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On the Representation Property of Kernels of Quasidifferentials

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In this paper the conjecture that the kernel of each quasidifferential always represents this quasidifferential is proved false in $\mathbb{R}^3$.

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Let X be a topological vector space over the field $\mathbb{R}$, and $\mathcal{K}(X)$ be the family of all nonempty compact convex subsets of X. If $A, B \in \mathcal{K}(X)$ then $A + B = \{a + b \mid a \in A,\, b \in B\}$ is the Minkowski sum of A and B. Let $A \vee B$ be the convex hull of $A \cup B$. If $f : X \to \mathbb{R}$ is a continuous functional, then $H_f A = \{a \in A \mid f(a) = \max_{b\in A} f(b)\}$. Let $(A, B) \sim (C, D)$ if and only if $A+D = B+C$, for $(A, B), (C, D) \in \mathcal{K}^2(X) = \mathcal{K}(X)\times\mathcal{K}(X)$. The relation "$\sim$" is a relation of equivalence in $\mathcal{K}^2(X)$. By $[A, B]$, we denote the quotient class of (A, B) in $\mathcal{K}^2(X)/_\sim$. We can identify quotient classes $[A, B]$, $(A, B) \in \mathcal{K}^2(X)$, with quasidifferentials [1]. Let $(A, B) \leq (C, D)$ if and only if $(A, B) \sim (C, D)$, $A \subset C$, and $B \subset D$. The pair (A, B) is called *minimal* if (A, B) is a minimal element in $([A, B], \leq)$.

The set $A \in \mathcal{K}(X)$ is called a *summand of* $B \in \mathcal{K}(X)$ if there exists $C \in \mathcal{K}(X)$ such that $A + C = B$.

The kernel

$$\Big(\bigcap_{(A,B)\in[A_0,B_0]} (A - B), \bigcap_{(A,B)\in[A_0,B_0]} (B - B) \Big)$$

of quasidifferential $[A_0, B_0]$ was studied in [10] and [3].

Proposition. *Let $A_0, B_0 \in \mathcal{K}(\mathbb{R}^n)$. Then the sets*

$$\bigcap_{(A,B)\in[A_0,B_0]} (A - B), \quad \bigcap_{(A,B)\in[A_0,B_0]} (B - B)$$

are nonempty.

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Proof. Since $0 \in B - B$, $B \in \mathcal{K}(\mathbb{R}^n)$, then $\bigcap_{(A,B)\in[A_0,B_0]}(B-B) \neq \emptyset$.

Let $f \in (\mathbb{R}^n)^*, f \not\equiv 0$. Notice that if $(A,B) \in [A_0,B_0]$ then $(H_f A, H_f B) \in [H_f A_0, H_f B_0]$. Take $f_k : \mathbb{R}^n \to \mathbb{R}$, $f_k(x_1,\ldots,x_n) = x_k$, $k = 1,\ldots,n$. Then

$$(H_{f_1}A, H_{f_1}B) \in [H_{f_1}A_0, H_{f_2}B_0] \quad \text{and} \quad \dim H_{f_2}B \leq n-1.$$

Also $(H_{f_2}H_{f_1}A, H_{f_2}H_{f_1}B) \in [H_{f_2}H_{f_1}A_0, H_{f_2}H_{f_1}B_0]$ and

$$\dim H_{f_2}H_{f_1}A,\ \dim H_{f_2}H_{f_1}B \leq n-2.$$

In n steps we obtain $(H_{f_n}\ldots H_{f_1}A, H_{f_n}\ldots H_{f_1}B) \in [H_{f_n}\ldots H_{f_1}A_0, H_{f_n}\ldots H_{f_1}B_0]$ and $\dim H_{f_n}\ldots H_{f_1}A$, $\dim H_{f_n}\ldots H_{f_1}B = 0$. Then there exist $a, b, a_0, b_0 \in \mathbb{R}^n$ such that

$$H_{f_n}\ldots H_{f_1}A = \{a\},\ \ H_{f_n}\ldots H_{f_1}B = \{b\},$$

$$H_{f_n}\ldots H_{f_1}A_0 = \{a_0\},\ H_{f_n}\ldots H_{f_1}B_0 = \{b_0\}.$$

Hence $a + b_0 = a_0 + b$ and $a_0 - b_0 = a - b \in A - B$. Therefore $a_0 - b_0 \in \bigcap_{(A,B)\in[A_0,B_0]}(A-B)$, and $\bigcap_{(A,B)\in[A_0,B_0]}(A-B)$ is nonempty. □

