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A Necessary and Sufficient Optimality Condition for a Class of Nonconvex Scalar Variational Problems

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This article studies the minimization of the functional $u \mapsto \int_0^1 f(\dot{u})$ among all convex functions u that satisfy the additional obstacle constraint $u \geq \underline{u}$, $u(0) = \underline{u}(0)$, $u(1) = \underline{u}(1)$ where $\underline{u}$ is a given convex function. We first show that this nonconvex problem is in fact equivalent to a linear programming problem. This enables us to establish a necessary and sufficient optimality condition.

Keywords: Convexity constraint, monotone rearrangements, duality

1. Introduction

Given a Lipschitz convex function $\underline{u}$ on $[0,1]$ (the obstacle) and a continuous function $f : \mathbb{R} \to \mathbb{R}$, we consider the optimization problem:

$$\inf_{u \in \mathcal{A}} J(u) := \int_0^1 f(\dot{u}(t))dt \tag{1}$$

where:

$$\mathcal{A} := \{u : [0,1] \to \mathbb{R},\ u \text{ is convex},\ u \geq \underline{u},\ u(0) = \underline{u}(0),\ u(1) = \underline{u}(1)\}. \tag{2}$$

Variational problems subject to a convexity constraint have received a lot of attention in recent years because of their applications in economics [9], Newton's least resistance problem [2] and connections with optimal transportation [1].

In Section 2, we show that solving (1) actually amounts to solving a linear problem. In Section 3, using this linear formulation a necessary and sufficient optimality condition is given. In Section 4, we introduce a dual formulation. Finally, Section 5, is devoted to some applications and examples.

2. Reformulation of the problem

Let us first recall some basic facts about convex functions and set some notations. First, we recall that a convex function $u : [0,1] \to \mathbb{R}$ is right and left-differentiable at any $t \in (0,1)$ and that it is differentiable except perhaps on an at most countable subset of $(0,1)$. In what follows we shall denote by $\dot{u}(t_+)$ (respectively $\dot{u}(t_-)$) the right (respectively left) derivative of u at $t \in [0,1)$ (respectively $t \in (0,1]$) with the convention that $\dot{u}(0_+)$

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(respectively $\dot{u}(1_-)$) may take the value $+\infty$. We also recall that the classical subdifferential of convex analysis of u at $t \in (0,1)$ is given by $\partial u(t) = [\dot{u}(t_-), \dot{u}(t_+)]$. Slightly abusing notations, we shall simply write $\dot{u}$ instead of the left-continuous selection of ∂u. Finally, we define $\alpha := \dot{\underline{u}}(0_+)$ and $\beta := \dot{\underline{u}}(1_-)$, those two numbers being finite by our Lipschitz assumption on the obstacle.

2.1. Preliminaries

Substracting an affine function to $\underline{u}$ if necessary, we may assume without loss of generality that $\underline{u}(0) = \underline{u}(1) = 0$. From now on, we shall then assume $\underline{u}(0) = \underline{u}(1) = 0$. For further reference, let us note that if $u \in \mathcal{A}$ then $u \in W^{1,\infty}$ and more precisely $\alpha \leq \dot{u}(t) \leq \beta$ for all $t \in [0,1]$; this implies in particular that the infimum of (1) is finite.

Lemma 2.1. *The minimization problem* (1) *admits at least one solution.*

Proof. Let $(u_n) \in \mathcal{A}^{\mathbb{N}}$ be a minimizing sequence of (1), the sequence $(\dot{u}_n)$ is a uniformly bounded sequence of nondecreasing functions. Helly's Theorem implies that a subsequence again denoted $(\dot{u}_n)$ converges pointwise except on an at most countable set to some nondecreasing function v. It is obvious that the function defined by $u(t) := \int_0^t v(s)ds$ for all $t \in [0,1]$ satisfies $u \in \mathcal{A}$. Finally by Lebesgue's dominated convergence theorem $J(u_n)$ converges to $J(u)$ which proves that u is a solution of (1). $\square$

Lemma 2.2. *Let u be a convex function on $[0,1]$ such that $u(0) = 0$, then $u \in \mathcal{A}$ if and only if:*

$$\int_0^1 \psi(\dot{u}) \leq \int_0^1 \psi(\dot{\underline{u}}) \text{ for every convex function, } \psi : \mathbb{R} \to \mathbb{R}. \tag{3}$$

