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A Variational Approach to a Class of Singular Semilinear Elliptic Equations

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We provide a variational approach to singular semilinear elliptic equations of the form $-\Delta u = u^{-\beta} + g(x,u)$, for every $\beta > 0$.

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1. Introduction

Since the pioneering papers of Crandall, Rabinowitz and Tartar [6] and Stuart [18], singular semilinear elliptic problems of the form

$$\begin{cases} u > 0 & \text{in } \Omega\,, \\ -\Delta u = u^{-\beta} + g(x,u) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1}$$

where Ω is a bounded open subset of $\mathbb{R}^n$, $\beta > 0$ and g satisfies suitable growth conditions, have been considered by several authors (see e.g. [10, 13, 15, 16, 20] and the references therein). Let us also mention [5, 9], where the case in which the singular term $u^{-\beta}$ has the opposite sign is treated.

However, in spite of the fact that (1) is formally the Euler equation of the functional

$$f(u) = \frac{1}{2}\int_\Omega |Du|^2\,dx + \int_\Omega \Phi(u)\,dx - \int_\Omega \int_0^{u(x)} g(x,s)\,ds dx\,, \qquad u \in W_0^{1,2}(\Omega)\,,$$

where

$$\Phi(s) = \begin{cases} -\displaystyle\int_1^s t^{-\beta}\,dt & \text{if } s \geq 0\,, \\ +\infty & \text{if } s < 0\,, \end{cases} \tag{2}$$

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few existence and multiplicity results for (1) have been so far obtained through a direct variational approach. Let us mention [12], where the case in which $\beta < 3$ and g has critical growth is studied by minmax techniques. The restriction $\beta < 3$ is due to the fact that, according to [16, Theorem 2]), the functional f is identically $+\infty$, if $\beta \geq 3$.

The main purpose of this paper is to provide a variational approach to (1) also when $\beta \geq 3$. Actually, our results apply for any $\beta > 0$, but the novelty concerns the case $\beta \geq 3$.

More precisely, in this paper we provide first of all a variational approach to the problem

$$\begin{cases} u > 0 & \text{in } \Omega\,, \\ -\Delta u = u^{-\beta} + w & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{3}$$

with $w \in W^{-1,2}(\Omega)$. To this aim, consider first the case $w = 0$. We denote by $\Phi_k : \mathbb{R} \to \mathbb{R}$ the primitive of the function

$$\begin{cases} \max\left\{-s^{-\beta}, -k\right\} & \text{if } s > 0\,, \\ -k & \text{if } s \leq 0\,, \end{cases}$$

such that $\Phi_k(1) = 0$ and we define a proper, lower semicontinuous, strictly convex functional $\hat{f}_{0,k} : L^2(\Omega) \to]-\infty, +\infty]$ as

$$\hat{f}_{0,k}(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |Du|^2\,dx + \int_\Omega \Phi_k(u)\,dx & \text{if } u \in W_0^{1,2}(\Omega)\,, \\ +\infty & \text{if } u \in L^2(\Omega) \setminus W_0^{1,2}(\Omega)\,. \end{cases}$$

Since a primitive is naturally defined up to an additive constant, to prevent a possible unhappy choice we pass to consider $f_{0,k} : L^2(\Omega) \to]-\infty, +\infty]$, defined as

$$f_{0,k}(u) = \hat{f}_{0,k}(u) - \min \hat{f}_{0,k} = \hat{f}_{0,k}(u) - \hat{f}_{0,k}(u_{0,k})\,,$$

where $u_{0,k} \in W_0^{1,2}(\Omega)$ is the minimum of $\hat{f}_{0,k}$.

More generally, for every $w \in W^{-1,2}(\Omega)$, we define $f_{w,k} : L^2(\Omega) \to]-\infty, +\infty]$ as

$$f_{w,k}(u) = \begin{cases} f_{0,k}(u) - \langle w, u - u_{0,k}\rangle & \text{if } u \in W_0^{1,2}(\Omega)\,, \\ +\infty & \text{if } u \in L^2(\Omega) \setminus W_0^{1,2}(\Omega)\,. \end{cases}$$

Our first step is to prove that the sequence $(f_{w,k})$ is still equicoercive in $L^2(\Omega)$ and is now Γ-convergent (see [1, 7, 8]) as $k \to \infty$ to a *proper*, lower semicontinuous, strictly convex functional $f_w : L^2(\Omega) \to]-\infty, +\infty]$, whose effective domain $\{u \in L^2(\Omega) : \; f_w(u) < +\infty\}$ is independent of w. Moreover, if u_0 is the minimum of f_0 (the Γ-limit functional corresponding to $w = 0$), then the effective domain of f_w is contained in $u_0 + W_0^{1,2}(\Omega)$ and an explicit description of f_w can be provided.

The second step is to study the Euler equation associated with f_w. If $w \in L^1_{loc}(\Omega) \cap W^{-1,2}(\Omega)$, then (3) is just the Euler equation of f_w, provided that the boundary condition $u = 0$ *on* $\partial\Omega$ has a suitable relaxed meaning. Moreover, if Ω has smooth boundary and w is Hölder continuous on $\overline{\Omega}$, then the minimum of f_w is just the solution in $C(\overline{\Omega}) \cap C^2(\Omega)$

of (3) already considered in [6]. In general, if $w \in W^{-1,2}(\Omega)$ then the minimum of f_w is characterized by a variational inequality.

Finally, the variational description of (1) is obtained by considering the sum of a convex, lower semicontinuous functional and a functional of class C^1 taking into account the term $g(x,u)$. For such a class of functionals, minmax techniques have been developed in [19].

2. On the equation $-\Delta u = u^{-\beta}$

Let Ω be a bounded open subset of $\mathbb{R}^n$ and let $\beta > 0$. In the following, we will denote by $L^\infty_c(\Omega)$ the space of L^∞-functions with compact support in Ω. We will also denote by $\| \, \|_p$ the usual norm in $L^p(\Omega)$ and by $\| \, \|_{-1,2}$ the norm in $W^{-1,2}(\Omega)$ dual to the norm $\|Du\|_2$ in $W^{1,2}_0(\Omega)$.

Definition 2.1. Let $u \in W^{1,2}_{loc}(\Omega)$. We say that $u \leq 0$ *on* $\partial\Omega$ if, for every $\varepsilon > 0$, the function $(u-\varepsilon)^+$ belongs to $W^{1,2}_0(\Omega)$.

It is readily seen that, if $u \in W^{1,2}_0(\Omega)$, then $u \leq 0$ on $\partial\Omega$. The same fact holds if $u \in C(\overline{\Omega}) \cap W^{1,2}_{loc}(\Omega)$ and $u(x) \leq 0$ for every $x \in \partial\Omega$.

