A Variational Approach to a Class of Singular Semilinear Elliptic Equations

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Received May 30, 2003

We provide a variational approach to singular semilinear elliptic equations of the form $-\Delta u = u^{-\beta} + g(x,u)$, for every $\beta > 0$.

Keywords: Singular semilinear elliptic equations, variational methods, convex functionals

1991 Mathematics Subject Classification: 35J65, 49J40

1. Introduction

Since the pioneering papers of Crandall, Rabinowitz and Tartar [6] and Stuart [18], singular semilinear elliptic problems of the form

$$\begin{cases}
    u > 0 & \text{in } \Omega, \\
    -\Delta u = u^{-\beta} + g(x,u) & \text{in } \Omega, \\
    u = 0 & \text{on } \partial \Omega,
\end{cases}$$

(1)

where $\Omega$ is a bounded open subset of $\mathbb{R}^n$, $\beta > 0$ and $g$ satisfies suitable growth conditions, have been considered by several authors (see e.g. [10, 13, 15, 16, 20] and the references therein). Let us also mention [5, 9], where the case in which the singular term $u^{-\beta}$ has the opposite sign is treated.

However, in spite of the fact that (1) is formally the Euler equation of the functional

$$f(u) = \frac{1}{2} \int_\Omega |Du|^2 \, dx + \int_\Omega \Phi(u) \, dx - \int_\Omega \int_0^{u(x)} g(x,s) \, ds \, dx,$$

where $u \in W_0^{1,2}(\Omega)$,

$$\Phi(s) = \begin{cases}
    -\int_1^s t^{-\beta} \, dt & \text{if } s \geq 0, \\
    +\infty & \text{if } s < 0,
\end{cases}$$

(2)

*The research of the authors was partially supported by the MIUR project “Variational and topological methods in the study of nonlinear phenomena” (PRIN 2003) and by Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (INdAM)
few existence and multiplicity results for (1) have been so far obtained through a direct variational approach. Let us mention [12], where the case in which \( \beta < 3 \) and \( g \) has critical growth is studied by minmax techniques. The restriction \( \beta < 3 \) is due to the fact that, according to [16, Theorem 2]), the functional \( f \) is identically \(+\infty\), if \( \beta \geq 3 \).

The main purpose of this paper is to provide a variational approach to (1) also when \( \beta \geq 3 \). Actually, our results apply for any \( \beta > 0 \), but the novelty concerns the case \( \beta \geq 3 \).

More precisely, in this paper we provide first of all a variational approach to the problem

\[
\begin{cases}
  u > 0 & \text{in } \Omega, \\
  -\Delta u = u^{-\beta} + w & \text{in } \Omega, \\
  u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(3)

with \( w \in W^{-1,2}(\Omega) \). To this aim, consider first the case \( w = 0 \). We denote by \( \Phi_k : \mathbb{R} \to \mathbb{R} \) the primitive of the function

\[
\begin{cases}
  \max \{ -s^{-\beta}, -k \} & \text{if } s > 0, \\
  -k & \text{if } s \leq 0,
\end{cases}
\]

such that \( \Phi_k(1) = 0 \) and we define a proper, lower semicontinuous, strictly convex functional \( \hat{f}_{0,k} : L^2(\Omega) \to ] - \infty, +\infty] \) as

\[
\hat{f}_{0,k}(u) = \begin{cases}
  \frac{1}{2} \int_{\Omega} |Du|^2 \, dx + \int_{\Omega} \Phi_k(u) \, dx & \text{if } u \in W^{1,2}_0(\Omega), \\
  +\infty & \text{if } u \in L^2(\Omega) \setminus W^{1,2}_0(\Omega).
\end{cases}
\]

Since a primitive is naturally defined up to an additive constant, to prevent a possible unhappy choice we pass to consider \( f_{0,k} : L^2(\Omega) \to ] - \infty, +\infty] \), defined as

\[
f_{0,k}(u) = \hat{f}_{0,k}(u) - \min \hat{f}_{0,k} = \hat{f}_{0,k}(u) - \hat{f}_{0,k}(u_{0,k}),
\]

where \( u_{0,k} \in W^{1,2}_0(\Omega) \) is the minimum of \( \hat{f}_{0,k} \).

More generally, for every \( w \in W^{-1,2}(\Omega) \), we define \( f_{w,k} : L^2(\Omega) \to ] - \infty, +\infty] \) as

\[
f_{w,k}(u) = \begin{cases}
  f_{0,k}(u) - \langle w, u - u_{0,k} \rangle & \text{if } u \in W^{1,2}_0(\Omega), \\
  +\infty & \text{if } u \in L^2(\Omega) \setminus W^{1,2}_0(\Omega).
\end{cases}
\]

Our first step is to prove that the sequence \((f_{w,k})\) is still equicoercive in \( L^2(\Omega) \) and is now \( \Gamma \)-convergent (see [1, 7, 8]) as \( k \to \infty \) to a proper, lower semicontinuous, strictly convex functional \( f_w : L^2(\Omega) \to ] - \infty, +\infty] \), whose effective domain \( \{ u \in L^2(\Omega) : f_w(u) < +\infty \} \) is independent of \( w \). Moreover, if \( u_0 \) is the minimum of \( f_0 \) (the \( \Gamma \)-limit functional corresponding to \( w = 0 \)), then the effective domain of \( f_w \) is contained in \( u_0 + W^{1,2}_0(\Omega) \) and an explicit description of \( f_w \) can be provided.

The second step is to study the Euler equation associated with \( f_w \). If \( w \in L^1_{loc}(\Omega) \cap W^{-1,2}(\Omega) \), then (3) is just the Euler equation of \( f_w \), provided that the boundary condition \( u = 0 \) on \( \partial \Omega \) has a suitable relaxed meaning. Moreover, if \( \Omega \) has smooth boundary and \( w \) is Hölder continuous on \( \bar{\Omega} \), then the minimum of \( f_w \) is just the solution in \( C(\bar{\Omega}) \cap C^2(\Omega) \).
of (3) already considered in [6]. In general, if \( w \in W^{-1,2}(\Omega) \) then the minimum of \( f_w \) is characterized by a variational inequality.

Finally, the variational description of (1) is obtained by considering the sum of a convex, lower semicontinuous functional and a functional of class \( C^1 \) taking into account the term \( g(x, u) \). For such a class of functionals, minmax techniques have been developed in [19].

2. On the equation \(-\Delta u = u^{-\beta}\)

Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \) and let \( \beta > 0 \). In the following, we will denote by \( L^\infty_c(\Omega) \) the space of \( L^\infty \)-functions with compact support in \( \Omega \). We will also denote by \( \| \cdot \|_p \) the usual norm in \( L^p(\Omega) \) and by \( \| \cdot \|_{-1,2} \) the norm in \( W^{-1,2}(\Omega) \) dual to the norm \( \| Du \|_2 \) in \( W_0^{1,2}(\Omega) \).

