^2 dx - \int_\Omega f(x)u_\varepsilon(x,t)dx\\ &\qquad = \frac{1}{2}\int_\Omega \rho_\varepsilon(x)\left|v_0(x)\right|^2 dx + \frac{1}{2}\int_\Omega a_\varepsilon|\nabla \varphi_0(x)|^2 dx - \int_\Omega f(x)\varphi_0(x)dx < C,\end{aligned} \tag{43}$$

and deduce (25).

The second step consists of the construction of an appropriate sequence of test functions Φ_ε by which we will multiply (5), then pass to the limit as $\varepsilon \to 0$ in accordance with the convergences stated in Proposition 3.1, to get a weak formulation of the limit problem. To that aim, we fix two regular functions $\varphi, \psi \in C^\infty(\overline{\Omega \times (0,T)})$ such that $\varphi = \psi = 0$ on $\partial\Omega\times]0,T]$, and introduce the following subset B_ε of Ω consisting of e_3-parallel tubes surrounding the fibers of outer radius R_ε:

$$\begin{aligned}&B_\varepsilon := \{x \in \Omega, \quad r_\varepsilon < d_\varepsilon(x') < R_\varepsilon\},\\ &x' := (x_1, x_2), \quad d_\varepsilon(x') := \mathrm{dist}(x', \{\varepsilon i, i \in Z^2\}).\end{aligned} \tag{44}$$

The sequence of test functions (Φ_ε) is defined as follows:

$$\Phi_\varepsilon(x,t) := (1-\theta_\varepsilon(x'))\varphi(x,t) + \theta_\varepsilon(x')(\varphi_\varepsilon(x,t) + \psi_\varepsilon(x,t)), \tag{45}$$

where

$$\varphi_\varepsilon(x,t) := \sum_{i\in I_\varepsilon}\left(⨍_{D^i_\varepsilon}\varphi(.,.,x_3,t)\right)1_{Y^i_\varepsilon}(x'), \quad \psi_\varepsilon(x,t) := \sum_{i\in I_\varepsilon}\left(⨍_{D^i_\varepsilon}\psi(.,.,x_3,t)\right)1_{Y^i_\varepsilon}(x'), \tag{46}$$

and $\theta_\varepsilon : \mathbb{R}^2 \to [0,1]$ is given by

$$\theta_\varepsilon(x') := 1 \ \text{ if } d_\varepsilon(x') < r_\varepsilon; \ \ \theta_\varepsilon(x') := 0 \ \text{ if } d_\varepsilon(x') > R_\varepsilon; \ \ \theta_\varepsilon(x') := \frac{\ln \frac{d_\varepsilon(x')}{R_\varepsilon}}{\ln \frac{r_\varepsilon}{R_\varepsilon}} \ \text{ otherwise.} \tag{47}$$

Let us multiply (5) by Φ_ε and integrate it by parts over $\Omega \times (0,T)$. We get, if $n = 1$:

$$-\int_\Omega \rho_\varepsilon(x)\varphi_0(x)\Phi_\varepsilon(x,0)dx - \int_0^T\!\!\int_\Omega \rho_\varepsilon(x)u_\varepsilon \frac{\partial \Phi_\varepsilon}{\partial t} dxdt + \int_0^T\!\!\int_\Omega a_\varepsilon \nabla u_\varepsilon \nabla \Phi_\varepsilon dxdt = \int_0^T\!\!\int_\Omega f\Phi_\varepsilon dxdt, \tag{48}$$

and if $n = 2$:

$$\begin{aligned} -\int_\Omega \rho_\varepsilon(x)v_0(x)\Phi_\varepsilon(x,0)dx + \int_\Omega \rho_\varepsilon(x)\varphi_0(x)\frac{\partial \Phi_\varepsilon}{\partial t}(x,0)dx \\ + \int_0^T\!\!\int_\Omega \rho_\varepsilon(x)u_\varepsilon \frac{\partial^2 \Phi_\varepsilon}{\partial t^2} dxdt + \int_0^T\!\!\int_\Omega a_\varepsilon \nabla u_\varepsilon \nabla \Phi_\varepsilon dxdt = \int_0^T\!\!\int_\Omega f\Phi_\varepsilon dxdt. \end{aligned} \tag{49}$$

As the sequence (Φ_ε) converges strongly to φ in $L^2(\Omega \times (0,T))$ and since, by (7), (11), (22) and (45), the sequence $(\rho_\varepsilon(x)\Phi_\varepsilon(x,0))$ convergences star-weakly in $\mathcal{M}_b(\Omega)$ to the function $x \to (\overline{\rho}_0\varphi + \overline{\rho}_1(\varphi+\psi))(x,0)$, taking (15) into account, we deduce

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_\Omega \rho_\varepsilon(x)\varphi_0(x)\Phi_\varepsilon(x,0)dx = \int_\Omega \varphi_0(x)(\overline{\rho}_0\varphi + \overline{\rho}_1(\varphi+\psi))(x,0)dx \quad (n=1), \\ &\lim_{\varepsilon\to 0} \int_\Omega \rho_\varepsilon(x)v_0(x)\Phi_\varepsilon(x,0)dx = \int_\Omega v_0(x)(\overline{\rho}_0\varphi + \overline{\rho}_1(\varphi+\psi))(x,0)dx \quad (n=2), \\ &\lim_{\varepsilon\to 0} \int_{t=0}^T \int_\Omega f\Phi_\varepsilon \, dx \, dt = \int_{t=0}^T \int_\Omega f\varphi \, dx \, dt. \end{aligned} \tag{50}$$

