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On the Exact Value of Packing Spheres in a Class of Orlicz Function Spaces

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Main result: the Packing constants of Orlicz function spaces $L^{(\Phi)}[0,1]$ and $L^{\Phi}[0,1]$ with Luxemburg and Orlicz norm have the exact value.

(*i*) If $F_\Phi(t) = t\varphi(t)/\Phi(t)$ is decreasing, $1 < C_\Phi < 2$, then

$$P(L^{(\Phi)}[0,1]) = P(L^{\Phi}[0,1]) = \frac{2^{1/C_\Phi}}{2 + 2^{1/C_\Phi}};$$

(*ii*) If $F_\Phi(t)$ is increasing, $C_\Phi > 2$, then

$$P(L^{(\Phi)}[0,1]) = P(L^{\Phi}[0,1]) = \frac{1}{1 + 2^{1/C_\Phi}},$$

where $C_\Phi = \lim\limits_{t\to\infty} F_\Phi(t)$.

Keywords: Orlicz space, Packing constants, Kottman constants

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1. Introduction

Definition 1.1 ([5] [10]). The *packing constant* $P(X)$ of a Banach spaces X is

$$P(X) = \sup\{\ r > 0 : \ \text{infinitely many balls of radius } r \text{ are packed into the unit ball of } X\}.$$

The *Kottman constant* of a infinite dimensional Banach space X is defined [5] as

$$K(X) = \sup\left\{\inf_{i\neq j} \|x_i - x_j\| : \{x_i\}_1^\infty \subset S(X)\right\},$$

where $S(X)$ is the unit sphere of X.

Clearly, $1 \le K(X) \le 2$. The following relationship was offered by Kottman [5] (cf. Ye [18]):

Proposition 1.2. *For a infinite dimensional Banach space X, one has*

$$P(X) = \frac{K(X)}{2 + K(X)}. \tag{1}$$

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Hudzik [4] verified that $K(X) = 2$ if X is a nonreflexive Banach lattice, therefore $P(X) = \frac{1}{2}$.

If X is an infinite dimensional Hilbert space, then Rankin [8] established

$$P(X) = \frac{1}{1+\sqrt{2}}.$$

Slightly later, Burlack, Rankin and Robertson [1] generalized it as:

$$P(l^p) = \frac{1}{1+2^{1-\frac{1}{p}}}, \qquad 1 < p < \infty.$$

Much later, in 1975, Wells and Williams [15] showed

$$P(L^p[0,1]) = \begin{cases} \frac{1}{1+2^{1-\frac{1}{p}}}, & 1 < p \leq 2, \\ \frac{1}{1+2^{\frac{1}{p}}}, & 2 \leq p < \infty \end{cases}$$

and that for $p = 1$ and $p = \infty$, the value is $1/2$. However, the calculation of packing constants of general Banach spaces is a difficult problem. Researchers turned to study the Kottman constants in Orlicz spaces.

Let

$$\Phi(u) = \int_0^{|u|} \varphi(t)dt \text{ and } \Psi(v) = \int_0^{|v|} \psi(s)ds$$

be a pair of complementary $N-$functions, i.e., $\varphi(t)$ is right continuous, $\varphi(0) = 0$, and $\varphi(t) \nearrow \infty$ as $t \nearrow \infty$(to simplify the discussion, we assume φ being continuously differential). We call $\Phi \in \Delta_2(\infty)$, if there exist $u_0 > 0$ and $k > 2$ such that $\Phi(2u) \leq k\Phi(u)$ for $u \geq u_0$. The Orlicz function space $L^\Phi[0,1]$ is defined as

$$L^\Phi[0,1] = \left\{ x(t) : \rho_\Phi(\lambda x) = \int_0^1 \Phi(\lambda|x(t)|)dt < \infty \text{ for some } \lambda > 0 \right\}.$$

