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The Generalized Minkowski Functional with Applications in Approximation Theory

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We give a systematic and thorough study of geometric notions and results connected to Minkowski's measure of symmetry and the extension of the well-known Minkowski functional to arbitrary, not necessarily symmetric convex bodies K on any (real) normed space X. Although many of the notions and results we treat in this paper can be found elsewhere in the literature, they are scattered and possibly hard to find. Further, we are not aware of a systematic study of this kind and we feel that several features, connections and properties – e.g. the connections between many equivalent formulations – are new, more general and they are put in a better perspective now. In particular, we prove a number of fundamental properties of the extended Minkowski functional $\alpha(K, \mathbf{x})$, including convexity, global Lipschitz boundedness, linear growth and approximation of the classical Minkowski functional of the central symmetrization of the body K. Our aim is to present how in the recent years these notions proved to be surprisingly relevant and effective in problems of approximation theory.

Keywords: Convex body, support function, supporting hyperplanes, halfspaces and layers, Minkowski functional, convex functions in normed spaces, Lipschitz bounds, central symmetrization, centroid, cone of convex bodies, measure of symmetry, width of K in a direction, homothetic transformations, existence of minima of continuous convex functions in normed spaces, separation of convex sets, multivariate polynomials, Bernstein and Chebyshev type extremal problems for multivariate polynomials

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1. Introduction

The present work grew out of our studies on inequalities and extremal problems for multivariate polynomials, especially multivariate polynomials defined on infinite dimensional normed spaces. For a general introduction to polynomials on infinite dimensional spaces we refer to the comprehensive first chapter of Dineen's monograph [5]; as for the discussion of the particular multivariate polynomial extremal problems connected to generalized Minkowski functional, see Section 9 of this article.

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A quite general question in these type of problems presents itself in the following setting. Suppose we are given a (continuous) polynomial p of known degree n and satisfying some boundedness condition on a reasonably well-shaped set, say on a convex body $K \subset X$, where X is a normed space. If $\mathcal{P}_n(X)$ is the space of polynomials $p: X \to \mathbb{R}$ of degree not exceeding n, the "initial conditions" of a usual extremal problem are $p \in \mathcal{P}_n(X)$ and $\|p\|_K \le 1$, where $\|p\|_K := \sup_K |p|$. The aim is to find best possible (or at least of a correct order of magnitude) estimates on some expressions — say, a linear functional — of p, which depend on a point $\mathbf{x} \in X$. For example, the "Bernstein problem" is to estimate $\|\operatorname{grad} p(\mathbf{x})\|$; the "point value Chebyshev problem" is to estimate $\big|p(\mathbf{x})\big|$. To solve these questions, apart from functional analysis and approximation theory, a crucial role is played by the geometric features of the configuration of K and $\mathbf{x}$. In particular, any such estimate has to use some quantitative formulation of the location of $\mathbf{x}$ with respect to the body K. Obvious choices — like the metric distance of $\mathbf{x} \notin K$ from K, or $\mathbf{x} \in K$ from the boundary ∂K — yield only essentially trivial but too rough estimates.

If the *convex body* $K \subset X$ (i.e., K is a bounded, convex and closed set with nonempty interior) is also *centrally symmetric* with respect to the origin, then there is a norm $\|\cdot\|_{(K)}$ on X — equivalent to the original norm $\|\cdot\|$ of X in view of $B_{X,\|\cdot\|}(\mathbf{0}, r) \subset K \subset B_{X,\|\cdot\|}(\mathbf{0}, R)$ — which can be used successfully in approximation theoretic questions as well. For this norm $\|\cdot\|_{(K)}$ the unit ball of X will be K itself, $B_{X,\|\cdot\|_{(K)}}(\mathbf{0}, 1) = K$, while for any $\mathbf{x} \in X$ the *Minkowski functional* or *(Minkowski) distance function* [10, p. 57] or *gauge* [28, p. 28] or *Minkowski gauge functional* [22, 1.1(d)]

$$\varphi_K(\mathbf{x}) := \inf\{\lambda > 0 : \ \mathbf{x} \in \lambda K\} \tag{1}$$

will serve as definition of $\|\mathbf{x}\|_{(K)}$. Clearly (1) is a norm on X if and only if the convex body K is centrally symmetric with respect to the origin.

In case K is nonsymmetric, (1) still can be used. But then, even the choice of the homothetic centre is questionable since the use of any alternative gauge functional

$$\varphi_{K,\mathbf{x}_0}(\mathbf{x}) := \inf\big\{\lambda > 0 : \ \mathbf{x} \in \lambda(K - \mathbf{x}_0)\big\}$$

is equally well justified. Moreover, neither is good enough for the applications.

There are two possible approaches here. The first one, which to us seems to be more easily understood, is to define some "continuous", increasing, absorbing set function $t \to K_t$, and to derive the corresponding generalized functional analogously to (1). That will be pursued in this Introduction. On the other hand, another general approach is described e.g. by Grünbaum in connection to the construction of measures of symmetry in general, see [7, p. 245]. In this approach, one defines first a functional $f(K, \mathbf{x})$ by directly using some geometric features of the configuration of K and $\mathbf{x}$. Then a measure of symmetry is defined by considering the extremal value of f over the whole of K (or X). Depending on the geometric properties we start with, corresponding approaches lead to valuable information. In Section 4 we examine how this second approach can be used to arrive at equivalent forms and thus unifying a number of seemingly different geometric constructions.

An interesting construction of the first type, which resembles somewhat to our upcoming choice of subject, is the so called "floating body" construction[1] (see [4]), which was successfully applied in a number of geometric problems (c.f. e.g. the recent survey article [3]).

Skipping the precise details of formulation, let us describe heuristically this definition. If $0 < \beta < 1$, imagine that the closed surface of ∂K is a container which contains a liquid of volume $\beta \operatorname{vol}(K)/2$. Rotating the body shell (in the fixed gravitational field) to all possible positions, the liquid flows freely around. Then the body will consist of points which became "wet" during the rotation, and of "dry" points which went under water in neither position. Cutting down the wet points, the set of the dry points form a smaller set $K(\beta)$ which is easily seen to be convex, and, naturally, form a monotone set sequence of the continuous parameter β.

The crucial feature is that, although K is not symmetric, a natural, geometrical definition is given to a reasonable set sequence $K(\beta)$, which reflects the geometric characteristics (in our case the boundary shape in particular). If we want, similarly to (1), a "floating distance" can be defined:

$$\psi_K(\mathbf{x}) := \inf\big\{\lambda > 0 : \ \mathbf{x} \in K(1-\lambda)\big\},$$

although this is not usual in the applications of the floating body construction.

Now this is very much like what we want, but at least two properties constitute some major drawback. First, the definition is restricted to points $\mathbf{x} \in K$, and second, the reference to the volume excludes infinite dimensional spaces. Moreover, we do not know any particular applications in approximation theory which could be well described using the floating body notion.

It is worth noting that for $\mathbf{x} \notin K$ there is an alternative notion, the "illumination body" construction, quite similar in nature. See for instance [37]. However, this also relies on use of the volume, hence is restricted to finite dimensional spaces.

So, our choice will be another construction of a set function which was considered first by Hammer [8]. The use of an equivalent geometric notion (the somewhat implicit use of $\lambda(K, \mathbf{x})$, see (31) below) has already appeared in approximation theory in the work of Rivlin and Shapiro [27] dealing with the Chebyshev problem for finite dimensional spaces. Although Rivlin–Shapiro, as well as Kroó–Schmidt [17] later on, have worked with exterior points of K, it was realized in [16] that the same definition works for points of K as well. Also, in [16] a few observations and questions were mentioned concerning the generalized Minkowski functional $\alpha(K, \mathbf{x})$ (see definition (14) below).

However, to the best of our knowledge this present work is the first attempt in bringing together what geometers on the one hand, and approximation theorists on the other hand have contributed in the subject. For this reason, new results are accompanied by quite a number of facts which were already known to either geometers or approximation theorists, but not in our setting and without the connections our investigation reveals. Lacking the drive of accommodating to application needs of approximation theory, some geometric facts, which were accessible to experts well before, seem to surface or to be explored

[1] We are grateful to Professor Apostolos Giannopoulos for calling our attention to this notion and providing us the corresponding references.

the first time here. In particular, not even the fundamental properties of the generalized Minkowski functional, like convexity, Lipschitz boundedness, or linear speed growth, seem to have been established before. Moreover, in the present work we investigate the subject in the full generality of normed spaces.

A careful checking reveals that essentially $\alpha(K,\mathbf{x})$ does not depend on the initial norm of X, at least in the sense that any equivalent norm can be considered. Actually the sets K^λ of (13), and hence $\alpha(K,\mathbf{x})$, depend only on geometry and not on the norm. Therefore, this notion can be equally well introduced in topological vector spaces (t.v.s) where the (topological) dual X^*, the support function and every necessary ingredient is available. However, if X is only a t.v.s, then the mere existence of the convex body K leads to a topologically equivalent norm, the unit ball of which is the central symmetrization C (c.f. (2)) of K. As we will show in the sequel, actually $\|\cdot\|_{(C)} = \varphi_C$ and $\alpha(K,\cdot)$ have only a very small — globally bounded — difference, and so the seemingly easy generalization to t.v.s. is essentially void.

A word of warning concerns the use of the inner product space structure, which is justified in $\mathbb{R}^d$ in view of the equivalence of all norms, but is not available for normed spaces in general. Hence, here we must carefully distinguish the space X, its unit ball $B = B_X(\mathbf{0},1)$ and its unit sphere $S = S_X(\mathbf{0},1) = \partial B$ on one hand, and X^* its corresponding unit ball B^* and its unit sphere S^* on the other hand. The value of a linear functional $\mathbf{x}^* \in X^*$ at a vector $\mathbf{x} \in X$ will be denoted by $\langle \mathbf{x}^*, \mathbf{x}\rangle$.

2. Notations, definitions, and general background

The *central symmetrization* of a set K in a normed space X is

$$C := C(K) := \frac{1}{2}(K-K) := \left\{ \frac{1}{2}(\mathbf{x}-\mathbf{y}) : \ \mathbf{x},\mathbf{y} \in K \right\}. \tag{2}$$

The central symmetrization of K is centrally symmetric with respect to the origin. In case K is a convex body, we also have $\mathbf{0} \in \operatorname{int} C$ [2]. On the other hand, even though K is assumed to be closed, C is not necessarily closed (c.f. Section 6 of [25]), hence C is not a convex body in general. Nevertheless, the closure $\overline{C}$ of C is a symmetric convex body, which is also *fat*, and $\operatorname{int} C \subset C \subset \overline{\operatorname{int} C} = \overline{C}$.

The “ *maximal chord* " of K in direction of $\mathbf{v} \neq \mathbf{0}$ is

$$\begin{aligned} \tau(K,\mathbf{v}) : &= \sup\{\lambda \geq 0 : \exists\ \mathbf{y},\, \mathbf{z} \in K \ \text{ s.t. } \ \mathbf{z} = \mathbf{y} + \lambda \mathbf{v}\} \\ &= \sup\big\{\lambda \geq 0 : \ K \cap (K + \lambda\mathbf{v}) \neq \emptyset\big\} \\ &= \sup\{\lambda \geq 0 : \ \lambda\mathbf{v} \in K - K\} = 2 \sup\big\{\lambda > 0 : \ \lambda\mathbf{v} \in C\big\} \\ &= 2 \max\big\{\lambda \geq 0 : \ \lambda\mathbf{v} \in \overline{C}\big\} = \tau\big(C,\mathbf{v}\big). \end{aligned} \tag{3}$$

Usually $\tau(K,\mathbf{v})$ is not a “maximal" chord length, but only a supremum, however we shall use the familiar finite dimensional terminology (see for example [36]).

[2] Throughout the paper we denote when convenient $C(K)$, $\tau(K,\mathbf{v})$, $\alpha(K,\mathbf{x})$, $w(K,\mathbf{v}^*)$, etc. by C, τ, α, w, etc., respectively.

The *support function* to K, where K can be an arbitrary set, is defined for all $\mathbf{v}^* \in X^*$ (sometimes only for $\mathbf{v}^* \in S^*$) as

$$h(K, \mathbf{v}^*) := \sup_K \mathbf{v}^* = \sup \big\{ \langle \mathbf{v}^*, \mathbf{x} \rangle : \ \mathbf{x} \in K \big\}, \tag{4}$$

and the *width of K in direction* $\mathbf{v}^* \in X^*$ (or $\mathbf{v}^* \in S^*$) is

$$\begin{aligned} w(K, \mathbf{v}^*) : &= h(K, \mathbf{v}^*) + h(K, -\mathbf{v}^*) = \sup_K \mathbf{v}^* + \sup_K (-\mathbf{v}^*) \\ &= \sup \big\{ \langle \mathbf{v}^*, \mathbf{x} - \mathbf{y} \rangle : \ \mathbf{x}, \mathbf{y} \in K \big\} = 2h\big(C, \mathbf{v}^*\big) = w\big(C, \mathbf{v}^*\big). \end{aligned} \tag{5}$$

Let us introduce the notations

$$X_t(\mathbf{v}^*) := \big\{ \mathbf{x} \in X : \ \langle \mathbf{v}^*, \mathbf{x} \rangle \le t \big\}, \qquad X(K, \mathbf{v}^*) := X_{h(K,\mathbf{v}^*)}(\mathbf{v}^*). \tag{6}$$

Clearly the closed halfspace $X(K, \mathbf{v}^*)$ contains K and the hyperplane

$$H(K, \mathbf{v}^*) := H_{h(K,\mathbf{v}^*)}(\mathbf{v}^*), \quad H_t(\mathbf{v}^*) := \big\{ \mathbf{x} \in X : \ \langle \mathbf{v}^*, \mathbf{x} \rangle = t \big\} = \partial X_t(\mathbf{v}^*) \tag{7}$$

is a supporting hyperplane[3] to K.