By (10), (11), (27) and (28) we get

$$\rho_\varepsilon(x)u_\varepsilon 1_{T_\varepsilon} \overset{\star}{\rightharpoonup} \overline{\rho}_1 v \ \text{ in } \mathcal{M}_b(\Omega \times (0,T)) \ \text{ and } \ \rho_\varepsilon(x)u_\varepsilon 1_{\Omega\setminus T_\varepsilon} \rightharpoonup \overline{\rho}_0 u \ \text{ weakly in } L^2(\Omega \times (0,T)),$$

and deduce, by the estimate

$$\left| \frac{\partial^n \Phi_\varepsilon}{\partial t^n} - \frac{\partial^n(\varphi+\psi)}{\partial t^n} \right|(x,t) \leq Cr_\varepsilon \quad \text{for } x \in T_\varepsilon,$$

and by the strong convergence in $L^2(\Omega \times (0,T))$ of the sequence $\frac{\partial^n \Phi_\varepsilon}{\partial t^n} 1_{\Omega\setminus T_\varepsilon}$ to $\frac{\partial^n \varphi}{\partial t^n}$, that

$$\lim_{\varepsilon\to 0} \int_{t=0}^T \int_\Omega \rho_\varepsilon(x)u_\varepsilon \frac{\partial^n \Phi_\varepsilon}{\partial t^n} \, dx \, dt = \int_{t=0}^T \int_\Omega \left(\overline{\rho}_0 u \frac{\partial^n \varphi}{\partial t^n} + \overline{\rho}_1 v \frac{\partial^n(\varphi+\psi)}{\partial t^n} \right) dx \, dt. \tag{51}$$

Next we split the remaining term of (48) (or (49)) into the following sum of three terms:

$$\begin{aligned} &\int_{t=0}^T \int_\Omega a_\varepsilon \nabla u_\varepsilon \nabla \Phi_\varepsilon \, dx \, dt = I_{1\varepsilon} + I_{2\varepsilon} + I_{3\varepsilon}, \ I_{1\varepsilon} := \int_{t=0}^T \int_{\Omega\setminus(B_\varepsilon \cup T_\varepsilon)} \nabla u_\varepsilon \nabla \varphi dxdt, \\ &I_{2\varepsilon} := \int_{t=0}^T \int_{B_\varepsilon} \nabla u_\varepsilon \nabla \Phi_\varepsilon \, dx \, dt, \quad I_{3\varepsilon} := \int k_\varepsilon \nabla u_\varepsilon \nabla \Phi_\varepsilon d\mu_\varepsilon. \end{aligned} \tag{52}$$

By (27) and by the strong convergence in L^2 of $\nabla\varphi\,\mathbf{1}_{\Omega\setminus(B_\varepsilon\cup T_\varepsilon)}$ to $\nabla\varphi$, we obtain

$$\lim_{\varepsilon\to 0} I_{1\varepsilon} = \int_{t=0}^{T}\int_{\Omega} \nabla u \nabla\varphi\, dx\, dt. \tag{53}$$

With regard to $I_{3\varepsilon}$, let us notice that since $\dfrac{\partial\Phi_\varepsilon}{\partial x_1} = \dfrac{\partial\Phi_\varepsilon}{\partial x_2} = 0$ on $T_\varepsilon\times(0,T)$, we have

$$I_{3\varepsilon} = \int k_\varepsilon \frac{\partial u_\varepsilon}{\partial x_3}\frac{\partial(\varphi_\varepsilon+\psi_\varepsilon)}{\partial x_3} d\mu_\varepsilon, \tag{54}$$

and by (26) and (46) the following estimate holds

$$\left|\int \frac{\partial u_\varepsilon}{\partial x_3}\frac{\partial(\varphi_\varepsilon+\psi_\varepsilon)}{\partial x_3} d\mu_\varepsilon - \int \frac{\partial u_\varepsilon}{\partial x_3}\frac{\partial(\varphi+\psi)}{\partial x_3} d\mu_\varepsilon\right| \leq C r_\varepsilon. \tag{55}$$

Let us assume that $k<+\infty$. It follows then from (7) and (28) that

$$\lim_{\varepsilon\to 0} I_{3\varepsilon} = \int_{t=0}^{T}\int_{\Omega} k\frac{\partial v}{\partial x_3}\frac{\partial(\varphi+\psi)}{\partial x_3}\, dx\, dt. \tag{56}$$

In order to study the asymptotic behaviour of $I_{2\varepsilon}$, we first notice that (see below)

$$\lim_{\varepsilon\to 0}\left(I_{2\varepsilon} - \int_{0}^{T}\int_{B_\varepsilon} \psi_\varepsilon \nabla u_\varepsilon \nabla\theta_\varepsilon\, dx\, dt\right) = 0. \tag{57}$$

By using cylindrical coordinates on each tubular set B^i_ε ($\theta_\varepsilon(r,\theta) = \frac{\ln\frac{r}{R_\varepsilon}}{\ln\frac{r_\varepsilon}{R_\varepsilon}}$) we obtain:

$$\begin{aligned}
&\int_0^T\int_{B^i_\varepsilon} \psi_\varepsilon\nabla u_\varepsilon\nabla\theta_\varepsilon r\, dr\, d\theta\, dt\\
&\quad = \frac{1}{\ln\frac{r_\varepsilon}{R_\varepsilon}}\int_0^T\int_{x_3=0}^{L}\int_{\theta=0}^{2\pi}\int_{r=r_\varepsilon}^{R_\varepsilon}\frac{\partial u_\varepsilon}{\partial r}(r,\theta,x_3,t)\psi_\varepsilon(x_3)\, dr\, d\theta\, dx_3\, dt\\
&\quad = \frac{1}{\ln\frac{r_\varepsilon}{R_\varepsilon}}\int_{t=0}^T\int_{x_3=0}^{L}\int_{\theta=0}^{2\pi}(u_\varepsilon(R_\varepsilon,\theta,x_3)-u_\varepsilon(r_\varepsilon,\theta,x_3))\psi_\varepsilon(x_3)\, d\theta\, dx_3\, dt\\
&\quad = \frac{2\pi}{\varepsilon^2\ln\frac{r_\varepsilon}{R_\varepsilon}}\int_0^T\int_{Y^i_\varepsilon\times(0,L)}(\tilde u_\varepsilon-\tilde v_\varepsilon)\psi_\varepsilon dx\, dt,
\end{aligned} \tag{58}$$

where $\tilde u_\varepsilon$ and $\tilde v_\varepsilon$ are given by (23), (24). By adding up over $i\in I_\varepsilon$ we get

$$\int_0^T\int_{B_\varepsilon}\psi_\varepsilon\nabla u_\varepsilon\nabla\theta_\varepsilon\, dx\, dt = \frac{2\pi}{\varepsilon^2\left|\ln\frac{r_\varepsilon}{R_\varepsilon}\right|}\int_{t=0}^T\int_\Omega(\tilde v_\varepsilon-\tilde u_\varepsilon)\psi_\varepsilon\, dx\, dt. \tag{59}$$