Let us state the main result of this section.

Theorem 2.2. *There exists one and only one $u_0 \in C^\infty(\Omega)$ satisfying*

$$\begin{cases} u_0 > 0 & \text{in } \Omega\,, \\ -\Delta u_0 = u_0^{-\beta} & \text{in } \Omega\,, \\ u_0 \leq 0 & \text{on } \partial\Omega\,. \end{cases} \tag{4}$$

Moreover, if $u_1 \in W^{1,2}_0(\Omega) \cap C^\infty(\Omega)$ satisfies $-\Delta u_1 = 1$ in Ω, then

$$\|u_1\|_\infty^{-\frac{\beta}{\beta+1}} u_1 \leq u_0 \leq ((\beta+1)u_1)^{\frac{1}{\beta+1}} \qquad \text{in } \Omega\,. \tag{5}$$

Remark 2.3. If $\partial\Omega$ is sufficiently smooth, then much sharper estimates than (5) have been proved in [6, 16].

Corollary 2.4. *There exists one and only one $u_0 \in C(\overline{\Omega}) \cap C^\infty(\Omega)$ such that*

$$\begin{cases} u_0 > 0 & \text{in } \Omega\,, \\ -\Delta u_0 = u_0^{-\beta} & \text{in } \Omega\,, \\ u_0 = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{6}$$

if and only if each $x \in \partial\Omega$ satisfies the Wiener criterion [11].

Proof. Of course, the uniqueness in Theorem 2.2 implies the uniqueness in Corollary 2.4.

Let u_0 be given by Theorem 2.2. By (5), we have that u_0 is a $C(\overline{\Omega})$-solution of (6) if and only if u_1 belongs to $C(\overline{\Omega})$ and satisfies $u_1 = 0$ on $\partial\Omega$. It is quite standard to show that in turn this holds if and only if each $x \in \partial\Omega$ satisfies the Wiener criterion. For the reader's convenience, we give a proof of this fact in the Appendix. □

The remaining part of the section will be devoted to the proof of Theorem 2.2.

Definition 2.5. Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, let $w \in W^{-1,2}(\Omega)$ and let $\varphi \in W^{1,2}_{loc}(\Omega)$. We say that φ is a (*local*) *subsolution* of the equation

$$-\Delta u = g(x, u) + w\,, \tag{7}$$

if $g(x, \varphi) \in L^1_{loc}(\Omega)$ and

$$\int_\Omega D\varphi Dv\, dx \le \int_\Omega g(x, \varphi)v\, dx + \langle w, v\rangle \qquad \forall v \in W^{1,2}_0(\Omega) \cap L^\infty_c(\Omega) \text{ with } v \ge 0 \text{ a.e. in } \Omega\,.$$

We say that φ is a (*local*) *supersolution* of (7), if $g(x, \varphi) \in L^1_{loc}(\Omega)$ and

$$\int_\Omega D\varphi Dv\, dx \ge \int_\Omega g(x, \varphi)v\, dx + \langle w, v\rangle \qquad \forall v \in W^{1,2}_0(\Omega) \cap L^\infty_c(\Omega) \text{ with } v \ge 0 \text{ a.e. in } \Omega\,.$$

Definition 2.6. Let $w \in W^{-1,2}(\Omega)$ and $\varphi \in W^{1,2}_{loc}(\Omega)$. We say that φ is a (*local*) *subsolution* of the variational inequality

$$\int_\Omega DuD(v-u)\, dx \ge \int_\Omega u^{-\beta}(v-u)\, dx + \langle w, v-u\rangle \qquad \forall v \ge 0\,, \tag{8}$$

if $\varphi > 0$ a.e. in Ω, $\varphi^{-\beta} \in L^1_{loc}(\Omega)$ and

$$\int_\Omega D\varphi Dv\, dx \le \int_\Omega \varphi^{-\beta}v\, dx + \langle w, v\rangle \quad \forall v \in W^{1,2}_0(\Omega) \cap L^\infty_c(\Omega) \text{ with } 0 \le v \le \varphi \text{ a.e. in } \Omega\,.$$

We say that φ is a (*local*) *supersolution* of (8), if $\varphi > 0$ a.e. in Ω, $\varphi^{-\beta} \in L^1_{loc}(\Omega)$ and

$$\int_\Omega D\varphi Dv\, dx \ge \int_\Omega \varphi^{-\beta}v\, dx + \langle w, v\rangle \quad \forall v \in W^{1,2}_0(\Omega) \cap L^\infty_c(\Omega) \text{ with } v \ge 0 \text{ a.e. in } \Omega\,.$$

Lemma 2.7. *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying*

$$\forall S > 0 : \sup_{|s| \le S} |g(\cdot, s)| \in L^1_{loc}(\Omega)\,,$$

let $w \in W^{-1,2}(\Omega)$ and let $\varphi, u, \psi \in W^{1,2}_{loc}(\Omega)$. Assume that φ is a subsolution of (7), ψ is a supersolution of (7), $\varphi \le u \le \psi$ a.e. in Ω, $g(x, u) \in L^1_{loc}(\Omega)$ and

$$\int_\Omega DuD(v-u)\, dx \ge \int_\Omega g(x, u)(v-u)\, dx + \langle w, v-u\rangle$$
$$\forall v \in u + \left(W^{1,2}_0(\Omega) \cap L^\infty_c(\Omega)\right) \text{ with } \varphi \le v \le \psi \text{ a.e. in } \Omega\,.$$

Then $-\Delta u = g(x, u) + w$ in $\mathcal{D}'(\Omega)$.

Proof. Let $\vartheta \in C^\infty_c(\mathbb{R})$ with $0 \le \vartheta \le 1$ on $\mathbb{R}$, $\vartheta = 1$ on $[-1, 1]$ and $\vartheta = 0$ outside $]-2, 2[$. Let $v \in C^\infty_c(\Omega)$ with $v \ge 0$, let $k \ge 1$, $t > 0$ and let

$$v_k = \vartheta\left(\frac{u}{k}\right)v\,, \qquad v_{k,t} = \min\left\{u + tv_k, \psi\right\}\,.$$