A *layer* (sometimes also called *strip* in the literature) is the region of X enclosed by two parallel hyperplanes, i.e.

$$L_{r,s}(\mathbf{v}^*) := \big\{ \mathbf{x} \in X : \ r \le \langle \mathbf{v}^*, \mathbf{x} \rangle \le s \big\} = X_s(\mathbf{v}^*) \cap X_{-r}(-\mathbf{v}^*), \tag{8}$$

while the *supporting layer* or *fitting layer* of K with normal $\mathbf{v}^*$ is

$$\begin{aligned} L(K, \mathbf{v}^*) : &= X(K, \mathbf{v}^*) \cap X(K, -\mathbf{v}^*) = L_{-h(K,-\mathbf{v}^*),\, h(K,\mathbf{v}^*)}(\mathbf{v}^*) \\ &= \big\{ \mathbf{x} \in X : \ -h(K, -\mathbf{v}^*) \le \langle \mathbf{v}^*, \mathbf{x} \rangle \le h(K, \mathbf{v}^*) \big\}. \end{aligned} \tag{9}$$

By convexity, K is the intersection of its “ supporting halfpaces " $X(K, \mathbf{v}^*)$, and grouping opposite normals we get

$$K = \bigcap_{\mathbf{v}^* \in S^*} X(K, \mathbf{v}^*) = \bigcap_{\mathbf{v}^* \in S^*} L(K, \mathbf{v}^*). \tag{10}$$

Any layer (8) can be homothetically dilated with quotient $\lambda \ge 0$ at any of its symmetry centers lying on the symmetry hyperplane $H_{\frac{r+s}{2}}(\mathbf{v}^*)$ to obtain

$$L^{\lambda}_{r,s}(\mathbf{v}^*) := \left\{ \mathbf{x} \in X : \ \frac{\lambda+1}{2} r - \frac{\lambda-1}{2} s \le \langle \mathbf{v}^*, \mathbf{x} \rangle \le \frac{\lambda+1}{2} s - \frac{\lambda-1}{2} r \right\}. \tag{11}$$

In particular, we have also defined

$$L^{\lambda}(K, \mathbf{v}^*) = L^{\lambda}_{-h(K,-\mathbf{v}^*),\, h(K,\mathbf{v}^*)}(\mathbf{v}^*) \tag{12}$$

[3]Note that throughout the paper we mean “supporting" in the weak sense, that is, we do not require $K \cap H(K, \mathbf{v}^*) \ne \emptyset$, but only $\operatorname{dist}\big(K, H(K, \mathbf{v}^*)\big) = 0$. The same convention is in effect for other supporting objects as halfspaces, layers etc.

and by using (12) one can even define

$$K^\lambda := \bigcap_{\mathbf{v}^* \in S^*} L^\lambda(K, \mathbf{v}^*). \tag{13}$$

Note that K^λ can be empty for small values of λ. Using the convex, closed, bounded, increasing and (as easily seen, c.f. Proposition 3.3) even absorbing set system $\{K^\lambda\}_{\lambda \geq 0}$, the *generalized Minkowski functional* or *gauge functional* is defined as

$$\alpha(K, \mathbf{x}) := \inf\{\lambda \geq 0 : \ \mathbf{x} \in K^\lambda\}. \tag{14}$$

Denoting the cone of convex bodies in X by $\mathcal{K} := \mathcal{K}(X)$, in analogy to e.g. [10, Chapter 1.9, p. 302], we see that α is a mapping from $\mathcal{K} \times X \to \mathbb{R}_+$. Speaking of topological or analytical features of α, we always understand $\mathcal{K}$ as the topological space equipped with the (Pompeiu–)Hausdorff(–Blaschke) metric $\delta = \delta^H$ (c.f. [10], p. 307)

$$\delta(K, M) := \max\left\{ \sup_{\mathbf{x} \in K} \inf_{\mathbf{y} \in M} \|\mathbf{x} - \mathbf{y}\|, \ \sup_{\mathbf{y} \in M} \inf_{\mathbf{x} \in K} \|\mathbf{x} - \mathbf{y}\| \right\}, \tag{15}$$

while $\mathbb{R}_+ = [0, \infty)$ is considered with its usual metric. Note that (15) may be defined for any subsets of X. We shall also use the notation

$$\alpha_K := \inf \alpha(K, \cdot) = \inf\{\lambda \geq 0 : \ K^\lambda \neq \emptyset\}. \tag{16}$$

For further use, we introduce the notation

$$\mathcal{P} : \ \mathcal{K} \times \mathbb{R}_+ \to \mathcal{K} \tag{17}$$

$$\mathcal{P}(K, \lambda) := \mathcal{P}^\lambda(K) := K^\lambda$$

for the formation of the " λ-powered set " of K. To avoid notational complications, here and elsewhere if necessary, we identify $\mathcal{K}$ and its (metric) complete closure of not necessarily proper (i.e., the set of all convex, closed, bounded, nonempty, but not necessarily $\operatorname{int} K \neq \emptyset$) convex bodies. Cauchy sequences in $\mathcal{K}$ converge.

In reference to the notations (2)–(5), we also use the notions of the *width of* K, the *diameter of* K and the *inradius* of C which are defined as

$$w := w(K) := \inf\big\{w(K, \mathbf{v}^*) : \ \mathbf{v}^* \in S^*\big\}, \tag{18}$$

$$d := d(K) := \operatorname{diam}(K) := \sup\big\{\|\mathbf{x} - \mathbf{y}\| : \ \mathbf{x}, \mathbf{y} \in K\big\}, \tag{19}$$

and

$$r := r\big(C\big) := \sup\big\{r : \ B(\mathbf{0}, r) \subset C\big\}, \tag{20}$$

respectively.

Finally, we introduce the notation

$$D := D(K) := \sup_{\mathbf{x} \in K} \|\mathbf{x}\|. \tag{21}$$

In view of (2)–(5) we have

$$w(K) = w\big(C\big), \tag{22}$$

and by taking into consideration the notation (21), we also have

$$d(K) = d(C) = 2D(C). \tag{23}$$

Observe also that $C \subset \overline{C}$ implies $r(C) \le r(\overline{C})$. Moreover, we claim that

$$r(C) = \max\{r: \ B(\mathbf{0}, r) \subset \overline{C}\} = r(\overline{C}). \tag{24}$$

Indeed, the maximum exists because the ball $B(\mathbf{0}, r(\overline{C}))$ is the closure of the union of all balls in $\overline{C}$ and obviously it is contained in $\overline{C}$ by closedness. But $r(\overline{C})$ can not exceed $r(C)$ since by fatness $\operatorname{int} B(\mathbf{0}, r(\overline{C})) \subset \operatorname{int} \overline{C} \subset C$. Whence $r(\overline{C}) = r(C)$ proving (24).

Usually maximal chord length and width are considered in finite dimensional spaces, equipped with the inner product structure, and so they can be compared (after identifying X and X^*). See e.g. [36], Th. 7.6.1. For infinite dimensional spaces this is not possible, but some of the intrinsic connections can be proved.

Lemma 2.1. $w(K) = 2r(C)$.

Proof. In view of (22) is enough to show that $w(C) = 2r(C)$. Also by fatness we may assume that C is closed, that is $\overline{C} = C$. So, for $C = \overline{C}$ take now $B_r = B(\mathbf{0}, r(C)) \subset C$ (c.f. (24)). In view of (5) and (18) we have to prove that $r(C) = \inf\{h(C, \mathbf{v}^*): \ \mathbf{v}^* \in S^*\}$. The " $\le$ " direction is obvious as $B_r \subset C$, $\|\mathbf{v}^*\| = \sup_B \mathbf{v}^* = \frac{1}{r}\sup_{B_r} \mathbf{v}^*$ and (4) yields $r(C) = \sup_{B_r} \mathbf{v}^* \le h(C, \mathbf{v}^*)$, for all $\mathbf{v}^* \in S^*$.

On the other hand, by using the definition (20) of $r(C)$, for any $\varepsilon > 0$ there exists some point $\mathbf{x} \notin C$ with $\|\mathbf{x}\| \le r + \varepsilon$. Applying the separation theorem to this $\mathbf{x}$ and C, we conclude that $r + \varepsilon \ge \|\mathbf{x}\| \ge \langle \mathbf{v}^*, \mathbf{x}\rangle \ge h(C, \mathbf{v}^*)$, for some $\mathbf{v}^* \in S^*$. Taking infimum we conclude also the " $\ge$ " part of the assertion.

Corollary 2.2. $w(K) = \inf\{\tau(K, \mathbf{v}): \ \mathbf{v} \in S\}$.

Proof. Since by (3) $\partial C = \left\{\frac{\tau(K,\mathbf{v})}{2}\mathbf{v}: \ \mathbf{v} \in S\right\}$, the inradius $r(C)$ is the infimum of $\tau(K, \mathbf{v})/2$ over all $\mathbf{v} \in S$. Therefore a reference to Lemma 2.1 proves the statement.

Note that again by the definition of $\|\mathbf{v}^*\|$, for any $\mathbf{v}^* \in S^*$, one can always choose a $\mathbf{v} \in S$ so that $\langle \mathbf{v}^*, \mathbf{v}\rangle > 1 - \varepsilon$. Hence, the obvious relation

$$w(K, \mathbf{v}^*) \ge \tau(K, \mathbf{v})|\langle \mathbf{v}^*, \mathbf{v}\rangle| \tag{25}$$

also implies the " $\ge$ " part of Corollary 2.2, while the other direction can be also obtained from the following result.

Lemma 2.3. *For any $\mathbf{v} \in S$ there exists $\mathbf{v}^* \in S^*$ such that*

$$w(K, \mathbf{v}^*) = \tau(K, \mathbf{v}) \cdot \langle \mathbf{v}^*, \mathbf{v}\rangle. \tag{26}$$

Proof. Consider a normal vector $\mathbf{v}^* \in S^*$ to C at $\frac{\tau}{2}\mathbf{v} \in \partial C$. Clearly $h(C, \mathbf{v}^*) = \frac{\tau}{2}\langle \mathbf{v}^*, \mathbf{v}\rangle$. Now (26) follows from the fact that $w(C, \mathbf{v}^*) = w(K, \mathbf{v}^*)$ (see (5)).

At this stage we can formulate the generalization of [36] Theorem 7.6.1 to infinite dimensional spaces.

Proposition 2.4. *For all $\mathbf{v} \in S$, $\mathbf{v}^* \in S^*$ we have the following sharp estimates*

$$w(K) \le \tau(K, \mathbf{v}) \le d(K), \quad w(K) \le w(K, \mathbf{v}^*) \le d(K).$$

Proof. The lower bounds follow from Corollary 2.2 and the definition (18), respectively. The upper bound $\tau(K, \mathbf{v}) \le d(K)$ is an easy consequence of definitions (19) and (3). These three inequalities are obviously sharp.

Now consider inequality (25) and take the supremum over S^* on both sides. Since $\sup\{\langle \mathbf{v}^*, \mathbf{v}\rangle : \mathbf{v}^* \in S^*\} = \|\mathbf{v}\| = 1$, we get $\tau(K, \mathbf{v}) \le \sup\{w(K, \mathbf{v}^*) : \mathbf{v}^* \in S^*\}$. In view of $d(K) = \sup\{\tau(K, \mathbf{v}) : \mathbf{v} \in S\}$, we have $d(K) \le \sup\{w(K, \mathbf{v}^*) : \mathbf{v}^* \in S^*\}$. To prove the converse inequality, by (3)–(5) is enough to consider C in place of K. But this is trivial since $w(C, \mathbf{v}^*) = 2h(C, \mathbf{v}^*) = 2\sup_{\partial C} \mathbf{v}^* = 2\sup\{\langle \mathbf{v}^*, \frac{\tau}{2}\mathbf{v}\rangle : \mathbf{v} \in S\} \le 2\sup_S \tau/2 = d(K)$.

Example 2.5. Let ℓ^2 be the real Hilbert space of square summable sequences and let

$$K := \left\{ \mathbf{x} = (x_n)_{n=1}^{\infty} \in \ell^2 : \sum_{n=1}^{\infty} w_n x_n^2 \le 1 \right\},$$

where the weights w_n satisfy $1 < w_n < 2$ $(n \in \mathbb{N})$. Then K is a closed, bounded, convex set in ℓ^2 with $\mathbf{0}$ in its interior. However,

(i) with $w_n := 1 + \frac{1}{n}$ $(n \in \mathbb{N})$, we have $d(K) = 2$, but the supremum in (19) is not attained;

(ii) with $w_n := 2 - \frac{1}{n}$ $(n \in \mathbb{N})$, we have $w(K) = \sqrt{2}$, but the infimum in (18) is not attained.

Proof. By definition, K is closed. Since $B(\mathbf{0}, 1/\sqrt{2}) \subset K \subset B(\mathbf{0}, 1)$, we have $\sqrt{2} \le w(K) \le d(K) \le 2$. Also $\mathbf{0} \in \operatorname{int} B(\mathbf{0}, 1/\sqrt{2}) \subset \operatorname{int} K$ and K is obviously convex. Moreover, K is a centrally symmetric convex body and therefore $w(K) = 2\inf\{\|\mathbf{x}\| : \mathbf{x} \in \partial K\}$ and $d(K) = 2\sup\{\|\mathbf{x}\| : \mathbf{x} \in \partial K\}$. Let $(\mathbf{e}_n)_{n=1}^{\infty}$ be the standard unit vector basis of ℓ^2 where $\mathbf{e}_n := (0, \ldots, 0, 1, 0, \ldots, 0, \ldots)$, with 1 at the n^{th} place. In case (i), $\frac{1}{\sqrt{w_n}}\mathbf{e}_n \in \partial K$, hence $d(K) \ge \frac{2}{\sqrt{w_n}}$ $(n \in \mathbb{N})$ and we find $d(K) = 2$. On the other hand, for $w_n > 1$ $(n \in \mathbb{N})$ it is clear that any $\mathbf{x} \in K \setminus \{\mathbf{0}\}$ satisfies $\sum_n x_n^2 < \sum_n w_n x_n^2 \le 1$, hence $\|\mathbf{x}\| < 1$, and $d(K)$ is not attained.

Similarly, in case (ii), for any $\mathbf{v}^* = \mathbf{v} = (v_n) \in S^*_{\ell_2} = S_{\ell_2}$ we get

$$h(K, \mathbf{v}^*) \ge \left\langle \mathbf{v}^*, \frac{1}{\sqrt{\sum_n w_n v_n^2}} \mathbf{v} \right\rangle = \frac{1}{\sqrt{\sum_n w_n v_n^2}} > \frac{1}{\sqrt{2}\|\mathbf{v}^*\|_2} = \frac{1}{\sqrt{2}},$$

and hence $w(K, \mathbf{v}^*) > \sqrt{2}$. Now since $w(K, \mathbf{e}_n) = 2\left\|\frac{1}{\sqrt{w_n}} \cdot \mathbf{e}_n\right\| = \frac{2}{\sqrt{w_n}} \to \sqrt{2}$, we have $w(K) = \sqrt{2}$. But $w(K) = \sqrt{2}$ is not attained.

3. Basic properties

Lemma 3.1. *The mapping $\mathcal{P}$, defined in* (17), *has the following monotonicity properties:*

i) *For $K \in \mathcal{K}$ fixed, $\mathcal{P}^\lambda(K) = K^\lambda$ is increasing on $\mathbb{R}_+$.*
ii) *For $\lambda \geq 1$ fixed, $\mathcal{P}^\lambda(K) = K^\lambda$ is increasing on $\mathcal{K}$.*
iii) *$\mathcal{P}$ is increasing on $\mathcal{K} \times [1, \infty)$.*
iv) *For $\lambda < 1$ fixed, $\mathcal{P}^\lambda(K)$ is not monotonic.*

Proof. i) By definition (13), it suffices to show the assertion for an arbitrary layer L which is obvious.
ii) Again, it suffices to show the assertion for layers. Let $L \subset L'$; then they have to be parallel and in view of (8) we must have $L = L_{r,s}(\mathbf{v}^*)$, $L' = L_{r',s'}(\mathbf{v}^*)$ and $[r, s] \subset [r', s']$, for some $\mathbf{v}^* \in S^*$ and $r, s, r', s' \in \mathbb{R}$. Now (11) obviously gives $L^\lambda \subset L'^\lambda$, as stated. Note that $\mathbf{v}^*$ brings the question for layers to a question for real intervals. Moreover, it is easy to see e.g. from (11) that actually for any layer L and any $\lambda \geq 0$ we have

$$L^\lambda = \frac{\lambda+1}{2} L + \frac{\lambda-1}{2}(-L) = \lambda \cdot C(L) + \mathbf{x}_0 \text{ with } L - \mathbf{x}_0 = -(L - \mathbf{x}_0) \tag{27}$$

i.e. with any symmetry center $\mathbf{x}_0$ of L. (In fact (27) is valid for any centrally symmetric convex set).
(iii) Follows from i) and ii).
iv) Let $L \subset L'$ with common normal vector $\mathbf{v}^*$, say. If $r + s = r' + s'$, (where $L = L_{r,s}(\mathbf{v}^*)$, $L' = L'_{r',s'}(\mathbf{v}^*)$), then $L^\lambda \subset L'^\lambda$ for any $\lambda \geq 0$. But if $\lambda < 1$, $r = r' = 0$, $s = 1$ and $s' > \frac{1+\lambda}{1-\lambda}$, then L^λ and L'^λ are disjoint.

Lemma 3.2. $K^\lambda = \bigcap\limits_{\mu > \lambda} K^\mu$.

Proof. We have $\bigcap\limits_{\mu>\lambda} K^\mu = \bigcap\limits_{\mu>\lambda} \bigcap\limits_{\mathbf{v}^* \in S^*} L^\mu(K, \mathbf{v}^*) = \bigcap\limits_{\mathbf{v}^* \in S^*} \bigcap\limits_{\mu>\lambda} L^\mu(K, \mathbf{v}^*) = \bigcap\limits_{\mathbf{v}^* \in S^*} L^\lambda(K, \mathbf{v}^*) = K^\lambda$, as stated.