Proof. Let $u \in \mathcal{A}$ and ψ be convex $\mathbb{R} \to \mathbb{R}$. Define $p(.)$ such that $p(t) \in \partial\psi(\dot{u}(t))$ for all $t \in [0,1]$, by convexity we have:

$$\int_0^1 \psi(\dot{\underline{u}}(t))dt - \int_0^1 \psi(\dot{u}(t))dt \geq \int_0^1 p(t)(\dot{\underline{u}}(t) - \dot{u}(t))dt.$$

Since p is nondecreasing its derivative in the sense of distributions is a nonnegative measure μ and the right-hand side of the previous inequality can be integrated by parts in Stieltjès sense:

$$\int_0^1 p(t)(\dot{\underline{u}}(t) - \dot{u}(t))dt = \int_0^1 (u(t) - \underline{u}(t))d\mu(t)$$

and since $u \geq \underline{u}$ and $\mu \geq 0$ this proves the desired inequality.

Conversely, let u be convex satisfy (3) and $u(0) = 0$, taking $\psi(v) = v$ and $\psi(v) = -v$ in (3) first yields $u(1) = 0$. Taking ψ the distance function to $[\alpha, \beta]$ in (3) we then get $\dot{u}(t) \in [\alpha, \beta]$ for all $t \in [0,1]$.

Let $\eta \in L^1(0;1)$, $\eta \geq 0$ and $g(t) := \int_0^t \eta$ for $t \in [0,1]$. Define then

$$\gamma(v) := \sup\{g(t) : \dot{\underline{u}}(t) \leq v\}, \text{ for all } v \in [\alpha, \beta]$$

note that γ is nondecreasing and that by construction for all $t \in (0,1)$:

$$\lim_{\varepsilon \to 0^+} \gamma(\dot{\underline{u}}(t) - \varepsilon) \leq g(t) \leq \lim_{\varepsilon \to 0^+} \gamma(\dot{\underline{u}}(t) + \varepsilon).$$

Define finally:

$$\psi(v) := \int_\alpha^v \gamma(s)ds, \text{ for all } v \in [\alpha, \beta]$$

so that $\psi(\dot{u})$ is well defined. By convexity of ψ and the fact that $g(t) \in \partial\psi(\underline{\dot{u}}(t))$ for all t, we get with (3):

$$\begin{aligned} 0 &\geq \int_0^1 \psi(\dot{u}(t))dt - \int_0^1 \psi(\underline{\dot{u}}(t))dt \\ &\geq \int_0^1 g(t)(\dot{u}(t) - \underline{\dot{u}}(t))dt = \int_0^1 \eta(t)(\underline{u}(t) - u(t))dt \end{aligned}$$

since the previous inequality holds for all $\eta \geq 0$ we have $u \geq \underline{u}$ hence $u \in \mathcal{A}$. □

Remark. Note that Lemma 2.2 implies at once that if f is concave then $\underline{u}$ solves (1). By Jensen's inequality note also that, if f is convex, then $u \equiv 0$ solves (1).

The "if" part of Lemma 2.2 was first proved by Lachand-Robert and Peletier in [4] (without our one-dimensional space restriction). Lemma 2.2 may therefore be viewed (in one dimension again) as a kind of converse of Lachand-Robert and Peletier's result.

2.2. Reformulation as a linear programming problem

Given a measurable function g from $[0,1]$ to $[\alpha, \beta]$, we define the Radon probability measure ν_g on $[\alpha, \beta]$ by:

$$\int_\alpha^\beta \varphi(v)d\nu_g(v) := \int_0^1 \varphi(g(t))dt, \text{ for all } \varphi \in C^0([\alpha, \beta], \mathbb{R})$$

Our aim is now to characterize the set:

$$\mathcal{B} := \{\nu_{\dot{u}},\, u \in \mathcal{A}\}. \tag{4}$$

Let us also define $\underline{\nu} \in \mathcal{B}$ by:

$$\underline{\nu} := \nu_{\underline{\dot{u}}}. \tag{5}$$

It is a well-known fact from the rearrangement literature (or optimal transportation) that a (right or left-continuous) nondecreasing function g is fully determined by the measure ν_g. For the sake of completeness, we recall the classical result (see [7] for instance):