The Luxemburg norm and Orlicz norm are expressed as

$$\|x\|_{(\Phi)} = \inf\{c > 0 : \rho_\Phi(\frac{x}{c}) \leq 1\}$$

and

$$\|x\|_\Phi = \inf_{k>0} \frac{1}{k} \left[1 + \rho_\Phi(kx)\right],$$

respectively. For the Orlicz sequence spaces equipped with Luxemburg norm and Orlicz norm with Φ satisfying the Δ_2-condition, Wang [14] and Ye [18] gave the expressions for Kottman constants. Latter on, the author [17] gave the formulae for real computation and answered Rao and Ren's [10] open problem which concerning the exact value of some Orlicz sequence spaces. However, formulae for Orlicz function spaces is still unknown in spite of the work of Cleaver [2], Ren [11], [12]. This paper is trying to give a formula for packing constants in a class of Orlicz function spaces equipped with both norms.

In what follows, we will use Semenove and Simonenko indices of $\Phi(u)$:

$$\alpha_\Phi = \liminf_{u\to\infty} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \qquad \beta_\Phi = \limsup_{u\to\infty} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \tag{2}$$

$$A_\Phi = \liminf_{t\to\infty} \frac{t\varphi(t)}{\Phi(t)}, \qquad B_\Phi = \limsup_{t\to\infty} \frac{t\varphi(t)}{\Phi(t)}, \tag{3}$$

The same indices can be applied to $\Psi(v)$. The author [17] obtained

$$2\alpha_\Phi\beta_\Psi = 1 = 2\alpha_\Psi\beta_\Phi. \tag{4}$$

Rao and Ren [8] gave the following interrelations:

$$2^{-\frac{1}{A_\Phi}} \leq \alpha_\Phi \leq \beta_\Phi \leq 2^{-\frac{1}{B_\Phi}}, \tag{5}$$

If the index function $F_\Phi(t) = \frac{t\varphi(t)}{\Phi(t)}$ is monotonic (increase or decrease) at a neighborhood of ∞, then the limit $C_\Phi = \lim\limits_{t\to+\infty} \frac{t\varphi(t)}{\Phi(t)}$ must exist, and hence

$$\alpha_\Phi = \beta_\Phi = 2^{-\frac{1}{C_\Phi}}. \tag{6}$$

Define $G_\Phi(c,u) = \Phi^{-1}(u)/\Phi^{-1}(cu),\ c > 1$, and $G_\Phi = G_\Phi(2,u)$. The author [17] proved:

Proposition 1.3. *F_Φ is increasing (decreasing) on $[\Phi^{-1}(u_0), +\infty)$ if and only if $G_\Phi(c,u)$ is increasing (decreasing) on $[u_0/c, +\infty)$ for any $c > 1$.*

2. Main results

We need only to observe the Kottman constants for $L^{(\Phi)}[0,1]$ and $L^\Phi[0,1]$ being reflexive, or equivalently, $\Phi \in \triangle_2(\infty) \cap \nabla_2(\infty)$, since otherwise, the Kottman constants of a non-reflexive space is 2 and hence the Packing sphere constants must be 1/2. Cleaver [2] and Ren [11], [12] obtained the following results:

$$\max\left(\frac{1}{\alpha_\Phi}, 2\beta_\Phi\right) \leq K(L^{(\Phi)}[0,1]), \tag{7}$$

$$\max\left(\frac{1}{\alpha_\Psi}, 2\beta_\Psi\right) \leq K(L^{\Phi}[0,1]), \tag{8}$$

and

$$\left\{K(L^{(\Phi_s)}[0,1]), K(L^{\Phi_s}[0,1])\right\} \leq 2^{1-\frac{s}{2}} \tag{9}$$

for the interpolation of Orlicz spaces. In view of the author's result (4), the left sides of (7) and (8) are indeed the same and greater than or equal to $\sqrt{2}$. Therefore, the above results can be refined for interpolation of spaces as follows:

Proposition 2.1 ([2], [12], [9]). *Let Φ be an $N-$function, $\Phi_0(u) = u^2$, and let Φ_s be the inverse of*

$$\Phi_s^{-1}(u) = \left[\Phi^{-1}(u)\right]^{1-s}\left[\Phi_0^{-1}(u)\right]^s, \quad 0 < s \leq 1, u \geq 0.$$

Then

$$\max\left(\frac{1}{\alpha_{\Phi_s}}, 2\beta_{\Phi_s}\right) \leq \left\{K(L^{(\Phi_s)}[0,1]), K(L^{\Phi_s}[0,1])\right\} \leq 2^{1-\frac{s}{2}}. \tag{10}$$

For any $N-$function Φ with $C_\Phi \neq 2$, we produce a function M such that

$$\Phi^{-1}(u) = \left[M^{-1}(u)\right]^{1-s}\left[\Phi_0^{-1}(u)\right]^{s} \tag{11}$$

for some $0 < s < 1$, where $\Phi_0(u) = u^2$. Then M is determined by

$$M^{-1}(u) = u^{-s/2(1-s)}\left[\Phi^{-1}(u)\right]^{1/(1-s)}.$$

If $1 < C_\Phi < 2$, we take l such that $1 < l < C_\Phi$ and let $s = 2(C_\Phi - l)/C_\Phi(2-l)$, then $0 < s < 1$. If $C_\Phi > 2$, we take l such that $C_\Phi < l < \infty$ and let $s = 2(C_\Phi - l)/C_\Phi(2-l)$, then we also have $0 < s < 1$. It is important to show that M is an $N-$function under some conditions.

Theorem 2.2. *Let Φ be an $N-$function,*

(i) *If $F_\Phi(t)$ is decreasing, $1 < C_\Phi < 2$, then the function M determined by* (11) *satisfies:*
 (A) $\lim\limits_{t\to+\infty} M(t)/t = +\infty$.
 (B) *M is convex.*

(ii) *If $F_\Phi(t)$ is increasing, $C_\Phi > 2$, then M determined by* (11) *also satisfies* $(A), (B)$ *in* (i).

Proof. (i) In this case, there is a u_0 such that $u^{C_\Phi} \leq \Phi(u)$ for $u \geq u_0$, or equivalently, $\Phi^{-1}(u) \leq u^{1/C_\Phi}$. Therefore,

$$\frac{u}{M^{-1}(u)} = \left(\frac{u^{\frac{2-s}{2}}}{\Phi^{-1}(u)}\right)^{\frac{1}{1-s}} \geq \left(\frac{u^{\frac{2-s}{2}}}{u^{\frac{1}{C_\Phi}}}\right)^{\frac{1}{1-s}} = \left(u^{\frac{(C_\Phi-2)(l-1)}{C_\Phi(l-2)}}\right)^{\frac{1}{1-s}} \to \infty$$

as $u \to +\infty$. Let $M^{-1}(u) = t$, then $u = M(t)$ and hence $\lim\limits_{t\to\infty} M(t)/t = +\infty$.