We deduce from (30), (31) and the estimate $|\psi-\psi_\varepsilon|\leq C\varepsilon$ that

$$\lim_{\varepsilon\to 0}\int_{t=0}^T\int_\Omega(\tilde v_\varepsilon-\tilde u_\varepsilon)\psi_\varepsilon\, dx\, dt = \int_{t=0}^T\int_\Omega(v-u)\psi\, dx\, dt. \tag{60}$$

Let us assume that $0 \leq \gamma < +\infty$. Then by (11), (57), (59) and (60) we have

$$\lim_{\varepsilon \to 0} I_{2\varepsilon} = 2\pi\gamma \int_{t=0}^{T} \int_{\Omega} (v-u)\psi \, dx \, dt. \tag{61}$$

Collecting (48), (50), (51), (52), (53), (56) and (61) we infer, if $n = 1$, that

$$\begin{aligned} &- \int_{\Omega} \varphi_0(x)(\overline{\rho}_0 \varphi + \overline{\rho}_1(\varphi+\psi))(x,0)dx - \int_0^T \int_{\Omega} \left(\overline{\rho}_0 u \frac{\partial \varphi}{\partial t} + \overline{\rho}_1 v \frac{\partial(\varphi+\psi)}{\partial t} \right) dx \, dt \\ &\quad + \int_0^T \int_{\Omega} \nabla u \nabla \varphi \, dx \, dt + k \int_{t=0}^{T} \int_{\Omega} \frac{\partial v}{\partial x_3} \frac{\partial(\varphi+\psi)}{\partial x_3} \, dx \, dt + 2\pi\gamma \int_{t=0}^{T} \int_{\Omega} (v-u)\psi \, dx \, dt \\ &= \int_{t=0}^{T} \int_{\Omega} f\varphi \, dx \, dt, \end{aligned} \tag{62}$$

and if $n = 2$,

$$\begin{aligned} &- \int_{\Omega} v_0(x)(\overline{\rho}_0 \varphi + \overline{\rho}_1(\varphi+\psi))(x,0)dx + \int_{\Omega} \varphi_0(x)(\overline{\rho}_0 \varphi + \overline{\rho}_1(\varphi+\psi))(x,0)dx \\ &\quad + \int_0^T \int_{\Omega} \left(\overline{\rho}_0 u \frac{\partial^2 \varphi}{\partial t^2} + \overline{\rho}_1 v \frac{\partial^2(\varphi+\psi)}{\partial t^2} \right) dx \, dt + \int_0^T \int_{\Omega} \nabla u \nabla \varphi \, dx \, dt \\ &\quad + k \int_{t=0}^{T} \int_{\Omega} \frac{\partial v}{\partial x_3} \frac{\partial(\varphi+\psi)}{\partial x_3} \, dx \, dt + 2\pi\gamma \int_{t=0}^{T} \int_{\Omega} (v-u)\psi \, dx \, dt \\ &= \int_{t=0}^{T} \int_{\Omega} f\varphi \, dx \, dt. \end{aligned} \tag{63}$$

By choosing successively $\psi = 0$ and $\varphi = 0$ in (62), (63) we deduce from the arbitrary nature of φ, ψ, from the standart regularity results (see [16]) and from (29) that the couple (u, v) is solution of:

$$\begin{cases} \overline{\rho}_0 \dfrac{\partial^n u}{\partial t^n} - \Delta u + 2\pi\gamma(u-v) = f & \text{on } \Omega \times (0,T), \\ \overline{\rho}_1 \dfrac{\partial^n v}{\partial t^n} - k \dfrac{\partial^2 v}{\partial x_3^2} + 2\pi\gamma(v-u) = 0 & \text{on } \Omega \times (0,T), \\ (u,v) \in \mathcal{D}_n^{\text{eff}}, \end{cases} \tag{64}$$

where $\mathcal{D}_n^{\text{eff}}$ is defined by (12). The proof of Theorem 2.2 is achieved in the case $0 \leq \gamma < +\infty$, $k < +\infty$. □

Assuming now $\gamma = +\infty$ (and $k < +\infty$) , we first notice that by (48), (50), (51),(52), (53) and (56), the sequence $I_{2\varepsilon}$ admits a finite limit as $\varepsilon \to 0$. Since γ is infinite, we deduce from (57), (59) and (60) that $\int_{t=0}^{T} \int_{\Omega} (v-u)\psi dx dt = 0$, hence $u = v$. Next we substitute $\psi = 0$ in (45) and pass to the limit as $\varepsilon \to 0$ in (48) and (49). By (50), (51), (53), (56), (57), we obtain for $n = 1$

$$-\int_\Omega \varphi_0(x)(\overline{\rho}_0+\overline{\rho}_1)\varphi(x,0)dx - \int_0^T\int_\Omega(\overline{\rho}_0+\overline{\rho}_1)u\frac{\partial\varphi}{\partial t}\,dxdt + \int_0^T\int_\Omega \nabla u\nabla\varphi dx\,dt + k\int_0^T\int_\Omega \frac{\partial u}{\partial x_3}\frac{\partial\varphi}{\partial x_3}dx\,dt = \int_{t=0}^T\int_\Omega f\varphi dx\,dt,$$

and for $n = 2$,

$$-\int_\Omega \varphi_0(x)(\overline{\rho}_0+\overline{\rho}_1)\varphi(x,0)dx + \int_\Omega v_0(x)(\overline{\rho}_0+\overline{\rho}_1)\varphi(x,0)dx + \int_0^T\int_\Omega(\overline{\rho}_0+\overline{\rho}_1)u\frac{\partial\varphi}{\partial t}\,dxdt + \int_{t=0}^T\int_\Omega \nabla u\nabla\varphi dx\,dt + k\int_0^T\int_\Omega \frac{\partial u}{\partial x_3}\frac{\partial\varphi}{\partial x_3}dx\,dt = \int_0^T\int_\Omega f\varphi dx\,dt,$$

yielding by the same argument the case $\gamma = +\infty$, $k < +\infty$ of Theorem 2.2. $\square$