Since $u \leq v_{k,t} \leq \psi$ and $v_{k,t} - u \leq tv_k \leq tv$, we have

$$
\begin{aligned}
&\int_\Omega \Big(|D(v_{k,t}-u)|^2 - (g(x,v_{k,t}) - g(x,u))\,(v_{k,t}-u) \Big)\,dx \\
\leq\ & \int_\Omega (Dv_{k,t}D(v_{k,t}-u) - g(x,v_{k,t})\,(v_{k,t}-u))\,dx - \langle w, v_{k,t}-u\rangle \\
=\ & \int_\Omega (Dv_{k,t}D(v_{k,t}-u-tv_k) - g(x,v_{k,t})\,(v_{k,t}-u-tv_k))\,dx \\
& \qquad -\langle w, v_{k,t}-u-tv_k\rangle + t\int_\Omega (Dv_{k,t}Dv_k - g(x,v_{k,t})v_k)\,dx - t\langle w, v_k\rangle \\
=\ & \int_\Omega (D\psi D(v_{k,t}-u-tv_k) - g(x,\psi)\,(v_{k,t}-u-tv_k))\,dx \\
& \qquad -\langle w, v_{k,t}-u-tv_k\rangle + t\int_\Omega (Dv_{k,t}Dv_k - g(x,v_{k,t})v_k)\,dx - t\langle w, v_k\rangle \\
\leq\ & t\int_\Omega (Dv_{k,t}Dv_k - g(x,v_{k,t})v_k)\,dx - t\langle w, v_k\rangle\,,
\end{aligned}
$$

whence

$$
\int_\Omega (Dv_{k,t}Dv_k - g(x,v_{k,t})v_k)\,dx - \langle w, v_k\rangle \geq -\int_\Omega |g(x,v_{k,t}) - g(x,u)|\,|v_k|\,dx\,. \tag{9}
$$

Since

$$
|g(x,v_{k,t})|\,|v_k| \leq \left(\sup_{|s|\leq 2k+t\|v\|_\infty} |g(x,s)| \right) |v|\,,
$$

by Lebesgue theorem we can pass to the limit in (9) as $t \to 0^+$, obtaining

$$
\int_\Omega (DuDv_k - g(x,u)v_k)\,dx - \langle w, v_k\rangle \geq 0\,.
$$

Going to the limit as $k \to \infty$, it follows

$$
\int_\Omega (DuDv - g(x,u)v)\,dx - \langle w, v\rangle \geq 0\,. \tag{10}
$$

Let now $v \in C_c^\infty(\Omega)$ with $v \leq 0$, let $k \geq 1$, $t > 0$ and let

$$
v_k = \vartheta\left(\frac{u}{k}\right) v\,, \qquad v_{k,t} = \max\{u + tv_k, \varphi\}\,.
$$

Arguing as before, we find again (10).

Therefore, (10) holds for every $v \in C_c^\infty(\Omega)$ and the assertion follows, as we can exchange v in $-v$. □

Lemma 2.8. *Let $w \in W^{-1,2}(\Omega)$ and $\varphi, \psi \in W^{1,2}_{loc}(\Omega)$. Assume that φ is a subsolution of (8) with $\varphi \leq 0$ on $\partial\Omega$ and ψ a supersolution of (8).*

Then $\varphi \leq \psi$ a.e. in Ω.

Proof. Let $\varepsilon > 0$, let $k > \varepsilon^{-\beta}$ and let $\Phi_k : \mathbb{R} \to \mathbb{R}$ and $f_{w,k} : L^2(\Omega) \to]-\infty, +\infty]$ be defined as in the Introduction.

Let u be the minimum of the functional $f_{w,k}$ on the convex set

$$\mathbb{K} = \left\{ u \in W_0^{1,2}(\Omega) : 0 \leq u \leq \psi \text{ a.e. in } \Omega \right\} .$$

According to [14], we have

$$\int_\Omega DuD(v-u)\,dx \geq -\int_\Omega \Phi_k'(u)(v-u)\,dx + \langle w, v-u\rangle \qquad \forall v \in \mathbb{K}\,.$$

In particular, if $v \in C_c^\infty(\Omega)$ with $v \geq 0$ and $t > 0$, we can consider as test function $v_t = \min\{u + tv, \psi\}$. Since ψ is a supersolution also of the equation $-\Delta u = -\Phi_k'(u) + w$, arguing as in the proof of Lemma 2.7, we find that

$$\int_\Omega DuDv\,dx \geq -\int_\Omega \Phi_k'(u)v\,dx + \langle w, v\rangle\,. \tag{11}$$

It easily follows that (11) holds for every $v \in W_0^{1,2}(\Omega)$ with $v \geq 0$ a.e. in Ω.

In particular, since $u \geq 0$ we have $(\varphi - u - \varepsilon)^+ \in W_0^{1,2}(\Omega)$ and

$$\int_\Omega DuD(\varphi - u - \varepsilon)^+\,dx \geq -\int_\Omega \Phi_k'(u)(\varphi - u - \varepsilon)^+\,dx + \langle w, (\varphi - u - \varepsilon)^+\rangle\,. \tag{12}$$

Let now $v \in W_0^{1,2}(\Omega)$ such that $0 \leq v \leq \varphi$ a.e. in Ω and $D\varphi \in L^2(\{v > 0\})$. Let $(\hat{v}_k)$ be a sequence in $C_c^\infty(\Omega)$ converging to v in $W_0^{1,2}(\Omega)$ and let $v_k = \min\{\hat{v}_k^+, v\}$. We have

$$\int_\Omega D\varphi Dv_k\,dx \leq \int_\Omega \varphi^{-\beta} v_k\,dx + \langle w, v_k\rangle\,dx\,.$$

If $\varphi^{-\beta}v \in L^1(\Omega)$, going to the limit as $k \to \infty$, we get

$$\int_\Omega D\varphi Dv\,dx \leq \int_\Omega \varphi^{-\beta} v\,dx + \langle w, v\rangle\,dx\,. \tag{13}$$

If $\varphi^{-\beta}v \notin L^1(\Omega)$, formula (13) is obviously true. In particular, it follows

$$\int_\Omega D\varphi D(\varphi - u - \varepsilon)^+\,dx \leq \int_\Omega \varphi^{-\beta}(\varphi - u - \varepsilon)^+\,dx + \langle w, (\varphi - u - \varepsilon)^+\rangle\,. \tag{14}$$

Since $\varepsilon^{-\beta} < k$, from (12) and (14) we deduce that

$$\begin{aligned}
\int_\Omega |D(\varphi - u - \varepsilon)^+|^2\,dx &= \int_\Omega D(\varphi - u)D(\varphi - u - \varepsilon)^+\,dx \\
&\leq \int_\Omega \left(\varphi^{-\beta} + \Phi_k'(u)\right)(\varphi - u - \varepsilon)^+\,dx \\
&= \int_\Omega \left(-\Phi_k'(\varphi) + \Phi_k'(u)\right)(\varphi - u - \varepsilon)^+\,dx \leq 0\,,
\end{aligned}$$

whence $\varphi \leq u + \varepsilon \leq \psi + \varepsilon$. The assertion follows from the arbitrariness of ε. $\square$

Proof of Theorem 2.2. Let $\Phi_k : \mathbb{R} \to \mathbb{R}$ and $f_{0,k} : L^2(\Omega) \to]-\infty, +\infty]$ be defined as in the Introduction. Let also $u_1 \in W_0^{1,2}(\Omega) \cap C^\infty(\Omega)$ be the solution of $-\Delta u_1 = 1$ in Ω and let

$$\varphi = \|u_1\|_\infty^{-\frac{\beta}{\beta+1}} u_1 , \qquad \psi = ((\beta+1)u_1)^{\frac{1}{\beta+1}} .$$

Recall that $u_1 > 0$ in Ω. Then it turns out that $\varphi \le \psi$ and φ is a subsolution and ψ a supersolution of the equation $-\Delta u = -\Phi_k'(u)$, for any $k \ge \|u_1\|_\infty^{-\frac{\beta}{\beta+1}}$.