Proposition was also proved in Chinese in [2]. The following Theorem was proved as Theorem 2.1 in [3].

Theorem. *Let $A_0, B_0 \in \mathcal{K}(\mathbb{R}^2)$ and*

$$C = \bigcap_{(A,B)\in[A_0,B_0]}(A-B),\ \ D = \bigcap_{(A,B)\in[A_0,B_0]}(B-B).$$

Then $(C,D) \in [A_0,B_0]$. If (A_0,B_0) is a minimal pair then $C = A_0 - B_0$, $D = B_0 - B_0$.

Theorem holds true due to the uniqueness up to translation of equivalent minimal pairs of compact convex sets in $\mathbb{R}^2$ [4]. Theorem holds true also for $A_0, B_0 \in \mathcal{K}(\mathbb{R})$.

Now, we will show that Theorem cannot be extended to $\mathbb{R}^3$. In order to do this we need the following Lemma.

Lemma. *Let $A_0, B_0 \in \mathcal{K}(\mathbb{R}^3)$, $f \in (\mathbb{R}^3)^*$, $f \not\equiv 0$ and (A', B') be a minimal pair in $[H_f A_0, H_f B_0]$. Let*

$$C = \bigcap_{(A,B)\in[A_0,B_0]}(A-B),\ \ D = \bigcap_{(A,B)\in[A_0,B_0]}(B-B).$$

Then $A' - B' \subset C$ and $B' - B' \subset D$.

Proof. Let $x \in \mathbb{R}^3$ and $f(x) = 1$. We can assume that $\max f(A_0) = \max f(B_0) = 0$. Notice that if $(A,B) \sim (A_0,B_0)$ then

$$H_f A + H_f B_0 = H_f(A + B_0) = H_f(A_0 + B) = H_f A_0 + H_f B.$$

Hence

$$(H_f A - \max f(A) \cdot x,\ H_f B - \max f(A) \cdot x) \in [H_f A_0, H_f B_0] \in \mathcal{K}^2(Y)/\sim,$$

where $Y = f^{-1}(0)$. Applying Theorem, we obtain

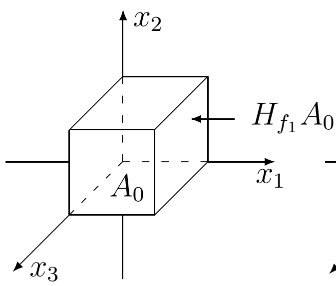
$$A' - B' = \bigcap_{(A,B) \in [H_f A_0, H_f B_0]} (A - B) \subset \bigcap_{(A,B) \in [A_0, B_0]} (H_f A - H_f B) \subset C$$

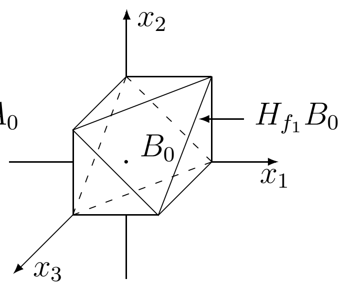
and

$$B' - B' = \bigcap_{(A,B) \in [H_f A_0, H_f B_0]} (B - B) \subset \bigcap_{(A,B) \in [A_0, B_0]} (H_f B - H_f B) \subset D.$$