Proposition 3.3. *$\{K^\lambda\}_{\lambda \geq 0}$ is absorbing, and thus $\alpha(K, \mathbf{x})$ is finitely defined all over X. Moreover, $\alpha(K, \mathbf{x})$ is bounded on bounded sets.*

Proof. Since $\operatorname{int} K \neq \emptyset$, there exists a ball $B(\mathbf{x}_0, r) \subset K$. By ii) of Lemma 3.1, for any $\lambda \geq 1$ we have $K^\lambda \supset B^\lambda(\mathbf{x}_0, r) = B(\mathbf{x}_0, \lambda r)$, and thus $\bigcup\limits_{\lambda \geq 0} K^\lambda \supset \bigcup\limits_{\lambda \geq 1} B(\mathbf{x}_0, \lambda r) = X$. Boundedness of $\alpha(K, \mathbf{x})$ on bounded sets follows from

$$B(\mathbf{0}, R) \subset B(\mathbf{x}_0, \lambda r) \subset K^\lambda \quad \left(\lambda \geq \max\left\{1, \frac{1}{r}\big(\|\mathbf{x}_0\| + R\big)\right\}\right).$$

Corollary 3.4. *For all $\mathbf{x} \in X$ we have $\mathbf{x} \in K^{\alpha(K,\mathbf{x})}$, and in fact* (14) *is a minimum:*

$$\alpha(K, \mathbf{x}) = \min\{\lambda \geq 0 : \ \mathbf{x} \in K^\lambda\}. \tag{28}$$

Proof. By Lemma 3.1 i), Proposition 3.3 implies that $\Lambda := \{\lambda \geq 0 : \ \mathbf{x} \in K^\lambda\}$ is a nonempty interval with right endpoint at ∞. By Lemma 3.2, Λ is closed, too.

Proposition 3.5. $\alpha(K, \mathbf{x}) \to \infty$ *as* $\|\mathbf{x}\| \to \infty$.

Proof. Since K is bounded, $K \subset B(\mathbf{0}, R)$ for some $0 < R < \infty$, whence for any $\lambda \geq 1$ by Lemma 3.1 ii) we have $K^\lambda \subset B(\mathbf{0}, \lambda R)$. It follows that for any $N \in \mathbb{N}$ and $\|\mathbf{x}\| > NR$ the point $\mathbf{x} \notin K^N$ and thus $\alpha(K, \mathbf{x}) \geq N$ $\big(\|\mathbf{x}\| > R_0(N) := NR\big)$.

Proposition 3.6. *For fixed $K \in \mathcal{K}$, $\alpha(K, \cdot)$ is a convex function on the whole space X.*

Proof. Let us denote $\alpha' := \alpha(K, \mathbf{x}')$, $\alpha'' := \alpha(K, \mathbf{x}'')$, $\beta := t\alpha' + (1-t)\alpha''$ and $\hat{\mathbf{x}} := t\mathbf{x}' + (1-t)\mathbf{x}''$, $0 \leq t \leq 1$. Moreover, for a given $\mathbf{v}^* \in S^*$ denote $L := L(K, \mathbf{v}^*) = L_{r,s}(\mathbf{v}^*)$, with $r := -h(K, -\mathbf{v}^*)$, $s := h(K, \mathbf{v}^*)$.

It is easy to see, e.g. by using (27), that

$$L^\beta(K, \mathbf{v}^*) = L^{t\alpha' + (1-t)\alpha''} = tL^{\alpha'} + (1-t)L^{\alpha''}. \tag{29}$$

In view of Corollary 3.4 we have $\mathbf{x}' \in K^{\alpha'} \subset L^{\alpha'}$, $\mathbf{x}'' \in K^{\alpha''} \subset L^{\alpha''}$ and therefore $\hat{\mathbf{x}} \in tL^{\alpha'} + (1-t)L^{\alpha''} = L^\beta(K, \mathbf{v}^*)$. Since this is true for all $\mathbf{v}^*$, it follows that $\hat{\mathbf{x}} \in \bigcap_{\mathbf{v}^* \in S^*} L^\beta(K, \mathbf{v}^*) = K^\beta$. Hence $\alpha(K, \hat{\mathbf{x}}) \leq \beta$ and the Proposition is proved.

Corollary 3.7. *For fixed $K \in \mathcal{K}$, $\alpha(K, \cdot)$ is continuous on X.*

Proof. If X is finite dimensional, say $X = \mathbb{R}^d$, convexity itself implies continuity, see for instance [28, Corollary 10.1.1.]. However, for infinite dimensional spaces finiteness and convexity do not imply continuity. Thus, here we have to invoke also local boundedness, furnished by the second half of Proposition 3.3. Now, a reference to [22, Proposition 1.6 and the following Remark] yields continuity; moreover, it follows that $\alpha(K, \cdot)$ is *locally Lipschitzian.* Instead of utilizing this stronger assertion, we postpone the question of Lipschitz bounds to later analysis. We will prove even more precise results in Proposition 5.3 and Theorem 5.5.

Corollary 3.8.

(*i*) $K^\lambda = \big\{\mathbf{x} \in X : \ \alpha(K, \mathbf{x}) \leq \lambda\big\}$.

(*ii*) $\operatorname{int} K^\lambda = \bigcup_{\mu < \lambda} K^\mu = \bigcup_{\mu < \lambda} \operatorname{int} K^\mu$.

(*iii*) $\alpha(K, \mathbf{x}) < \lambda$ *if and only if* $\mathbf{x} \in \operatorname{int} K^\lambda$.

(*iv*) $\alpha(K, \mathbf{x}) = \lambda$ *if and only if* $\mathbf{x} \in \partial K^\lambda$.

(*v*) $\alpha(K, \mathbf{x}) > \lambda$ *if and only if* $\mathbf{x} \in \operatorname{ext} K^\lambda := X \setminus K^\lambda$.

(*vi*) $\alpha_K < 1$ *for every* $K \in \mathcal{K}(X)$.

Proof. (i) The proof is obvious in view of (14) and Lemma 3.1 i).
(ii) If $\lambda > \mu$, by Lemma 3.1 i) $K^\lambda \supset K^\mu \supset \operatorname{int} K^\mu$. Hence $\operatorname{int} K^\lambda \supset \operatorname{int} K^\mu$ and the " $\supset$ " inclusion follows. On the other hand, for any $\mathbf{x} \in \operatorname{int} K^\lambda$ there exists a ball $B(\mathbf{x}, r) \subset K^\lambda = \big\{\mathbf{x} \in X : \ \alpha(K, \mathbf{x}) \leq \lambda\big\}$ in view of i). If the value of $\alpha(K, \mathbf{x})$ were λ, then α would have a local maximum at $\mathbf{x}$ which, in view of convexity, would imply that α is a constant multiple of λ. But this contradicts Proposition 3.5. Thus $\alpha(K, \mathbf{x}) < \lambda$, i.e. $\mathbf{x} \in \bigcup_{\mu < \lambda} K^\mu$, and the first equality follows. Equality of the middle and the right expression is obvious from the first identity.
(iii) follows from (i) and (ii), and (iv) follows from (i) and (iii), while (v) follows from (i) trivially. Finally, (ii) and and the fact that $\operatorname{int} K^1 = \operatorname{int} K \neq \emptyset$ imply (vi).

4. Relatives of $\alpha(K, \mathbf{x})$

There are various other geometric quantities, which are closely related to our definition of the Minkowski functional. Many of these equivalent forms and related quantities play a crucial role in approximation theoretic applications. Therefore, we find it appropriate to give a more or less comprehensive account of them here. The use of them will also be rather helpful in the forthcoming sections.

Let us start with the description of the original definition of Minkowski, [20], as presented in Grünbaum's article, c.f. [7, p. 246]. Denote

$$t := t(K, \mathbf{v}^*, \mathbf{x}) := \frac{2\langle \mathbf{v}^*, \mathbf{x}\rangle - h(K, \mathbf{v}^*) + h(K, -\mathbf{v}^*)}{w(K, \mathbf{v}^*)}. \tag{30}$$

For fixed $\mathbf{v}^*$ this function is an affine linear functional in $\mathbf{x} \in X$, while for fixed $\mathbf{x}$ it is a norm-continuous mapping from S^* (or $X^* \setminus \{\mathbf{0}\}$) to $\mathbb{R}$. In fact, for fixed $\mathbf{v}^* \in S^*$, t maps the layer $L(K, \mathbf{v}^*)$ to $[-1, 1]$, and $L^\eta(K, \mathbf{v}^*)$ to $[-\eta, \eta]$. Therefore, the two forms of the following definition are really equivalent;

$$\begin{aligned} \lambda : = \lambda(K, \mathbf{x}) &:= \sup\left\{\eta > 0 : \ \exists \mathbf{v}^* \in S^*, \ \mathbf{x} \in \partial L^\eta(K, \mathbf{v}^*)\right\} \\ &= \sup\left\{\left|t(K, \mathbf{v}^*, \mathbf{x})\right| : \ \mathbf{v}^* \in S^*\right\} = \sup\left\{t(K, \mathbf{v}^*, \mathbf{x}) : \ \mathbf{v}^* \in S^*\right\} \end{aligned} \tag{31}$$

Note that $t(K, \mathbf{v}^*, \mathbf{x}) = -t(K, -\mathbf{v}^*, \mathbf{x})$ and therefore we don't have to use the absolute value.

In fact, $\lambda(K, \mathbf{x})$ expresses the supremum of the ratios of the distances between the point $\mathbf{x}$ and the symmetry hyperplane $\frac{1}{2}(H(K, \mathbf{v}^*) + H(K, -\mathbf{v}^*))$ of any layer $L(K, \mathbf{v}^*)$ and the half-width $w(K, \mathbf{v}^*)/2$. Now Minkowski's definition was

$$\varphi(K, \mathbf{x}) := \inf\left\{\frac{\min\{\operatorname{dist}(\mathbf{x}, H(K, \mathbf{v}^*)), \operatorname{dist}(\mathbf{x}, H(K, -\mathbf{v}^*))\}}{\max\{\operatorname{dist}(\mathbf{x}, H(K, \mathbf{v}^*)), \operatorname{dist}(\mathbf{x}, H(K, -\mathbf{v}^*))\}} : \ \mathbf{v}^* \in S^*\right\},$$

which clearly implies the relation

$$\varphi(K, \mathbf{x}) = \frac{1 - \lambda(K, \mathbf{x})}{1 + \lambda(K, \mathbf{x})} \qquad (\mathbf{x} \in K).$$

Although this $\varphi(K, \mathbf{x})$ seems to be used traditionally only for $\mathbf{x} \in K$, extending the definition to arbitrary $\mathbf{x} \in X$ yields the similar relation

$$\varphi(K, \mathbf{x}) = \frac{|1 - \lambda(K, \mathbf{x})|}{1 + \lambda(K, \mathbf{x})} \qquad (\mathbf{x} \in X).$$

Proposition 4.1. *We have $\alpha(K, \mathbf{x}) = \lambda(K, \mathbf{x})$ $(\forall \mathbf{x} \in X)$.*

Proof. If $\mathbf{x} \in \partial L^\lambda(K, \mathbf{v}^*)$ for some $\lambda > 0$ and $\mathbf{v}^* \in S^*$, then obviously $\mathbf{x} \notin L^{\lambda-\varepsilon}(K, \mathbf{v}^*)$ for any $\varepsilon > 0$, and hence also $\mathbf{x} \notin K^{\lambda-\varepsilon}$ and $\alpha(K, \mathbf{x}) \geq \lambda$. This proves $\alpha(K, \mathbf{x}) \geq \lambda(K, \mathbf{x})$. Let now $\mathbf{v}^* \in S^*$ be arbitrary. Then, in view of (31) we have $\mathbf{x} \in \partial L^{|t|}(K, \mathbf{v}^*) \subset L^\lambda(K, \mathbf{v}^*)$ with t and λ defined in (30) and (31), respectively. Therefore $\mathbf{x} \in K^\lambda$ and thus $\alpha \leq \lambda$ follows.

We now define some geometric quantities using homothetic images. Let us define

$$\beta(K,\mathbf{x}) := \sup\big\{\lambda \ge 0 :\ \{-\lambda(K-\mathbf{x})+\mathbf{x}\} \subset K\big\} \qquad (\mathbf{x}\in K), \tag{32}$$

$$\varrho(K,\mathbf{x}) := \sup\Big\{0<\lambda<1 :\ K\cap\big\{\mathbf{x}+\lambda(K-\mathbf{x})\big\}=\emptyset\Big\} \qquad (\mathbf{x}\notin K), \tag{33}$$

$$\varrho^*(K,\mathbf{x}) := \sup\Big\{0<\lambda<1 \colon \operatorname{int}K\cap\big\{\mathbf{x}+\lambda(K-\mathbf{x})\big\}=\emptyset\Big\} \qquad (\mathbf{x}\notin K). \tag{34}$$

Let us mention that these geometric quantities were used in various ways for long. In particular, Schneider defines the set of parallel bodies of a body K with respect to another one, say D, c.f. [31, p. 134] or (82) below. When $D = K$, this reduces to parallel bodies of K closely related to the above β. In fact, in [31, Note 3.1.14, p. 141] Schneider mentions that $r(K,-K) = \max_K \beta$ is Minkowski's measure of symmetry. Moreover, the above defined $\rho(K,\mathbf{x})$ is closely connected to a well-known notion, the so-called "associated bodies" of Hammer, c.f. [31, pp. 141-142] or [8]. Thus it will be used to establish the connection of the parametric set sequence K^λ to Hammer's associated bodies.

In the following we also consider the set of straight lines

$$\begin{aligned}\mathcal{L} := \mathcal{L}(K,\mathbf{x}) &:= \big\{l=\{\mathbf{x}+t\mathbf{v} :\ t\in\mathbb{R}\} : \#\{K\cap l\}>1,\ \mathbf{v}\in S\big\} \\ &= \big\{l=\{\mathbf{x}+t\mathbf{v} :\ t\in\mathbb{R}\} :\ \mathbf{v}\in S,\ K\cap l=[\mathbf{a},\mathbf{b}],\ \mathbf{a}\ne\mathbf{b}\big\}.\end{aligned} \tag{35}$$

For $\mathbf{x}\in\operatorname{int}K$, $\mathcal{L}$ contains all straight lines through $\mathbf{x}$, but for $\mathbf{x}\notin K$ only some lines are included. When they need to be distinguished, we always assume that the notation of the endpoints of $K\cap l$ is chosen so that $\|\mathbf{x}-\mathbf{b}\|\le\|\mathbf{x}-\mathbf{a}\|$. With these in mind, we define

$$\mu(K,\mathbf{x}) := \inf\left\{\frac{\|2\mathbf{x}-\mathbf{a}-\mathbf{b}\|}{\|\mathbf{b}-\mathbf{a}\|} :\ l\in\mathcal{L}\right\} \qquad (\mathbf{x}\notin K), \tag{36}$$

$$\nu(K,\mathbf{x}) := \sup\left\{\frac{\|2\mathbf{x}-\mathbf{a}-\mathbf{b}\|}{\|\mathbf{b}-\mathbf{a}\|} :\ l\in\mathcal{L}\right\} \qquad (\mathbf{x}\in K),$$

$$\sigma(K,\mathbf{x}) := \inf\left\{\frac{\|\mathbf{b}-\mathbf{x}\|}{\|\mathbf{a}-\mathbf{x}\|} :\ l\in\mathcal{L}\right\} \qquad (\mathbf{x}\in X),$$

$$\omega(K,\mathbf{x}) := \sup\left\{\frac{\|\mathbf{a}-\mathbf{x}\|}{\|\mathbf{a}-\mathbf{b}\|} :\ l\in\mathcal{L}\right\} \qquad (\mathbf{x}\in K),$$

$$\gamma^2(K,\mathbf{x}) := \inf\left\{\frac{\|\mathbf{b}-\mathbf{x}\|\,\|\mathbf{a}-\mathbf{x}\|}{\left\|\frac{1}{2}(\mathbf{b}-\mathbf{a})\right\|^2} :\ l\in\mathcal{L}\right\} \qquad (\mathbf{x}\in K).$$

Note the various restrictions on the domains of the definition of these quantities. Note also that in fact $\sigma(K,\mathbf{x})$ is exactly the equivalent form of $\varphi(K,\mathbf{x})$ mentioned also in [7, p. 247]. This form is used by even more authors than the original one, hence we could refer to common knowledge for $\varphi(K,\mathbf{x})=\sigma(K,\mathbf{x})$. Still, we give proofs for all equivalences below.