To prove (B), it suffice to prove $M^{-1}(u)$ is concave. Observe that

$$M^{-1}(u) = \frac{(\Phi^{-1})^{\frac{1}{1-s}}}{u^{\frac{s}{2(1-s)}}},$$

$$\begin{aligned}
\frac{d}{du}M^{-1} &= \frac{\frac{1}{1-s}(\Phi^{-1})^{\frac{s}{1-s}} \cdot \frac{1}{\varphi} \cdot u^{\frac{s}{2(1-s)}} - \frac{s}{2(1-s)}u^{\frac{3s-2}{2(1-S)}}(\Phi^{-1})^{\frac{1}{1-s}}}{u^{\frac{s}{1-s}}} \\
&= \frac{u(\Phi^{-1})^{\frac{s}{1-s}} - \frac{s}{2}\varphi \cdot (\Phi^{-1})^{\frac{1}{1-s}}}{u^{\frac{2-s}{2(1-s)}}\varphi} \cdot \frac{1}{1-s} \\
&= \frac{1 - \frac{s}{2}\frac{\varphi \cdot \Phi^{-1}}{u}}{u^{\frac{2-s}{2(1-s)}}\varphi} \cdot \frac{u(\Phi^{-1})^{\frac{s}{1-s}}}{1-s} \geq \frac{1 - \frac{s}{2}C_\Phi}{u^{\frac{2-s}{2(1-s)}}\varphi} \cdot \frac{u(\Phi^{-1})^{\frac{s}{1-s}}}{1-s} \\
&= \frac{\frac{2-C_\Phi}{2-l}}{u^{\frac{2-s}{2(1-s)}}\varphi} \cdot \frac{u(\Phi^{-1})^{\frac{s}{1-s}}}{1-s} > 0,
\end{aligned}$$

and that

$$\frac{d^2}{du^2}M^{-1} = \frac{d}{du}\left[\frac{u(\Phi^{-1})^{\frac{s}{1-s}} - \frac{s}{2}\varphi\cdot(\Phi^{-1})^{\frac{1}{1-s}}}{u^{\frac{2-s}{2(1-s)}}\varphi}\right]\frac{1}{1-s}$$

$$= \left\{\left[\frac{s}{1-s}(\Phi^{-1})^{\frac{s}{1-s}-1}\cdot\frac{1}{\varphi}\cdot u + (\Phi^{-1})^{\frac{s}{1-s}} - \frac{s}{2}\cdot\frac{1}{1-s}(\Phi^{-1})^{\frac{s}{1-s}}\cdot\frac{1}{\varphi}\cdot\varphi\right.\right.$$
$$\left.-\frac{s}{2}(\Phi^{-1})^{\frac{1}{1-s}}\cdot\frac{\varphi'}{\varphi}\right]\varphi u^{\frac{2-s}{2(1-s)}} - \left[\frac{\varphi'}{\varphi}u^{\frac{2-s}{2(1-s)}} + \frac{2-s}{2(1-s)}\varphi u^{\frac{s}{2(1-s)}}\right]$$
$$\left.\left(u(\Phi^{-1})^{\frac{s}{1-s}} - \frac{s}{2}\varphi\cdot(\Phi^{-1})^{\frac{1}{1-s}}\right)\right\}\cdot\frac{1}{\left(\varphi\cdot u^{\frac{2-s}{2(1-s)}}\right)^2}\cdot\frac{1}{1-s}$$

$$= \left[\frac{s}{1-s}\left(u^2 - \Phi^{-1}\varphi\cdot u + \frac{2-s}{4}(\Phi^{-1})^2\varphi^2\right) - \frac{\varphi'}{\varphi}\Phi^{-1}\cdot u^2\right]\cdot$$
$$\frac{(\Phi^{-1})^{\frac{s}{1-s}-1}u^{\frac{2-s}{2(1-s)}-1}}{(1-s)\left(\varphi\cdot u^{\frac{2-s}{2(1-s)}}\right)^2}$$

$$= \left[\frac{2(C_\Phi-l)}{l(2-C_\Phi)}\left(1-\frac{\Phi^{-1}\varphi}{u}+\frac{C_\Phi+l-C_\Phi l}{2C_\Phi(2-l)}\left(\frac{\Phi^{-1}\varphi}{u}\right)^2\right)-\frac{\varphi'\Phi^{-1}}{\varphi}\right]\cdot$$
$$\frac{(\Phi^{-1})^{\frac{2s-1}{1-s}}}{(1-s)\varphi^2u^{\frac{s}{2(1-s)}}}.$$