**Definition 2.1.** Let \( u \in W^{1,2}(\Omega) \). We say that \( u \leq 0 \) on \( \partial \Omega \) if, for every \( \varepsilon > 0 \), the function \((u - \varepsilon)^+\) belongs to \( W^{1,2}_0(\Omega) \).

It is readily seen that, if \( u \in W_0^{1,2}(\Omega) \), then \( u \leq 0 \) on \( \partial \Omega \). The same fact holds if \( u \in C(\Omega) \cap W^{1,2}_0(\Omega) \) and \( u(x) \leq 0 \) for every \( x \in \partial \Omega \).

Let us state the main result of this section.

**Theorem 2.2.** There exists one and only one \( u_0 \in C^\infty(\Omega) \) satisfying

\[
\begin{aligned}
    u_0 &> 0 \quad \text{in } \Omega, \\
    -\Delta u_0 &= u_0^{-\beta} \quad \text{in } \Omega, \\
    u_0 &\leq 0 \quad \text{on } \partial \Omega. 
\end{aligned}
\]

Moreover, if \( u_1 \in W^{1,2}_0(\Omega) \cap C^\infty(\Omega) \) satisfies \(-\Delta u_1 = 1 \) in \( \Omega \), then

\[
\| u_1 \|^{-\frac{\beta}{\beta+1}}_\infty u_1 \leq u_0 \leq ((\beta + 1)u_1)^{\frac{1}{\beta+1}} \quad \text{in } \Omega. 
\]

**Remark 2.3.** If \( \partial \Omega \) is sufficiently smooth, then much sharper estimates than (5) have been proved in [6, 16].

**Corollary 2.4.** There exists one and only one \( u_0 \in C(\Omega) \cap C^\infty(\Omega) \) such that

\[
\begin{aligned}
    u_0 &> 0 \quad \text{in } \Omega, \\
    -\Delta u_0 &= u_0^{-\beta} \quad \text{in } \Omega, \\
    u_0 &= 0 \quad \text{on } \partial \Omega, 
\end{aligned}
\]

if and only if each \( x \in \partial \Omega \) satisfies the Wiener criterion [11].

**Proof.** Of course, the uniqueness in Theorem 2.2 implies the uniqueness in Corollary 2.4. Let \( u_0 \) be given by Theorem 2.2. By (5), we have that \( u_0 \) is a \( C(\Omega) \)-solution of (6) if and only if \( u_1 \) belongs to \( C(\Omega) \) and satisfies \( u_1 = 0 \) on \( \partial \Omega \). It is quite standard to show that in turn this holds if and only if each \( x \in \partial \Omega \) satisfies the Wiener criterion. For the reader’s convenience, we give a proof of this fact in the Appendix. □

The remaining part of the section will be devoted to the proof of Theorem 2.2.
Definition 2.5. Let \( g : \Omega \times \mathbb{R} \to \mathbb{R} \) be a Carathéodory function, let \( w \in W^{-1,2}(\Omega) \) and let \( \varphi \in W^{1,2}_{\text{loc}}(\Omega) \). We say that \( \varphi \) is a (local) subsolution of the equation

\[
-\Delta u = g(x,u) + w,
\]

if \( g(x,\varphi) \in L_{\text{loc}}^1(\Omega) \) and

\[
\int_\Omega D\varphi Dv \, dx \leq \int_\Omega g(x,\varphi)v \, dx + \langle w,v \rangle \quad \forall v \in W^{1,2}_0(\Omega) \cap L^\infty(\Omega) \text{ with } v \geq 0 \text{ a.e. in } \Omega.
\]

We say that \( \varphi \) is a (local) supersolution of (7), if \( g(x,\varphi) \in L_{\text{loc}}^1(\Omega) \) and

\[
\int_\Omega D\varphi Dv \, dx \geq \int_\Omega g(x,\varphi)v \, dx + \langle w,v \rangle \quad \forall v \in W^{1,2}_0(\Omega) \cap L^\infty(\Omega) \text{ with } v \geq 0 \text{ a.e. in } \Omega.
\]

Definition 2.6. Let \( w \in W^{-1,2}(\Omega) \) and \( \varphi \in W^{1,2}_{\text{loc}}(\Omega) \). We say that \( \varphi \) is a (local) subsolution of the variational inequality

\[
\int_\Omega DuD(v-u) \, dx \geq \int_\Omega \varphi^{-\beta}(v-u) \, dx + \langle w,v-u \rangle \quad \forall v \geq 0,
\]

if \( \varphi > 0 \text{ a.e. in } \Omega \), \( \varphi^{-\beta} \in L_{\text{loc}}^1(\Omega) \) and

\[
\int_\Omega D\varphi Dv \, dx \leq \int_\Omega \varphi^{-\beta}v \, dx + \langle w,v \rangle \quad \forall v \in W^{1,2}_0(\Omega) \cap L^\infty(\Omega) \text{ with } 0 \leq v \leq \varphi \text{ a.e. in } \Omega.
\]

We say that \( \varphi \) is a (local) supersolution of (8), if \( \varphi > 0 \text{ a.e. in } \Omega \), \( \varphi^{-\beta} \in L_{\text{loc}}^1(\Omega) \) and

\[
\int_\Omega D\varphi Dv \, dx \geq \int_\Omega \varphi^{-\beta}v \, dx + \langle w,v \rangle \quad \forall v \in W^{1,2}_0(\Omega) \cap L^\infty(\Omega) \text{ with } v \geq 0 \text{ a.e. in } \Omega.
\]

Lemma 2.7. Let \( g : \Omega \times \mathbb{R} \to \mathbb{R} \) be a Carathéodory function satisfying

\[
\forall S > 0 : \sup_{|s| \leq S} |g(\cdot,s)| \in L_{\text{loc}}^1(\Omega),
\]

let \( w \in W^{-1,2}(\Omega) \) and let \( \varphi,u,\psi \in W^{1,2}_{\text{loc}}(\Omega) \). Assume that \( \varphi \) is a subsolution of (7), \( \psi \) is a supersolution of (7), \( \varphi \leq u \leq \psi \text{ a.e. in } \Omega \), \( g(x,u) \in L_{\text{loc}}^1(\Omega) \) and

\[
\int_\Omega DuD(v-u) \, dx \geq \int_\Omega g(x,u)(v-u) \, dx + \langle w,v-u \rangle
\]

\[
\forall v \in u + \left( W^{1,2}_0(\Omega) \cap L^\infty(\Omega) \right) \text{ with } \varphi \leq v \leq \psi \text{ a.e. in } \Omega.
\]

Then \( -\Delta u = g(x,u) + w \in \mathcal{D}'(\Omega). \)