Proposition 4.2.

i) *In* (34)*, the sup is actually a maximum.*

ii) $\varrho(K,\mathbf{x})=\varrho^*(K,\mathbf{x})$.

iii) *There exists $\mathbf{v}^* \in S^*$ such that the supporting hyperplane* (7) *separates (weakly) K and $\mathbf{x} + \varrho(K - \mathbf{x})$, in other words*

$$h(K, \mathbf{v}^*) \leq (1 - \varrho)\langle \mathbf{v}^*, \mathbf{x}\rangle - \varrho h(K, -\mathbf{v}^*). \tag{37}$$

Proof. i) Consider for some scalar $0 < \sigma < 1$ the set $M := M_\sigma := \operatorname{int} K \cap \big(\mathbf{x} + \sigma(K - \mathbf{x})\big)$. Let us suppose first that $M \neq \emptyset$. As $\mathbf{x} + \sigma(K - \mathbf{x})$ is also a fat set, this entails that $\operatorname{int} M \neq \emptyset$, and so there exist $\mathbf{y} \in M$ and $r > 0$ such that even $B(\mathbf{y}, r) \subset M$. On the other hand, boundedness of K implies $K \subset B(\mathbf{x}, R)$, for R large enough. Take now $\varepsilon < r\sigma/R$ and consider $\mathbf{z} := \frac{\sigma-\varepsilon}{\sigma}(\mathbf{y} - \mathbf{x}) + \mathbf{x}$. Clearly $\|\mathbf{z} - \mathbf{y}\| = \frac{\varepsilon}{\sigma}\|\mathbf{y} - \mathbf{x}\| < r$, hence even $\mathbf{z} \in B(\mathbf{y}, r) \subset \operatorname{int} K$. Since $\mathbf{y} \in \mathbf{x} + \sigma(K - \mathbf{x})$, we have $\mathbf{z} \in \mathbf{x} + (\sigma - \varepsilon)(K - \mathbf{x})$ and this shows that together with $M_\sigma \neq \emptyset$ also $M_{\sigma-\varepsilon} \neq \emptyset$ follows with some $\varepsilon > 0$. Thus $M_{\varrho^*} \neq \emptyset$ would lead to a contradiction with (34), and so $M_{\varrho^*} = \emptyset$, as needed.
ii) According to part i) the convex sets $\operatorname{int} K$ and $\mathbf{x} + \varrho^*(K - \mathbf{x})$ are disjoint. Applying the separation theorem of convex sets (c.f. e.g. [29], Theorem 3.4.) we conclude the existence of a $\mathbf{v}^* \in S^*$ separating them. Since $\overline{\operatorname{int} K} = K$, $\mathbf{v}^*$ separates even K and $\mathbf{x} + \varrho^*(K - \mathbf{x})$ at least in the weak (i.e. $\leq$) sense. However, this entails that K and $\mathbf{x} + (\varrho^* - \varepsilon)(K - \mathbf{x})$ are separated by $\mathbf{v}^*$ even in the strong sense for any $\varepsilon > 0$. Hence, in particular $K \cap \big\{(\varrho^* - \varepsilon)(K - \mathbf{x}) + \mathbf{x}\big\} = \emptyset$, and $\varrho \geq \varrho^* - \varepsilon$ $(\forall \varepsilon > 0)$, which leads to $\varrho \geq \varrho^*$. On the other hand, it is clear that for all $\mathbf{x} \notin K$ we have $0 < \varrho \leq \varrho^* < 1$ and thus we arrive at $\varrho = \varrho^*$.
iii) Since $\rho = \rho^*$, by construction we see that $\mathbf{v}^*$ separates K and $\mathbf{x} + \varrho(K - \mathbf{x})$, and a simple calculation gives (37) as well.

Proposition 4.3. *For all $\mathbf{x} \notin K$ we have $\sigma(K, \mathbf{x}) = \varrho(K, \mathbf{x})$.*

Proof. Let $\lambda < \sigma := \sigma(K, \mathbf{x})$ be arbitrary. Then by definition (36) of σ, K and $\mathbf{x} + \lambda(K - \mathbf{x})$ are disjoint, hence also $\lambda \leq \varrho$, which implies $\sigma \leq \varrho$.

On the other hand, in view of Proposition 4.2 iii) there exists $\mathbf{v}^*$ separating K and $\mathbf{x} + \varrho(K - \mathbf{x})$, i.e. $K - \mathbf{x}$ and $\varrho(K - \mathbf{x})$, which implies even for arbitrary $\mathbf{a}, \mathbf{b} \in K$ (and not only for points on the same line l) that with some constant $c > 0$

$$\langle \mathbf{v}^*, \mathbf{b} - \mathbf{x}\rangle \geq c \geq \big\langle \mathbf{v}^*, \varrho(\mathbf{a} - \mathbf{x})\big\rangle \geq 0. \tag{38}$$

Clearly, separation occurs only if $\langle \mathbf{v}^*, \mathbf{y} - \mathbf{x}\rangle$ does not vanish for points $\mathbf{y} \in \operatorname{int} K$ at least. However, this yields $\|\mathbf{b} - \mathbf{x}\| \geq \varrho\|\mathbf{a} - \mathbf{x}\|$ from (38), and by definition (36) we get even $\sigma \geq \varrho$ after taking the infimum over $l \in \mathcal{L}$. Together with the first part this completes the proof.

Proposition 4.4. *For all $\mathbf{x} \notin K$ we have $\sigma(K, \mathbf{x}) = \frac{\mu(K,\mathbf{x})-1}{\mu(K,\mathbf{x})+1}$.*

Proof. For any $\mathbf{x} \notin K$ it is clear that $\mu > 1$. Moreover,

$$\begin{aligned}\frac{\mu - 1}{\mu + 1} &= 1 - \frac{2}{\mu + 1} = 1 - \frac{2}{\inf_{\mathcal{L}} \big(\|2\mathbf{x} - 2\mathbf{a}\|/\|\mathbf{b} - \mathbf{a}\|\big)} \\ &= 1 - \sup_{\mathcal{L}} \frac{\|\mathbf{b} - \mathbf{a}\|}{\|\mathbf{x} - \mathbf{a}\|} = \inf_{\mathcal{L}} \frac{\|\mathbf{b} - \mathbf{x}\|}{\|\mathbf{a} - \mathbf{x}\|} = \sigma,\end{aligned}$$

as asserted.

Proposition 4.5. *For all* $\mathbf{x} \in X \setminus K$ *we have* $\varrho(K,\mathbf{x}) = \frac{\alpha(K,\mathbf{x})-1}{\alpha(K,\mathbf{x})+1}$.

Proof. Suppose first that $\mathbf{x} \in K^\lambda = \bigcap_{\mathbf{u}^* \in S^*} L^\lambda(K,\mathbf{u}^*)$. Then a simple calculation leads to the inequality

$$\langle \mathbf{u}^*, \mathbf{x} \rangle \le \frac{\lambda}{2} w(K,\mathbf{u}^*) + \frac{1}{2}\big(h(K,\mathbf{u}^*) - h(K,-\mathbf{u}^*)\big) \quad (\forall \mathbf{u}^* \in S^*)$$

which is inherited by $\alpha = \inf\{\lambda : \ \mathbf{x} \in K^\lambda\}$ as well. Thus, also for the particular $\mathbf{v}^*$, furnished by Proposition 4.2 iii), we obtain

$$\langle \mathbf{v}^*, \mathbf{x} \rangle \le \frac{\alpha}{2} w(K,\mathbf{v}^*) + \frac{1}{2}\big(h(K,\mathbf{v}^*) - h(K,-\mathbf{v}^*)\big). \tag{39}$$

On combining (37) and (39) we are led to

$$h(K,\mathbf{v}^*) \le (1-\varrho)\left\{\frac{\alpha+1}{2} h(K,\mathbf{v}^*) + \frac{\alpha-1}{2} h(K,-\mathbf{v}^*)\right\} - \varrho h(K,-\mathbf{v}^*)$$

and some calculation yields

$$w(K,\mathbf{v}^*)\big((\alpha-1) - \varrho(\alpha+1)\big) \ge 0. \tag{40}$$

Since $w(K,\mathbf{v}^*) \ge w(K) > 0$, (40) entails $\varrho \le \frac{\alpha-1}{\alpha+1}$.

Let now $1 < \lambda < \alpha$, i.e. $\mathbf{x} \notin L^\lambda(K,\mathbf{u}^*)$ for some appropriate $\mathbf{u}^* \in S^*$. Then, a simple calculation shows that already $\mathbf{x} + \frac{\lambda-1}{\lambda+1}\big(L(K,\mathbf{u}^*) - \mathbf{x}\big)$ is disjoint from $L(K,\mathbf{u}^*)$, thus even more so $\left\{\mathbf{x} + \frac{\lambda-1}{\lambda+1}(K-\mathbf{x})\right\} \cap K = \emptyset$, and $\frac{\lambda-1}{\lambda+1} \le \varrho$ follows. Letting here $\lambda \to \alpha^-$ we even get $\frac{\alpha-1}{\alpha+1} \le \varrho$, and the opposite inequality being already proven, this concludes the proof.

Proposition 4.6. *For* $\mathbf{x} \in K$ *we have* $\sigma(K,\mathbf{x}) = \frac{1}{\omega(K,\mathbf{x})} - 1$.

Proof.

$$\sigma = \inf_{\mathcal{L}} \frac{||\mathbf{b}-\mathbf{x}||}{||\mathbf{a}-\mathbf{x}||} = \inf_{\mathcal{L}} \frac{||\mathbf{b}-\mathbf{a}||}{||\mathbf{a}-\mathbf{x}||} - 1 = \frac{1}{\sup_{\mathcal{L}} \frac{||\mathbf{a}-\mathbf{x}||}{||\mathbf{b}-\mathbf{a}||}} - 1 = \frac{1}{\omega} - 1.$$

Proposition 4.7. *For* $\mathbf{x} \in K$ *we have* $\sigma(K,\mathbf{x}) = \beta(K,\mathbf{x})$.

Proof. Let $\mathbf{a} \in \partial K$ be any boundary point. It is easy to see (by closedness of K) that (32) is actually a maximum, that is, $-\beta(K-\mathbf{x})+\mathbf{x} \subset K$, and therefore $\mathbf{a}' := -\beta(\mathbf{a}-\mathbf{x})+\mathbf{x} \in K$. Consider now an $l \in \mathcal{L}$, $K \cap l = [\mathbf{a},\mathbf{b}]$, $\|\mathbf{b}-\mathbf{x}\| \le \|\mathbf{a}-\mathbf{x}\|$, and estimate $\frac{\|\mathbf{b}-\mathbf{x}\|}{\|\mathbf{a}-\mathbf{x}\|} \ge \frac{\|\mathbf{a}'-\mathbf{x}\|}{\|\mathbf{a}-\mathbf{x}\|} = \beta$. We only need to take infimum over $l \in \mathcal{L}$ to obtain $\sigma \ge \beta$.

For the other direction let us consider the set $K' := -\sigma(K-\mathbf{x})+\mathbf{x}$. Let $\mathbf{y} \in K$ be arbitrary: we show that also $\mathbf{z} := -\sigma(\mathbf{y}-\mathbf{x})+\mathbf{x} \in K$, leading to $K' \subset K$ and $\sigma \le \beta$, which will prove Proposition 4.7.

Take now the straight line l through $\mathbf{x}$ and $\mathbf{y}$ ($\mathbf{y} \ne \mathbf{x}$, as for $\mathbf{y} = \mathbf{x}$ also $\mathbf{z} = \mathbf{x}$ and there remains nothing to prove). Let $\mathbf{a}$ be the endpoint of $K \cap l$ on the side of $\mathbf{y}$, — i.e. $\mathbf{y} \in [\mathbf{x},\mathbf{a}]$ and $\mathbf{a} \in \partial_{rel} K \cap l$. Then by definition (36) we must have for the other endpoint $\mathbf{b} \in K \cap l$, on the other side of $\mathbf{x}$, the inequality $\|\mathbf{b}-\mathbf{x}\| \ge \sigma\|\mathbf{a}-\mathbf{x}\| \ge \sigma\|\mathbf{y}-\mathbf{x}\| = \|\mathbf{z}-\mathbf{x}\|$. This shows that $\mathbf{z} \in [\mathbf{x},\mathbf{b}] \subset K$ and hence $\mathbf{z} \in K$ as needed.

Proposition 4.8. *For $\mathbf{x} \in K$ we have $\beta(K, \mathbf{x}) = \frac{1-\lambda(K,\mathbf{x})}{1+\lambda(K,\mathbf{x})}$.*

Proof. For arbitrary $\mathbf{v}^* \in S^*$ consider the layer $L := L(K, \mathbf{v}^*)$. By (30) a simple calculation gives $\{-\frac{1-|t|}{1+|t|}(L-\mathbf{x})+\mathbf{x}\} \subset L \; \big(t := t(K, \mathbf{v}^*, \mathbf{x})\big)$. Since the function $\varphi(t) := \frac{1-t}{1+t}$ $(0 \le t \le 1)$ is strictly decreasing, this last relation is also true for $\lambda = \sup_{S^*} |t|$ defined by (31). Thus $-\frac{1-\lambda}{1+\lambda}(L - \mathbf{x}) + \mathbf{x} \subset L$ ($\mathbf{v}^* \in S^*$), and hence

$$K = \bigcap_{\mathbf{v}^* \in S^*} L(K, \mathbf{v}^*) \supset \bigcap_{\mathbf{v}^* \in S^*} \left\{-\frac{1-\lambda}{1+\lambda}\big(L(K, \mathbf{v}^*) - \mathbf{x}\big) + \mathbf{x}\right\} = -\frac{1-\lambda}{1+\lambda}(K - \mathbf{x}) + \mathbf{x}, \tag{41}$$

proving that $\beta \ge \frac{1-\lambda}{1+\lambda}$.

To deduce the converse, consider now $-\beta(K - \mathbf{x}) + \mathbf{x} \subset K$. Forming supporting layers normal to $\mathbf{v}^* \in S^*$, we find

$$-\beta\big(L(K, \mathbf{v}^*) - \mathbf{x}\big) + \mathbf{x} = L\Big(\big(- \beta(K - \mathbf{x}) + \mathbf{x}\big), \mathbf{v}^*\Big) \subset L(K, \mathbf{v}^*),$$

that is, $-\beta(L - \mathbf{x}) + \mathbf{x} \subset L$ ($\forall \mathbf{v}^* \in S^*$). For any given $\mathbf{v}^* \in S^*$, however, the maximal scalar to satisfy this relation is exactly $\frac{1-|t|}{1+|t|}$, hence $\beta \le \frac{1-|t|}{1+|t|}$ ($\mathbf{v}^* \in S^*$), and taking infimum on the right hand side now gives $\beta \le \frac{1-\lambda}{1+\lambda}$, referring to the decreasing shape of $\varphi(t)$ again.