Proof of (57). By the definitions (44-47), we check easily that

$$\int_0^T\int_{B_\varepsilon}|\nabla\theta_\varepsilon|^2\,dx\,dt \le \frac{C}{\varepsilon^2\left|\ln\frac{r_\varepsilon}{R_\varepsilon}\right|}, \quad |\varphi-\varphi_\varepsilon| < CR_\varepsilon \text{ on } B_\varepsilon\times(0,T),$$
$$|B_\varepsilon\times(0,T)| \le C\frac{R_\varepsilon^2}{\varepsilon^2}, \quad |\nabla(\varphi-\varphi_\varepsilon+\psi_\varepsilon)| < C \text{ on } B_\varepsilon\times(0,T),\ 0\le\theta_\varepsilon\le 1.$$

Since $\nabla\Phi_\varepsilon - \nabla\theta_\varepsilon\psi_\varepsilon = \nabla\varphi + \nabla\theta_\varepsilon(\varphi-\varphi_\varepsilon) + \theta_\varepsilon\nabla(\varphi-\varphi_\varepsilon+\psi_\varepsilon)$, we infer

$$\int_0^T\int_{B_\varepsilon}|\nabla\Phi_\varepsilon-\nabla\theta_\varepsilon\psi_\varepsilon|^2dxdt \le C\frac{R_\varepsilon^2}{\varepsilon^2} + C\frac{R_\varepsilon^2}{\varepsilon^2\ln\frac{r_\varepsilon}{R_\varepsilon}} \longrightarrow_{\varepsilon\to0} 0,$$

which proves (57), because ∇u_ε is bounded in $L^2(\Omega\times(0,T))$. $\square$

Now we assume

$$k = +\infty, \quad n = 1. \tag{65}$$

First we choose $\psi = 0$ in the definition (45) of Φ_ε and deduce from (57) that $\lim_{\varepsilon\to0} I_{2\varepsilon} = 0$. By (48), (50), (51),(52) and (53), we infer that $I_{3\varepsilon}$ admits a finite limit as ε tends to 0 and then, by (28), (54), (55) and (65), that

$$\lim_{\varepsilon\to0}\int\frac{\partial u_\varepsilon}{\partial x_3}\frac{\partial\varphi}{\partial x_3}d\mu_\varepsilon = \int_{\Omega\times(0,T)}\frac{\partial v}{\partial x_3}\frac{\partial\varphi}{\partial x_3}dxdt = 0.$$

Hence $\dfrac{\partial^2 v}{\partial x_3^2} = 0$ in the sense of distributions on $\Omega\times(0,T)$. It follows then from (29) that

$$v = 0. \tag{66}$$

We modify now the test functions Φ_ε, setting

$$\begin{aligned}&\Phi_\varepsilon := (1-\eta_\varepsilon)\varphi + \eta_\varepsilon(1-\theta_\varepsilon)\varphi_\varepsilon,\\ &\eta_\varepsilon(x') := 1 \text{ if } d_\varepsilon(x') < R_\varepsilon; \quad \eta_\varepsilon(x') := 0 \text{ if } d_\varepsilon(x') > 2R_\varepsilon;\\ &\eta_\varepsilon(x') := \left(\ln\frac{d_\varepsilon(x')}{2R_\varepsilon}\right)\left(\ln\frac{1}{2}\right)^{(-1)} \quad \text{otherwise,}\end{aligned} \tag{67}$$

where d_ε, φ_ε and θ_ε are given by (44), (46), (47). By the same argument as before we get (48) and

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_\Omega \rho_\varepsilon(x)\varphi_0(x)\Phi_\varepsilon(x,0)dx = \int_\Omega \varphi_0(x)\overline{\rho}_0\varphi(x,0)dx,\\
&\lim_{\varepsilon\to 0}\int_{t=0}^T\int_\Omega f\Phi_\varepsilon\,dx\,dt = \int_{t=0}^T\int_\Omega f\varphi\,dx\,dt,\\
&\lim_{\varepsilon\to 0}\int_{t=0}^T\int_\Omega \rho_\varepsilon(x)u_\varepsilon\frac{\partial\Phi_\varepsilon}{\partial t}\,dx\,dt = \int_{t=0}^T\int_\Omega \overline{\rho}_0 u\frac{\partial\varphi}{\partial t}\,dx\,dt.
\end{aligned}\tag{68}$$

Then we split the remaining term of (48) as follows:

$$\begin{aligned}
&\int_{t=0}^T\int_\Omega a_\varepsilon\nabla u_\varepsilon\nabla\Phi_\varepsilon dxdt = J_{1\varepsilon}+J_{2\varepsilon}+J_{3\varepsilon};\ J_{1\varepsilon}:=\int_0^T\int_{\Omega\setminus(C_\varepsilon\cup B_\varepsilon\cup T_\varepsilon)}\nabla u_\varepsilon\nabla\Phi_\varepsilon dxdt,\\
&J_{2\varepsilon}:=\int_{t=0}^T\int_{C_\varepsilon}\nabla u_\varepsilon\nabla\Phi_\varepsilon\,dx\,dt,\quad J_{3\varepsilon}:=\int_{t=0}^T\int_{B_\varepsilon}\nabla u_\varepsilon\nabla\Phi_\varepsilon\,dx\,dt,
\end{aligned}\tag{69}$$

where

$$C_\varepsilon := \{x\in\Omega,\quad R_\varepsilon < d_\varepsilon(x) < 2R_\varepsilon\}.$$