Proof. Let \( \vartheta \in C_c^\infty(\mathbb{R}) \) with \( 0 \leq \vartheta \leq 1 \text{ on } \mathbb{R}, \vartheta = 1 \text{ on } [-1,1] \text{ and } \vartheta = 0 \text{ outside } ]-2,2[. \)

Let \( v \in C_c^\infty(\Omega) \) with \( v \geq 0 \), let \( k \geq 1 \), \( t > 0 \) and let

\[
v_k = \vartheta \left( \frac{u}{k} \right) v, \quad v_{k,t} = \min\{u + tv_k, \psi\}.
\]
Since $u \leq v_{k,t} \leq \psi$ and $v_{k,t} - u \leq tv_k \leq tv$, we have
\[
\int_{\Omega} \left( |D(v_{k,t} - u)|^2 - (g(x, v_{k,t}) - g(x, u))(v_{k,t} - u) \right) dx 
\leq \int_{\Omega} (Dv_{k,t}D(v_{k,t} - u) - g(x, v_{k,t})(v_{k,t} - u)) dx - \langle w, v_{k,t} - u \rangle 
= \int_{\Omega} (Dv_{k,t}D(v_{k,t} - u - tv_k) - g(x, v_{k,t})(v_{k,t} - u - tv_k)) dx 
- \langle w, v_{k,t} - u - tv_k \rangle + t \int_{\Omega} (Dv_{k,t}Dv_k - g(x, v_{k,t})v_k) dx - t\langle w, v_k \rangle 
\leq t \int_{\Omega} (Dv_{k,t}Dv_k - g(x, v_{k,t})v_k) dx - t\langle w, v_k \rangle ,
\]
whence
\[
\int_{\Omega} (Dv_{k,t}Dv_k - g(x, v_{k,t})v_k) dx - \langle w, v_k \rangle \geq - \int_{\Omega} |g(x, v_{k,t}) - g(x, u)||v_k| dx .
\tag{9}
\]
Since
\[
|g(x, v_{k,t})||v_k| \leq \left( \sup_{|s| \leq 2k + t\|v\|_{\infty}} |g(x, s)| \right) |v| ,
\]
by Lebesgue theorem we can pass to the limit in (9) as $t \to 0^+$, obtaining
\[
\int_{\Omega} (DuDv_k - g(x, u)v_k) dx - \langle w, v_k \rangle \geq 0 .
\]
Going to the limit as $k \to \infty$, it follows
\[
\int_{\Omega} (DuDv - g(x, u)v) dx - \langle w, v \rangle \geq 0 .
\tag{10}
\]
Let now $v \in C_c^\infty(\Omega)$ with $v \leq 0$, let $k \geq 1$, $t > 0$ and let
\[
v_k = v \left( \frac{u}{k} \right) v , \quad v_{k,t} = \max \{ u + tv_k, \varphi \} .
\]
Arguing as before, we find again (10).

Therefore, (10) holds for every $v \in C_c^\infty(\Omega)$ and the assertion follows, as we can exchange $v$ in $-v$. \qed

**Lemma 2.8.** Let $w \in W^{-1,2}(\Omega)$ and $\varphi, \psi \in W^{1,2}_{loc}(\Omega)$. Assume that $\varphi$ is a subsolution of (8) with $\varphi \leq 0$ on $\partial \Omega$ and $\psi$ a supersolution of (8).

Then $\varphi \leq \psi$ a.e. in $\Omega$. 

In particular, if $v$ a.e. in $\Omega$.

Let $u$ be the minimum of the functional $f_{w,k}$ on the convex set

$$K = \{ u \in W_0^{1,2}(\Omega) : 0 \leq u \leq \psi \text{ a.e. in } \Omega \}.$$  

According to [14], we have

$$\int_\Omega DuD(v-u) \, dx \geq - \int_\Omega \Phi_k(u)(v-u) \, dx + \langle w, v-u \rangle \quad \forall v \in K.$$  

In particular, if $v \in C_0^\infty(\Omega)$ with $v \geq 0$ and $t > 0$, we can consider as test function $v_t = \min\{u+tv, \psi\}$. Since $\psi$ is a supersolution also of the equation $-\Delta u = -\Phi_k(u) + w$, arguing as in the proof of Lemma 2.7, we find that

$$\int_\Omega DuDv \, dx \geq - \int_\Omega \Phi_k(u)v \, dx + \langle w, v \rangle.$$  

(11)

It easily follows that (11) holds for every $v \in W_0^{1,2}(\Omega)$ with $v \geq 0$ a.e. in $\Omega$.

In particular, since $u \geq 0$ we have $(\varphi - u - \varepsilon)^+ \in W_0^{1,2}(\Omega)$ and

$$\int_\Omega DuD(\varphi - u - \varepsilon)^+ \, dx \geq - \int_\Omega \Phi_k(u)(\varphi - u - \varepsilon)^+ \, dx + \langle w, (\varphi - u - \varepsilon)^+ \rangle.$$  

(12)

Let now $v \in W_0^{1,2}(\Omega)$ such that $0 \leq v \leq \varphi$ a.e. in $\Omega$ and $D\varphi \in L^2(\{v > 0\})$. Let $(\hat{v}_k)$ be a sequence in $C_0^\infty(\Omega)$ converging to $v$ in $W_0^{1,2}(\Omega)$ and let $v_k = \min\{\hat{v}_k^+, v\}$. We have

$$\int_\Omega D\varphi Dv_k \, dx \leq \int_\Omega \varphi^{-\beta}v_k \, dx + \langle w, v_k \rangle \, dx.$$  

If $\varphi^{-\beta}v \in L^1(\Omega)$, going to the limit as $k \to \infty$, we get

$$\int_\Omega D\varphi Dv \, dx \leq \int_\Omega \varphi^{-\beta}v \, dx + \langle w, v \rangle \, dx.$$  

(13)

If $\varphi^{-\beta}v \notin L^1(\Omega)$, formula (13) is obviously true. In particular, it follows

$$\int_\Omega D\varphi D(\varphi - u - \varepsilon)^+ \, dx \leq \int_\Omega \varphi^{-\beta}(\varphi - u - \varepsilon)^+ \, dx + \langle w, (\varphi - u - \varepsilon)^+ \rangle.$$  

(14)

Since $\varepsilon^{-\beta} < k$, from (12) and (14) we deduce that

$$\int_\Omega |D(\varphi - u - \varepsilon)^+|^2 \, dx = \int_\Omega D(\varphi - u)D(\varphi - u - \varepsilon)^+ \, dx \leq \int_\Omega \left( \varphi^{-\beta} + \Phi_k'(u) \right)(\varphi - u - \varepsilon)^+ \, dx = \int_\Omega (-\Phi_k'(\varphi) + \Phi_k'(u))(\varphi - u - \varepsilon)^+ \, dx \leq 0,$$

whence $\varphi \leq u + \varepsilon \leq \psi + \varepsilon$. The assertion follows from the arbitrariness of $\varepsilon$.  

\[\square\]
Proof of Theorem 2.2. Let $\Phi_k : \mathbb{R} \to \mathbb{R}$ and $f_{0,k} : L^2(\Omega) \to [-\infty, +\infty]$ be defined as in the Introduction. Let also $u_1 \in W^{1,2}_0(\Omega) \cap C^\infty(\Omega)$ be the solution of $-\Delta u_1 = 1$ in $\Omega$ and let

$$\varphi = \|u_1\|_{-\infty}^\frac{1}{\beta} u_1, \quad \psi = ((\beta + 1)u_1)^\frac{1}{\beta}.$$ \(15\)

Recall that $u_1 > 0$ in $\Omega$. Then it turns out that $\varphi \leq \psi$ and $\varphi$ is a subsolution and $\psi$ a supersolution of the equation $-\Delta u = -\Phi'_k(u)$, for any $k \geq \|u_1\|_{-\infty}^\frac{1}{\beta}$.