Proposition 4.9. *For arbitrary $\mathbf{x} \in K$ we have $\nu(K, \mathbf{x}) = \frac{1-\sigma(K,\mathbf{x})}{1+\sigma(K,\mathbf{x})}$.*

Proof. Taking care of the distinction between $\mathbf{a}$ and $\mathbf{b}$ in the notation for $K \cap l = [\mathbf{a}, \mathbf{b}]$ prescribed for $l \in \mathcal{L}$, we write with $\varphi(t) := \frac{1-t}{1+t}$

$$\nu = \sup_{\mathcal{L}} \frac{\|2\mathbf{x} - \mathbf{a} - \mathbf{b}\|}{\|\mathbf{b} - \mathbf{a}\|} = \sup_{\mathcal{L}} \frac{\|\mathbf{x} - \mathbf{a}\| - \|\mathbf{b} - \mathbf{x}\|}{\|\mathbf{x} - \mathbf{a}\| + \|\mathbf{b} - \mathbf{x}\|} = \sup_{\mathcal{L}} \frac{1 - \frac{\|\mathbf{b}-\mathbf{x}\|}{\|\mathbf{a}-\mathbf{x}\|}}{1 + \frac{\|\mathbf{b}-\mathbf{x}\|}{\|\mathbf{a}-\mathbf{x}\|}} = \sup_{\mathcal{L}} \varphi\Big(\frac{\|\mathbf{b} - \mathbf{x}\|}{\|\mathbf{a} - \mathbf{x}\|}\Big) = \frac{1 - \inf_{\mathcal{L}} \frac{\|\mathbf{b}-\mathbf{x}\|}{\|\mathbf{a}-\mathbf{x}\|}}{1 + \inf_{\mathcal{L}} \frac{\|\mathbf{b}-\mathbf{x}\|}{\|\mathbf{a}-\mathbf{x}\|}} = \frac{1-\sigma}{1+\sigma}\,.$$

Proposition 4.10. *For $\mathbf{x} \in K$ we have $\gamma^2(K, \mathbf{x}) = 4\omega(K, \mathbf{x})\,(1 - \omega(K, \mathbf{x}))$.*

Proof. Working in a similar fashion as in the proof of Proposition 4.9, but using here also the decreasing shape of $\psi(s) := 4s(1 - s)$ on $[\frac{1}{2}, 1]$, we can write

$$\gamma^2 = \inf_{\mathcal{L}} \frac{\|\mathbf{x} - \mathbf{a}\| \cdot \|\mathbf{b} - \mathbf{x}\|}{\big(\frac{1}{2}\|\mathbf{b} - \mathbf{a}\|\big)^2} = \inf_{\mathcal{L}} \left\{4(1 - s)s : \; \frac{1}{2} \le s = \frac{\|\mathbf{a} - \mathbf{x}\|}{\|\mathbf{b} - \mathbf{a}\|} \le 1\right\} = \psi\left(\sup_{\mathcal{L}} \frac{\|\mathbf{a} - \mathbf{x}\|}{\|\mathbf{a} - \mathbf{b}\|}\right) = \psi\,(\omega) = 4\omega(1 - \omega).$$

Corollary 4.11. *We have the following relations for $\alpha(K, \mathbf{x})$:*

$$\alpha(K, \mathbf{x}) = \lambda(K, \mathbf{x}) = \begin{cases} \dfrac{1 - \sigma(K, \mathbf{x})}{1 + \sigma(K, \mathbf{x})} = \dfrac{1 - \varphi(K, \mathbf{x})}{1 + \varphi(K, \mathbf{x})} & (\mathbf{x} \in K) \\[2ex] \dfrac{1 + \sigma(K, \mathbf{x})}{1 - \sigma(K, \mathbf{x})} = \dfrac{1 + \varphi(K, \mathbf{x})}{1 - \varphi(K, \mathbf{x})} & (\mathbf{x} \notin K) \end{cases} \quad \textit{for} \quad \mathbf{x} \in X, \tag{i}$$

(ii) $$\alpha(K,\mathbf{x}) = \nu(K,\mathbf{x}) = \frac{1-\beta(K,\mathbf{x})}{1+\beta(K,\mathbf{x})} = \sqrt{1-\gamma^2(K,\mathbf{x})} = 2\omega(K,\mathbf{x})-1 \qquad \text{for} \quad \mathbf{x}\in K,$$

(iii) $$\alpha(K,\mathbf{x}) = \mu(K,\mathbf{x}) = \frac{1+\varrho(K,\mathbf{x})}{1-\varrho(K,\mathbf{x})} = \frac{1+\varrho^*(K,\mathbf{x})}{1-\varrho^*(K,\mathbf{x})} \qquad \text{for} \quad \mathbf{x}\notin K.$$

5. Lipschitz bounds and linear growth

Lemma 5.1. *Let L be any layer and let $0<\beta<\gamma$ be arbitrary. Then we have*

$$\operatorname{dist}\{\partial L^\beta, \partial L^\gamma\} = \frac{\gamma-\beta}{2} w(L).$$

Proof. This distance formula follows easily from (5) and (27).

Lemma 5.2. *Let $\mathbf{x}\neq\mathbf{y}\in X$ and let $\mathbf{v} := \frac{1}{\|\mathbf{x}-\mathbf{y}\|}(\mathbf{x}-\mathbf{y})$. Then*

$$\big|\alpha(K,\mathbf{x})-\alpha(K,\mathbf{y})\big| \leq \frac{2\|\mathbf{x}-\mathbf{y}\|}{\tau(K,\mathbf{v})}\,, \tag{42}$$

with $\tau(K,\mathbf{v})$ defined by (3). Moreover, (42) is best possible.

Proof. For an arbitrary $\mathbf{v}^*\in S^*$ let $L := L(K,\mathbf{v}^*)$. There exist unique scalars $\beta,\gamma\geq 0$, so that $\mathbf{x}\in\partial L^\beta$ and $\mathbf{y}\in\partial L^\gamma$. If we take $\delta := \|\mathbf{x}-\mathbf{y}\|$, then it is obvious that $\operatorname{dist}\{\partial L^\beta,\partial L^\gamma\} = \delta\cdot|\langle\mathbf{v}^*,\mathbf{v}\rangle|$. On the other hand, by Lemma 5.1 this equals to $\frac{|\gamma-\beta|}{2}w(L) = \frac{|\gamma-\beta|}{2}w(K,\mathbf{v}^*)$. Hence, in view of (25) we find

$$\gamma \leq \beta + \frac{2\delta|\langle\mathbf{v}^*,\mathbf{v}\rangle|}{w(K,\mathbf{v}^*)} \leq \beta + \frac{2\delta}{\tau(K,\mathbf{v})}\,. \tag{43}$$

Now if $\mathbf{x}\in\partial L^\beta$ and $\alpha(K,\mathbf{x})=\alpha$, then by definition (31) and Proposition 4.1 we must have $\beta\leq\alpha$. But then (43) implies $\gamma\leq\alpha+2\delta/\tau$, whence $\mathbf{y}\in L^{\alpha+2\delta/\tau}$ for all $\mathbf{v}^*\in S^*$ and so $\mathbf{y}\in K^{\alpha+2\delta/\tau}$ and $\alpha(K,\mathbf{y})\leq\alpha+2\delta/\tau$. By symmetry we also have $\alpha\leq\alpha(K,\mathbf{y})+2\delta/\tau$. These last two inequalities yield (42).

To show that (42) is sharp, consider $\mathbf{x}=(s+\delta)\mathbf{v}$, $\mathbf{y}=s\mathbf{v}$ (with s large enough). For the given vector $\mathbf{v}\in S$, take now a $\mathbf{v}^*\in S^*$ furnished by Lemma 2.3 satisfying (26). Consider the point $\mathbf{y}=s\mathbf{v}$, the affine linear functional (30) and the quantity (31). By Proposition 4.1 we can write

$$f(s) := \alpha(K,s\mathbf{v}) = \lambda(K,s\mathbf{v}) \geq t(K,\mathbf{v}^*,s\mathbf{v}). \tag{44}$$

Now, by taking into consideration (30) and (26), (44) easily leads to

$$\lim_{s\to\infty} f(s)/s \geq \frac{2\langle\mathbf{v}^*,\mathbf{v}\rangle}{w(K,\mathbf{v}^*)} = \frac{2}{\tau(K,\mathbf{v})}. \tag{45}$$

Since the function $f(s)=\alpha(K,s\mathbf{v})$ is convex on $\mathbb{R}$, we get

$$\begin{aligned}\sup\{\alpha(K,\mathbf{x})-\alpha(K,\mathbf{y}) : \mathbf{x},\mathbf{y}\in X, \mathbf{x}-\mathbf{y}=\delta\mathbf{v}\}\\ \geq \limsup_{s\to\infty}\big(f(s+\delta)-f(s)\big) \geq \delta\lim_{s\to\infty} f(s)/s\,.\end{aligned} \tag{46}$$

It follows from (45) and (46) that (42) is sharp.

Proposition 5.3. *We have the uniform Lipschitz bound*

$$\left|\alpha(K,\mathbf{x})-\alpha(K,\mathbf{y})\right| \le \frac{2}{w(K)}\|\mathbf{x}-\mathbf{y}\| \quad (\mathbf{x},\mathbf{y}\in X). \tag{47}$$

Proof. Combining Lemma 5.2 and Proposition 2.4 yields (47).

Lemma 5.4. *Let $K, K' \in \mathcal{K}$, and $\mathbf{x} \in X$ be arbitrary with the Hausdorff distance $\delta = \delta(K,K') < \frac{1}{2}w(K)$. Then we have*

$$\left|\alpha(K,\mathbf{x})-\alpha(K',\mathbf{x})\right| \le \frac{\delta(K,K')\big(4\|\mathbf{x}\|+4D(K)+2w(K)\big)}{w(K)\big(w(K)-2\delta(K,K')\big)}. \tag{48}$$

Proof. For an arbitrary $\mathbf{v}^* \in S^*$ consider the affine linear functional t defined by (30) and similarly the affine linear functional t', where

$$t' := t(K',\mathbf{v}^*,\mathbf{x}) := \frac{2\langle \mathbf{v}^*,\mathbf{x}\rangle - h(K',\mathbf{v}^*)+h(K',-\mathbf{v}^*)}{w(K',\mathbf{v}^*)}. \tag{49}$$

By definition (15) of the Hausdorff distance we have $\left|h(K,\mathbf{v}^*)-h(K',\mathbf{v}^*)\right| \le \delta$, $|w-w'| \le 2\delta\,(< w)$, and thus

$$\begin{aligned}
|t-t'| &\le 2\left|\langle \mathbf{v}^*,\mathbf{x}\rangle\right|\frac{2\delta}{ww'}+\frac{2\delta}{w}+2D(K')\frac{2\delta}{ww'}\\
&\le \frac{4\delta\big(\|\mathbf{x}\|+D(K)+\delta\big)}{w(w-2\delta)}+\frac{2\delta(w-2\delta)}{w(w-2\delta)}\\
&= \frac{\delta\big(4\|\mathbf{x}\|+4D(K)+2w\big)}{w(w-2\delta)}\,.
\end{aligned} \tag{50}$$

In view of the definition (31), (50) implies

$$\left|\lambda(K,\mathbf{x})-\lambda(K',\mathbf{x})\right| \le \frac{\delta\big(4\|\mathbf{x}\|+4D(K)+2w\big)}{w(w-2\delta)}\,. \tag{51}$$

Now a reflection to Proposition 4.1 completes the proof of the Lemma.

Theorem 5.5. *Let $(K,\mathbf{x}),(M,\mathbf{y}) \in \mathcal{K}\times X$ with $\delta(K,M) < \frac{1}{2}w(K)$. Then we have*

$$\left|\alpha(K,\mathbf{x})-\alpha(M,\mathbf{y})\right| \le \frac{\delta(K,M)\big(4\|\mathbf{y}\|+4D(K)+2w(K)\big)}{w(K)\big(w(K)-2\delta(K,M)\big)}+\frac{2\|\mathbf{x}-\mathbf{y}\|}{w(K)}. \tag{52}$$

Proof. On applying the triangle inequality $\left|\alpha(K,\mathbf{x})-\alpha(M,\mathbf{y})\right| \le \left|\alpha(K,\mathbf{x})-\alpha(K,\mathbf{y})\right| + \left|\alpha(K,\mathbf{y})-\alpha(M,\mathbf{y})\right|$, we can use Proposition 5.3 for the first term and Lemma 5.4 for the second to obtain (52).

Already in Proposition 3.5 we have seen that $\alpha(K,\mathbf{x}) \to \infty$ $\big(\|\mathbf{x}\|\to\infty\big)$, and (42) with (46) suggest a linear speed growth. We can make this observation much more precise.

Lemma 5.6. *Denoting the classical Minkowski functional (c.f. (1), (2)) of $\overline{C}$ as $\varphi_{\overline{C}} = \|\cdot\|_{\overline{C}}$, we have*

$$\varphi_{\overline{C}}(\mathbf{x}) = \frac{2\|\mathbf{x}\|}{\tau\left(K, \frac{\mathbf{x}}{\|\mathbf{x}\|}\right)} \quad (\mathbf{0} \neq \mathbf{x} \in X). \tag{53}$$

Proof. In view of (3) and the fatness of C we see that $\tau := \tau(K, \mathbf{v}) = \tau\big(C, \mathbf{v}\big) = \tau\big(\overline{C}, \mathbf{v}\big)$. Choosing here $\mathbf{v} := \mathbf{x}/\|\mathbf{x}\|$, central symmetry of $\overline{C}$ shows $\overline{C} \cap \{\lambda\mathbf{v} : \ \lambda \in \mathbb{R}\} = \big[-\frac{\tau}{2}\mathbf{v}, \frac{\tau}{2}\mathbf{v}\big]$. Hence definition (1), applied to $\overline{C}$ in place of K, gives $\varphi_{\overline{C}}(\mathbf{x}) = \frac{2}{\tau}\|\mathbf{x}\|$, as stated.

Theorem 5.7. *For every $\mathbf{x} \in X$, $\mathbf{x} \neq \mathbf{0}$, we have*

$$|\alpha(K, \mathbf{x}) - \varphi_{\overline{C}}(\mathbf{x})| = \left|\alpha(K, \mathbf{x}) - \frac{2\|\mathbf{x}\|}{\tau(K, \mathbf{x}/\|\mathbf{x}\|)}\right| \leq \frac{2D(K)}{w(K)} - 1. \tag{54}$$

Proof. Lemma 5.6 implies the equality part of (54). Now let $\mathbf{v}^* \in S^*$ be arbitrary and consider the affine linear functional (30) for both K and $\overline{C}$. As in Lemma 5.6, we can easily see that $h\big(\overline{C}, \mathbf{v}^*\big) = h\big(\overline{C}, -\mathbf{v}^*\big)$ by symmetry, and that $w(K, \mathbf{v}^*) = w\big(\overline{C}, \mathbf{v}^*\big)$ by (5) and fatness of C. Therefore we find

$$\begin{aligned} t(K, \mathbf{v}^*, \mathbf{x}) - t\big(\overline{C}, \mathbf{v}^*, \mathbf{x}\big) &= \frac{-h(K, \mathbf{v}^*) + h(K, -\mathbf{v}^*)}{w(K, \mathbf{v}^*)} \\ = \frac{2h(K, -\mathbf{v}^*) - w(K, \mathbf{v}^*)}{w(K, \mathbf{v}^*)} &= 2\frac{h(K, -\mathbf{v}^*)}{w(K, \mathbf{v}^*)} - 1 \leq \frac{2D(K)}{w(K, \mathbf{v}^*)} - 1. \end{aligned} \tag{55}$$

Taking into account the similar calculation

$$t\big(\overline{C}, \mathbf{v}^*, \mathbf{x}\big) - t(K, \mathbf{v}^*, \mathbf{x}) = \frac{2h(K, \mathbf{v}^*)}{w(K, \mathbf{v}^*)} - 1 \leq \frac{2D(K)}{w(K, \mathbf{v}^*)} - 1,$$

now we get by (18)

$$\Big|t(K, \mathbf{v}^*, \mathbf{x}) - t\big(\overline{C}, \mathbf{v}^*, \mathbf{x}\big)\Big| \leq \frac{2D(K)}{w(K, \mathbf{v}^*)} - 1 \leq \frac{2D(K)}{w(K)} - 1. \tag{56}$$

By definition (31) we obtain from (56)

$$\big|\lambda(K, \mathbf{x}) - \lambda\big(\overline{C}, \mathbf{x}\big)\big| \leq \frac{2D(K)}{w(K)} - 1. \tag{57}$$

By using Proposition 4.1 and the fact that $\alpha\big(\overline{C}, \mathbf{x}\big) = \varphi_{\overline{C}}(\mathbf{x})$, (57) gives (54).