Let $\Phi^{-1}(u) = t$, then $u = \Phi(t)$. It remains to check that

$$f(t) := \frac{2(C_\Phi-l)}{l(2-C_\Phi)}\left(1-\frac{t\varphi}{\Phi}+\frac{C_\Phi+l-C_\Phi l}{2C_\Phi(2-l)}\left(\frac{t\varphi}{\Phi}\right)^2\right)-\frac{t\varphi'}{\varphi} < 0. \tag{12}$$

We first prove

$$h(l) = \frac{2(C_\Phi-l)}{l(2-C_\Phi)}\left(1-C_\Phi+\frac{C_\Phi+l-C_\Phi l}{2C_\Phi(2-l)}(C_\Phi)^2\right)-(C_\Phi-1) < 0, \tag{13}$$

for any definite $1 < l < C_\Phi$, or equivalently,

$$h(l) = \frac{C_\Phi\left[-(C_\Phi)^2l+4C_\Phi l+(C_\Phi)^2-4C_\Phi-4l+4\right]}{l(2-l)(2-C_\Phi)} < 0.$$

Let $l \to 1^+$, then $h(l) \to 0$. On the other hand, since

$$h'(l) = \frac{C_\Phi}{2-C_\Phi}\cdot\frac{-\left[l(C_\Phi-2)+2\right]^2-8C_\Phi(l-1)-4}{(2l-l^2)^2} < 0,$$

we see that $h(l)$ is decreasing on $(0, C_\Phi)$, and hence we deduce that $h(l) < h(1^+) = 0$ on $(0, C_\Phi)$.

Secondly, note that the function

$$g(x) := \frac{2(C_\Phi-l)}{l(2-C_\Phi)}\left(1-x+\frac{C_\Phi+l-C_\Phi l}{2C_\Phi(2-l)}x^2\right)-(x-1)$$
$$= \frac{(C_\Phi-l)}{l(2-C_\Phi)}\left[\frac{C_\Phi+l-C_\Phi l}{C_\Phi(2-l)}x^2-\frac{C_\Phi(2-l)}{C_\Phi-l}(x-1)\right]$$

is decreasing for $x \leq [C_\Phi(l-2)]^2/[2(C_\Phi + l - C_\Phi l)(C_\Phi - l)]$. If $C_\Phi < 2$, then $C_\Phi < [C_\Phi(l-2)]^2/[2(C_\Phi + l - C_\Phi l)(C_\Phi - l)]$ for a sufficiently small $l > 1$ (we will let $l \to 1+$ in the following context). It follows that

$$\begin{aligned} g(F_\Phi(t)) &\leq \frac{(C_\Phi - l)}{l(2-C_\Phi)}\left[\frac{C_\Phi + l - C_\Phi l}{C_\Phi(2-l)}(C_\Phi)^2 - \frac{C_\Phi(2-l)}{C_\Phi - l}(C_\Phi - 1)\right] \\ &= h(l) < 0 \end{aligned}$$

since $F_\Phi(t) \geq C_\Phi$ at a neighborhood of ∞.

Thirdly, By L'Hospital's theorem, we obtain

$$\lim_{t\to+\infty}\frac{t\varphi'(t)}{\varphi(t)} = \lim_{t\to+\infty}\frac{t\varphi(t)}{\Phi(t)} - 1 = C_\Phi - 1.$$

Therefore, for $0 < \varepsilon < -h(l)$, there is a sufficiently large u_0 such that

$$\frac{t\varphi'(t)}{\varphi(t)} > \frac{t\varphi(t)}{\Phi(t)} - 1 - \varepsilon \tag{14}$$

for $u \geq u_0$. Consequently, we have from (13) and (14) that

$$f(t) = g(F_\Phi(t)) - \left[\frac{t\varphi'}{\varphi} - \left(\frac{t\varphi(t)}{\Phi(t)} - 1\right)\right] \leq h(l) + \varepsilon < 0$$

and hence, we proved M is convex.