Let $u_{0,k} \in W^{1,2}_0(\Omega)$ be the minimum of $f_{0,k}$, namely the weak solution of

$$\begin{cases}
-\Delta u = -\Phi'_k(u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases} \quad (15)$$

Of course, $f_{0,k}$ admits one and only one minimum also on the convex set

$$\{u \in W^{1,2}_0(\Omega) : \varphi \leq u \leq \psi \text{ a.e. in } \Omega\}$$

and such a minimum is a solution of (15) by Lemma 2.7. It follows that $\varphi \leq u_{0,k} \leq \psi$ a.e. in $\Omega$. Since $u_{0,k}$ is a subsolution of $-\Delta u = -\Phi'_k(u)$, a similar argument shows that $u_{0,k} \leq u_{0,k+1}$ a.e. in $\Omega$. On the other hand, for every $\varepsilon > 0$ there exists $\bar{k} > \varepsilon^{-\beta}$. For every $k$, it follows

$$-\Delta(u_{0,\bar{k}} + \varepsilon) = -\Phi'_k((u_{0,\bar{k}} + \varepsilon) - \varepsilon) \geq -\Phi'_k(u_{0,\bar{k}} + \varepsilon),$$

namely $u_{0,\bar{k}} + \varepsilon$ is a supersolution of $-\Delta u = -\Phi'_k(u)$. Therefore $u_{0,k} \leq u_{0,\bar{k}} + \varepsilon$, namely $(u_{0,k})$ is a Cauchy sequence in $L^\infty(\Omega)$.

Therefore $(u_{0,k})$ is increasing and convergent, as $k \to \infty$, to some $u_0$ in $L^\infty(\Omega)$. Moreover, we have $\varphi \leq u_0 \leq \psi$, hence $u_0^{-\beta} \in L^\infty_{loc}(\Omega)$.

Given $\varepsilon > 0$, we have

$$\int_\Omega |D(u_{0,k} - \varepsilon)^+|^2 \, dx = -\int_\Omega \Phi'_k(u_{0,k})(u_{0,k} - \varepsilon)^+ \, dx \leq \varepsilon^{-\beta} \int_\Omega (u_{0,k} - \varepsilon)^+ \, dx.$$ \(15\)

It follows that $(u_{0,k} - \varepsilon)^+$ is bounded in $W^{1,2}_0(\Omega)$ as $k \to \infty$, so that

$$\forall \varepsilon > 0 : (u_0 - \varepsilon)^+ \in W^{1,2}_0(\Omega).$$

Since $u_{0,k} \geq \varphi$, we deduce that $u_0 \in W^{1,2}_{loc}(\Omega)$, $u_0 \leq 0$ on $\partial \Omega$ and $(D u_{0,k})$ is weakly convergent to $D u_0$ in $L^2(K)$ for any compact set $K$ in $\Omega$.

Then from (15) it follows that $-\Delta u_0 = u_0^{-\beta}$ in $\mathcal{D}'(\Omega)$. From the interior regularity theory, we infer that $u_0 \in C^\infty(\Omega)$ (see also [6, 16]).

The uniqueness of $u_0$ follows from Lemma 2.8. \(\square\)

3. The $\Gamma$-limit functional and the associated Euler equation

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ and let $\beta > 0$. Let $w \in W^{-1,2}(\Omega)$ and let $\Phi : \mathbb{R} \to [-\infty, +\infty]$, $\Phi_k : \mathbb{R} \to \mathbb{R}$ and $f_{w,k} : L^2(\Omega) \to [-\infty, +\infty]$ be defined as in the
Introduction. Let also $u_0 \in C^\infty(\Omega)$ be the solution of (4). According to (5), we have $u_0 \in L^\infty(\Omega)$.

Let $G_0 : \Omega \times \mathbb{R} \to [0, +\infty]$ be the Borel function defined as

$$G_0(x, s) = \Phi(u_0(x) + s) - \Phi(u_0(x)) + s u_0^{-\beta}(x).$$

Then $G_0(x, 0) = 0$ and $G_0(x, \cdot)$ is convex and lower semicontinuous for any $x \in \Omega$. Moreover, $G_0(x, \cdot)$ is of class $C^1$ on $]-u_0(x), +\infty[$ with

$$D_s G_0(x, s) = u_0^{-\beta}(x) - (u_0(x) + s)^{-\beta}.$$

Define a functional $f_w : L^2(\Omega) \to ]-\infty, +\infty]$ by

$$f_w(u) = \begin{cases} 
\frac{1}{2} \int_\Omega |D(u - u_0)|^2 \, dx + \int_\Omega G_0(x, u - u_0) \, dx - \langle w, u - u_0 \rangle & \text{if } u \in u_0 + W^{-1,2}_0(\Omega), \\
+\infty & \text{otherwise.}
\end{cases}$$

Then $f_w$ is strictly convex, lower semicontinuous and coercive, with $f_w(u_0) = 0$. Moreover, the effective domain of $f_w$ is

$$\{ u \in u_0 + W^{-1,2}_0(\Omega) : G_0(x, u - u_0) \in L^1(\Omega) \} \subseteq W^{1,2}_{loc}(\Omega),$$

independently of $w$. In the case $w = 0$, it is clear that $u_0$ is just the minimum of $f_0$.

Let us recall from [1, 7, 8, 17] the following

**Definition 3.1.** Let $X$ be a topological space, $f_k : X \to [-\infty, +\infty]$ a sequence of functions and $f : X \to [-\infty, +\infty]$ a function.

We say that

$$f = \Gamma(X^-) - \lim_k f_k$$

if the following facts hold:

(a) for every sequence $(u_k)$ convergent to $u$ in $X$, we have

$$f(u) \leq \liminf_k f_k(u_k);$$

(b) for every $u \in X$ there exists a sequence $(u_k)$ in $X$ convergent to $u$ satisfying

$$f(u) \geq \limsup_k f_k(u_k).$$

When $X$ is a Banach space, we say that $(f_k)$ is convergent to $f$ in the sense of Mosco ($M$-convergent, for short), if (a) holds with respect to the weak topology of $X$ and (b) with respect to the strong topology.