Let $u_{0,k} \in W_0^{1,2}(\Omega)$ be the minimum of $f_{0,k}$, namely the weak solution of

$$\begin{cases} -\Delta u = -\Phi_k'(u) & \text{in } \Omega , \\ u = 0 & \text{on } \partial\Omega . \end{cases} \tag{15}$$

Of course, $f_{0,k}$ admits one and only one minimum also on the convex set

$$\left\{ u \in W_0^{1,2}(\Omega) : \ \varphi \le u \le \psi \text{ a.e. in } \Omega \right\}$$

and such a minimum is a solution of (15) by Lemma 2.7. It follows that $\varphi \le u_{0,k} \le \psi$ a.e. in Ω. Since $u_{0,k}$ is a subsolution of $-\Delta u = -\Phi_{k+1}'(u)$, a similar argument shows that $u_{0,k} \le u_{0,k+1}$ a.e. in Ω. On the other hand, for every $\varepsilon > 0$ there exists $\overline{k} > \varepsilon^{-\beta}$. For every k, it follows

$$-\Delta(u_{0,\overline{k}} + \varepsilon) = -\Phi_{\overline{k}}'((u_{0,\overline{k}} + \varepsilon) - \varepsilon) \ge -\Phi_k'(u_{0,\overline{k}} + \varepsilon) ,$$

namely $u_{0,\overline{k}} + \varepsilon$ is a supersolution of $-\Delta u = -\Phi_k'(u)$. Therefore $u_{0,k} \le u_{0,\overline{k}} + \varepsilon$, namely $(u_{0,k})$ is a Cauchy sequence in $L^\infty(\Omega)$.

Therefore $(u_{0,k})$ is increasing and convergent, as $k \to \infty$, to some u_0 in $L^\infty(\Omega)$. Moreover, we have $\varphi \le u_0 \le \psi$, hence $u_0^{-\beta} \in L^\infty_{loc}(\Omega)$.

Given $\varepsilon > 0$, we have

$$\int_\Omega |D(u_{0,k} - \varepsilon)^+|^2 \, dx = - \int_\Omega \Phi_k'(u_{0,k})(u_{0,k} - \varepsilon)^+ \, dx \le \varepsilon^{-\beta} \int_\Omega (u_{0,k} - \varepsilon)^+ \, dx .$$

It follows that $(u_{0,k} - \varepsilon)^+$ is bounded in $W_0^{1,2}(\Omega)$ as $k \to \infty$, so that

$$\forall \varepsilon > 0 : \ (u_0 - \varepsilon)^+ \in W_0^{1,2}(\Omega) .$$

Since $u_{0,k} \ge \varphi$, we deduce that $u_0 \in W^{1,2}_{loc}(\Omega)$, $u_0 \le 0$ on $\partial\Omega$ and $(Du_{0,k})$ is weakly convergent to Du_0 in $L^2(K)$ for any compact set K in Ω.

Then from (15) it follows that $-\Delta u_0 = u_0^{-\beta}$ in $\mathcal{D}'(\Omega)$. From the interior regularity theory, we infer that $u_0 \in C^\infty(\Omega)$ (see also [6, 16]).

The uniqueness of u_0 follows from Lemma 2.8. □

3. The Γ-limit functional and the associated Euler equation

Let Ω be a bounded open subset of $\mathbb{R}^n$ and let $\beta > 0$. Let $w \in W^{-1,2}(\Omega)$ and let $\Phi : \mathbb{R} \to]-\infty, +\infty]$, $\Phi_k : \mathbb{R} \to \mathbb{R}$ and $f_{w,k} : L^2(\Omega) \to]-\infty, +\infty]$ be defined as in the

Introduction. Let also $u_0 \in C^\infty(\Omega)$ be the solution of (4). According to (5), we have $u_0 \in L^\infty(\Omega)$.

Let $G_0 : \Omega \times \mathbb{R} \to [0, +\infty]$ be the Borel function defined as

$$G_0(x,s) = \Phi(u_0(x) + s) - \Phi(u_0(x)) + s\, u_0^{-\beta}(x)\,.$$

Then $G_0(x,0) = 0$ and $G_0(x,\cdot)$ is convex and lower semicontinuous for any $x \in \Omega$. Moreover, $G_0(x,\cdot)$ is of class C^1 on $]-u_0(x), +\infty[$ with

$$D_s G_0(x,s) = u_0^{-\beta}(x) - (u_0(x)+s)^{-\beta}\,.$$

Define a functional $f_w : L^2(\Omega) \to]-\infty, +\infty]$ by

$$f_w(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |D(u-u_0)|^2\,dx + \int_\Omega G_0(x, u-u_0)\,dx - \langle w, u-u_0\rangle & \text{if } u \in u_0 + W_0^{1,2}(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{16}$$

Then f_w is strictly convex, lower semicontinuous and coercive, with $f_w(u_0) = 0$. Moreover, the effective domain of f_w is

$$\left\{u \in u_0 + W_0^{1,2}(\Omega) : \, G_0(x, u-u_0) \in L^1(\Omega)\right\} \subseteq W^{1,2}_{loc}(\Omega)\,,$$

independently of w. In the case $w = 0$, it is clear that u_0 is just the minimum of f_0.

Let us recall from [1, 7, 8, 17] the following

Definition 3.1. Let X be a topological space, $f_k : X \to [-\infty, +\infty]$ a sequence of functions and $f : X \to [-\infty, +\infty]$ a function.

We say that

$$f = \Gamma(X^-) - \lim_k f_k$$

if the following facts hold:

(a) for every sequence (u_k) convergent to u in X, we have

$$f(u) \le \liminf_k f_k(u_k)\,;$$

(b) for every $u \in X$ there exists a sequence (u_k) in X convergent to u satisfying

$$f(u) \ge \limsup_k f_k(u_k)\,.$$

When X is a Banach space, we say that (f_k) is convergent to f *in the sense of Mosco* (*M-convergent*, for short), if (a) holds with respect to the weak topology of X and (b) with respect to the strong topology.