(ii) If $F_\Phi(t)$ is increasing and $C_\Phi > 2$, then for every sufficiently small $\varepsilon > 0$, there is u_0, such that $u^{1/C_\Phi} \leq \Phi^{-1}(u) < u^{1/C_\Phi + \varepsilon}$ for $u > u_0$. Therefore,

$$\frac{u}{M^{-1}(u)} = \left(\frac{u^{\frac{2-s}{2}}}{\Phi^{-1}(u)}\right)^{\frac{1}{1-s}} \geq \left(\frac{u^{\frac{2-s}{2}}}{u^{\frac{1}{C_\Phi}+\varepsilon}}\right)^{\frac{1}{1-s}} = \left(u^{\frac{(C_\Phi-2)(l-1)}{C_\Phi(l-2)}-\varepsilon}\right)^{\frac{1}{1-s}} \to \infty$$

as $u \to +\infty$. Therefore, $\lim\limits_{t\to\infty} M(t)/t = +\infty$, that is (A) holds.

To prove (B), we are reduced to prove (12). Since $F_\Phi(t) = t\varphi(t)/\Phi(t)$ is increasing from C_Φ, we have

$$F'_\Phi(t) = \frac{(t\varphi' + \varphi)\Phi - t\varphi^2}{\Phi^2} = \frac{\varphi\left(\frac{t\varphi'}{\varphi} + 1 - \frac{t\varphi}{\Phi}\right)}{\Phi} \geq 0.$$

Therefore,

$$\frac{t\varphi'}{\varphi} + 1 - \frac{t\varphi}{\Phi} \geq 0,$$

or

$$\frac{t\varphi'}{\varphi} \geq \frac{t\varphi}{\Phi} - 1.$$

Thus,

$$f(t) \leq \frac{2(C_\Phi - l)}{l(2-C_\Phi)}\left(1 - \frac{t\varphi}{\Phi} + \frac{C_\Phi + l - C_\Phi l}{2C_\Phi(2-l)}\left(\frac{t\varphi}{\Phi}\right)^2\right) - \left(\frac{t\varphi}{\Phi} - 1\right) = g(F_\Phi(t)). \tag{15}$$

Observing $h(l)$ defined in (13), we found that $\lim_{l\to+\infty} h(l) = 0$, and $h'(l) > 0$. Therefore, $h(l) < 0$ for $l > C_\Phi$.

Since the function $g(x)$ is increasing for $x \geq [C_\Phi(l-2)]^2/[2(C_\Phi + l - C_\Phi l)(C_\Phi - l)]$. If $C_\Phi > 2$, then $C_\Phi > [C_\Phi(l-2)]^2/[2(C_\Phi + l - C_\Phi l)(C_\Phi - l)]$ for a sufficiently big l. We deduce that

$$\begin{aligned} f(t) &\leq g(F_\Phi(t)) \leq \frac{(C_\Phi - l)}{l(2 - C_\Phi)} \left[\frac{C_\Phi + l - C_\Phi l}{C_\Phi(2-l)} (C_\Phi)^2 - \frac{C_\Phi(2-l)}{C_\Phi - l}(C_\Phi - 1) \right] \\ &= h(l) < 0 \end{aligned}$$

since $F_\Phi(t) \leq C_\Phi$ at a neighborhood of ∞. Thus, we proved M is convex. The proof is finished. □

Theorem 2.3. *Let Φ be an $N-$function.*

(i) *If $F_\Phi(t) = t\varphi(t)/\Phi(t)$ is decreasing, $1 < C_\Phi < 2$, then*

$$K(L^{(\Phi)}[0,1]) = K(L^\Phi[0,1]) = 2^{\frac{1}{C_\Phi}}. \tag{16}$$

(ii) *If $F_\Phi(t) = t\varphi(t)/\Phi(t)$ is increasing, $C_\Phi > 2$, then*

$$K(L^{(\Phi)}[0,1]) = K(L^\Phi[0,1]) = 2^{1-\frac{1}{C_\Phi}}. \tag{17}$$

Proof. (i) When $F_\Phi(t)$ is decreasing and $1 < C_\Phi < 2$, it follows from (10) and (11) that

$$\max\left(\frac{1}{\alpha_\Phi}, 2\beta_\Phi\right) \leq \{K(L^{(\Phi)}[0,1]), K(L^\Phi[0,1])\} \leq 2^{1-\frac{s}{2}} \tag{18}$$