In the same way as (53) we get

$$\lim_{\varepsilon\to 0} J_{1\varepsilon} = \int_{t=0}^T\int_\Omega \nabla u\nabla\varphi\,dx\,dt.\tag{70}$$

From the following estimate, easily deduced from (67),

$$\int_{t=0}^T\int_{C_\varepsilon}|\nabla\Phi_\varepsilon|^2dxdt \le CR_\varepsilon^2\int_{t=0}^T\int_{C_\varepsilon}|\nabla\eta_\varepsilon|^2dxdt + C|C_\varepsilon| \le C\frac{R_\varepsilon^2}{\varepsilon^2}\to_{\varepsilon\to 0} 0,$$

we infer

$$\lim_{\varepsilon\to 0} J_{2\varepsilon} = 0.\tag{71}$$

It is easy to check that

$$\lim_{\varepsilon\to 0}\left(J_{3\varepsilon} + \int_0^T\int_{B_\varepsilon}\varphi_\varepsilon\nabla\theta_\varepsilon\nabla u_\varepsilon dxdt\right) = 0.\tag{72}$$

By repeating the computation (58), we obtain

$$\int_0^T\int_{B_\varepsilon}\varphi_\varepsilon\nabla u_\varepsilon\nabla\theta_\varepsilon\,dx\,dt = \frac{2\pi}{\varepsilon^2\ln\frac{r_\varepsilon}{R_\varepsilon}}\int_{t=0}^T\int_\Omega(\tilde u_\varepsilon - \tilde v_\varepsilon)\varphi_\varepsilon\,dx\,dt.\tag{73}$$

Assuming first $0\le\gamma<+\infty$ we deduce from (11), (30), (31), (46), (66), (72) and (73) that $\lim_{\varepsilon\to 0} J_{3\varepsilon} = 2\pi\gamma\int_{t=0}^T\int_\Omega u\varphi\,dx\,dt$, and then by (48), (68), (69) (70) and (71) that

$$\begin{aligned}
&-\int_\Omega\varphi_0(x)\overline{\rho}_0\varphi(x,0)dx - \int_0^T\int_\Omega\overline{\rho}_0 u\frac{\partial\varphi}{\partial t}\,dxdt\\
&\quad+\int_0^T\int_\Omega\nabla u\nabla\varphi dx\,dt + 2\pi\gamma\int_{t=0}^T\int_\Omega u\varphi\,dx\,dt = \int_{t=0}^T\int_\Omega f\varphi dx\,dt,
\end{aligned}$$

yielding the case $k=+\infty$, $\gamma<+\infty$ of Theorem 2.2.

Finally let us assume that $k=+\infty$ and $\gamma=+\infty$. Since, by (48), (68), (69), (70) and (71), the sequence $J_{3\varepsilon}$ has a finite limit as $\varepsilon\to 0$, we infer from (11), (30), (31), (66), (72) and (73) that $\displaystyle\int_{t=0}^{T}\int_{\Omega} u\varphi dx dt = 0$. Hence $u=0$ and recalling (66), the proof of Theorem 2.2 is achieved. □

Proof of Theorem 2.5. The relative compactness of the sequences (u_ε), (v_ε) results from the estimates (42) and (43) stated above (corresponding to $n=1,2$) and from the following one (proved in [5], p. 20, in a more general context)

$$\int_{\Omega}|w|^2 dx \leq C\int_{\Omega\setminus T_\varepsilon}\varepsilon^2|\nabla w|^2 dx + C\int_{T_\varepsilon}|\nabla w|^2 dx, \quad \forall w\in H^1_0(\Omega),$$

which, applied for a.e. t to $w=u_\varepsilon(.,t)$ and integrated over $t\in(0,T)$, yields

$$\int_{t=0}^{T}\int_{\Omega}|u_\varepsilon|^2\, dx\, dt \leq C\int_{t=0}^{T}\int_{\Omega} a_\varepsilon|\nabla u_\varepsilon|^2 dx\, dt.$$

We easily deduce from (42), (43) and the last inequation, that the sequences (u_ε), (v_ε), $(\varepsilon\nabla u_\varepsilon)$, $(\nabla u_\varepsilon 1_{T_\varepsilon\times(0,T)})$ are bounded in $L^2(\Omega\times(0,T))$. Moreover if $n=2$ we get that u_ε is bounded in $L^2(\Omega,H^1(0,T))$. Therefore up to a subsequence, the convergences (38) ((38) and (40) if $n=2$) hold where the various limits, in accordance with Proposition 3.3, are identified by (39) (by (39) and (41) if $n=2$). Moreover the sequences of measures $u_\varepsilon\mu_\varepsilon$ and $\frac{\partial u_\varepsilon}{\partial x_3}\mu_\varepsilon$ are bounded and by repeating the argument of Proposition 3.1, we get (28) and (29).