Theorem 3.2. *For every $w \in W^{-1,2}(\Omega)$, the sequence $(f_{w,k})$ is equicoercive in $L^2(\Omega)$ and we have*

$$f_w = \Gamma(L^2(\Omega)^-) - \lim_k f_{w,k}\,.$$

Proof. Let $u_{0,k} \in W_0^{1,2}(\Omega)$ be the minimum of $f_{0,k}$. According to the proof of Theorem 2.2, $(u_{0,k})$ is convergent to u_0 in $L^\infty(\Omega)$.

Then, since $\hat{f}_{0,k}$ is of class C^1 on $W_0^{1,2}(\Omega)$, for every $u \in W_0^{1,2}(\Omega)$ we have

$$\begin{aligned} f_{w,k}(u) &= \hat{f}_{0,k}(u) - \hat{f}_{0,k}(u_{0,k}) - \langle w, u - u_{0,k}\rangle \\ &= \hat{f}_{0,k}(u) - \hat{f}_{0,k}(u_{0,k}) - \langle \hat{f}'_{0,k}(u_{0,k}), u - u_{0,k}\rangle - \langle w, u - u_{0,k}\rangle \\ &= \frac{1}{2}\int_\Omega |D(u-u_{0,k})|^2\,dx + \int_\Omega \left(\Phi_k(u) - \Phi_k(u_{0,k}) - \Phi'_k(u_{0,k})(u-u_{0,k})\right)\,dx \\ &\qquad -\langle w, u - u_{0,k}\rangle\,. \end{aligned}$$

Since Φ_k is convex, for every $c \in \mathbb{R}$ the set

$$\bigcup_{k\in\mathbb{N}} \left\{u - u_{0,k} :\, u \in L^2(\Omega),\ f_{w,k}(u) \le c\right\}$$

is bounded in $W_0^{1,2}(\Omega)$. In particular, the sequence $(f_{w,k})$ is equicoercive in $L^2(\Omega)$.

Let now (u_k) be a sequence convergent to u in $L^2(\Omega)$. If $\liminf_k f_{w,k}(u_k) = +\infty$, it is obvious that

$$f_w(u) \le \liminf_k f_{w,k}(u_k)\,. \tag{17}$$

Otherwise, up to a subsequence $(u_k - u_{0,k})$ is bounded in $W_0^{1,2}(\Omega)$ and convergent to u a.e. in Ω. It follows that $u \in u_0 + W_0^{1,2}(\Omega)$ and $(u_k - u_{0,k})$ is weakly convergent to $u - u_0$ in $W_0^{1,2}(\Omega)$.

Since $u_0 > 0$ in Ω, it is clear that $\Phi_k(u_k) - \Phi_k(u_{0,k}) - \Phi'_k(u_{0,k})(u_k - u_{0,k})$ is convergent to $G_0(x, u-u_0)$ a.e. in Ω. Then (17) easily follows also in this case.

Finally, let $u \in L^2(\Omega)$. If $f_w(u) = +\infty$ it is obvious that (b) of Definition 3.1 holds. Otherwise, let $u \in u_0 + W_0^{1,2}(\Omega)$ with $u \ge 0$ a.e. in Ω and $G_0(x, u-u_0) \in L^1(\Omega)$. Let $(\hat{v}_m)$ be a sequence in $C_c^\infty(\Omega)$ convergent to $u - u_0$ in $W_0^{1,2}(\Omega)$ and let

$$v_m = \max\left\{\hat{v}_m, -(u-u_0)^-\right\}\,.$$

Then $v_m \in W_0^{1,2}(\Omega) \cap L_c^\infty(\Omega)$ and is strongly convergent to $u - u_0$ in $W_0^{1,2}(\Omega)$ with $v_m \ge -(u-u_0)^-$ and $(G_0(x, v_m))$ is strongly convergent to $G_0(x, u-u_0)$ in $L^1(\Omega)$. Therefore, given $\varepsilon > 0$, there exists $\overline{v} \in W_0^{1,2}(\Omega)\cap L_c^\infty(\Omega)$ with $\overline{v} \ge -(u-u_0)^-$, $\|D\overline{v} - D(u-u_0)\|_2 < \varepsilon$ and $\|G_0(x,\overline{v}) - G_0(x, u-u_0)\|_1 < \varepsilon$. Let $\vartheta \in C_c^\infty(\Omega)$ with $\vartheta \ge 0$ in Ω and $\vartheta = 1$ where $\overline{v} \ne 0$. If we set $v = \overline{v} + \delta\vartheta$ with $\delta > 0$ small enough, then $v \in W_0^{1,2}(\Omega) \cap L_c^\infty(\Omega)$ with $\|Dv - D(u-u_0)\|_2 < \varepsilon$, $\|G_0(x,v) - G_0(x,u-u_0)\|_1 < \varepsilon$ and

$$\operatorname*{ess\,inf}_{\{v\ne 0\}} (u_0 + v) > 0\,.$$

Then it is easy to see that

$$\lim_k \left\|(\Phi_k(u_{0,k}+v) - \Phi_k(u_{0,k}) - \Phi'_k(u_{0,k})v) - G_0(x, u-u_0)\right\|_1 < \varepsilon\,.$$

In particular, there exists a sequence (v_k) strongly convergent to $u - u_0$ in $W_0^{1,2}(\Omega)$ with

$$\lim_k \left\|(\Phi_k(u_{0,k}+v_k) - \Phi_k(u_{0,k}) - \Phi'_k(u_{0,k})v_k) - G_0(x, u-u_0)\right\|_1 = 0\,.$$

If we set $u_k = u_{0,k} + v_k$, then (u_k) is strongly convergent to u in $L^2(\Omega)$ with $(f_{w,k}(u_k))$ convergent to $f_w(u)$. $\square$

Remark 3.3. From the proof of Theorem 3.2 it follows that:

(a) if we define $\tilde{f}_{w,k} : W_0^{1,2}(\Omega) \to \mathbb{R}$ and $\tilde{f}_w : W_0^{1,2}(\Omega) \to]-\infty, +\infty]$ as $\tilde{f}_{w,k}(v) = f_{w,k}(u_{0,k} + v)$, $\tilde{f}_w(v) = f_w(u_0 + v)$, then $(\tilde{f}_{w,k})$ is M-convergent to $\tilde{f}_w$;

(b) if $n = 1$, then the restriction of $(f_{w,k})$ to $L^\infty(\Omega)$ is M-convergent to the corresponding restriction of f_w;

(c) if $n \geq 2$, $2 \leq p < \infty$ and $p(n-2) \leq 2n$, then the restriction of $(f_{w,k})$ to $L^p(\Omega)$ is M-convergent to the corresponding restriction of f_w.