Corollary 5.8. *For $\mathbf{x} \to \infty$, $\alpha(K, \mathbf{x})$ grows linearly. In particular,*

$$\lim_{s\to\infty} \frac{\alpha(K, s\mathbf{v})}{s} = \frac{2}{\tau(K, \mathbf{v})} \quad (\mathbf{v} \in X, \ \|\mathbf{v}\| = 1) \tag{58}$$

uniformly on S, and also

$$\frac{2}{d(K)} = \varliminf_{\mathbf{x}\to\infty} \frac{\alpha(K, \mathbf{x})}{\|\mathbf{x}\|} \leq \varlimsup_{\mathbf{x}\to\infty} \frac{\alpha(K, \mathbf{x})}{\|\mathbf{x}\|} = \frac{2}{w(K)}. \tag{59}$$

Proof. By the second term of (54), we can replace $\alpha(K,\mathbf{x})$ by $2\|\mathbf{x}\|/\tau$, $\big(\tau = \tau(K,\mathbf{x}/\|\mathbf{x}\|)\big)$ in (58)–(59). This gives (58) immediately, while (59) follows in view of the first part of Proposition 2.4.

Corollary 5.9. *For the Hausdorff distance δ, defined by* (15)*, we have*

i) $\delta\big(K^\lambda, \lambda C\big)$ *is bounded uniformly for* $\lambda > \alpha_K$ *and*

ii) $\frac{1}{\lambda}K^\lambda \to C$ $(\lambda \to +\infty)$.

Proof. By convexity, $\delta\big(K^\lambda, \lambda C\big) = \delta\big(\partial K^\lambda, \partial\lambda C\big)$. Applying Corollary 3.8 iv) and Lemma 5.6 we obtain

$$\begin{aligned}\delta\big(K^\lambda, \lambda C\big) &= \delta\big(\partial K^\lambda, \partial\lambda C\big) \\ &\le \sup\left\{\left\|s\mathbf{v} - \lambda\frac{\tau(K,\mathbf{v})}{2}\mathbf{v}\right\| :\ \mathbf{v}\in S,\ 0 < s = s(\lambda,\mathbf{v}) \quad \text{s.t.} \quad \alpha(K, s\mathbf{v}) = \lambda\right\}.\end{aligned} \tag{60}$$

For any particular $\mathbf{v}$, $s(\lambda,\mathbf{v})$ exists provided the ray $\{s\mathbf{v} :\ s > 0\}$ intersects K^λ; this is certainly the case when $\lambda > \alpha(K,\mathbf{0})$, since then $\mathbf{0} \in \operatorname{int} K^\lambda$, see Corollary 3.8 iii). Hence, there exists some $s = s(\lambda,\mathbf{v})$ as described in (60). Moreover, again by local boundedness (see Proposition 3.3), $s \to \infty$ as $\lambda \to \infty$.

Here we refer to Theorem 5.7, using (54) with $\mathbf{x} = s\mathbf{v}$ to obtain

$$\left|\alpha(K, s\mathbf{v}) - \frac{2s}{\tau(K,\mathbf{v})}\right| \le \frac{2D(K)}{w(K)} - 1. \tag{61}$$

Comparing (60) and (61), with $\lambda = \alpha(K, s\mathbf{v})$, we get

$$\delta\big(K^\lambda, \lambda C\big) \le \sup_{\mathbf{v}\in S} \frac{\tau(K,\mathbf{v})}{2}\left(\frac{2D(K)}{w(K)} - 1\right) < d(K)\cdot\frac{D(K)}{w(K)}, \tag{62}$$

in view of Proposition 2.4. Clearly (62) implies i) and ii) follows easily from i).

6. Global properties on $\mathcal{K} \times X$

Perhaps the most important consequence of Theorem 5.5 is the next result.

Corollary 6.1. *The generalized Minkowski functional α is continuous on $\mathcal{K} \times X$.*

Proposition 6.2. *Let $K, M \in \mathcal{K}$. Then we have*

$$\alpha_{K+M} \le \max\{\alpha_K, \alpha_M\}, \tag{63}$$

where α_{K+M}, α_K and α_M are defined by (16).

Proof. It is easy to see that for any $\mathbf{v}^* \in S^*$

$$L(K+M, \mathbf{v}^*) = L(K,\mathbf{v}^*) + L(M,\mathbf{v}^*), \tag{64}$$

hence also for any $\lambda \ge 0$

$$L^\lambda(K+M, \mathbf{v}^*) = L^\lambda(K,\mathbf{v}^*) + L^\lambda(M,\mathbf{v}^*). \tag{65}$$

But (65) implies that

$$\begin{aligned}(K+M)^\lambda = \bigcap_{\mathbf{v}^*\in S^*} L^\lambda(K+M,\mathbf{v}^*) \supset \\ \supset \bigcap_{\mathbf{v}^*\in S^*} L^\lambda(K,\mathbf{v}^*) + \bigcap_{\mathbf{v}^*\in S^*} L^\lambda(M,\mathbf{v}^*) = K^\lambda + M^\lambda.\end{aligned} \tag{66}$$

Therefore, if $\mathbf{y}\in K^\lambda$, $\mathbf{z}\in M^\lambda$ and $\mathbf{x}=\mathbf{y}+\mathbf{z}$, then $\mathbf{x}\in(K+M)^\lambda$ and thus $\alpha(K+M,\mathbf{x})\le\lambda$ which implies $\alpha_{K+M}\le\lambda$. As the argument applies to all $\lambda>\max\{\alpha_K,\alpha_M\}$ (for then $K^\lambda\neq\emptyset$ and $M^\lambda\neq\emptyset$), we can conclude (63).

In the following we investigate the case when $\alpha_K=0$. We will show that this characterizes centrally symmetric domains.

Lemma 6.3. *Suppose $\alpha_K=0$. Then $K^{\alpha_K}=K^0=\{\mathbf{x}_0\}$, that is, K^0 is a one-point set.*

Proof. We have

$$\begin{aligned} d(K^\lambda) &= \sup_{\mathbf{v}^*\in S^*} w(K^\lambda,\mathbf{v}^*) \le \sup_{\mathbf{v}^*\in S^*} w\big(\big(L^\lambda(K,\mathbf{v}^*),\mathbf{v}^*\big) \\ &= \sup_{\mathbf{v}^*\in S^*} \lambda w\big(L(K,\mathbf{v}^*),\mathbf{v}^*\big) = \lambda \sup_{\mathbf{v}^*\in S^*} w(K,\mathbf{v}^*) = \lambda d(K),\end{aligned} \tag{67}$$

hence in case $\lambda\to0^+$ we also have $d(K^\lambda)\to0$. Since K^λ are closed sets and K^λ $(\lambda\ge0)$ forms a monotone system, all conditions of a Cantor-type intersection theorem are satisfied and by using also Lemma 3.2 we get

$$K^0=\bigcap_{\lambda>0}K^\lambda\neq\emptyset.$$

Since $d(K^\lambda)\to0$, $|K^0|\le1$ is obvious.

Question 6.4. For any $0\le\alpha_K<1$, is it true in general that the minimal K^{α_K} of K^λ is nonempty? In other words, does $\alpha(K,\mathbf{x})$ attain its infimum α_K? Compare also to Remarks 8.2.

In the next two theorems we recover the fact that Minkowski's measure of symmetry is indeed a measure of symmetry in the sense defined by Grünbaum, see [7] or [10, p. 362]. That goes back to Minkowski [20] and Radon [23] and is a well-known fact for finite dimensional spaces. Note that already Grünbaum points out, c.f. [7, p. 234], that together with f also $h\circ f$ must be measure of symmetry for any homeomorphism h of [0,1] onto itself with $h(1)=1$. Thus, the equivalent formulations listed in Corollary 4.11 lead to a measure of symmetry simultaneously and the next two theorems are equivalent to Section 6 of [7] at least for finite dimensional spaces.

On the other hand, we are not aware of any other publications extending these to infinite dimensional spaces. Note that some of the best examples of measures of symmetry — e.g. the "affine-invariant volume ratio", cf. [10, p. 362] — use finite dimensional notions like volume, and thus they cannot be extended to infinite dimensional spaces. Therefore it is of some interest that the same Minkowski functional and measure of symmetry can be used in that generality.

Theorem 6.5. *Let $K\in\mathcal{K}$. Then $\alpha_K=0$ if and only if K is centrally symmetric. In this case $K^0=\{\mathbf{x}_0\}$ and the symmetry center of K is $\mathbf{x}_0$.*

Proof. If K is centrally symmetric with respect to a certain point $\mathbf{x}_0$, then it is easy to see that $K^0 = \{\mathbf{x}_0\}$ and of course $\alpha_K = 0$.

Assume now that $\alpha_K = 0$. Then according to Lemma 6.3 we have $K^0 \neq \emptyset$ and $K^0 = \{\mathbf{x}_0\}$ has exactly one element. But this means that $\mathbf{x}_0 \in L^0(K, \mathbf{v}^*)$ for all $\mathbf{v}^* \in S^*$, i.e. $\mathbf{x}_0$ is a symmetry center for all $L(K, \mathbf{v}^*)$ $(\mathbf{v}^* \in S^*)$. Since $K = K^1 = \bigcap_{\mathbf{v}^* \in S^*} L(K, \mathbf{v}^*)$, and all $L(K, \mathbf{v}^*)$ are symmetric w.r.t. $\mathbf{x}_0$, we obtain that also K is symmetric w.r.t. $\mathbf{x}_0$, as asserted.

Theorem 6.6. *The mapping $f : \mathcal{K}(X) \to (0, 1]$, $f(K) := 1 - \alpha_K$ is a measure of symmetry. That is, we have*

(*i*) $f(K) = 1$ *if and only if K is centrally symmetric;*
(*ii*) f *is continuous (w.r.t. δ on $\mathcal{K}$);*
(*iii*) f *satisfies the "superminimality condition"*

$$f(K_1 + K_2) \geq \min\big\{f(K_1),\, f(K_2)\big\} \quad (K_1, K_2 \in \mathcal{K}).$$

Proof. (i) is equivalent to Theorem 6.5, and (ii) is a consequence of Theorem 5.5. Finally, (iii) is the reformulation of our Proposition 6.2 above.

Theorem 6.7. *Let $K = M \times N \subset X = Y \times Z$, $M \subset Y$, $N \subset Z$, where X, Y, Z are normed spaces and K, M, N are convex bodies. Then we have*

(*i*) $K^\lambda = \{\mathbf{x} = (\mathbf{y}, \mathbf{z}) :\ \mathbf{y} \in M^\lambda,\ \mathbf{z} \in N^\lambda\} = M^\lambda \times N^\lambda$,
(*ii*) $\alpha(K, \mathbf{x}) = \max\big\{\alpha(M, \mathbf{y}),\, \alpha(N, \mathbf{z})\big\} \quad \big(\mathbf{x} = (\mathbf{y}, \mathbf{z})\big)$.

Proof. The two assertions are clearly equivalent, hence it suffices to show (ii). Although a direct calculation is possible, we take advantage of the proven equivalent forms of $\alpha(K, \mathbf{x})$.

Suppose first that $\mathbf{y} \in M$ and $\mathbf{z} \in N$. Since homothetic reduction can be written coordinatewise in M and N, definition (32) immediately provides

$$\beta(K, \mathbf{x}) = \beta\big(M \times N, (\mathbf{y}, \mathbf{z})\big) = \min\big\{\beta(M, \mathbf{y}), \beta(N, \mathbf{z})\big\},$$

which implies (ii) in view of Corollary 4.11 ii) and the fact that $\frac{1-t}{1+t}$ is decreasing.

Suppose now that $\mathbf{x} \notin K$, and apply (33). Obviously $K \cap \big(\mathbf{x} + \lambda(K - \mathbf{x})\big) = \emptyset$ if and only if either $M \cap \big(\mathbf{y} + \lambda(M - \mathbf{y})\big) = \emptyset$, or $N \cap \big(\mathbf{z} + \lambda(N - \mathbf{z})\big) = \emptyset$. First, for $\mathbf{x} \notin M$ and $\mathbf{z} \notin N$ we find

$$\varrho(K, \mathbf{x}) = \varrho\big(M \times N,\ (\mathbf{y}, \mathbf{z})\big) = \max\big\{\varrho(M, \mathbf{y}),\ \varrho(N, \mathbf{z})\big\}.$$

Here now we refer to Corollary 4.11 iii) and the fact that the function $\phi(t) = \frac{1+t}{1-t}$ is increasing $\big(t \in (0, 1)\big)$ to conclude (ii). Finally, if for example $\mathbf{y} \notin M$ and $\mathbf{z} \in N$ we similarly have $\varrho(K, \mathbf{x}) = \varrho(M, \mathbf{y})$ which by $\alpha(N, \mathbf{z}) \leq 1 < \alpha(M, \mathbf{y})$ yields (ii). The proof is complete.

Corollary 6.8. *For $K = M \times N \subset X = Y \times Z$, $M \subset Y$, $N \subset Z$, where X, Y, Z are normed spaces and K, M, N are convex bodies, we have*

$$\alpha_K = \max\{\alpha_M, \alpha_N\}.$$

The special case of Corollary 6.8, where $\dim Z < \infty$, M is centrally symmetric but N is not and $|N^{\alpha_N}| = 1$, has been mentioned in [9].

7. More on the sets K^λ

While presenting the above results in Anogia-Crete in the summer of 2001, Professor Rolf Schneider [32] communicated us the following nice formula which enables us to directly obtain some of the above results as well. (Note that this is also connected to a general theorem of Schneider, c.f. [10] p. 305 Theorem 2.)

Proposition 7.1 (R. Schneider). *For $\lambda \geq 1$ we have*

$$K^\lambda = \overline{\frac{\lambda+1}{2}K + \frac{\lambda-1}{2}(-K)} = \overline{(\lambda-1)C + K}. \tag{68}$$

Proof. Let us check the last equality first. Note that by convexity of K we have $\mu K + \varrho K = (\mu+\varrho)K$ $(\forall \mu, \varrho \geq 0)$; hence

$$\frac{\lambda+1}{2}K + \frac{\lambda-1}{2}(-K) = K + \frac{\lambda-1}{2}K + \frac{\lambda-1}{2}(-K) = K + (\lambda-1)C,$$

as needed.

Next observe that the asserted formula (68) holds for all centrally symmetric bodies (even if their center of symmetry is not at $\mathbf{0}$). In particular, for layers this was already observed in (27). Therefore, by using (27) and the fact that $L(\mu A + \nu B, \mathbf{v}^*) = \mu L(A, \mathbf{v}^*) + \nu L(B, \mathbf{v}^*)$ we get

$$\begin{aligned} L^\lambda(K, \mathbf{v}^*) &= \frac{\lambda+1}{2}L(K, \mathbf{v}^*) + \frac{\lambda-1}{2}\big(L(-K, \mathbf{v}^*)\big) \\ &= L\left(\frac{\lambda+1}{2}K + \frac{\lambda-1}{2}(-K), \mathbf{v}^*\right). \end{aligned} \tag{69}$$

Thus applying the representation (10) now to $\overline{\frac{\lambda+1}{2}K + \frac{\lambda-1}{2}(-K)}$ as well, we are led to

$$\begin{aligned} K^\lambda &= \bigcap_{\mathbf{v}^* \in S^*} L^\lambda(K, \mathbf{v}^*) \qquad (70) \\ &= \bigcap_{\mathbf{v}^* \in S^*} L\left(\frac{\lambda+1}{2}K + \frac{\lambda-1}{2}(-K), \mathbf{v}^*\right) = \overline{\frac{\lambda+1}{2}K + \frac{\lambda-1}{2}(-K)}. \end{aligned}$$

Note that for $X = \mathbb{R}^d$, compactness ensures closedness and there is no need to take closure in formula (68).

Corollary 7.2. *For all $\lambda \geq 1$ and $\mathbf{v}^* \in S^*$ we have the relations*

i) $h(K^\lambda, \mathbf{v}^*) = h\big(L^\lambda(K, \mathbf{v}^*), \mathbf{v}^*\big) = \frac{\lambda+1}{2}h(K, \mathbf{v}^*) + \frac{\lambda-1}{2}h(K, -\mathbf{v}^*)$,

ii) $w(K^\lambda, \mathbf{v}^*) = w\big(L^\lambda(K, \mathbf{v}^*), \mathbf{v}^*\big) = \lambda w(K, \mathbf{v}^*)$,

iii) $L(K^\lambda, \mathbf{v}^*) = L^\lambda(K, \mathbf{v}^*)$.