**Theorem 3.2.** For every $w \in W^{-1,2}(\Omega)$, the sequence $(f_{w,k})$ is equicoercive in $L^2(\Omega)$ and we have

$$f_w = \Gamma(L^2(\Omega)^-) - \lim_k f_{w,k}.$$
**Proof.** Let \( u_{0,k} \in W^{1,2}_0(\Omega) \) be the minimum of \( f_{0,k} \). According to the proof of Theorem 2.2, \((u_{0,k})\) is convergent to \( u_0 \) in \( L^\infty(\Omega) \).

Then, since \( \hat{f}_{0,k} \) is of class \( C^1 \) on \( W^{1,2}_0(\Omega) \), for every \( u \in W^{1,2}_0(\Omega) \) we have

\[
\begin{align*}
 f_{w,k}(u) &= \hat{f}_{0,k}(u) - \hat{f}_{0,k}(u_{0,k}) - \langle w, u - u_{0,k} \rangle \\
 &= \hat{f}_{0,k}(u) - \hat{f}_{0,k}(u_{0,k}) - \langle \hat{f}'_{0,k}(u_{0,k}), u - u_{0,k} \rangle - \langle w, u - u_{0,k} \rangle \\
 &= \frac{1}{2} \int_{\Omega} |D(u - u_{0,k})|^2 \, dx + \int_{\Omega} (\Phi_k(u) - \Phi_k(u_{0,k}) - \Phi'_k(u_{0,k})(u - u_{0,k})) \, dx \\
 & \quad - \langle w, u - u_{0,k} \rangle.
\end{align*}
\]

Since \( \Phi_k \) is convex, for every \( c \in \mathbb{R} \) the set

\[
\bigcup_{k \in \mathbb{N}} \{ u - u_{0,k} : u \in L^2(\Omega), f_{w,k}(u) \leq c \}
\]

is bounded in \( W^{1,2}_0(\Omega) \). In particular, the sequence \((f_{w,k})\) is equicoercive in \( L^2(\Omega) \).

Let now \((u_k)\) be a sequence convergent to \( u \) in \( L^2(\Omega) \). If \( \liminf_{k} f_{w,k}(u_k) = +\infty \), it is obvious that

\[
f_{w}(u) \leq \liminf_{k} f_{w,k}(u_k).
\]  (17)

Otherwise, up to a subsequence \((u_k - u_{0,k})\) is bounded in \( W^{1,2}_0(\Omega) \) and convergent to \( u \) a.e. in \( \Omega \). It follows that \( u \in u_0 + W^{1,2}_0(\Omega) \) and \((u_k - u_{0,k})\) is weakly convergent to \( u - u_0 \) in \( W^{1,2}_0(\Omega) \).

Since \( u_0 > 0 \) in \( \Omega \), it is clear that \( \Phi_k(u_k) - \Phi_k(u_{0,k}) - \Phi'_k(u_{0,k})(u_k - u_{0,k}) \) is convergent to \( G_0(x, u - u_0) \) a.e. in \( \Omega \). Then (17) easily follows also in this case.

Finally, let \( u \in L^2(\Omega) \). If \( f_{w}(u) = +\infty \) it is obvious that \((b)\) of Definition 3.1 holds. Otherwise, let \( u \in u_0 + W^{1,2}_0(\Omega) \) with \( u \geq 0 \) a.e. in \( \Omega \) and \( G_0(x, u - u_0) \in L^1(\Omega) \). Let \((\bar{v}_m)\) be a sequence in \( C_c^\infty(\Omega) \) convergent to \( u - u_0 \) in \( W^{1,2}_0(\Omega) \) and let

\[
v_m = \max \{ \bar{v}_m, -(u - u_0)^- \}.
\]

Then \( v_m \in W^{1,2}_0(\Omega) \cap L^\infty_c(\Omega) \) and is strongly convergent to \( u - u_0 \) in \( W^{1,2}_0(\Omega) \) with \( v_m \geq -(u - u_0)^- \) and \((G_0(x, v_m))\) is strongly convergent to \( G_0(x, u - u_0) \) in \( L^1(\Omega) \). Therefore, given \( \varepsilon > 0 \), there exists \( \bar{v} \in W^{1,2}_0(\Omega) \cap L^\infty(\Omega) \) with \( \bar{v} \geq -(u - u_0)^- \), \( ||D\bar{v} - D(u - u_0)||_2 < \varepsilon \) and \( ||G_0(x, \bar{v}) - G_0(x, u - u_0)||_1 < \varepsilon \). Let \( \vartheta \in C_c^\infty(\Omega) \) with \( \vartheta \geq 0 \) in \( \Omega \) and \( \vartheta = 1 \) where \( \vartheta \neq 0 \). If we set \( \bar{\vartheta} = \vartheta + \delta \vartheta \) with \( \delta > 0 \) small enough, then \( v \in W^{1,2}_0(\Omega) \cap L^\infty(\Omega) \) with \( ||Dv - D(u - u_0)||_2 < \varepsilon \), \( ||G_0(x, v) - G_0(x, u - u_0)||_1 < \varepsilon \) and

\[
\text{ess inf}_{\{v \neq 0\}} (u_0 + v) > 0.
\]

Then it is easy to see that

\[
\lim_{k} \| (\Phi_k(u_{0,k} + v) - \Phi_k(u_{0,k}) - \Phi'_k(u_{0,k})v) - G_0(x, u - u_0) \|_1 < \varepsilon.
\]

In particular, there exists a sequence \((v_k)\) strongly convergent to \( u - u_0 \) in \( W^{1,2}_0(\Omega) \) with

\[
\lim_{k} \| (\Phi_k(u_{0,k} + v_k) - \Phi_k(u_{0,k}) - \Phi'_k(u_{0,k})v_k) - G_0(x, u - u_0) \|_1 = 0.
\]
If we set \( u_k = u_{0,k} + v_k \), then \((u_k)\) is strongly convergent to \( u \) in \( L^2(\Omega) \) with \( (f_{w,k}(u_k)) \) convergent to \( f_w(u) \).

**Remark 3.3.** From the proof of Theorem 3.2 it follows that:

(a) if we define \( \tilde{f}_{w,k} : W^{1,2}_0(\Omega) \to \mathbb{R} \) and \( \tilde{f}_w : W^{1,2}_0(\Omega) \to ]-\infty, +\infty[ \) as \( \tilde{f}_{w,k}(v) = f_{w,k}(u_{0,k} + v), \tilde{f}_w(v) = f_w(u_0 + v) \), then \((\tilde{f}_{w,k})\) is \( M \)-convergent to \( f_w \);

(b) if \( n = 1 \), then the restriction of \((f_{w,k})\) to \( L^\infty(\Omega) \) is \( M \)-convergent to the corresponding restriction of \( f_w \);

(c) if \( n \geq 2, 2 \leq p < \infty \) and \( p(n - 2) \leq 2n \), then the restriction of \((f_{w,k})\) to \( L^p(\Omega) \) is \( M \)-convergent to the corresponding restriction of \( f_w \).