Now we consider the associated Euler equation.

Theorem 3.4. *The following facts hold:*

(a) *for every $w \in W^{-1,2}(\Omega)$ and $u \in W^{1,2}_{loc}(\Omega)$, we have that u is the minimum of f_w if and only if u satisfies*

$$\begin{cases} u > 0 \ \text{a.e. in } \Omega \text{ and } u^{-\beta} \in L^1_{loc}(\Omega)\,, \\ \displaystyle\int_\Omega DuD(v-u)\,dx - \int_\Omega u^{-\beta}(v-u)\,dx \geq \langle w, v-u\rangle \\ \qquad\qquad \forall v \in u + \left(W_0^{1,2}(\Omega) \cap L_c^\infty(\Omega)\right) \text{ with } v \geq 0 \ \text{a.e. in } \Omega\,, \\ u \leq 0 \ \text{on } \partial\Omega\,; \end{cases} \tag{18}$$

in particular, for every $w \in W^{-1,2}(\Omega)$ problem (18) admits one and only one solution $u \in W^{1,2}_{loc}(\Omega)$;

(b) *if $w_1, w_2 \in W^{-1,2}(\Omega)$ and $u_1, u_2 \in W^{1,2}_{loc}(\Omega)$ are the corresponding solutions of (18), we have $u_1 - u_2 \in W_0^{1,2}(\Omega)$ and*

$$\|D(u_1 - u_2)\|_2 \leq \|w_1 - w_2\|_{-1,2}\,.$$

Proof. (a) Given $w \in W^{-1,2}(\Omega)$, there exists one and only one minimum $u \in u_0 + W_0^{1,2}(\Omega)$ of f_w. According to [2], we have $G_0(x, u - u_0) \in L^1(\Omega)$, hence $u \geq 0$ a.e. in Ω, and

$$\begin{cases} \left(u_0^{-\beta} - u^{-\beta}\right)(v-u) \in L^1(\Omega)\,, \\ \displaystyle\int_\Omega D(u-u_0)D(v-u)\,dx + \int_\Omega \left(u_0^{-\beta} - u^{-\beta}\right)(v-u)\,dx \geq \langle w, v-u\rangle\,, \end{cases} \tag{19}$$

for every $v \in u_0 + W_0^{1,2}(\Omega)$ with $G_0(x, v - u_0) \in L^1(\Omega)$ (here we agree that $0^{-\beta} = +\infty$).

In particular, we have

$$\left(u_0^{-\beta} - u^{-\beta}\right) v \in L^1(\Omega) \qquad \text{for every } v \in C_c^\infty(\Omega) \text{ with } v \geq 0\,,$$

whence $u > 0$ a.e. in Ω and $u^{-\beta} \in L^1_{loc}(\Omega)$.

Let now $\varepsilon, \sigma > 0$ and let

$$v = \min\left\{u - u_0, \varepsilon - (u_0 - \sigma)^+\right\}\,.$$

Clearly $v \in W_0^{1,2}(\Omega)$. Moreover, we have a.e. either $v = u - u_0$ or $\varepsilon = v \le u - u_0$ or $v = \varepsilon + \sigma - u_0$ with $u_0 \ge \sigma$. It follows $G_0(x, v) \in L^1(\Omega)$, hence

$$((u_0 - \sigma)^+ + u - u_0 - \varepsilon)^+ = u - u_0 - v \in W_0^{1,2}(\Omega)\,,$$
$$\left(u_0^{-\beta} - u^{-\beta}\right)(u_0 + v - u) \in L^1(\Omega)$$

and

$$\int_\Omega D(u-u_0)D(u-u_0-v)\,dx \le -\int_\Omega \left(u_0^{-\beta} - u^{-\beta}\right)(u-u_0-v)\,dx + \langle w, u-u_0-v\rangle\,. \quad (20)$$

In particular, since $u \ne u_0 + v$ implies $u > \varepsilon$, we have that both $u^{-\beta}(u - u_0 - v)$ and $u_0^{-\beta}(u - u_0 - v)$ belong to $L^1(\Omega)$.

On the other hand, we also have

$$\int_\Omega D(u_0 - \sigma)^+ D\varphi\,dx \le \int_\Omega u_0^{-\beta}\varphi\,dx \qquad \text{for every } \varphi \in C_c^\infty(\Omega) \text{ with } \varphi \ge 0\,.$$

Arguing as in [3], it follows

$$\int_\Omega D(u_0 - \sigma)^+ D\varphi\,dx \le \int_\Omega u_0^{-\beta}\varphi\,dx \qquad \text{for every } \varphi \in W_0^{1,2}(\Omega) \text{ with } \varphi \ge 0 \text{ a.e. in } \Omega\,.$$

In particular, we have

$$\int_\Omega D(u_0 - \sigma)^+ D(u - u_0 - v)\,dx \le \int_\Omega u_0^{-\beta}(u - u_0 - v)\,dx\,,$$

which yields, combined with (20),

$$\begin{aligned}
\int_\Omega |D(u - u_0 - v)|^2\,dx &= \int_\Omega D\left((u_0 - \sigma)^+ + u - u_0\right) D(u - u_0 - v)\,dx \\
&\le \int_\Omega u^{-\beta}(u - u_0 - v)\,dx + \langle w, u - u_0 - v\rangle \\
&\le \varepsilon^{-\beta}\int_\Omega (u - u_0 - v)\,dx + \langle w, u - u_0 - v\rangle\,.
\end{aligned}$$

Therefore, for any $\varepsilon > 0$, we have that $((u_0 - \sigma)^+ + u - u_0 - \varepsilon)^+$ is bounded in $W_0^{1,2}(\Omega)$ as $\sigma \to 0^+$. It follows that $(u - \varepsilon)^+ \in W_0^{1,2}(\Omega)$, namely that $u \le 0$ on $\partial\Omega$.

Let now $v \in u + \left(W_0^{1,2}(\Omega) \cap L_c^\infty(\Omega)\right)$ with $v \ge 0$ a.e. in Ω. Let $v_0 \in C_c^\infty(\Omega)$ with $v_0 \ge 0$ in Ω and $v_0 = 1$ where $v \ne u$. Then, for every $\varepsilon > 0$, we have $G_0(x, v + \varepsilon v_0 - u_0) \in L^1(\Omega)$, whence

$$\int_\Omega D(u - u_0)D(v + \varepsilon v_0 - u)\,dx + \int_\Omega \left(u_0^{-\beta} - u^{-\beta}\right)(v + \varepsilon v_0 - u)\,dx \ge \langle w, v + \varepsilon v_0 - u\rangle\,.$$

From the arbitrariness of ε it follows

$$\int_\Omega D(u - u_0)D(v - u)\,dx + \int_\Omega \left(u_0^{-\beta} - u^{-\beta}\right)(v - u)\,dx \ge \langle w, v - u\rangle\,.$$

Since

$$\int_\Omega Du_0 D(v-u)\,dx = \int_\Omega u_0^{-\beta}(v-u)\,dx\,,$$

it turns out that u satisfies (18).