Observe that the relation "$\subset$ " in (iii), or "$\leq$ " in the first equality of (ii), are direct consequences of the definition (13) even for arbitrary $\lambda \geq 0$.

An analogous formula to (68) was pointed out to us, even for $0 \leq \lambda \leq 1$, by E. Makai [18]. To formulate this, one uses the Minkowski difference

$$A \sim B := \{\mathbf{x} \in X : \mathbf{x} + B \subset A\}. \tag{71}$$

Proposition 7.3 (E. Makai). *For $0 \le \lambda \le 1$ and $K \in \mathcal{K}$ we have*

$$K^\lambda = K \sim (1-\lambda)\overline{C} = \frac{1+\lambda}{2} K \sim \frac{1-\lambda}{2}(-K). \tag{72}$$

The key in proving Proposition 7.3 is a well-known lemma, see e.g. formula (3.1.16) of [31], extended to infinite dimensional spaces.

Lemma 7.4. *For $K, M \in \mathcal{K}$ we have*

$$K \sim M = \bigcap_{\mathbf{u}^* \in S^*} \{\mathbf{x} \in X : < \mathbf{u}^*, \mathbf{x} > \le h(K, \mathbf{u}^*) - h(M, \mathbf{u}^*)\}. \tag{73}$$

Proof. Just follow the argument in [31, p. 134], where the restriction to finite dimensional spaces is of no relevance.

Proof of Proposition 7.3. By (73) and (5) we have

$$\begin{aligned} K \sim\ & (1-\lambda)\overline{C} \\ =\ & \bigcap_{\mathbf{v}^* \in S^*} \{\mathbf{x} \in X : \quad < \mathbf{v}^*, \mathbf{x} > \le h(K, \mathbf{v}^*) - \tfrac{1-\lambda}{2} w(K, \mathbf{v}^*)\} \\ =\ & \bigcap_{\mathbf{v}^* \in S^*} \{\mathbf{x} \in X : \quad < \mathbf{v}^*, \mathbf{x} > \le \tfrac{1+\lambda}{2} h(K, \mathbf{v}^*) - \tfrac{1-\lambda}{2} h(K, -\mathbf{v}^*)\}. \end{aligned} \tag{74}$$

Grouping together opposite pairs $\mathbf{v}^*, -\mathbf{v}^* \in S^*$ and reflecting back to (11)–(13) yields the first equation of (72). Since $h(-K, \mathbf{v}^*) = h(K, -\mathbf{v}^*)$, it is even easier to see that

$$\begin{aligned} & \frac{1+\lambda}{2} K \sim \frac{1-\lambda}{2}(-K) \\ = & \bigcap_{\mathbf{v}^* \in S^*} \left\{ \mathbf{x} \in X : \quad < \mathbf{v}^*, \mathbf{x} > \le \frac{1+\lambda}{2} h(K, \mathbf{v}^*) - \frac{1-\lambda}{2} h(K, -\mathbf{v}^*) \right\}. \end{aligned} \tag{75}$$

This proves the second equation of (72).

At this stage we point out that the "level sets" K^λ were known in geometry under an equivalent definition. In fact, for any convex body $K \in \mathcal{K}$ Hammer [8] has introduced the associated bodies

$$C(K, \rho) := \begin{cases} \bigcap_{\mathbf{x} \in \partial K} \{\mathbf{x} + \rho(K - \mathbf{x})\}, & 0 \le \rho \le 1 \\ \overline{\bigcup_{\mathbf{x} \in \partial K} \{\mathbf{x} + \rho(K - \mathbf{x})\}}, & \rho \ge 1, \end{cases} \tag{76}$$

naturally without the closure, since Hammer has considered only finite dimension.

Hammer's definition (76) is closely related to the functional $\sigma(K, \mathbf{x})$ of (36); in fact, Hammer has defined these associated bodies as level sets of the functional $\sigma(K, \mathbf{x})$ in Minkowski's measure of symmetry, c.f. [7, pp. 246-247]. Thus, one can expect that there is an equivalence, like for the functionals, also even for the level sets as well. Using Corollary 4.11 (the relations between $\alpha(K, \mathbf{x})$ and $\sigma(K, \mathbf{x})$), it is not hard to obtain this equivalence for $\dim X < \infty$. In fact, Hammer's Theorem 5 can be regarded as a predecessor to our Corollary 4.11 with several versions of the equivalent extremal ratios mentioned there. However, there is another way to get the equivalence. In [31, p. 141] one finds the finite dimensional version of

$$C(K,\rho)=\begin{cases} K\sim 2(1-\rho)\overline{C}\ , & 0\le\rho\le 1\\ K+2(\rho-1)\overline{C}\ , & \rho\ge 1\end{cases}\tag{77}$$

without the closure; the extension to $\dim(X)=\infty$ is standard. But comparing (77) with Propositions 7.1 and 7.3, we are led to

$$C(K,\rho)=K^{2\rho-1}.\tag{78}$$

Let us continue with a general statement, obtained in a different way by Hörmander [12] in 1955.

Lemma 7.5 (Hörmander). *If K and K' are arbitrary non-empty bounded convex sets in X, their Hausdorff distance satisfies*

$$\delta(K,K')=\sup_{\mathbf{v}^*\in S^*}|h(K,\mathbf{v}^*)-h(K',\mathbf{v}^*)|.\tag{79}$$

Proof. Recall that already in the proof of Lemma 5.4 we have mentioned the " $\ge$ " direction of the stated formula as a trivial consequence of definition (15) of the Hausdorff distance. To prove the other direction, it suffices to restrict ourselves to closed convex sets K and K' since by taking the closure does not change either sides of the stated inequality.

First, let's assume that $\mathbf{x}\in K$ and $B(\mathbf{x},r)\cap K'=\emptyset$, with $r>0$. Since both $B(\mathbf{x},r)$ and K' are convex, and $\operatorname{int}B(\mathbf{x},r)\ne\emptyset$, by the separation theorem of convex sets (c.f. [29], Theorem 3.4) we can find a $\mathbf{v}^*\in S^*$ such that

$$h(K',\mathbf{v}^*)=\sup_{K'}\mathbf{v}^*\le\inf_{B(\mathbf{x},r)}\mathbf{v}^*=\langle\mathbf{v}^*,\mathbf{x}\rangle-r\le\sup_{K}\mathbf{v}^*-r=h(K,\mathbf{v}^*)-r.\tag{80}$$

Second, let us consider the case when $\mathbf{y}\in K'$ and $B(\mathbf{y},R)\cap K=\emptyset$, where $R>0$. Working as above, for some appropriate $\mathbf{u}^*\in S^*$ we have

$$h(K,\mathbf{u}^*)\le h(K',\mathbf{u}^*)-R.\tag{81}$$

By (15) we have for any $\epsilon>0$ either some $\mathbf{x}\in K$ and $r>\delta-\epsilon$, or some $\mathbf{y}\in K'$ and $R>\delta-\epsilon$ with the above properties; hence either $h(K',\mathbf{v}^*)\le h(K,\mathbf{v}^*)+\epsilon-\delta$, or $h(K,\mathbf{u}^*)\le h(K',\mathbf{u}^*)+\epsilon-\delta$. In either case we conclude $\sup_{S^*}|h(K,\cdot)-h(K',\cdot)|\ge\delta$, as needed.

As a first application, we present a sharp form of Corollary 5.9.

Corollary 7.6. *For $\lambda\ge 1$ we have*

$$\delta\big(K^\lambda,\lambda C\big)\le D(K)-\frac{1}{2}w(K).\tag{82}$$

Proof. Since $\lambda\ge 1$, we can apply Schneider's Formula, or its Corollary 7.2 (i) which is more suitable to us here. For an arbitrary $\mathbf{v}^*\in S^*$ and by using (5) we obtain

$$h(K^\lambda,\mathbf{v}^*)-h(\lambda C,\mathbf{v}^*)=\frac{\lambda+1}{2}h(K,\mathbf{v}^*)+\frac{\lambda-1}{2}h(K,-\mathbf{v}^*)-\lambda h(C,\mathbf{v}^*)$$
$$=\frac{1}{2}(h(K,\mathbf{v}^*)-h(K,-\mathbf{v}^*))\le h(K,\mathbf{v}^*)-\frac{1}{2}w(K)\le D(K)-\frac{1}{2}w(K).$$

An analogous calculation gives $h(\lambda C, \mathbf{v}^*) - h(K^\lambda, \mathbf{v}^*) \leq D(K) - \frac{1}{2}w(K)$. Since by Lemma 7.5 we have $\delta\big(K^\lambda, \lambda C\big) = \sup_{S^*} |h(K^\lambda, \cdot) - h(\lambda C, \cdot)|$, the proof of the Corollary follows.

Remarks 7.7. (i) Corollary 7.6 is quite sharp. For instance, let K be a ball $B(\mathbf{a}, r)$ of radius r. Then $w(K) = d(K) = 2r$, $C = C = B(\mathbf{0}, r)$, $K = C + \mathbf{a}$ and $D(K) = a + r$, where $a = \|\mathbf{a}\|$. For this K and for all $\lambda \geq 0$ we have

$$\delta\big(K^\lambda, \lambda C\big) = a = D(K) - \frac{1}{2}w(K).$$

(ii) Remarkably, (82) does not hold for $\lambda < 1$. To construct an example, consider in $X = \mathbb{R}^2$ the triangle $K := \operatorname{conv}\{(10, 10); (16, 10); (10, 16)\}$. In this case $w(K) = 3\sqrt{2}$ and $D(K) = \sqrt{10^2 + 16^2} = 18.86...$, while $D(K) - \frac{1}{2}w(K) = 16.74....$ On the other hand

$$C(K) = \operatorname{conv}\{(3, 0); (0, 3); (-3, 3); (-3, 0); (0, -3); (3, -3)\},$$

and for $\lambda = \frac{1}{3}$ we have $K^\lambda = \{(12, 12)\}$. By taking $\mathbf{y} := (0, -1) \in \lambda C$, we can calculate that $\operatorname{dist}(\mathbf{y}, K^\lambda) = \sqrt{12^2 + 13^2} = 17.69...$ which exceeds the above value of $D(K) - \frac{1}{2}w(K) = 16.74....$ Although all computed examples suggest that $D(K)$ should be a generally valid upper bound for all $\lambda > \alpha_K$, we are not aware of a sharp bound on $\delta(K^\lambda, \lambda C)$ in the case $\lambda < 1$.

Corollary 7.8. *Let $1 \leq \lambda, \mu$ be arbitrary. Then we have*

$$\big(K^\lambda\big)^\mu = K^{\lambda\mu}. \tag{83}$$

Proof.

$$\begin{aligned}
\big(K^\lambda\big)^\mu &= \overline{(\mu - 1)C(K^\lambda) + K^\lambda} \\
&= \overline{(\mu - 1)C\big(\overline{(\lambda - 1)C + K}\big) + \overline{(\lambda - 1)C(K) + K}} \\
&= \overline{(\mu - 1)\big\{(\lambda - 1)C(K) + C\big\} + (\lambda - 1)C + K} \\
&= \overline{(\mu\lambda - 1)C + K} = K^{\lambda\mu}.
\end{aligned}$$

Remark 7.9. Note that $\big(K^\lambda\big)^\mu \subset K^{\lambda\mu}$ ($\lambda > 0$, $\mu \geq 1$), but not for all $\lambda, \mu > 0$.

In fact, a further generalization of the above can be found in [31] where the relative parallel bodies of K with respect to $D \in \mathcal{K}$ are defined as

$$K_{\rho,D} := \begin{cases} \overline{K + \rho D}\ , & \rho \geq 0 \\ K \sim (-\rho D)\ , & \rho < 0\ . \end{cases} \tag{84}$$

Here we use closure in (84) only in view of the case $\dim X = \infty$. Clearly, $K_{\rho,D}$ is a convex body for $\rho > \rho_0$, with some $\rho_0 > 0$ and, in analogy to Section 2, basic properties of this parametric set sequence can be easily seen.

Lemma 7.10. *For $K, M, D \in \mathcal{K}$ and $\rho, \sigma \in \mathbb{R}$ we have*

$$K_{\rho,D} + M_{\sigma,D} \subset \overline{(K + M)_{\rho+\sigma,D}}. \tag{85}$$

Proof. Taking into consideration (73), the obvious analogy for the closure of the Minkowski sum, and by using the fact that the convexity of K, M and D implies that $K_{\rho,D}+M_{\sigma,D}$ is also convex, we obtain

$$\begin{aligned} K_{\rho,D}+M_{\sigma,D} &\subset \bigcap_{\mathbf{v}^*\in S^*} \{X_{h(K,\mathbf{v}^*)+\rho h(D,\mathbf{v}^*)}(\mathbf{v}^*)+X_{h(M,\mathbf{v}^*)+\sigma h(D,\mathbf{v}^*)}(\mathbf{v}^*)\} \\ &= \bigcap_{\mathbf{v}^*\in S^*} X_{h(K+M,\mathbf{v}^*)+(\rho+\sigma)h(D,\mathbf{v}^*)}(\mathbf{v}^*) = (\overline{K+M})_{\rho+\sigma,D}. \end{aligned} \tag{86}$$

This approach gives rise to the more general definition

$$\alpha_D(K,\mathbf{x}) := \inf\{\rho\in\mathbb{R} : \mathbf{x}\in K_{\rho,D}\}. \tag{87}$$

Note that with this extension $\alpha(K,\mathbf{x}) = 1+\alpha_{\overline{C}}(K,\mathbf{x})$. Therefore, a number of properties of K^λ and $\alpha(K,\mathbf{x})$ can be obtained from more general statements for $K_{\rho,D}$ and $\alpha_D(K,\mathbf{x})$. However, we don't need them for our present applications and we leave this subject now.

8. Small values of α and the critical set of K

Here is an interesting question in geometry: what kind of quasi-centre can be found for a nonsymmetric convex K? A usual choice is the centroid, but there are other important definitions like e.g. the *Santaló point* (c.f. [10], p. 165), the *Steiner point* (c.f. [31], p. 42), or the *analytic center* (c.f. [10], p. 661) of a convex body.

The general notion of a center of a convex body with respect to the *center function* $\Delta:\ \mathcal{K}^d\times\mathbb{R}^d\to\mathbb{R}_+$ was introduced by Huard in 1967, see [13] or [10, p. 660]. According to Huard, a general center function has to satisfy

$$\begin{aligned} &\text{(i)} \quad \Delta(K,\mathbf{x}') = \min_{\mathbf{x}\in K}\Delta(K,\mathbf{x}) \quad (\forall \mathbf{x}'\in\partial K,\ K\in\mathcal{K}^d) \\ &\text{(ii)} \quad \Delta(K,\mathbf{x}') > \min_{\mathbf{x}\in K}\Delta(K,\mathbf{x}) \quad (\forall \mathbf{x}'\in \operatorname{int} K,\ K\in\mathcal{K}^d) \\ &\text{(iii)} \quad \Delta(K,\mathbf{x}) \geq \Delta(K',\mathbf{x}) \quad (\forall \mathbf{x}\in K'\subset K,\ K,K'\in\mathcal{K}^d) \\ &\text{(iv)} \quad \max_{\mathbf{x}\in K}\Delta(K,\mathbf{x}) = \Delta(K,\mathbf{x}^*) \text{ for a unique point } \mathbf{x}^*\in K\ (K\in\mathcal{K}^d). \end{aligned} \tag{88}$$

Webster (see [36], p. 319) presents another approach. Starting from the observation that the symmetry center of a centrally symmetric convex body bisects every chord of the body through the centre, Webster is looking for a point in K which divides all chords through it "relatively evenly". Theorem 7.1.5 in [36] states that for all $K\in\mathcal{K}^d$ there exists a point $\mathbf{c}=\mathbf{c}_K\in\operatorname{int}K$, such that for any chord $l\cap K=[\mathbf{a},\mathbf{b}]\ni\mathbf{c}$, the segments always satisfy $\frac{\|\mathbf{b}-\mathbf{c}\|}{\|\mathbf{a}-\mathbf{b}\|}\leq\frac{d}{d+1}$. In other words, we have $\omega(K,\mathbf{c})\leq\frac{d}{d+1}$. This translates to the inequality $\alpha(K,\mathbf{c})\leq\frac{d-1}{d+1}$, see Corollary 4.11 (ii).