Since $F_\Phi(t)$ is decreasing, $\frac{1}{\alpha_\Phi} = 2^{\frac{1}{C_\Phi}}$ by (6). On the other hand, in (18) let $l \to 1+$, then $(2-s)/2 \to 1/C_\Phi$. Therefore, (16) holds.

(ii) Similar to (i), (17) follows from

$$2^{\lim\limits_{l\to\infty} 1-\frac{s}{2}} = 2^{1-\frac{1}{C_\Phi}} = 2\beta_\Phi.$$

□

Remark 2.4. In view of the equation (1), (16) and (17) are equivalent to

$$P(L^{(\Phi)}[0,1]) = P(L^\Phi[0,1]) = \frac{2^{1/C_\Phi}}{2 + 2^{1/C_\Phi}}$$

and

$$P(L^{(\Phi)}[0,1]) = P(L^\Phi[0,1]) = \frac{1}{1 + 2^{1/C_\Phi}},$$

respectively. Therefore, we obtained the expression of Packing constants in a class of Orlicz function spaces. It is obvious to see that (16) also holds for $C_\Phi = 1$, and (17) holds for $C_\Phi = \infty$ since the spaces generated by Φ is nonreflexive. One can easily deduces the Kottman constants of $L^p[0,1]$, that is,

$$K(L^p[0,1]) = \max\left(2^{1-\frac{1}{p}}, 2^{\frac{1}{p}}\right).$$

since $C_\Phi = p$ for $\Phi(u) = |u|^p$.

The author have used the methods in this paper to study the other geometric constants in Orlicz function spaces as well as sequence spaces. However, whether the formulae (16) and (17) is fit for $C_\Phi = 2$ is still unknown.

Corollary 2.5. *Let Φ be an $N-$function.*

(*i*) *If $F_\Phi(t) = t\varphi(t)/\Phi(t)$ is decreasing, φ is concave, then* (16) *holds;*

(*ii*) *If $F_\Phi(t)$ is increasing, φ is convex, then* (17) *holds.*

Proof. (i) Note that $\varphi(0) = 0$ since Φ is an $N-$function. When φ is concave, $\varphi(t) = \int_0^t \varphi'(s)ds + \varphi(0) = \int_0^t \varphi'(s)ds \geq t\varphi'(t)$. Therefore,

$$[t\varphi(t) - 2\Phi(t)]' = t\varphi' - \varphi \leq 0,$$

and hence $t\varphi(t) - 2\Phi(t) \leq 0$, in other word, $F_\Phi(t) \leq 2$ which means $C_\Phi \leq 2$. If $C_\Phi < 2$, then (16) holds by Theorem 2.3. If $C_\Phi = 2$, then $t\varphi(t)/\Phi(t) \geq 2$ since $F_\Phi(t)$ is decreasing from at a neighborhood of ∞, and hence,$t\varphi(t)/\Phi(t) = 2$ at a neighborhood of ∞. This means that $\Phi(t) = at^2 (a > 0)$, which generates the Hilbert space $L^2[0,1]$, so (17) holds by the well known result. Analogously we can prove (ii). □

Example 2.6. Let $N-$function be $\Phi(u) = 2|u|^p + |u|^{2p},\ p > 1$. Then

$$F_\Phi(t) = \frac{t\Phi'(t)}{\Phi(t)} = 2p\left(\frac{t^p+1}{t^p+2}\right) \leq 2p$$

and $C_\Phi = 2p$. Therefore, we have

$$K(L^{(\Phi)}[0,1]) = K(L^\Phi[0,1]) = 2^{1-\frac{1}{C_\Phi}} = 2^{1-\frac{1}{2p}}. \tag{19}$$