Conversely, let u be a solution of (18) and let $\hat{u} \in W^{1,2}_{loc}(\Omega)$ be the minimum of f_w. We already know that $\hat{u}$ also is a solution of (18). In particular, u and $\hat{u}$ are both a subsolution and a supersolution of (8). From Lemma 2.8 it follows that $u = \hat{u}$, namely u is the minimum of f_w.

(*b*) If $w_1, w_2 \in W^{-1,2}(\Omega)$ and $u_1, u_2 \in W^{1,2}_{loc}(\Omega)$ are the corresponding minima of f_{w_1} and f_{w_2}, from (19) it follows that

$$\begin{aligned} \int_\Omega |D(u_1-u_2)|^2\,dx &\le \int_\Omega \left(u_1^{-\beta} - u_2^{-\beta}\right)(u_1-u_2)\,dx + \langle w_1 - w_2, u_1 - u_2\rangle \\ &\le \langle w_1 - w_2, u_1 - u_2\rangle\,, \end{aligned}$$

whence $\|D(u_1-u_2)\|_2 \le \|w_1 - w_2\|_{-1,2}$. □

Theorem 3.5. *Let $w \in W^{-1,2}(\Omega)$ and $u \in W^{1,2}_{loc}(\Omega)$. If u satisfies*

$$\begin{cases} u > 0 \text{ a.e. in } \Omega & \text{and } u^{-\beta} \in L^1_{loc}(\Omega)\,, \\ -\Delta u - u^{-\beta} = w & \text{in } \mathcal{D}'(\Omega)\,, \\ u \le 0 & \text{on } \partial\Omega\,, \end{cases} \tag{21}$$

then u is the solution of (18)*. If $w \in L^1_{loc}(\Omega) \cap W^{-1,2}(\Omega)$, then* (18) *and* (21) *are equivalent.*

Proof. If u satisfies (21), a simple approximation argument shows that

$$\int_\Omega DuDv\,dx - \int_\Omega u_0^{-\beta} v\,dx = \langle w, v\rangle$$

for every $v \in W^{1,2}_0(\Omega) \cap L^\infty_c(\Omega)$. Then u satisfies (18).

Assume now that $w \in L^1_{loc}(\Omega) \cap W^{-1,2}(\Omega)$ and that u is the solution of (18). It is readily seen that, for every $v \in C^\infty_c(\Omega)$ with $v \ge 0$,

$$\int_\Omega DuDv\,dx \ge \int_\Omega u^{-\beta} v\,dx + \int_\Omega wv\,dx\,. \tag{22}$$

Let now $v \in C^\infty_c(\Omega)$ with $v \le 0$, let $t > 0$ and let $v_t = (u+tv)^+$. Since $|v_t - u| \le t|v|$, we have

$$\begin{aligned} \int_{\{u+tv>0\}} DuDv\,dx &\ge -\frac{1}{t}\int_{\{u+tv\le 0\}} |Du|^2\,dx + \int_{\{u+tv>0\}} DuDv\,dx \\ &= \int_\Omega DuD\left(\frac{v_t-u}{t}\right)dx \ge \int_\Omega u^{-\beta}\frac{v_t-u}{t}\,dx + \int_\Omega w\frac{v_t-u}{t}\,dx\,. \end{aligned}$$

Going to the limit as $t \to 0^+$, we get

$$\int_\Omega DuDv\,dx \ge \int_\Omega u^{-\beta} v\,dx + \int_\Omega wv\,dx$$

also in this case. Therefore u satisfies (21). □

Example 3.6. Let $0 < \beta < 2$ and let $\Omega =]-\pi, \pi[$. Let $u(x) = |\sin x|^\alpha$, where $1/2 < \alpha < 1/\beta$ and let $w = -u'' - u^{-\beta} - \delta_0$, where δ_0 denotes the Dirac measure at 0.

Then $u \in W_0^{1,2}(\Omega)$, $w \in W^{-1,2}(\Omega)$ and (18) is satisfied, even if u is not a solution of (21). Since the solution of (18) is unique, this means that (21) has no solution at all. Thus, if w is merely in $W^{-1,2}(\Omega)$, we can solve the variational inequality (18), but not the equation (21), in general.

Corollary 3.7. *Assume that each $x \in \partial\Omega$ satisfies the Wiener criterion (for instance, Ω has Lipschitz boundary) and that $w \in L^\infty(\Omega)$. Let $u \in W_{loc}^{1,2}(\Omega)$ be the solution of* (21) *given by Theorems* 3.4 *and* 3.5.

Then $u \in C(\overline{\Omega}) \cap W_{loc}^{2,p}(\Omega)$ for any $p < \infty$ and satisfies

$$\begin{cases} u > 0 & \text{in } \Omega\,, \\ -\Delta u - u^{-\beta} = w & \text{a.e. in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{23}$$

Moreover, we have

$$t_w\, u_0 \le u \le T_w\, u_0 \qquad \text{in } \overline{\Omega}$$

for some $0 < t_w \le T_w < +\infty$.

Proof. According to Corollary 2.4, we have $u_0 \in C(\overline{\Omega})$ and $u_0 = 0$ on $\partial\Omega$. Since $w \in L^\infty(\Omega)$, it is readily seen that there exist $T_w, t_w > 0$ such that $t_w u_0$ is a subsolution and $T_w u_0$ a supersolution of (8). From Lemma 2.8 we deduce that $t_w u_0 \le u \le T_w u_0$ a.e. in Ω. Then $u^{-\beta} \in L_{loc}^\infty(\Omega)$ and the assertion follows from standard regularity theory (see e.g. [14]). □

4. C^1 perturbations

Let Ω be a bounded open subset of $\mathbb{R}^n$ and let $\beta > 0$. For the sake of simplicity, we suppose here that $n \ge 3$. In the cases $n = 1, 2$, simple adaptations are required for the growth condition (24) below. Let also $u_0 \in L^\infty(\Omega) \cap C^\infty(\Omega)$ be the solution of (4) and let $\tilde{f}_0 : W_0^{1,2}(\Omega) \to]-\infty, +\infty]$ be the lower semicontinuous, convex functional defined in (a) of Remark 3.3 when $w = 0$.