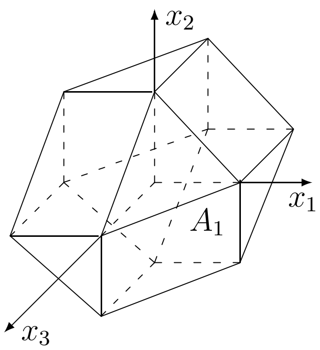
$\square$

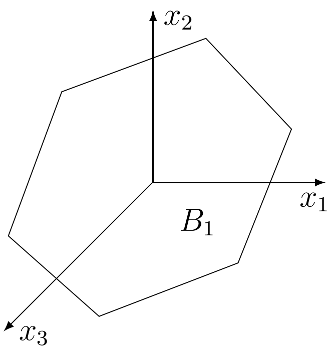
Example. Let $T = (1,0,0) \vee (0,1,0) \vee (0,0,1)$ and $a = (1,1,1)$. Let $A_0 = [0,1]^3 \subset \mathbb{R}^3$ and $B_0 = T \vee (a - T)$.





Let $A_1 = (-T) \vee (T - T) \vee T$ and $B_1 = T - T$.





We have $(A_0, B_0) \sim (A_1 - B_1, B_1 - B_1)$. Notice that

$$B_1 = (B_0 - B_0) \cap (B_1 - B_1) \supset \bigcap_{(A,B) \in [A_0, B_0]} (B - B) = D.$$

Let us consider $f_1 : \mathbb{R}^3 \to \mathbb{R}$, $f_1(x_1, x_2, x_3) = x_1$. Let $B' = (1,1,0) \vee (1,0,1)$, $A' = B' \vee (1,1,1)$. The pair (A', B') is minimal and belongs to the quotient class $[H_{f_1} A_0, H_{f_1} B_0]$. According to Lemma,

$$A' - B' \subset \bigcap_{(A,B) \in [A_0, B_0]} (A - B) = C.$$

Then $(0,0,1),(0,1,0),(0,1,-1),(0,-1,1) \in C$. Replacing f_1 with $f_2 : \mathbb{R}^3 \to \mathbb{R}$, $f_2(x_1,x_2,x_3) = x_2$, $f_3 : \mathbb{R}^3 \to \mathbb{R}$, $f_3(x_1,x_2,x_3) = x_3$, $-f_1$, $-f_2$ and $-f_3$ and A', B' with suitable sets, we prove that $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,-1,0)$, $(-1,1,0)$, $(1,0,-1)$, $(-1,0,1)$, $(0,1,-1)$, $(0,-1,1)$, $(-1,0,0)$, $(0,-1,0)$, $(0,0,-1) \in C$. Therefore, all the vertices of A_1 belong to C. Hence $A_1 \subset C$. Take any $(A,B) \in [A_0,B_0]$. Then $A_1 \subset C \subset A - B$ and $(A_1,B_1) \sim (A,B) \sim (A-B,B-B)$. Applying the order law of cancellation [9], we obtain $B_1 \subset B - B$. Hence $B_1 \subset D \subset B_1$. Therefore, $D = B_1$.

Let us denote

$$E = \bigcap_{(A,B)\in[B_0,A_0]} (A-B), \quad F = \bigcap_{(A,B)\in[B_0,A_0]} (B-B).$$

Assume that $(C,D) \in [A_0,B_0]$ and $(E,F) \in [B_0,A_0]$. Since $D = B_1$ and $(C,D) \sim (A_1,B_1)$ then $C = A_1$ and $E = -C = A_1$. Since $(E,F) \sim (B_1,A_1)$ then $2A_1 = E + A_1 = F + B_1$. But the fact that B_1 is not a summand of $2A_1$ contradicts our assumption. Therefore, $(C,D) \notin [A_0,B_0]$ or $(E,F) \notin [B_0,A_0]$. In fact, it can be proved that $C = A_1$. Hence $(E,F) \notin [B_0,A_0]$. Let us notice that, incidently, $C = E$ because the set A_1 is symmetric. The set D here and generally is not equal to F.

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