Proposition 2.3. *Let ν be a Radon probability measure on $[\alpha, \beta]$, there exists a unique left-continuous nondecreasing function g on $[\alpha, \beta]$ such that $\nu = \nu_g$ and g is given by:*

$$g(t) = \inf\{\delta \in [\alpha, \beta] : \int_\alpha^\delta d\nu \geq t\} \text{ for all } t \in [0,1]. \tag{6}$$

Proof. Let g be defined by (6). Let us first remark that for all $(\delta, t) \in [\alpha, \beta] \times [0,1]$, one has:

$$\delta \geq g(t) \Leftrightarrow \int_\alpha^\delta d\nu \geq t. \tag{7}$$

And it is immediate to check that g is left-continuous nondecreasing.

Let us prove that $\nu_g = \nu$. Let $\delta \in [\alpha, \beta]$, and let φ be the indicator function of $[\alpha, \delta]$, by (7), one has:

$$\int_\alpha^\delta d\nu = \int_0^1 \varphi(g(t))dt$$

hence, by standard arguments, we get $\nu = \nu_g$.

Finally, let us prove uniqueness. Assume that h is a nondecreasing left-continuous function such that $\nu_h = \nu$. One may easily check that since h is nondecreasing and left-continuous it admits the representation:

$$h(t) = \inf\{\delta \in [\alpha, \beta] : u(\delta) \geq t\} \text{ for all } t \in [0, 1] \tag{8}$$

where:

$$u(\delta) = \inf\{t \in [0, 1] : h(t) > \delta)\} \text{ for all } \delta \in [\alpha, \beta]. \tag{9}$$

Once again one has $t \leq u(\delta) \Leftrightarrow h(t) \leq \delta$. Now since $\nu = \nu_h = \nu_g$ for all $\delta \in [\alpha, \beta]$, one has:

$$\int_\alpha^\delta d\nu = \int_{\{g \leq \delta\}} dt = \int_{\{h \leq \delta\}} dt = \int_0^{u(\delta)} dt = u(\delta).$$

By formulas (6) and (8) we then get $h = g$. □

In the sequel we shall denote by $\mathcal{M}([\alpha, \beta])$ (respectively $\mathcal{M}_+([\alpha, \beta])$) the space of scalar Radon measures (respectively the cone of nonnegative scalar Radon measures) on $[\alpha, \beta]$.

Lemma 2.4. *Defining $\mathcal{B}$ by (4) and $\underline{\nu} \in \mathcal{B}$ by (5), one has:*

$$\mathcal{B} = \{\nu \in \mathcal{M}_+([\alpha, \beta]) : \int_\alpha^\beta \varphi d\nu \geq \int_\alpha^\beta \varphi d\underline{\nu} \textit{ for all concave } \varphi \in C^0([\alpha, \beta], \mathbb{R})\}.$$

The proof directly follows from Proposition 2.3 and Lemma 2.2.

Remark. The previous result can be expressed in terms of *balayages*, a well-kown notion in probability theory, which literally means *sweeping* in french. Indeed, $\nu \in \mathcal{B}$ if and only if $\underline{\nu}$ is a *balayée* of ν (see Meyer [6], Chap. XI for properties of balayages) which, as noted by Meyer ([6], p. 279), expresses in some sense that $\underline{\nu}$ is closer to the boundary of $[\alpha, \beta]$ than ν. We shall not use however fine properties of balayages in the sequel.