Moreover, suppose that $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Assume that

$$\begin{cases} \text{there exist } a \in L^{\frac{2n}{n+2}}(\Omega) \text{ and } b \in \mathbb{R} \text{ such that} \\ |g(x,s)| \le a(x) + b|s|^{\frac{n+2}{n-2}} \text{ for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R}\,. \end{cases} \tag{24}$$

Define a new Carathéodory function $g_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ by $g_1(x,s) = g(x, u_0(x) + s)$. Since $u_0 \in L^\infty(\Omega)$, g_1 also satisfies (24). Let $G_1(x,s) = \int_0^s g_1(x,t)\,dt$ and let $f : W_0^{1,2}(\Omega) \to]-\infty, +\infty]$ be the functional defined as $f(u) = \tilde{f}_0(u) + \gamma(u)$, where γ is the functional of class C^1 defined as

$$\gamma(u) = -\int_\Omega G_1(x,u)\,dx\,.$$

According to [19], $u \in W_0^{1,2}(\Omega)$ is said to be a *critical point* of f, if $\tilde{f}_0(u) < +\infty$ and

$$\forall v \in W_0^{1,2}(\Omega) : \ \langle \gamma'(u), v - u \rangle + \tilde{f}_0(v) - \tilde{f}_0(u) \ge 0\,.$$

Theorem 4.1. *For every u, the following assertions are equivalent:*

(a) $u \in W^{1,2}_{loc}(\Omega) \cap L^{\frac{2n}{n-2}}(\Omega)$ *and we have*

$$\begin{cases} u > 0 \text{ a.e. in } \Omega \text{ and } u^{-\beta} \in L^1_{loc}(\Omega)\,, \\ -\Delta u = u^{-\beta} + g(x,u) \quad \text{in } \mathcal{D}'(\Omega)\,, \\ u \leq 0 \text{ on } \partial\Omega\,; \end{cases} \tag{25}$$

(b) $u \in u_0 + W^{1,2}_0(\Omega)$ *and $u - u_0$ is a critical point of f.*

Proof. If (a) holds, let $w = g(x,u) = g_1(x, u - u_0)$. By Theorems 3.4 and 3.5, we have that $u \in u_0 + W^{1,2}_0(\Omega)$ and minimizes f_w. This means that

$$\begin{aligned} \forall v \in W^{1,2}_0(\Omega) :\ \tilde{f}_0(v) &\geq \tilde{f}_0(u-u_0) + \langle w, u_0 + v - u\rangle \\ &= \tilde{f}_0(u-u_0) - \langle \gamma'(u-u_0), u_0 + v - u\rangle\,, \end{aligned}$$

namely $u - u_0$ is a critical point of f.

Conversely, assume that (b) holds. Then $u \in W^{1,2}_{loc}(\Omega) \cap L^{\frac{2n}{n-2}}(\Omega)$ and

$$w := g(x,u) = g_1(x, u-u_0) \in L^1_{loc}(\Omega) \cap W^{-1,2}(\Omega)\,.$$

From Theorems 3.4 and 3.5 it follows that u is a solution of (25). □

Corollary 4.2. *Assume that each $x \in \partial\Omega$ satisfies the Wiener criterion and that*

$$\begin{cases} \text{there exists } b \in \mathbb{R} \text{ such that} \\ |g(x,s)| \leq b(1 + |s|^{\frac{n+2}{n-2}}) \text{ for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R}\,. \end{cases} \tag{26}$$

Let $u \in W^{1,2}_{loc}(\Omega) \cap L^{\frac{2n}{n-2}}(\Omega)$ be a solution of (25).
Then $u \in C(\overline{\Omega}) \cap W^{2,p}_{loc}(\Omega)$ for any $p < \infty$ and satisfies

$$\begin{cases} u > 0 \text{ in } \Omega\,, \\ -\Delta u = u^{-\beta} + g(x,u) \quad \text{a.e. in } \Omega\,, \\ u = 0 \text{ on } \partial\Omega\,. \end{cases}$$

Proof. Let $z = (u-1)^+$. Then $z \in W^{1,2}_0(\Omega)$ and is a subsolution of the equation

$$-\Delta v = \hat{g}(x,v) + w\,,$$

where $\hat{g}(x,s) = g(x,s+1)\chi_{\{u>1\}}$ and $w = u^{-\beta}\chi_{\{u>1\}} \in L^\infty(\Omega)$. Then it is standard to show (see in particular [4, Theorem 2.3]) that $z \in L^\infty(\Omega)$, whence $u \in L^\infty(\Omega)$.

Since in turn $g(x,u) \in L^\infty(\Omega)$, the assertion follows from Corollary 3.7. □

5. Appendix

In this appendix we prove the following result.

Theorem 5.1. *Let Ω be a bounded open subset of $\mathbb{R}^n$ and let $u_1 \in W_0^{1,2}(\Omega) \cap C^\infty(\Omega)$ be such that $-\Delta u_1 = 1$ in Ω.*

Then $u_1 \in C(\overline{\Omega})$ with $u_1 = 0$ on $\partial\Omega$ if and only if each $x \in \partial\Omega$ satisfies the Wiener criterion.

Proof. Assume that each $x \in \partial\Omega$ satisfies the Wiener criterion. Let $u_2(x) = x_1^2/2$, so that $\Delta u_2 = 1$. According to [11], there exists $u \in C(\overline{\Omega}) \cap C^\infty(\Omega)$ with $u = u_2$ on $\partial\Omega$ and $\Delta u = 0$ in Ω. Then it is easily seen that $u - u_2 \in W_0^{1,2}(\Omega) \cap C^\infty(\Omega)$ and satisfies $-\Delta(u - u_2) = 1$ in Ω, whence $u - u_2 = u_1$.

Assume now that $u_1 \in C(\overline{\Omega})$ with $u_1 = 0$ on $\partial\Omega$. According to [11], it is enough to show that, for every $v \in C(\partial\Omega)$ there exists $u \in C(\overline{\Omega}) \cap C^\infty(\Omega)$ with $u = v$ on $\partial\Omega$ and $\Delta u = 0$ in Ω. Let (v_k) be a sequence in $C^\infty(\mathbb{R}^n)$ converging to v uniformly on $\partial\Omega$. By the weak maximum principle, it is enough to show the assertion for v_k instead of v. Let $z_k \in W_0^{1,2}(\Omega) \cap C^\infty(\Omega)$ be such that $-\Delta z_k = \Delta v_k$ in Ω. There exists $M_k > 0$ such that $|\Delta v_k| \leq M_k$ on $\overline{\Omega}$, whence $|z_k| \leq M_k u_1$. Therefore $z_k \in C(\overline{\Omega})$ with $z_k = 0$ on $\partial\Omega$, namely $u = v_k + z_k$ satisfies $u = v_k$ on $\partial\Omega$ and $\Delta u = 0$ in Ω. $\square$

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