As before we next determine a weak formulation of the limit problem by passing to the limit as $\varepsilon\to 0$ in the equation (48) if $n=1$ and in (49) if $n=2$, using the suitable sequence of test functions Φ_ε defined as follows: fixing two functions $\varphi,\psi\in C^\infty(\overline{\Omega\times(0,T)})$ such that $\varphi=\psi=0$ on $\partial\Omega\times]0,T]$ and two Y-periodic functions $w,\eta\in C^\infty(\mathbb{R}^2)$ such that $1-w=\eta=1$ on D and $\eta=0$ on ∂Y, we set

$$\begin{aligned}\Phi_\varepsilon(x,t) &:= w\left(\frac{x'}{\varepsilon}\right)\varphi(x,t)+\left(1-w\left(\frac{x'}{\varepsilon}\right)\right)\overline{\psi}_\varepsilon(x,t),\\ \overline{\psi}_\varepsilon(x,t) &:= \left(1-\eta\left(\frac{x'}{\varepsilon}\right)\right)\psi(x,t)+\eta\left(\frac{x'}{\varepsilon}\right)\psi_\varepsilon(x,t),\end{aligned}\tag{74}$$

where ψ_ε is defined by (46). In order to compute the limit as $\varepsilon\to 0$ of each term of (48), (49) we notice that by the following estimates

$$\left|\rho_\varepsilon(x)\Phi_\varepsilon(x,t)-\left(\frac{1}{1-|D|}\overline{\rho}_0 w\left(\frac{x'}{\varepsilon}\right)(\varphi(x,t)-\psi(x,t))+\rho\left(\frac{x'}{\varepsilon}\right)\psi(x,t)\right)\right|\leq C\varepsilon,$$

$$\left|\rho_\varepsilon(x)\frac{\partial\Phi_\varepsilon}{\partial t}(x,t)-\left(\frac{1}{1-|D|}\overline{\rho}_0 w\left(\frac{x'}{\varepsilon}\right)\frac{\partial}{\partial t}(\varphi(x,t)-\psi(x,t))+\rho\left(\frac{x'}{\varepsilon}\right)\frac{\partial}{\partial t}\psi(x,t)\right)\right|\leq C\varepsilon,$$

where

$$\rho(y) := \frac{1}{1-|D|}\overline{\rho}_0 1_{Y\setminus D}(y)+\frac{1}{|D|}\overline{\rho}_1 1_D(y),\tag{75}$$

and by (38) (resp (38) and (40) if $n = 2$), there holds

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int_\Omega \rho_\varepsilon(x)\varphi_0(x)\Phi_\varepsilon(x,0)dx = \int_{\Omega\times Y}\varphi_0(x)\Big(\frac{\overline{\rho}_0}{1-|D|}w(y)(\varphi-\psi)(x,0)+\rho(y)\psi(x,0)\Big)dxdy,\\
&\lim_{\varepsilon\to 0}\int_\Omega \rho_\varepsilon(x)v_0(x)\Phi_\varepsilon(x,0)dx =\\
&\qquad\int_{\Omega\times Y} v_0(x)\Big(\frac{1}{1-|D|}\overline{\rho}_0 w(y)(\varphi-\psi)(x,0)+\rho(y)\psi(x,0)\Big)dxdy,\ (n=2),\\
&\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \rho_\varepsilon(x)u_\varepsilon\frac{\partial\Phi_\varepsilon}{\partial t}dxdt =\\
&\qquad\int_0^T\int_{\Omega\times Y} u_0(x,t,y)\Big(\frac{\overline{\rho}_0}{1-|D|}w(y)\frac{\partial(\varphi-\psi)}{\partial t}(x,t)+\rho(y)\frac{\partial\psi}{\partial t}(x,t)\Big)\,dxdtdy,\ \ (n=1),\\
&\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \rho_\varepsilon(x)\frac{\partial u_\varepsilon}{\partial t}\frac{\partial\Phi_\varepsilon}{\partial t}dxdt=\\
&\qquad\int_0^T\int_{\Omega\times Y}\frac{\partial u_0}{\partial t}\Big(\frac{\overline{\rho}_0}{1-|D|}w(y)\frac{\partial(\varphi-\psi)}{\partial t}(x,t)+\rho(y)\frac{\partial\psi}{\partial t}(x,t)\Big)\,dxdtdy,\quad (n=2),\\
&\lim_{\varepsilon\to 0}\int_0^T\int_\Omega f\Phi_\varepsilon\,dx\,dt = \int_0^T\int_{\Omega\times Y} f\Big(w(y)(\varphi-\psi)(x,t)+\psi(x,t)\Big)\,dx\,dtdy.
\end{aligned}
\tag{76}
$$

With regard to the remaining term of (48) (or (49)), we write

$$
\begin{aligned}
&\int_{t=0}^T\int_\Omega a_\varepsilon\nabla u_\varepsilon\nabla\Phi_\varepsilon\,dx\,dt = I_{1\varepsilon}+I_{2\varepsilon};\\
&I_{1\varepsilon} := \int_{(\Omega\setminus T_\varepsilon)\times(0,T)}\varepsilon\nabla u_\varepsilon.\varepsilon\nabla\Phi_\varepsilon;\ \ I_{2\varepsilon} := \int_{(T_\varepsilon)\times(0,T)} k\nabla u_\varepsilon\nabla\Phi_\varepsilon d\mu_\varepsilon,
\end{aligned}
\tag{77}
$$

and noticing that

$$
\begin{aligned}
&\left|\varepsilon\nabla\Phi_\varepsilon - \left(\nabla w\left(\frac{x'}{\varepsilon}\right)(\varphi(x,t)-\psi(x,t))\right)\right| < C\varepsilon,\\
&I_{2\varepsilon} = \int k\frac{\partial u_\varepsilon}{\partial x_3}\frac{\partial\psi_\varepsilon}{\partial x_3}d\mu_\varepsilon,\quad \left|\frac{\partial\psi}{\partial x_3}-\frac{\partial\psi_\varepsilon}{\partial x_3}\right| < C\varepsilon,
\end{aligned}
$$

we infer from (28), (38) and (39)

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} I_{1\varepsilon} = \int_{t=0}^T\int_{\Omega\times Y}\nabla_y u_0\nabla w(y)(\varphi(x,t)-\psi(x,t))dx\,dy\,dt,\\
&\lim_{\varepsilon\to 0} I_{2\varepsilon} = \int_{t=0}^T\int_\Omega k\frac{\partial v}{\partial x_3}\frac{\partial\psi}{\partial x_3}\,dx\,dt.
\end{aligned}
$$