Finally, formulating problem (1) in terms of $\nu_{\dot{u}}$ rather than in u, we are able to transform (1) into an infinite-dimensional linear programming problem. More precisely, we have:

Proposition 2.5. *$u \in \mathcal{A}$ is a solution of (1) if and only if $\nu_{\dot{u}}$ is a solution of the linear programming problem:*

$$\inf_{\nu \in \mathcal{B}} \int_\alpha^\beta f(v)d\nu(v). \tag{10}$$

Proof. This follows at once from Proposition 2.3 and the identity:

$$\int_0^1 f(\dot{u}(t))dt = \int_\alpha^\beta f(v)d\nu_{\dot{u}}(v).$$

□

3. A necessary and sufficient condition

Theorem 3.1. *$\nu \in \mathcal{B}$ is a solution of (10) if and only if there exists a continuous concave function on (α,β), $\underline{\varphi}$ such that $\underline{\varphi}$ is $\underline{\nu}$-integrable and:*

$$f \geq \underline{\varphi} \text{ on } (\alpha,\beta),\ f = \underline{\varphi}\ \nu\text{-a.e. and } \int_\alpha^\beta f d\nu = \int_\alpha^\beta \underline{\varphi} d\underline{\nu}. \tag{11}$$

Proof. Note that if φ is concave, only continuous on (α,β) (that is φ may tend to $-\infty$ at either α or β) and φ is $\underline{\nu}$-integrable, then there exists a nonincreasing sequence of Lipschitz concave functions $(\varphi_k)_{k\in\mathbb{N}}$ that converges pointwise to φ. Let $\mu \in \mathcal{B}$, for all k, one has:

$$\int_\alpha^\beta \varphi_k d\mu \geq \int_\alpha^\beta \varphi_k d\underline{\nu}.$$

On the one hand, since $\varphi_0 \geq \varphi_k \geq \varphi$, by Lebesgue's dominated convergence theorem we first obtain that $\int_\alpha^\beta \varphi_k d\underline{\nu}$ converges to $\int_\alpha^\beta \varphi d\underline{\nu}$. On the other hand, using the monotone convergence theorem, we obtain that φ is μ-integrable and that $\int_\alpha^\beta \varphi_k d\mu$ converges monotonically to $\int_\alpha^\beta \varphi d\mu$, hence we have:

$$\int_\alpha^\beta \varphi d\mu \geq \int_\alpha^\beta \varphi d\underline{\nu}.$$

Assume that $\nu \in \mathcal{B}$ satisfies (11) with $\underline{\varphi}$ $\underline{\nu}$-integrable; then, for all $\mu \in \mathcal{B}$, $\underline{\varphi}$ is μ-integrable and one has:

$$\int_\alpha^\beta f d\mu \geq \int_\alpha^\beta \underline{\varphi} d\mu \geq \int_\alpha^\beta \underline{\varphi} d\underline{\nu} = \int_\alpha^\beta f d\nu$$

so that ν solves (10).

Conversely assume that ν is a solution of (10). Defining $\lambda := \int_\alpha^\beta f d\nu$ and $g := f - \lambda$ we have $\int_\alpha^\beta g d\mu \geq 0$ for all $\mu \in \mathcal{B}$ which can be expressed as:

$$g \in \mathcal{B}^+ := \{h \in C^0([\alpha,\beta],\mathbb{R}) : \int_\alpha^\beta h d\mu \geq 0 \text{ for all } \mu \in \mathcal{B}\}.$$

Now note that $\mathcal{B}^+ = (\mathbb{R}_+\mathcal{B})^+$ and that $\mathbb{R}_+\mathcal{B}$ can be written as:

$$\mathbb{R}_+\mathcal{B} = (C^0_+)^+ \cap \Gamma_0^+$$

where C^0_+ is the cone of nonnegative continuous functions on $[\alpha,\beta]$, where:

$$\Gamma_0 := \{\varphi \text{ concave, continuous on } [\alpha,\beta] : \int_\alpha^\beta \varphi d\underline{\nu} = 0\}$$

and $(C^0_+)^+$ (respectively Γ_0^+) is the closed convex cone consisting of those $\mu \in \mathcal{M}([\alpha,\beta])$ for which $\int_\alpha^\beta h d\mu \geq 0$ for all $h \in C^0_+$ (respectively for all $h \in \Gamma_0$). Hahn-Banach's theorem implies then that $\mathcal{B}^+$ is the closure in the C^0-norm of $C^0_+ + \Gamma_0$. Hence there exists a sequence $(\Lambda_k, \varphi_k, \varepsilon_k) \in (C^0_+ \times \Gamma_0 \times C^0([\alpha,\beta],\mathbb{R}))^{\mathbb{N}}$ such that for all k:

$$g = \varphi_k + \Lambda_k + \varepsilon_k \text{ with } \varepsilon_k \to 0 \text{ uniformly on } [\alpha,\beta]. \tag{12}$$

Since $\Lambda_k \geq 0$, this implies in particular $\varphi_k \leq M$ for all k and some constant M. On the other hand, by Jensen's inequality, and since $\int_\alpha^\beta v d\underline{\nu}(v) = 0$, one has for all k:

$$\varphi_k(0) \geq \int_\alpha^\beta \varphi_k d\underline{\nu} = 0$$

so that $\varphi_k(0)$ is bounded.