Now we consider the associated Euler equation.

**Theorem 3.4.** The following facts hold:

(a) for every \( w \in W^{-1,2}(\Omega) \) and \( u \in W^{1,2}_0(\Omega) \), we have that \( u \) is the minimum of \( f_w \) if and only if \( u \) satisfies

\[
\begin{align}
  u &> 0 \text{ a.e. in } \Omega \text{ and } u^{-\beta} \in L^1_{\text{loc}}(\Omega), \\
  \int_\Omega DuD(v - u) \, dx - \int_\Omega u^{-\beta}(v - u) \, dx &\geq \langle w, v - u \rangle \\
  u &\leq 0 \text{ on } \partial\Omega;
\end{align}
\]

in particular, for every \( w \in W^{-1,2}(\Omega) \) problem (18) admits one and only one solution \( u \in W^{1,2}_0(\Omega) \);

(b) if \( w_1, w_2 \in W^{-1,2}(\Omega) \) and \( u_1, u_2 \in W^{1,2}_0(\Omega) \) are the corresponding solutions of (18), we have \( u_1 - u_2 \in W^{1,2}_0(\Omega) \) and

\[
\|D(u_1 - u_2)\|_2 \leq \|w_1 - w_2\|_{-1,2}.
\]

**Proof.** (a) Given \( w \in W^{-1,2}(\Omega) \), there exists one and only one minimum \( u \in u_0 + W^{1,2}_0(\Omega) \) of \( f_w \). According to [2], we have \( G_0(x, u - u_0) \in L^1(\Omega) \), hence \( u \geq 0 \) a.e. in \( \Omega \), and

\[
\begin{align}
  \left( u_0^{-\beta} - u^{-\beta} \right)(v - u) &\in L^1(\Omega), \\
  \int_\Omega D(u - u_0)D(v - u) \, dx + \int_\Omega \left( u_0^{-\beta} - u^{-\beta} \right)(v - u) \, dx &\geq \langle w, v - u \rangle,
\end{align}
\]

for every \( v \in u_0 + W^{1,2}_0(\Omega) \) with \( G_0(x, v - u_0) \in L^1(\Omega) \) (here we agree that \( 0^{-\beta} = +\infty \)).

In particular, we have

\[
\left( u_0^{-\beta} - u^{-\beta} \right) v \in L^1(\Omega) \quad \text{for every } v \in C^\infty_c(\Omega) \text{ with } v \geq 0,
\]

whence \( u > 0 \) a.e. in \( \Omega \) and \( u^{-\beta} \in L^1_{\text{loc}}(\Omega) \).

Let now \( \varepsilon, \sigma > 0 \) and let

\[
v = \min \{ u - u_0, \varepsilon - (u_0 - \sigma)^+ \}.\]
Clearly $v \in W^{1,2}_0(\Omega)$. Moreover, we have a.e. either $v = u - u_0$ or $\varepsilon = v \leq u - u_0$ or $v = \varepsilon + \sigma - u_0$ with $u_0 \geq \sigma$. It follows $G_0(x,v) \in L^1(\Omega)$, hence

\[
((u_0 - \sigma)^+ + u - u_0 - \varepsilon)^+ = u - u_0 - v \in W^{1,2}_0(\Omega),
\]

\[
(\mu_0^\beta - u^\beta)(u_0 + v - u) \in L^1(\Omega)
\]

and

\[
\int_\Omega D(u-u_0)D(u-u_0-v) \, dx \leq - \int_\Omega \left( \mu_0^\beta - u^\beta \right) (u-u_0-v) \, dx + \langle w, u-u_0-v \rangle. \tag{20}
\]

In particular, since $u \neq u_0 + v$ implies $u > \varepsilon$, we have that both $u^{-\beta}(u - u_0 - v)$ and $u_0^{-\beta}(u - u_0 - v)$ belong to $L^1(\Omega)$.

On the other hand, we also have

\[
\int_\Omega D(u_0 - \sigma)^+ D\varphi \, dx \leq \int_\Omega \mu_0^{-\beta} \varphi \, dx \quad \text{for every } \varphi \in C_0^\infty(\Omega) \text{ with } \varphi \geq 0.
\]

Arguing as in [3], it follows

\[
\int_\Omega D(u_0 - \sigma)^+ D\varphi \, dx \leq \int_\Omega \mu_0^{-\beta} \varphi \, dx \quad \text{for every } \varphi \in W^{1,2}_0(\Omega) \text{ with } \varphi \geq 0 \text{ a.e. in } \Omega.
\]

In particular, we have

\[
\int_\Omega D(u_0 - \sigma)^+ D(u-u_0-v) \, dx \leq \int_\Omega \mu_0^{-\beta}(u-u_0-v) \, dx,
\]

which yields, combined with (20),

\[
\int_\Omega |D(u-u_0-v)|^2 \, dx = \int_\Omega D \left( (u_0 - \sigma)^+ + u - u_0 \right) D(u-u_0-v) \, dx \\
\leq \int_\Omega u^{-\beta}(u-u_0-v) \, dx + \langle w, u-u_0-v \rangle \\
\leq \varepsilon^{-\beta} \int_\Omega (u-u_0-v) \, dx + \langle w, u-u_0-v \rangle.
\]

Therefore, for any $\varepsilon > 0$, we have that $((u_0 - \sigma)^+ + u - u_0 - \varepsilon)^+$ is bounded in $W^{1,2}_0(\Omega)$ as $\sigma \to 0^+$. It follows that $(u - \varepsilon)^+ \in W^{1,2}_0(\Omega)$, namely that $u \leq 0$ on $\partial\Omega$.

Let now $v \in u + (W^{1,2}_0(\Omega) \cap L^\infty(\Omega))$ with $v \geq 0$ a.e. in $\Omega$. Let $v_0 \in C_0^\infty(\Omega)$ with $v_0 \geq 0$ in $\Omega$ and $v_0 = 1$ where $v \neq u$. Then, for every $\varepsilon > 0$, we have $G_0(x,v+\varepsilon v_0 - u_0) \in L^1(\Omega)$, whence

\[
\int_\Omega D(u-u_0)D(v+\varepsilon v_0 - u) \, dx + \int_\Omega \left( \mu_0^{-\beta} - u^{-\beta} \right) (v+\varepsilon v_0 - u) \, dx \geq \langle w, v+\varepsilon v_0 - u \rangle.
\]

From the arbitrariness of $\varepsilon$ it follows

\[
\int_\Omega D(u-u_0)D(v-u) \, dx + \int_\Omega \left( \mu_0^{-\beta} - u^{-\beta} \right) (v-u) \, dx \geq \langle w, v-u \rangle.
\]
Since
\[ \int_{\Omega} Du_0 D(v - u) \, dx = \int_{\Omega} u_0^{-\beta}(v - u) \, dx, \]
it turns out that \( u \) satisfies (18).