Let $0 < s \leq 1$, then we can produce Φ_s by

$$\Phi_s^{-1}(u) = (\sqrt{u+1}-1)^{\frac{1-s}{p}} u^{\frac{s}{2}}.$$

We have the exact value:

$$K(L^{(\Phi_s)}[0,1]) = K(L^{\Phi_s}[0,1]) = 2^{1-\frac{1}{C_{\Phi_s}}} = 2^{1-\frac{1-s}{2p}-\frac{s}{2}}. \tag{20}$$

In fact,

$$\alpha_{\Phi_s} = \beta_{\Phi_s} = \lim_{u\to\infty}\frac{\Phi_s^{-1}(u)}{\Phi_s^{-1}(2u)} = \lim_{u\to\infty}\left(\frac{\sqrt{u+1}-1}{\sqrt{2u+1}-1}\right)^{\frac{1-s}{p}}\cdot\left(\frac{1}{2}\right)^{\frac{s}{2}} = \left(\frac{1}{2}\right)^{\frac{1-s}{2p}+\frac{s}{2}}.$$

In view of Proposition 1.3 we see that $F_{\Phi_s}(t)$ is increasing on $(0,+\infty)$ although it is impossible to express, since it is easy to check that the function $(\sqrt{u+1}-1)/(\sqrt{cu+1}-1)$ is increasing on $(0,+\infty)$ for any $c > 1$. Thus,

$$C_{\Phi_s} = \frac{1}{\frac{1-s}{2p}+\frac{s}{2}} \in (2,+\infty).$$

Particularly, let $s = 1/2$, then we obtain the $N-$function defined as:

$$\Phi_{1/2}^{-1}(u) = (\sqrt{u+1} - 1)^{\frac{1}{2p}} u^{\frac{1}{4}}.$$

We have the value:

$$K(L^{(\Phi_{1/2})}[0,1]) = K(L^{\Phi_{1/2}}[0,1]) = 2^{1-\frac{1}{4p}-\frac{1}{4}}. \tag{21}$$

Example 2.7. Let

$$\Phi_p(u) = \frac{|u|^p}{\ln(e+|u|)}, \qquad p > 2$$

defining an $N-$function. Then

$$K(L^{(\Phi_p)}[0,1]) = K(L^{\Phi_p}[0,1]) = 2^{1-\frac{1}{p}}. \tag{22}$$

Indeed, we have

$$F_{\Phi_p} = p - \frac{t}{(e+t)\ln(e+t)} \nearrow p = C_{\Phi_p}, \qquad (t \to +\infty).$$

Example 2.8. Consider the $N-$function

$$\Phi_{p,r}(u) = |u|^p \ln^r(1+|u|),\ \ 1 < p < 2.$$

Then

$$K(L^{(\Phi_{p,r})}[0,1]) = K(L^{\Phi_{p,r}}[0,1]) = 2^{\frac{1}{p}}. \tag{23}$$

Again for $t > 0$, we have

$$F_{\Phi_{p,r}} = p + \frac{rt}{(1+t)\ln(1+t)} \searrow p = C_{\Phi_{p,r}} \qquad (t \to +\infty).$$

Example 2.9. Let

$$\Phi_p(u) = |u|^p(C + |\ln|u||), \qquad 1 < p < 2, c \geq \frac{2p-1}{p(p-1)}$$

It was introduced by Gribanow and revised by Maligranda. Since

$$F_{\Phi_p} = p + \frac{1}{C + \ln t} > p,\ \ t > 1,$$

we have

$$C_{\Phi_p} = \lim_{t\to+\infty} F_{\Phi_p} = p.$$

Consequently,

$$K(L^{(\Phi_p)}[0,1]) = K(L^{\Phi_p}[0,1]) = 2^{\frac{1}{p}}. \tag{24}$$

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