Let us prove that φ_k is uniformly bounded on $[\alpha+\varepsilon, \beta-\varepsilon]$ for fixed $\varepsilon \in (0, \min(-\alpha,\beta))$. Note that by concavity one has:

$$\min_{[\alpha+\varepsilon,\beta-\varepsilon]} \varphi_k = \min(\varphi_k(\alpha+\varepsilon), \varphi_k(\beta-\varepsilon)).$$

Assume for instance that $\varphi_k(\alpha+\varepsilon) \to -\infty$ then for k large enough $\varphi_k(\alpha+\varepsilon) < \varphi_k(0)$ so that, by concavity, φ_k is nondecreasing on $[\alpha, \alpha+\varepsilon]$ hence $\varphi_k \leq \varphi_k(\alpha+\varepsilon)$ on $[\alpha,\alpha+\varepsilon]$. By definition of α, the set $\{t \in [0,1] : \underline{\dot{u}}(t) \leq \alpha+\varepsilon\}$ is an interval of the form $[0, t_\varepsilon]$ with $t_\varepsilon > 0$ so that $\underline{\nu}([\alpha,\alpha+\varepsilon]) = t_\varepsilon > 0$, this yields:

$$\int_\alpha^\beta \varphi_k d\underline{\nu} \leq M\underline{\nu}((\alpha+\varepsilon,\beta]) + \varphi_k(\alpha+\varepsilon)\underline{\nu}([\alpha,\alpha+\varepsilon]) \to -\infty$$

a contradiction with $\varphi_k \in \Gamma_0$.

Since each φ_k is concave, and φ_k is uniformly bounded on $[\alpha+\varepsilon/2, \beta-\varepsilon/2]$, φ_k is uniformly Lipschitz on $[\alpha+\varepsilon, \beta-\varepsilon]$, Ascoli's theorem implies then that (up to a subsequence not relabeled), φ_k converges uniformly on $[\alpha+\varepsilon,\beta-\varepsilon]$ to some concave function. By a standard diagonal extraction argument, we may assume that φ_k converges uniformly on compact subsets of (α,β) to some concave function φ. This implies that Λ_k also converges uniformly on compact subsets of (α,β) to some $\Lambda \geq 0$ so that passing to the limit in (12) yields:

$$g = \varphi + \Lambda \geq \varphi \text{ on } (\alpha,\beta)$$

Extending φ (possibly with value $-\infty$) to α and β, it is obvious that $\varphi_k(\alpha)$ and $\varphi_k(\beta)$ converge respectively to $\varphi(\alpha)$ and $\varphi(\beta)$. Since $\varphi_k \leq M$ and $\varphi_k \in \Gamma_0$ for all k, Fatou's Lemma implies that φ is $\underline{\nu}$-integrable and:

$$\int_\alpha^\beta \varphi d\underline{\nu} \geq 0$$

this implies that φ is ν-integrable and since $g \geq \varphi$ and $\nu \in \mathcal{B}$ one has:

$$\int_\alpha^\beta g d\nu = 0 \geq \int_\alpha^\beta \varphi d\nu \geq \int_\alpha^\beta \varphi d\underline{\nu} \tag{13}$$

which proves that

$$\int_\alpha^\beta \varphi d\underline{\nu} = \int_\alpha^\beta \varphi d\nu = 0. \tag{14}$$

Since $g \geq \varphi$ and $\int_\alpha^\beta g d\nu = 0 = \int_\alpha^\beta \varphi d\nu$ we also have $g = \varphi$ ν-a.e..