Conversely, let \( u \) be a solution of (18) and let \( \hat{u} \in W^{1,2}_{\text{loc}}(\Omega) \) be the minimum of \( f_w \). We already know that \( \hat{u} \) also is a solution of (18). In particular, \( u \) and \( \hat{u} \) are both a subsolution and a supersolution of (8). From Lemma 2.8 it follows that \( u = \hat{u} \), namely \( u \) is the minimum of \( f_w \).

(b) If \( w_1, w_2 \in W^{-1,2}(\Omega) \) and \( u_1, u_2 \in W^{1,2}_{\text{loc}}(\Omega) \) are the corresponding minima of \( f_{w_1} \) and \( f_{w_2} \), from (19) it follows that
\[
\int_{\Omega} |D(u_1 - u_2)|^2 \, dx \leq \int_{\Omega} (u_1^{-\beta} - u_2^{-\beta})(u_1 - u_2) \, dx + \langle w_1 - w_2, u_1 - u_2 \rangle,
\]
whence \( \|D(u_1 - u_2)\|_2 \leq \|w_1 - w_2\|_{-1,2} \).

**Theorem 3.5.** Let \( w \in W^{-1,2}(\Omega) \) and \( u \in W^{1,2}_{\text{loc}}(\Omega) \). If \( u \) satisfies
\[
\begin{cases}
  u > 0 \text{ a.e. in } \Omega & \text{and } u^{-\beta} \in L^1_{\text{loc}}(\Omega), \\
  -\Delta u - u^{-\beta} = w & \text{in } D'(\Omega), \\
  u \leq 0 & \text{on } \partial\Omega,
\end{cases}
\]
then \( u \) is the solution of (18). If \( w \in L^1_{\text{loc}}(\Omega) \cap W^{-1,2}(\Omega) \), then (18) and (21) are equivalent.

**Proof.** If \( u \) satisfies (21), a simple approximation argument shows that
\[
\int_{\Omega} Du \, dv \, dx - \int_{\Omega} u_0^{-\beta} v \, dx = \langle w, v \rangle
\]
for every \( v \in W^{1,2}_0(\Omega) \cap L^\infty_c(\Omega) \). Then \( u \) satisfies (18).

Assume now that \( w \in L^1_{\text{loc}}(\Omega) \cap W^{-1,2}(\Omega) \) and that \( u \) is the solution of (18). It is readily seen that, for every \( v \in C^\infty_c(\Omega) \) with \( v \geq 0 \),
\[
\int_{\Omega} Du \, dv \, dx \geq \int_{\Omega} u^{-\beta} v \, dx + \int_{\Omega} w v \, dx.
\]
Let now \( v \in C^\infty_c(\Omega) \) with \( v \leq 0 \), let \( t > 0 \) and let \( v_t = (u + tv)^+ \). Since \( |v_t - u| \leq t|v| \), we have
\[
\int_{\{u + tv > 0\}} Du \, dv \, dx \geq \frac{1}{t} \int_{\{u + tv \leq 0\}} |Du|^2 \, dx + \int_{\{u + tv > 0\}} Du \, dv \, dx \\
= \int_{\Omega} DuD \left( \frac{v_t - u}{t} \right) \, dx \geq \int_{\Omega} u^{-\beta} \frac{v_t - u}{t} \, dx + \int_{\Omega} w \frac{v_t - u}{t} \, dx.
\]
Going to the limit as \( t \to 0^+ \), we get
\[
\int_{\Omega} Du \, dv \, dx \geq \int_{\Omega} u^{-\beta} v \, dx + \int_{\Omega} w v \, dx
\]
also in this case. Therefore \( u \) satisfies (21).
Example 3.6. Let $0 < \beta < 2$ and let $\Omega = \mathbb{R}$. Let $u(x) = |\sin x|^\alpha$, where $1/2 < \alpha < 1/\beta$ and let $w = -u'' - u^{-\beta} - \delta_0$, where $\delta_0$ denotes the Dirac measure at 0.

Then $u \in W_0^{1,2}(\Omega)$, $w \in W^{-1,2}(\Omega)$ and (18) is satisfied, even if $u$ is not a solution of (21). Since the solution of (18) is unique, this means that (21) has no solution at all. Thus, if $w$ is merely in $W^{-1,2}(\Omega)$, we can solve the variational inequality (18), but not the equation (21), in general.

Corollary 3.7. Assume that each $x \in \partial \Omega$ satisfies the Wiener criterion (for instance, $\Omega$ has Lipschitz boundary) and that $w \in L^\infty(\Omega)$. Let $u \in W_0^{1,2}(\Omega)$ be the solution of (21) given by Theorems 3.4 and 3.5.

Then $u \in C(\overline{\Omega}) \cap W^{2,p}_a(\Omega)$ for any $p < \infty$ and satisfies

\[
\begin{cases}
  u > 0 & \text{in } \Omega, \\
  -\Delta u - u^{-\beta} = w & \text{a.e. in } \Omega, \\
  u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Moreover, we have

\[t_w u_0 \leq u \leq T_w u_0 \quad \text{in } \overline{\Omega}\]

for some $0 < t_w \leq T_w < +\infty$.

Proof. According to Corollary 2.4, we have $u_0 \in C(\overline{\Omega})$ and $u_0 = 0$ on $\partial \Omega$. Since $w \in L^\infty(\Omega)$, it is readily seen that there exist $T_w, t_w > 0$ such that $t_w u_0$ is a subsolution and $T_w u_0$ a supersolution of (8). From Lemma 2.8 we deduce that $t_w u_0 \leq u \leq T_w u_0$ a.e. in $\Omega$. Then $u^{-\beta} \in L^\infty(\Omega)$ and the assertion follows from standard regularity theory (see e.g. [14]).

4. $C^1$ perturbations

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ and let $\beta > 0$. For the sake of simplicity, we suppose here that $n \geq 3$. In the cases $n = 1, 2$, simple adaptations are required for the growth condition (24) below. Let also $u_0 \in L^\infty(\Omega) \cap C^\infty(\Omega)$ be the solution of (4) and let $f_0 : W_0^{1,2}(\Omega) \to ]-\infty, +\infty]$ be the lower semicontinuous, convex functional defined in (a) of Remark 3.3 when $w = 0$.