By (76) and (77), passing to the limit as $\varepsilon \to 0$ for $n = 1$ in (48), we obtain

$$
\begin{aligned}
&-\int_{\Omega\times Y} \varphi_0(x)\Big(\frac{1}{1-|D|}\overline{\rho}_0 w(y)(\varphi-\psi)(x,0)+\rho(y)\psi(x,0)\Big)dxdy\\
&\quad-\int_{t=0}^{T}\int_{\Omega\times Y} u_0(x,t,y)\left(\frac{1}{1-|D|}\overline{\rho}_0 w(y)\frac{\partial(\varphi-\psi)}{\partial t}(x,t)+\rho(y)\frac{\partial\psi}{\partial t}(x,t)\right)\,dx\,dt\,dy\\
&\quad+\int_{t=0}^{T}\int_{\Omega\times Y} \nabla_y u_0 \nabla w(y)(\varphi-\psi)(x,t)dx\,dt\,dy+\int_{t=0}^{T}\int_{\Omega} k\frac{\partial v}{\partial x_3}\frac{\partial \psi}{\partial x_3}\,dx\,dt\\
&=\int_{t=0}^{T}\int_{\Omega\times Y} f\Big(w(y)(\varphi-\psi)(x,t)+\psi(x,t)\Big)\,dx\,dtdy,
\end{aligned}
\tag{78}
$$

and, passing to the limit for $n = 2$ in (49),

$$
\begin{aligned}
&-\int_{\Omega\times Y} v_0(x)\Big(\frac{1}{1-|D|}\overline{\rho}_0 w(y)(\varphi-\psi)(x,0)+\rho(y)\psi(x,0)\Big)dxdy\\
&\quad-\int_{t=0}^{T}\int_{\Omega\times Y} \frac{\partial u_0}{\partial t}(x,t,y)\left(\frac{1}{1-|D|}\overline{\rho}_0 w(y)\frac{\partial(\varphi-\psi)}{\partial t}(x,t)+\rho(y)\frac{\partial\psi}{\partial t}(x,t)\right)\,dx\,dt\,dy\\
&\quad+\int_{t=0}^{T}\int_{\Omega\times Y} \nabla_y u_0 \nabla w(y)(\varphi-\psi)(x,t)dx\,dt\,dy+\int_{t=0}^{T}\int_{\Omega} k\frac{\partial v}{\partial x_3}\frac{\partial \psi}{\partial x_3}\,dx\,dt\\
&=\int_{t=0}^{T}\int_{\Omega\times Y} f\Big(w(y)(\varphi-\psi)(x,t)+\psi(x,t)\Big)\,dx\,dtdy.
\end{aligned}
\tag{79}
$$

By choosing first $\psi = 0$ in (78), (79) and denoting $\xi(t) : (x, y) \to w(y)\varphi(x, t)$, we obtain for $n = 1$

$$
\begin{aligned}
&-\int_{\Omega\times(Y\setminus D)} \frac{1}{1-|D|}\overline{\rho}_0\varphi_0\xi(0)dxdy-\int_{t=0}^{T}\int_{\Omega\times(Y\setminus D)}\frac{1}{1-|D|}\overline{\rho}_0 u_0\frac{\partial\xi}{\partial t}\,dx\,dt\,dy\\
&\quad+\int_{t=0}^{T}\int_{\Omega\times(Y\setminus D)}\nabla_y u_0\nabla_y\xi dx\,dt\,dy=\int_{t=0}^{T}\int_{\Omega\times(Y\setminus D)} f\xi\,dx\,dtdy,
\end{aligned}
\tag{80}
$$

and for $n = 2$

$$
\begin{aligned}
&-\int_{\Omega\times(Y\setminus D)} \frac{1}{1-|D|}\overline{\rho}_0 v_0\xi(0)dxdy-\int_{t=0}^{T}\int_{\Omega\times(Y\setminus D)}\frac{1}{1-|D|}\overline{\rho}_0 \frac{\partial u_0}{\partial t}\frac{\partial\xi}{\partial t}\,dx\,dt\,dy\\
&\quad+\int_{t=0}^{T}\int_{\Omega\times(Y\setminus D)}\nabla_y u_0\nabla_y\xi dx\,dt\,dy=\int_{t=0}^{T}\int_{\Omega\times(Y\setminus D)} f\xi\,dx\,dtdy.
\end{aligned}
\tag{81}
$$

Setting $H := L^2(\Omega \times (Y \setminus D))$, $V := \{\vartheta\lfloor_{\Omega\times(Y\setminus D)} |\ \vartheta \in L^2(\Omega, H^1_\sharp(Y))\}$ and $V_0 := \{\vartheta \in V,\ \vartheta = 0 \text{ on } \Omega \times D,\}$ associated with the norm $|\vartheta|^2_{V_0} := |\vartheta|^2_H + |\nabla_y\vartheta|^2_H$, we can rewrite

(80) and (81) as follows

$$\begin{aligned} &\int_0^T \Big[a(u_0(t),\xi(t)) - \frac{\overline{\rho}_0(u_0(t),\xi'(t))}{1-|D|}\Big]dt = \int_0^T (f(t),\xi(t))dt + \frac{\overline{\rho}_0(\varphi_0,\xi(0))}{1-|D|} \qquad (n=1),\\ &\int_0^T \Big[a(u_0(t),\xi(t)) - \frac{\overline{\rho}_0(u_0'(t),\xi'(t))}{1-|D|}\Big]dt = \int_0^T (f(t),\xi(t))dt + \frac{\overline{\rho}_0(v_0,\xi(0))}{1-|D|} \qquad (n=2), \end{aligned} \tag{82}$$

where $(.,.)$ denotes the scalar product in H and $a(.,.)$ is the continuous symetric bilinear form defined on $V_0 \times V_0$ by

$$a(\vartheta_1,\vartheta_2) := \int_{\Omega\times(Y\setminus D)} \nabla_y\vartheta_1\nabla_y\vartheta_2 dxdy.$$