Finally, defining the concave function $\underline{\varphi} := \varphi + \int_\alpha^\beta f d\nu$ one has $f \geq \underline{\varphi}$ and by (14) $\int_\alpha^\beta \underline{\varphi} d\underline{\nu} = \int f d\nu$ and $f = \underline{\varphi}$ ν-a.e.; this establishes (11). $\square$

Getting back to our initial problem (1), we have:

Corollary 3.2. *$u \in \mathcal{A}$ is a solution of* (1) *if and only if there exists a continuous concave function on (α, β), $\underline{\varphi}$ such that $\underline{\varphi} \circ \underline{\dot{u}}$ is Lebesgue-integrable and:*

$$\begin{aligned} f \geq \underline{\varphi} \text{ on } (\alpha,\beta),\ f(\dot{u}(t)) = \underline{\varphi}(\dot{u}(t)) \text{ a.e. } t \in [0,1]\ , \\ \text{and} \int_0^1 f(\dot{u}(t))dt = \int_0^1 \underline{\varphi}(\underline{\dot{u}}(t))dt. \end{aligned} \tag{15}$$

Remark. Let us remark that if $u \in \mathcal{A}$ is a solution of (1) and if $\underline{\varphi}$ is as in Corollary 3.2, then for every other solution $v \in \mathcal{A}$ of (1) one has $f(\dot{v}) = \underline{\varphi}(\dot{v})$ a.e.. Indeed, one has:

$$\begin{aligned} \int_0^1 f(\dot{u}(t))dt = \int_0^1 f(\dot{v}(t))dt \geq \int_0^1 \underline{\varphi}(\dot{v}(t))dt \\ \geq \int_0^1 \underline{\varphi}(\underline{\dot{u}}(t))dt = \int_0^1 f(\dot{u}(t))dt. \end{aligned}$$

4. Duality

Interpreting φ in Theorem 3.1 (respectively in Corollary 3.2) as a Lagrange multiplier associated to the constraint $\nu \in \mathcal{B}$ (respectively the constraint $u \in \mathcal{A}$), it is natural to look for a (dual) variational characterization of φ.

To that end, let us define:

$$\mathcal{C} := \{\psi : \underline{\nu}\text{-integrable and continuous concave on } (\alpha, \beta)\ \} \tag{16}$$

and

$$L(\mu, \psi) := \int_\alpha^\beta (f - \psi) d\mu + \int_\alpha^\beta \psi d\underline{\nu} \text{ for all } (\mu, \psi) \in \mathcal{M}_+ \times \mathcal{C}. \tag{17}$$

Problem (10) can then be written as the *inf-sup* problem:

$$\inf_{\mu \in \mathcal{M}_+} \sup_{\psi \in \mathcal{C}} L(\mu, \psi). \tag{18}$$

For $\psi \in \mathcal{C}$, one obviously has:

$$\inf_{\mu \in \mathcal{M}_+} L(\mu, \psi) = \begin{cases} \displaystyle\int_\alpha^\beta \psi d\underline{\nu} & \text{if } \psi \leq f \\ -\infty & \text{otherwise.} \end{cases}$$

We then define the *dual* (or *sup-inf*) problem of (10) by:

$$\sup_{\psi\in\mathcal{C},\ \psi\leq f}\int_\alpha^\beta \psi d\underline{\nu}. \tag{19}$$

The multiplier φ in Theorem 3.1 appears as a solution of the dual problem (19). More precisely, one has:

Proposition 4.1. *The following duality relation holds*

$$\inf_{\mu\in\mathcal{B}}\int_\alpha^\beta f d\mu = \sup_{\psi\in\mathcal{C},\ \psi\leq f}\int_\alpha^\beta \psi d\underline{\nu} \tag{20}$$

and both the infimum and the supremum in (20) *are achieved.*

Let $(\nu,\varphi)\in\mathcal{M}_+\times\mathcal{C}$*, then the following statements are equivalent:*