Hence, Webster's Theorem 7.1.5 is seen to be equivalent to the assertion that $\alpha_K\leq\frac{d-1}{d+1}$ and that $K^{\frac{d-1}{d+1}}\neq\emptyset$. This leads to the natural idea of considering K^{α_K} (or, equivalently the function $\Delta(K,\mathbf{x}):=1-\alpha(K,\mathbf{x})$) as a candidate for a center function. In fact, by an equivalent definition, the same set was already known in geometry for $\dim X<\infty$ as the *critical set* of K, see [8] or [31, p. 141].

In [16] the question of cardinality of K^{α_K} was raised. However, it turns out that the answer preceded this question by half a century. It is true that $|K^{\alpha_K}|=1$ for dimension

$d = 2$, see Neumann [21], but in general Sobczyk, see [8, p.792] and [9], has answered this question in the negative.

Example 8.1 (Sobczyk). Let $K = \Delta \times I = \{(x, y, z) \in \mathbb{R}^3 : \ 0 \le x, y \le x + y \le 1, -1 \le z \le 1\} \subset \mathbb{R}^3$. Then $\alpha_K = \frac{1}{3}$ and $K^{1/3} = \{(\frac{2}{3}, \frac{2}{3}, z) : \ |z| \le \frac{1}{3}\}$.

Indeed, for the triangle $\Delta = \{(x, y) : \ 0 \le x, y \le x + y \le 1\}$ we have $\alpha_\Delta = \frac{1}{3}$, while $\Delta^{1/3} = \{(\frac{2}{3}, \frac{2}{3})\}$. On the other hand $\alpha_I = 0$ and $I^{1/3} = [-\frac{1}{3}, \frac{1}{3}]$. Hence, by Theorem 6.7 (i) $K^{1/3} = \{(\frac{2}{3}, \frac{2}{3})\} \times [-\frac{1}{3}, \frac{1}{3}]$ and by Corollary 6.8 $\alpha_K = 1/3$.

Remarks 8.2. (i) Observe that in the general context the above construction is not a "center function" in the sense of Huard, as property (iv) is seen to fail by Example 8.1.
(ii) However, the most important fact is that (iii) of (88) does not hold. To see this just consider a large cube K and a small one K' in one of its corners. While for K' the generalized Minkowski functional $\alpha(K', \mathbf{x})$ assumes all ≤ 1 values on K', it is clear that only values close to 1 will be assumed by $\alpha(K, \mathbf{x})$ for $\mathbf{x} \in K'$.
(iii) Still, the size and properties of the critical set K^{α_K} are of interest. For the finite dimensional case Klee [14] has obtained a rather precise description of the critical set. For finite dimensional spaces Klee's estimate has as follows

$$\frac{r_K}{1 - r_K} + \dim K^{\alpha_K} \le \dim X,$$

where his r_K arises by using $\omega(K, \mathbf{x})$ and so by Corollary 4.11 one obtains $r_K := \inf \omega(K, \dot{)} = \frac{1+\alpha_K}{2}$. This is equivalent to

$$\frac{1 + \alpha_K}{1 - \alpha_K} \le \operatorname{codim} K^{\alpha_K}.$$

In this form the assertion can be interpreted even if $\dim X = \infty$. However, the whole argument of [14] relies essentially on compactness, so much so that even the objects he starts with do not exist in general. Hence, not only a description of the critical set is missing but even Question 6.4 seems to be complicated for $\dim X = \infty$.

Theorem 8.3 (Minkowski, Radon). *For any $K \subset \mathbb{R}^d$, $\alpha_K \le \frac{d-1}{d+1}$. Moreover, we always have $\mathbf{w}_K \in K^{\frac{d-1}{d+1}}$ for the centroid $\mathbf{w}_K$ of K, and $\alpha(K, \mathbf{w}_K) = \frac{d-1}{d+1}$ only if K is a finite cone.*

This result is well-known and was obtained already by Minkowski [20] for $d = 2, 3$ and Radon [23] for $d \in \mathbb{N}$. See also Section 6.2 (i) of [7] and the references therein. The reader can easily observe that Theorem 8.3 contains also the above mentioned Theorem 7.1.5 of [36].

Example 8.4. Consider any simplex $S \subset \mathbb{R}^d$. It is easy to see that $S^{\frac{d-1}{d+1}} = \{\mathbf{w}_S\}$ and thus $\alpha(S, \mathbf{w}_S) = \alpha_S = \frac{d-1}{d+1}$.

It is also known that in $\mathbb{R}^d$ simplices are the only extremal bodies with $\alpha_K = \frac{d-1}{d+1}$, see (4.6) of [14] or Section 6.2 (i) in [7].

Example 8.5 (Hammer). Let $d = 2$, and $K = \{(x, y) \in \mathbb{R}^2 : \ 0 \le y, \ x^2 + y^2 \le 1\}$

be the upper half of the unit disc. Then $\mathbf{w}_K = \left(0, \frac{4}{3\pi}\right)$, $\alpha_K = 3 - 2\sqrt{2} < \frac{1}{3}$, and $K^{\alpha_K} = \left\{(0, \sqrt{2} - 1)\right\}$. Thus $\mathbf{w}_K \notin K^{\alpha_K}$.

With respect to Theorem 8.3, it is worth noting that for vector spaces of infinite dimension no general upper bound, lower than 1, can be given to α_K of $K \in \mathcal{K}(X)$. Compare also to Corollary 3.8 (vi).

Proposition 8.6. *Let X be an infinite dimensional space. Then*

$$\sup\left\{\alpha_K : \ K \in \mathcal{K}(X)\right\} = 1.$$

Proof. Let $d \in \mathbb{N}$ be arbitrary and let Y be any d-dimensional subspace of X. Suppose $\mathbf{x}_1, \ldots, \mathbf{x}_d$ are linearly independent vectors of Y and let Z be a complement to Y in X. That is $X = Y \oplus Z$, where Z is a closed subspace such that $Y \cap Z = \{\mathbf{0}\}$, $Y + Z = \{\mathbf{y} + \mathbf{z} : \mathbf{y} \in Y, \mathbf{z} \in Z\} = X$. The existence of a complement to each finite dimensional subspace of a normed space X is a well known result in functional analysis, see for instance [29, Lemma 4.21]

Now consider the simplex $S = \mathrm{con}\{\mathbf{0}, \mathbf{x}_1, \ldots, \mathbf{x}_d\}$ in Y and define $K := S \oplus B_Z$, where B_Z is the unit ball of Z. This is the direct sum construction in Corollary 6.8. So, by Example 8.4 it can be easily seen that $\alpha_K = \max\{\alpha_S, \alpha_{B_Z}\} = \alpha_S = \frac{d-1}{d+1} \le \sup_{\mathcal{K}(X)} \alpha_K$ holds for all $d \in \mathbb{N}$, proving the assertion.

9. Applications in approximation theory

To illustrate the use of the above generalized Minkowski functional, we describe now two applications from the field of multivariate polynomial extremal problems. Our general setting here again will require that $K \subset X$ is a convex body, but we also introduce the space of continuous polynomials of degree at most n, denoted by $\mathcal{P}_n(X) = \mathcal{P}_n$, from X to $\mathbb{R}$. In infinite dimensional normed spaces, the space of continuous (or, equivalently, bounded) polynomials depends on the norm too. That is, for different, non-equivalent norms, even the space of polynomials (and in particular the space of continuous linear functionals) will be different. However, once K is a fixed convex *body*, that is, bounded and also containing a ball, we conclude that all such norms are equivalent (and actually equivalent to the norm generated by the central symmetrization C of K by means of its Minkowski functional φ_C). Since we always normalize the polynomials defined on K with respect to the sup-norm, that is

$$\|p\|_K := \sup_K |p| \qquad \left(p \in \mathcal{P}_n(X)\right), \tag{89}$$

it is clear that the resulting set of polynomials $\left\{p \in \mathcal{P}_n(X), \ \|p\|_K \le 1\right\}$ will always be the same, provided K is fixed.

To define, in general, polynomials we can follow e.g. Hörmander [11], Appendix A. We say that p *is a polynomial of degree at most* n if for arbitrary fixed $x, y \in X$ the function $p(x + ty) : \ \mathbb{R} \to \mathbb{R}$ is a real algebraic polynomial in $t \in \mathbb{R}$ of degree at most n.

A polynomial $p \in \mathcal{P}_n(X)$ is homogeneous of degree m, if it satisfies $p(t\mathbf{x}) = t^m p(\mathbf{x})$ $(\forall t \in \mathbb{R}, \forall \mathbf{x} \in X)$. It is easy to see that every polynomial $p \in \mathcal{P}_n(X)$ can be written as

$$p = \sum_{k=0}^{n} p_k \qquad (p_k \text{ is homogeneous of degree } k). \tag{90}$$

Note that in the previous expression the "leading term" p_n of p is a homogeneous polynomial of degree n.

Continuous polynomials are Fréchet differentiable everywhere and with any order of differentiation. In particular, the gradient of p at a point $\mathbf{x} \in X$ is

$$\operatorname{grad} p(\mathbf{x}) : \ X \to \mathbb{R}, \qquad \operatorname{grad} p(\mathbf{x}) \in X^*. \tag{91}$$

A pointwise estimate of $\operatorname{grad} p(\mathbf{x})$, for $\mathbf{x} \in K$, under normalization of the norm defined by (89) is the so-called *Bernstein problem* or *Bernstein type inequality.* For centrally symmetric convex bodies we have the following generalization of the classical one century old result of S. Bernstein.

Theorem 9.1 (Sarantopoulos, 1991 [30]). *Let K be $\mathbf{0}-$symmetric , $\mathbf{x} \in \operatorname{int}(K)$ and $p \in \mathcal{P}_n$. Then we have*

$$\left|\operatorname{grad} p(\mathbf{x})\right| \leq \frac{2n\|p\|_K}{w(K)\sqrt{1-\varphi_K^2(\mathbf{x})}}. \tag{92}$$

Here φ_K is the Minkowski functional of the $\mathbf{0}-$symmetric body K. Note that actually (92) was proved by Sarantopoulos in the equivalent setting when K is the unit ball, i.e. when $\|\cdot\| = \varphi_K$. Observe that in this case $w(K) = 2$. However, the two formulations are equivalent, so for further use we choose the above version.

For more general, not necessarily symmetric sets, various estimates have been proved, see [1], [2], [33], [34] and [35]. All involve some geometric techniques to quantify the location of $\mathbf{x}$ within K, i.e. the respective distance of $\mathbf{x}$ from the boundary. These "Bernstein factors" are sometimes quite complicated, and even the comparison of the various results is rather difficult. We quote here a result for the finite dimensional case.

Theorem 9.2 (Kroó–Révész, 1999 [16]). *Let $K \subset \mathbb{R}^d$ be a convex body, $\mathbf{x} \in \operatorname{int} K$, and $p \in \mathcal{P}_n$ be arbitrary. Then we have*

$$\left|\operatorname{grad} p(\mathbf{x})\right| \leq \frac{2n\|p\|_K}{w(K)\sqrt{1-\alpha(K,\mathbf{x})}}. \tag{93}$$

In [16] the possibility of extending the above result to arbitrary Banach spaces was mentioned, but the details were not worked out.

Note the similarity of (93) to (92), but α is not raised to the power 2. However, it is our conjecture that α should be raised to the power 2.

Conjecture 9.3. *If $K \subset X$ is a convex body and if $\mathbf{x} \in \operatorname{int} K$ and $p \in \mathcal{P}_n$ are arbitrary, then we have*

$$\left|\operatorname{grad} p(\mathbf{x})\right| \leq \frac{2n\|p\|_K}{w(K)\sqrt{1-\alpha(K,x)^2}}. \tag{94}$$

Let us turn our attention to another group of problems, the so called Chebyshev-type extremal problems concerning growth of real polynomials. The question has as follows: "How large can be a polynomial at a point $\mathbf{x} \in X$, or when $\mathbf{x} \to \infty$?" More precisely, we are interested in determining for arbitrary $\mathbf{v} \in X$ (say, with $\|\mathbf{v}\| = 1$)

$$A_n(K,\mathbf{v}) := \sup\left\{p_n(\mathbf{v}) : \ p \in \mathcal{P}_n \text{ satisfying (90)}, \|p\|_K \leq 1\right\}, \tag{95}$$

or for some fixed $\mathbf{x} \in X$

$$C_n(K,\mathbf{x}) := \sup\left\{p(\mathbf{x}) : p \in \mathcal{P}_n,\ \|p\|_K \le 1\right\}. \tag{96}$$

Note the appearance of the n-homogeneous part p_n in (95). Both problems are classical and fundamental in the theory of approximation, see e.g. [19] or [26] for the one and a half century old single variable result and its many consequences, variations and extensions.

As we have mentioned in the introduction, even the above formulation (13)-(14) of the definition of $\alpha(K,\mathbf{x})$ was applied first in work on these questions, particularly on (96), where a quantification of the position of $\mathbf{x}$ with respect to K is needed. After the results in $\mathbb{R}^d$ by Rivlin–Shapiro [27] and Kroó–Schmidt [17], we were able to settle the general case in a satisfactory way. We have the following result.

Theorem 9.4 (Révész–Sarantopoulos, 2001, [25]). *If $K \subset X$ is an arbitrary convex body and if $\mathbf{x} \in X \setminus K$ is arbitrary, then we have*

$$C_n(K,\mathbf{x}) = T_n\big(\alpha(K,\mathbf{x})\big). \tag{97}$$

Moreover, $C_n(K,\mathbf{x})$ is actually a maximum, attained by

$$P(\mathbf{x}) := T_n\big(t(K,\mathbf{v}^*,\mathbf{x})\big). \tag{98}$$

Here T_n is the classical Chebyshev polynomial

$$\begin{aligned} T_n(x) \; &:= 2^{n-1}\prod_{j=1}^{n}\left(x-\cos\left(\tfrac{(2j-1)\pi}{2n}\right)\right) = \\ &= \tfrac{1}{2}\left\{\left(x+\sqrt{x^2-1}\right)^n + \left(x-\sqrt{x^2-1}\right)^n\right\}, \end{aligned} \tag{99}$$

$t(K,\mathbf{v}^*,\mathbf{x})$ is the linear expression defined in (30), and $\mathbf{v}^*$ is some appropriately chosen linear functional from S^*.

Note that actually the restriction $\mathbf{x} \notin K$ is natural, as $p \equiv 1 \in \mathcal{P}_n$, and thus for $\mathbf{x} \in K$ we always have $C_n(K,\mathbf{x}) = 1$. Here we can observe that our results on the growth of $\alpha(K,\mathbf{x})$ together with (97) give strong indications even for the other Chebyshev problem, as we know that the Chebyshev polynomial itself has leading coefficient 2^{n-1}. Indeed, we have the following result.

Theorem 9.5 (Révész–Sarantopoulos, 2001 [25]). *Let $xK \subset X$ be an arbitrary convex body and let $\mathbf{v} \in X$. Then we have*

$$A_n(K,\mathbf{v}) = \frac{2^{2n-1}}{\tau(K,\mathbf{v})^n}, \tag{100}$$

and the supremum is actually a maximum attained by a polynomial of the form (98) with some appropriately chosen $\mathbf{v}^*$ satisfying (26).

Based on the determination of these extremal quantities, other related questions were already addressed in approximation theory, such as the uniqueness of the extremal polynomials, or the existence of the so-called universal majorant polynomials. These, in turn, have consequences e.g. concerning the approximation of convex bodies by convex hulls of algebraic surfaces. For further details we refer to [15] and [24].

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