Moreover, suppose that $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Assume that

\[
\begin{cases}
  \text{there exist } a \in L^{\frac{2n}{n+2}}(\Omega) \text{ and } b \in \mathbb{R} \text{ such that} \\
  |g(x,s)| \leq a(x) + b|s|^\frac{n+2}{n+2} \text{ for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R}.
\end{cases}
\]

Define a new Carathéodory function $g_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ by $g_1(x,s) = g(x,u_0(x) + s)$. Since $u_0 \in L^\infty(\Omega)$, $g_1$ also satisfies (24). Let $G_1(x,s) = \int_0^s g_1(x,t) \, dt$ and let $f : W_0^{1,2}(\Omega) \to ]-\infty, +\infty]$ be the functional defined as $f(u) = \tilde{f}_0(u) + \gamma(u)$, where $\gamma$ is the functional of class $C^1$ defined as

$$\gamma(u) = -\int_\Omega G_1(x,u) \, dx.$$ 

According to [19], $u \in W_0^{1,2}(\Omega)$ is said to be a critical point of $f$, if $\tilde{f}_0(u) < +\infty$ and

$$\forall v \in W_0^{1,2}(\Omega) : \langle \gamma'(u), v - u \rangle + \tilde{f}_0(v) - \tilde{f}_0(u) \geq 0.$$
Theorem 4.1. For every $u$, the following assertions are equivalent:

(a) $u \in W^{1,2}_{\text{loc}}(\Omega) \cap L^{\frac{2n}{n-2}}(\Omega)$ and we have

$$\begin{cases}
  u > 0 \text{ a.e. in } \Omega \text{ and } u^{-\beta} \in L^1_{\text{loc}}(\Omega), \\
  -\Delta u = u^{-\beta} + g(x,u) \text{ in } D'(\Omega), \\
  u \leq 0 \text{ on } \partial \Omega;
\end{cases} \quad (25)$$

(b) $u \in u_0 + W^{1,2}_0(\Omega)$ and $u - u_0$ is a critical point of $f$.

Proof. If (a) holds, let $w = g(x,u) = g_1(x,u - u_0)$. By Theorems 3.4 and 3.5, we have that $u \in u_0 + W^{1,2}_0(\Omega)$ and minimizes $\tilde{f}_w$. This means that

$$\forall v \in W^{1,2}_0(\Omega) : \tilde{f}_0(v) \geq \tilde{f}_0(u - u_0) + \langle w, u_0 + v - u \rangle = \tilde{f}_0(u - u_0) - \langle \gamma'(u - u_0), u_0 + v - u \rangle,$$

namely $u - u_0$ is a critical point of $f$.

Conversely, assume that (b) holds. Then $u \in W^{1,2}_{\text{loc}}(\Omega) \cap L^{\frac{2n}{n-2}}(\Omega)$ and $w := g(x,u) = g_1(x,u - u_0) \in L^1_{\text{loc}}(\Omega) \cap W^{-1,2}(\Omega)$.

From Theorems 3.4 and 3.5 it follows that $u$ is a solution of (25).

Corollary 4.2. Assume that each $x \in \partial \Omega$ satisfies the Wiener criterion and that

$$\begin{cases}
  \text{there exists } b \in \mathbb{R} \text{ such that} \\
  |g(x,s)| \leq b(1 + |s|^{\frac{n+2}{n-2}}) \text{ for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R}.
\end{cases} \quad (26)$$

Let $u \in W^{1,2}_{\text{loc}}(\Omega) \cap L^{\frac{2n}{n-2}}(\Omega)$ be a solution of (25).

Then $u \in C(\overline{\Omega}) \cap W^{2,p}_{\text{loc}}(\Omega)$ for any $p < \infty$ and satisfies

$$\begin{cases}
  u > 0 \text{ in } \Omega, \\
  -\Delta u = u^{-\beta} + g(x,u) \text{ a.e. in } \Omega, \\
  u = 0 \text{ on } \partial \Omega.
\end{cases}$$

Proof. Let $z = (u - 1)^+$. Then $z \in W^{1,2}_0(\Omega)$ and is a subsolution of the equation

$$-\Delta v = \hat{g}(x,v) + w,$$

where $\hat{g}(x,s) = g(x,s + 1)\chi_{\{s > 1\}}$ and $w = u^{-\beta}\chi_{\{u > 1\}} \in L^\infty(\Omega)$. Then it is standard to show (see in particular [4, Theorem 2.3]) that $z \in L^\infty(\Omega)$, whence $u \in L^\infty(\Omega)$.

Since in turn $g(x,u) \in L^\infty(\Omega)$, the assertion follows from Corollary 3.7.

5. Appendix

In this appendix we prove the following result.
Theorem 5.1. Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \) and let \( u_1 \in W^{1,2}_0(\Omega) \cap C^\infty(\Omega) \) be such that \(-\Delta u_1 = 1 \) in \( \Omega \).

Then \( u_1 \in C(\overline{\Omega}) \) with \( u_1 = 0 \) on \( \partial\Omega \) if and only if each \( x \in \partial\Omega \) satisfies the Wiener criterion.

Proof. Assume that each \( x \in \partial\Omega \) satisfies the Wiener criterion. Let \( u_2(x) = x_1^2/2 \), so that \( \Delta u_2 = 1 \). According to [11], there exists \( u \in C(\overline{\Omega}) \cap C^\infty(\Omega) \) with \( u = u_2 \) on \( \partial\Omega \) and \( \Delta u = 0 \) in \( \Omega \). Then it is easily seen that \( u - u_2 \in W^{1,2}_0(\Omega) \cap C^\infty(\Omega) \) and satisfies \(-\Delta(u - u_2) = 1 \) in \( \Omega \), whence \( u - u_2 = u_1 \).

Assume now that \( u_1 \in C(\overline{\Omega}) \) with \( u_1 = 0 \) on \( \partial\Omega \). According to [11], it is enough to show that, for every \( v \in C(\partial\Omega) \) there exists \( u \in C(\overline{\Omega}) \cap C^\infty(\Omega) \) with \( u = v \) on \( \partial\Omega \) and \( \Delta u = 0 \) in \( \Omega \). Let \( (v_k) \) be a sequence in \( C^\infty(\mathbb{R}^n) \) converging to \( v \) uniformly on \( \partial\Omega \). By the weak maximum principle, it is enough to show the assertion for \( v_k \) instead of \( v \). Let \( z_k \in W^{1,2}_0(\Omega) \cap C^\infty(\Omega) \) be such that \(-\Delta z_k = \Delta v_k \) in \( \Omega \). There exists \( M_k > 0 \) such that \( |\Delta v_k| \leq M_k \) on \( \overline{\Omega} \), whence \( |z_k| \leq M_k u_1 \). Therefore \( z_k \in C(\overline{\Omega}) \) with \( z_k = 0 \) on \( \partial\Omega \), namely \( u = v_k + z_k \) satisfies \( u = v_k \) on \( \partial\Omega \) and \( \Delta u = 0 \) in \( \Omega \). \( \square \)

References