1. $\nu\in\mathcal{B}$ *is a solution of* (10) *and* φ *is a solution of* (19)*,*
2. (ν,φ) *is a saddle-point of* L *on* $\mathcal{M}_+\times\mathcal{C}$*,*
3. $\nu\in\mathcal{B}$*,* $\varphi\leq f$ *on* (α,β)*,* $\varphi=f$ ν*-a.e. and:*

$$\int_\alpha^\beta f d\nu = \int_\alpha^\beta \varphi d\underline{\nu}.$$

Proof. Firstly, it is obvious that:

$$\inf_{\mu\in\mathcal{B}}\int_\alpha^\beta f d\mu \geq \sup_{\psi\in\mathcal{C},\ \psi\leq f}\int_\alpha^\beta \psi d\underline{\nu}.$$

Secondly, let $\nu\in\mathcal{B}$ be a solution of (10), by Theorem 3.1, there exists $\underline{\varphi}\in\mathcal{C}$, $\underline{\varphi}\leq f$ such that

$$\int_\alpha^\beta f d\nu = \int_\alpha^\beta \underline{\varphi} d\underline{\nu}$$

which proves (20) and that $\underline{\varphi}$ is a solution of (19).

Let $(\nu,\varphi)\in\mathcal{M}_+\times\mathcal{C}$, the equivalence between 1. and 2. is classical (see for instance [3]). Let us prove that 3. implies 1.: assume that (ν,φ) satisfies 3., then, by Theorem 3.1, ν is a solution of (10), and since:

$$\int_\alpha^\beta \varphi d\underline{\nu} = \int_\alpha^\beta f d\nu.$$

By (20) we deduce that φ is a solution of (19). Finally, let us assume that $\nu\in\mathcal{B}$ is a solution of (10) and φ is a solution of (19), then $\varphi\leq f$ and (20) yields:

$$\int_\alpha^\beta \varphi d\underline{\nu} = \int_\alpha^\beta f d\nu$$

since $\nu\in\mathcal{B}$ we have:

$$\int_\alpha^\beta \varphi d\nu \geq \int_\alpha^\beta \varphi d\underline{\nu} = \int_\alpha^\beta f d\nu$$

which finally implies $\varphi=f$ ν-a.e.. $\square$

5. Applications and examples

5.1. Remarks on uniqueness and on symmetry

Let us give two simple consequences of our reformulation stated in Proposition 2.5. We start with a generic uniqueness result (the assumptions on the obstacle $\underline{u}$ are the same as in Section 2).

Proposition 5.1. *There exists a G_δ dense subset of $C^0([\alpha,\beta],\mathbb{R})$, X, such that, if $f \in X$ then the problem:*

$$\inf_{u\in\mathcal{A}} \int_0^1 f(\dot{u}(t))dt \tag{21}$$

admits a unique solution.

Proof. By Propositions 2.3 and 2.5, the uniqueness of a minimizer for (21) is equivalent to the uniqueness of a minimizer of $\nu \mapsto \int_\alpha^\beta f d\nu$ on $\mathcal{B}$.

Generic uniqueness is then a consequence of Mazur's Theorem (see [5], see also [8, Theorem 1.20]) which states that the concave and Lipschitz functional

$$f \in C^0([\alpha,\beta],\mathbb{R}) \mapsto \inf_{\nu\in\mathcal{B}} \int_\alpha^\beta f d\nu$$

is Gâteaux-differentiable on a G_δ dense subset of $C^0([\alpha,\beta],\mathbb{R})$ so that the corresponding minimizing measure is unique. □

Proposition 2.5 may also be useful to derive a symmetry result.

Proposition 5.2. *If we assume that f is even and that $\underline{u}$ is symmetric: $\underline{u}(t) = \underline{u}(1-t)$ for all $t \in [0,1]$, then the problem:*

$$\inf_{u\in\mathcal{A}} \int_0^1 f(\dot{u}(t))dt$$

admits a symmetric solution u : $u(t) = u(1-t)$ for all $t \in [0,1]$.