By density (82) holds for any $\xi \in \mathcal{V}$, where

$$\mathcal{V} := \{\xi \in L^2(0,T;V_0),\ \xi' \in L^2(0,T,H),\ \xi(T)=0\}.$$

Since the form $a(.,.)$ satisfies $a(\vartheta,\vartheta) \geq |\vartheta|^2_{V_0} - |\vartheta|^2_H$, there exists a unique function $\xi_0 \in L^2(0,T;V_0)$ such that the following equality is satisfied for all $\xi \in \mathcal{V}$ (see [15] p. 44 or [16])

$$\begin{aligned} &\int_0^T \Big[a(\xi_0(t),\xi(t)) - \frac{\overline{\rho}_0(\xi_0(t),\xi'(t))}{1-|D|}\Big]dt = \int_0^T (f(t),\xi(t))dt + \frac{\overline{\rho}_0(\varphi_0,\xi(0))}{1-|D|} \qquad (n=1),\\ &\int_0^T \Big[a(\xi_0(t),\xi(t)) - \frac{\overline{\rho}_0(\xi_0'(t),\xi'(t))}{1-|D|}\Big]dt = \int_0^T (f(t),\xi(t))dt + \frac{\overline{\rho}_0(v_0,\xi(0))}{1-|D|} \qquad (n=2), \end{aligned} \tag{83}$$

Hence, by (39), u_0 is the unique function in $L^2(0,T;V)$ satisfying (82) for all $\xi \in \mathcal{V}$ and such that $u_0(x,y,t) = v(x,t)$ a.e. on $\Omega \times (0,T) \times D$. By localization, for a.e. $x \in \Omega$, the application $(t,y) \to u_0(x,t,y)$ is the unique solution of the elementary evolution problem (18) associated with $z(x,t) = v(x,t)$. By (19) we have:

$$u = \Phi_n(v). \tag{84}$$

Moreover there holds

$$\begin{cases} u_0 \in C([0,T],H),\ \dfrac{\partial u_0}{\partial t} \in L^2(0,T;V_0'),\ u_0(0)=\varphi_0, & \text{if } \overline{\rho}_0>0,\ n=1,\\ u_0 \in C([0,T],V)\cap C^1([0,T],H), & \\ \qquad \dfrac{\partial^2 u_0}{\partial t^2} \in L^2(0,T;V_0'),\ u_0(0)=\varphi_0,\ \dfrac{\partial u_0}{\partial t}(0)=v_0, & \text{if } \overline{\rho}_0>0,\ n=2. \end{cases} \tag{85}$$

Introducing the function

$$b(x,t) := ⨍_{Y\setminus D} u_0(x,t,y)dy, \tag{86}$$

it follows from (85) and (86) that

$$\begin{cases} b \in C([0,T],L^2(\Omega)),\ b(0)=\varphi_0, & \text{if } \overline{\rho}_0>0, n=1\\ b \in C^1([0,T],L^2(\Omega)),\ b(0)=\varphi_0,\ \dfrac{\partial b}{\partial t}(0)=v_0, & \text{if } \overline{\rho}_0>0, n=2. \end{cases} \tag{87}$$

Next we choose $\varphi = \psi$ and integrate (78) and (79) with respect to y. By taking (39), (75) into account, noticing that for almost every $(x,t) \in \Omega \times (0,T)$,

$$\int_Y u_0(x,t,y)\rho(y)dy = \overline{\rho}_0 b(x,t) + \overline{\rho}_1 v(x,t),$$

by integration with respect to the variable y we obtain for $n = 1$

$$\begin{aligned} -\int_\Omega (\overline{\rho}_0 + \overline{\rho}_1)\varphi_0(x)\psi(x,0)dx - \int_{t=0}^T \int_\Omega (\overline{\rho}_0 b(x,t) + \overline{\rho}_1 v(x,t)) \frac{\partial \psi}{\partial t}\, dx\, dt \\ + \int_{t=0}^T \int_\Omega k \frac{\partial v}{\partial x_3}\frac{\partial \psi}{\partial x_3}\, dx\, dt = \int_{t=0}^T \int_\Omega f\psi\, dx\, dt, \end{aligned}$$

and for $n = 2$

$$\begin{aligned} -\int_\Omega (\overline{\rho}_0 + \overline{\rho}_1) v_0(x)\psi(x,0)dx - \int_{t=0}^T \int_\Omega \left(\overline{\rho}_0 \frac{\partial b}{\partial t} + \overline{\rho}_1 \frac{\partial v}{\partial t}\right) \frac{\partial \psi}{\partial t}\, dx\, dt \\ + \int_{t=0}^T \int_\Omega k \frac{\partial v}{\partial x_3}\frac{\partial \psi}{\partial x_3}\, dx\, dt = \int_{t=0}^T \int_{\Omega\times Y} f\psi\, dx\, dt. \end{aligned}$$

By the arbitrary choice of ψ, taking (87) and (41) into account, we infer

$$\begin{cases} \overline{\rho}_1 \dfrac{\partial^n v}{\partial t^n} - k\dfrac{\partial^2 v}{\partial x_3^2} = f - \overline{\rho}_0 \dfrac{\partial^n b}{\partial t^n} \quad \text{in } \ \Omega \times (0,T), \\ v \in C^{n-1}([0,T], L^2(\Omega)),\ v(0) = \varphi_0, \left(\text{ and } \dfrac{\partial v}{\partial t}(0) = v_0 \text{ if } \ n = 2 \right) \quad \text{if } \ \overline{\rho}_1 > 0. \end{cases} \tag{88}$$

Since, by (39) and (86), there holds

$$u = (1 - |D|)b + |D|v, \tag{89}$$

we deduce from (87) and (88) that

$$u \in C^{n-1}([0,T], L^2(\Omega)),\ u(0) = \varphi_0, \left(\frac{\partial u}{\partial t}(0) = v_0 \text{ if } n = 2 \right), \qquad \text{if } \overline{\rho}_0.\overline{\rho}_1 > 0, \tag{90}$$

and by (84), (88), (89), (90), the proof of Theorem 2.5 is achieved. □

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