Proof. Once again with Propositions 2.3 and 2.5, we consider the linear problem:

$$\inf_{\nu\in\mathcal{B}} \int_\alpha^\beta f d\nu \tag{22}$$

With the symmetry assumptions of the proposition, one has $\alpha = -\beta$ and $\nu \in \mathcal{B} \Leftrightarrow \widetilde{\nu} \in \mathcal{B}$ where $\widetilde{\nu}$ is defined by:

$$\int_{-\beta}^\beta \varphi(v)d\widetilde{\nu}(v) = \int_{-\beta}^\beta \varphi(-v)d\nu(v), \text{ for all } \varphi \in C^0([-\beta,\beta],\mathbb{R}).$$

Let $\nu \in \mathcal{B}$ be a solution of (22), since f is even, $\widetilde{\nu}$ is also a solution of (22) and so is $\frac{1}{2}(\nu + \widetilde{\nu})$ by linearity. By Proposition 2.5, the convex function u that solves:

$$\nu_{\dot{u}} = \frac{1}{2}(\nu + \widetilde{\nu}),\ u(0) = 0$$

is a minimizer of the initial problem and clearly satisfies the desired symmetry property. □

5.2. Examples

Let us consider:

$$\inf_u J(u) := \int_0^1 \dot{u}^3 \, : \, u \text{ is convex, } u \geq \underline{u}, \; u(0) = u(1) = 0 \tag{23}$$

with:

$$\underline{u}(t) := |t - \frac{1}{2}| - \frac{1}{2}, \text{ for all } t \in [0,1]. \tag{24}$$

Proposition 5.3. *Problem* (23)-(24) *admits a unique solution u given by:*

$$u(t) = \begin{cases} -t & \text{if } t \in [0, 1/3] \\ \frac{1}{2}(t-1) & \text{if } t \in [1/3, 1]. \end{cases}$$

Proof. Let us define the affine function:

$$\underline{\varphi}(v) := \frac{3}{4}v - \frac{1}{4} \tag{25}$$

It is immediate to check that $\underline{\varphi}(v) \leq v^3$ for all $v \in [-1, 1]$ and that $\underline{\varphi}(\dot{u}) = \dot{u}^3$ a.e.. Since $\underline{\varphi}$ is affine, $\int_0^1 \dot{u}^3 = \int_0^1 \underline{\varphi}(\dot{u}) = \int_0^1 \underline{\varphi}(\dot{\underline{u}})$. By Corollary 3.2, u is then a solution of (23)-(24).

Let us finally prove uniqueness: if w is a solution of (23)-(24), then $\underline{\varphi}(\dot{w}) = \dot{w}^3$ a.e. (see the remark after Corollary 3.2) with $\underline{\varphi}$ defined by (25). Hence $\dot{w} \in \{-1, 1/2\}$ a.e. together with the convexity of w and $w(0) = w(1) = 0$ this implies $w \equiv u$. □

Let us now consider the same functional $u \mapsto \int_0^1 \dot{u}^3$ as in (23) but with a strictly convex obstacle:

$$\underline{u}(t) := t^2 - t, \text{ for all } t \in [0,1]. \tag{26}$$

Then we have:

Proposition 5.4. *Problem* (23)-(26) *admits a unique solution u given by:*

$$u(t) = \begin{cases} t^2 - t & \text{if } t \in [0, 1/4] \\ \frac{1}{4}(t-1) & \text{if } t \in [1/4, 1]. \end{cases}$$

Proof. Let us define the concave function:

$$\underline{\varphi}(v) := \begin{cases} v^3 & \text{if } v \in [-1, -1/2] \\ \frac{3}{16}v - \frac{1}{32} & \text{if } v \in [-1/2, 1]. \end{cases} \tag{27}$$

It is immediate to check that $\underline{\varphi}(v) \leq v^3$ for all $v \in [-1, 1]$ and that $\underline{\varphi}(\dot{u}) = \dot{u}^3$ a.e.. One readily checks $\int_0^1 \dot{u}^3 = \int_0^1 \underline{\varphi}(\dot{u}) = \int_0^1 \underline{\varphi}(\dot{\underline{u}})$. By Corollary 3.2, u is then a solution of (23)-(26).

Let us finally prove uniqueness: if w is a solution of (23)-(26), then $\underline{\varphi}(\dot{w}) = \dot{w}^3$ a.e. with $\underline{\varphi}$ defined by (27). Hence $\dot{w} \in [-1, -1/2] \cup \{1/4\}$ a.e.. Together with the convexity of w, $w \geq \underline{u}$, $w(0) = w(1) = 0$ and Lemma 2.2 this implies $w \equiv u$. □

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